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Torsion of Circular Sections

TABLE OF CONTENTS

	Page
§7.1. Introduction	7–3
§7.2. Problem Description	7–3
§7.2.1. Terminology, Notation, Coordinate Systems	7–3
§7.2.2. Deformation of Torqued Shaft	7–5
§7.2.3. Stress and Strain Distributions	7–5
§7.3. The Stress Formula	7–6
§7.4. The Twist Rate And Angle Formulas	7–7
§7.5. Torsion Failure Modes	7–8
§7.5.1. Principal Stresses and Maximum Shear	7–8
§7.5.2. Material Failure: Ductile Versus Brittle	7–9

§7.1. Introduction

The present lecture begins a set of three classes on *torsion*. This topic is presented now to provide timely theoretical support for the first experimental Lab, which will be demo'd on Friday Sept 21. The first lecture covers circular cross sections. These can be treated by elementary (Mechanics of Materials) methods that nonetheless provide the exact elasticity solution. The next two lectures deal with Thin Wall (TW) cross sections of important in Aerospace. For these approximate methods are necessary.

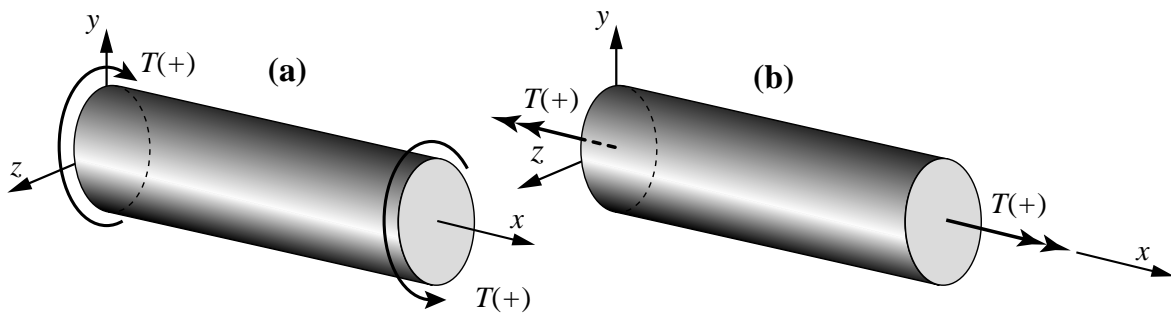


FIGURE 7.1. Torsion of a circular shaft: (a) Single-headed arrow (a.k.a. “harpoon”) representation and (b) double-headed arrow vector representation for torque. Both symbols are used in this course.

§7.2. Problem Description

This Section introduces the problem of a torqued member (shaft) of circular cross section (solid or annular), and presents the necessary terminology.

§7.2.1. Terminology, Notation, Coordinate Systems

We consider a bar-like, straight, prismatic structural member of constant cross section subjected to applied torque T about its long axis, as pictured in Figure 7.1(a). This kind of member, designed primarily to transmit torque, is called a *shaft*. Axis x is directed along the longitudinal shaft dimension. The applied torque is shown as a moment about that axis. It is positive when it acts as indicated, complying with the right-hand screw rule along the x direction. Axes y and z lie on a cross section conventionally taken as origin.

Figure 7.1(b) shows an alternative picture of the torque as a double-headed arrow vector. We will use this representation whenever it leads to a more convenient visualization.

An important use of shafts is to *transmit power* between parallel planes, as in automobile power trains, electrical machinery, aircraft engines or helicopter rotors. See Figure 7.2.

In this lecture we restrict consideration to shafts of circular cross section that may be either solid or hollow (annular, tubular), as pictured in Figure 7.3(a,b). If the shaft is solid, its radius is R . If hollow, the exterior and interior radius are denoted by R_e and R_i , respectively.

In addition to the (x, y, z) Rectangular Cartesian Coordinate (RCC) system of Figure 7.1 we shall also use often the cylindrical coordinate system (x, ρ, θ) depicted in Figure 7.3(c) for a typical

Lecture 7: TORSION OF CIRCULAR SECTIONS

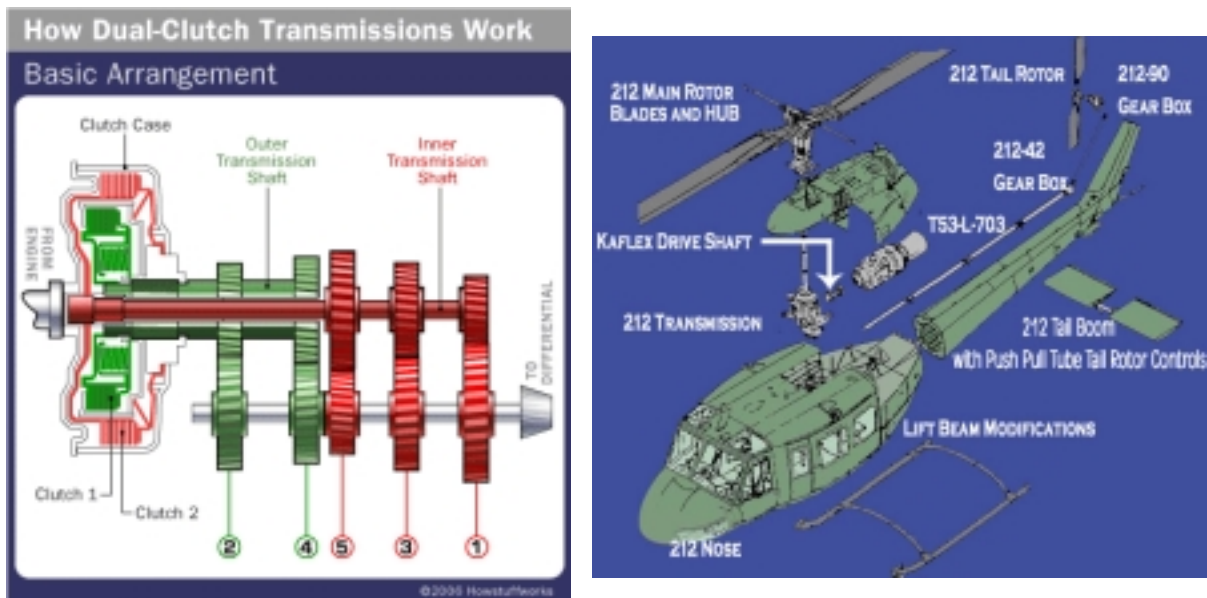
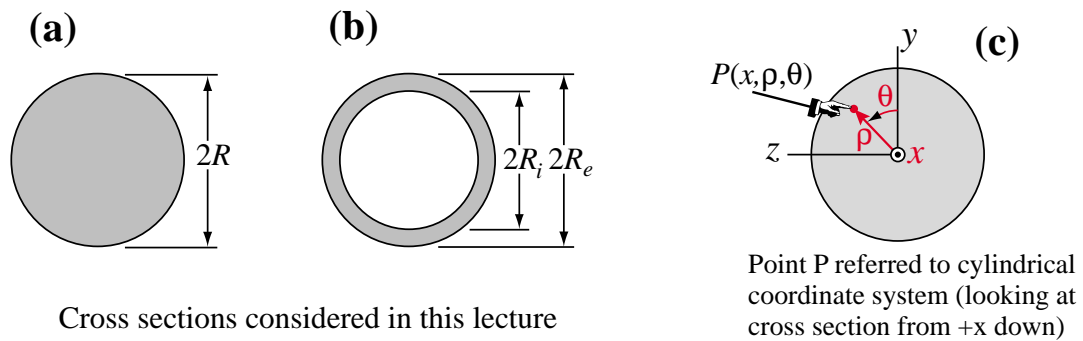


FIGURE 7.2. Transferring power between planes via torqued shafts.



Cross sections considered in this lecture

FIGURE 7.3. Circular cross sections considered in this Lecture, and coordinate systems.

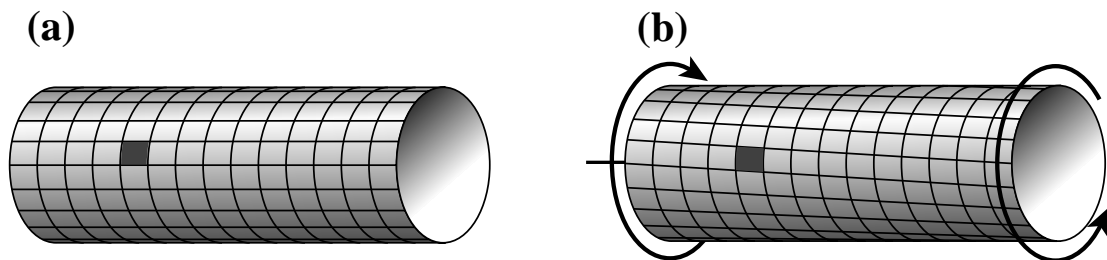


FIGURE 7.4. Visualizing deformations produced by torque.

cross section. The position coordinates of an arbitrary shaft point P are x, y, z in RCC and x, ρ, θ in cylindrical coordinates. Here ρ is the radial coordinate (not a happy notation choice because of potential confusion with density, but it is the symbol used in most Mechanics of Materials textbooks) and θ the angular coordinate, taken positive CCW from y as shown. Note that

$$y = \rho \cos \theta, \quad z = \rho \sin \theta. \quad (7.1)$$

§7.2.2. Deformation of Torqued Shaft

To visualize what happens when a circular shaft is torqued, consider first the undeformed state under zero torque. A rectangular-like mesh is marked on the surface as shown in Figure 7.4(a). The rectangles have sides parallel to the longitudinal axis x and the circumferential direction θ .

Apply torque to the end sections. Rectangles marked in Figure 7.4(b) shear into parallelograms, as illustrated in Figure 7.4(b). This diagram displays the main effect: shear stresses and associated shear strains develop over the "cylinders" $\rho = \text{constant}$. Next subsection goes into more details.

§7.2.3. Stress and Strain Distributions

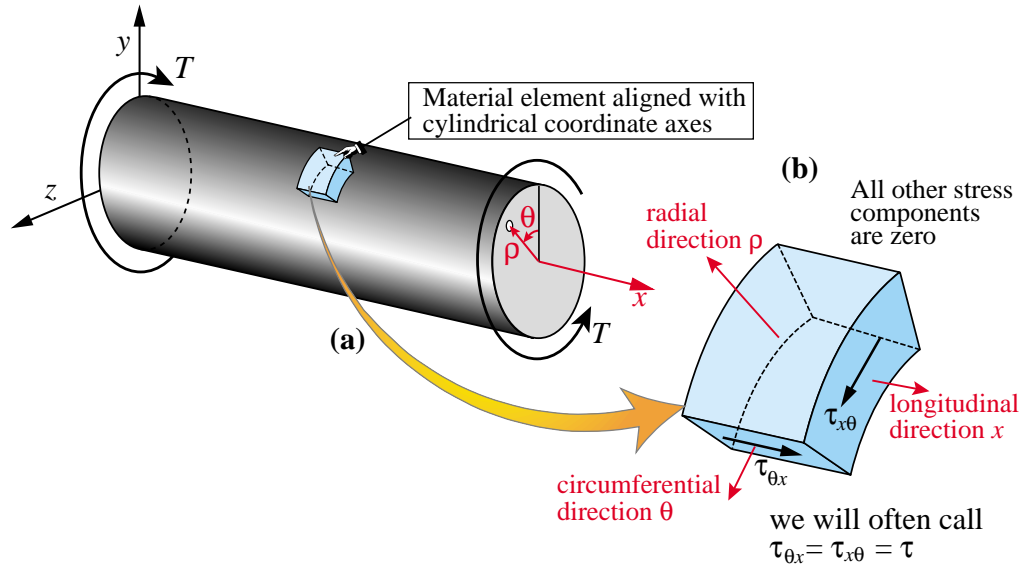


FIGURE 7.5. Extracting a (x, ρ, θ) -aligned material element. The only nonzero stresses are $\tau_{\theta x} = \tau_{x\theta}$.

A material element aligned with the cylindrical coordinate axes is cut out as shown in Figure 7.5(a). The element is magnified in Figure 7.5(b), which plainly shows that the only nonzero stress component on its faces is the shear stress $\tau_{x\theta} = \tau_{\theta x}$. It follows that the stress and strain matrices referred to the cylindrical coordinate system have the form

$$\begin{bmatrix} 0 & \tau_{x\theta} & 0 \\ \tau_{\theta x} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Leftarrow \text{Hooke's law} \Rightarrow \begin{bmatrix} 0 & \gamma_{x\theta} & 0 \\ \gamma_{\theta x} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (7.2)$$

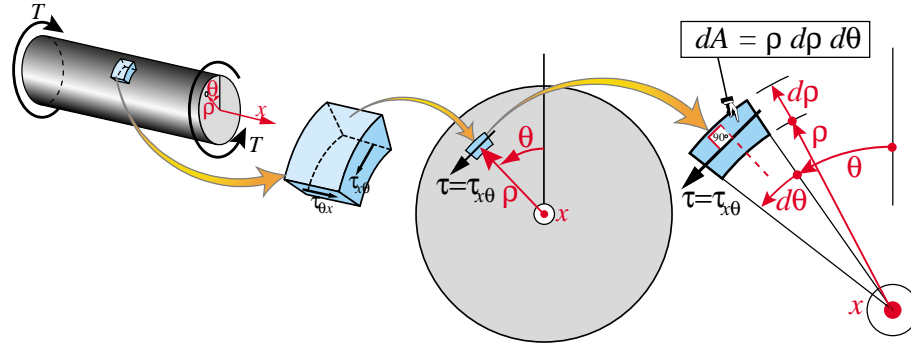


FIGURE 7.6. Integration over the cross section of the elementary moment produced by the shear stress component $\tau = \tau_{x\theta}$ will balance the internal torque T .

The axes have been reordered as (x, θ, ρ) for convenience in identifying with plane stress and strain later. In the sequel we often call $\tau = \tau_{x\theta} = \tau_{\theta x}$, and $\gamma = \gamma_{x\theta} = \gamma_{\theta x}$ to reduce subscript clutter. Stresses and strains in (7.2) are connected at each material point by Hooke's shear law

$$\tau = G \gamma, \quad \gamma = \frac{\tau}{G}, \quad (7.3)$$

in which G is the shear modulus. Because of circular symmetry τ and γ do not depend on θ . They may depend on x if either the cross section radius or the internal torque change along the shaft length, but that variation will not be generally made explicit. We will assume (subject to *a posteriori* verification) that τ and γ depend linearly on ρ , and we scale that dependence as follows:

$$\boxed{\tau = \frac{\rho}{R} \tau_{max}, \quad \gamma = \frac{\rho}{R} \gamma_{max}} \quad (7.4)$$

Here τ_{max} and γ_{max} are maximum values of the shear stress and strain, respectively, over the cross section. These occur at $\rho = R$, the radius of the shaft. (For an annular shaft, $R = R_e$, the exterior or outer radius)

§7.3. The Stress Formula

Call T the internal torque acting on a cross section $x = \text{constant}$. The shear stress $\tau = \tau_{x\theta}$ acting on an elementary cross section area dA produces an elementary force $dF = \tau dA$ and an elementary moment $dM = dF \rho = (\tau dA) \rho$ about the x axis. See Figure 7.6 for the geometric details. Integration of the elementary moment over the cross section must balance the internal torque:

$$\begin{aligned} T &= \int_A (\text{shear stress} \times \text{elemental area}) \times \text{lever-arm} \\ &= \int_A (\tau dA) \rho = \int_A \frac{\rho}{R} \tau_{max} \rho dA = \frac{\tau_{max}}{R} \int_A \rho^2 dA \\ &= \frac{\tau_{max}}{R} \int_A \rho^2 (\rho d\rho d\theta) = \frac{\tau_{max}}{R} \int_A \rho^3 d\rho d\theta = \frac{\tau_{max}}{R} J, \end{aligned} \quad (7.5)$$

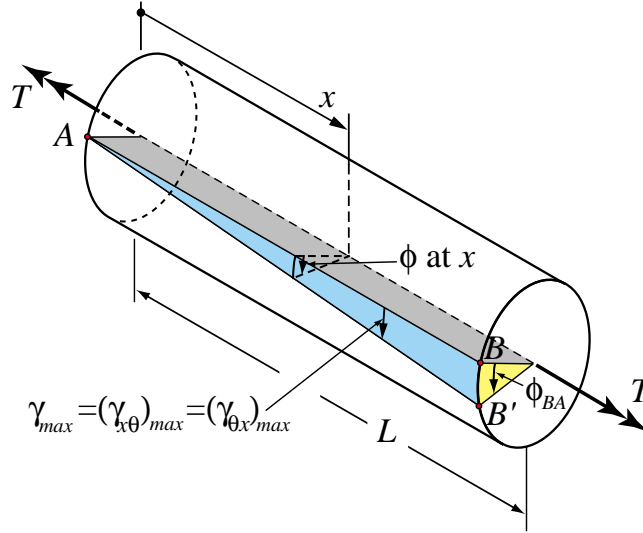


FIGURE 7.7. Twist deformation of a torqued circular shaft. Deformations and rotations greatly exaggerated for visualization convenience.

in which

$$J = \int_A \rho^2 dA = \int_A \rho^3 d\rho d\theta, \quad (7.6)$$

is the *polar moment of inertia* of the cross section about its center. Solving for τ_{max} gives the **stress formula**:

$$\tau_{max} = \frac{T R}{J}, \quad \tau = \frac{\rho}{R} \tau_{max} = \frac{T \rho}{J} \quad (7.7)$$

For a solid circular section of radius R , $J = \frac{1}{2} \pi R^4$. For an annular cross section of exterior radius R_e and interior radius R_i , $J = \frac{1}{2} \pi (R_e^4 - R_i^4)$.

§7.4. The Twist Rate And Angle Formulas

A torqued shaft produces a relative rotation of one end respect to the other; see visualization in Figure 7.4(b). This relative rotation is called the *twist angle* ϕ . The twist angle per unit of x -length is called the *twist rate* $d\phi/dx$. Often the design called for limitation on this rate or angle. It is thus of interest to have expressions for both of them in terms of the applied torque and the properties of the shaft. Geometric quantities used for working out those expressions are defined in Figure 7.7.

Note that *infinitesimal* strains and rotations are assumed; deformations depicted in Figure 7.7 are greatly exaggerated for visibility convenience. For small angles $BB' \approx R \phi_{BA} \approx L \gamma_{max}$. At an arbitrary distance x from the fixed end, $R \phi \approx x \gamma_{max}$ whence $\phi = x \gamma_{max}/R$. Consequently the twist rate is

$$\frac{d\phi}{dx} = \frac{\gamma_{max}}{R} = \frac{\gamma}{\rho} \quad \text{so} \quad \gamma = \rho \frac{d\phi}{dx} \quad (7.8)$$

But according to Hooke's law (7.3)

$$\gamma = \frac{\tau}{G} = \frac{T \rho}{G J} = \rho \frac{d\phi}{dx} \quad (7.9)$$

Lecture 7: TORSION OF CIRCULAR SECTIONS

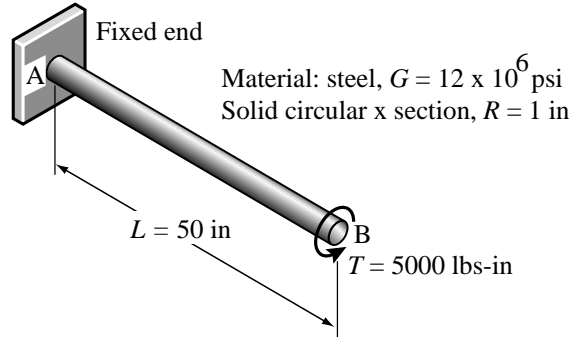


FIGURE 7.8. Example calculation of maximum shear stress and twist angle.

which yields

$$\frac{d\phi}{dx} = \frac{T}{GJ} \quad (7.10)$$

This is the **twist rate formula**. To find the twist angle ϕ_{BA} , where subscripts identify the angle measurement endpoints, integrate along the length of the shaft:

$$\phi_{BA} = \phi_B - \phi_A = \phi_B - 0 = \phi_B = \int_0^L d\phi = \int_0^L \frac{d\phi}{dx} dx = \int_0^L \frac{T}{GJ} dx. \quad (7.11)$$

If T , G and J are constant along the shaft:

$$\phi_{BA} = \frac{T}{GJ} \int_0^L dx = \frac{TL}{GJ} \quad (7.12)$$

This is the **twist angle formula** for a *prismatic* shaft.

Example 7.1. Data for this example is provided in Figure 7.8. Find: maximum shear stress τ_{max} , maximum shear strain γ_{max} and relative twist angle ϕ_{BA} .

$$\tau_{max} = \frac{TR}{J} = \frac{TR}{\frac{1}{2}\pi R^4} = \frac{5000}{1.57} = 3183 \text{ psi} \quad (7.13)$$

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{3200 \text{ psi}}{12 \times 10^6 \text{ psi}} = 265 \mu \text{ (microradians)} \quad (7.14)$$

$$\phi_{BA} = \frac{TL}{GJ} = \frac{5000 \times 50}{12 \times 10^6 \times (\frac{1}{2}\pi 1^4)} = 0.0133 \text{ rad} = 0.76^\circ \quad (7.15)$$

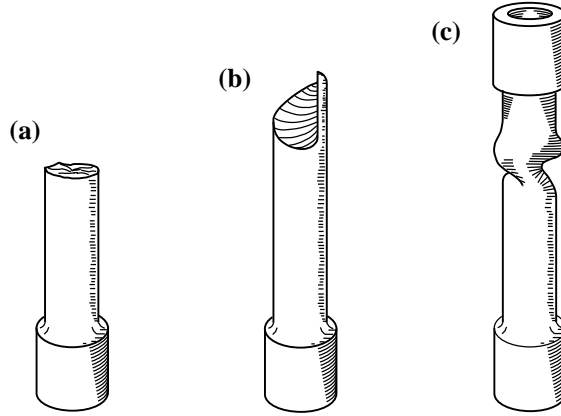


FIGURE 7.9. Three failure modes of a torqued specimen: (a,b): material failure in solid shaft, (c): wall buckling in thin walled tubular shaft.

§7.5. Torsion Failure Modes

Shafts should be designed against possible *failure modes*. Three practically important ones are sketched in Figure 7.9. Specimen (a) and (b) display two classical *material failure* modes in a solid shaft, whereas (c) illustrates a wall buckling failure that may occur for *hollow* shafts if the wall is too thin. Only the first two modes are discussed here, since the last one requires knowledge of shell buckling, which falls outside of the scope of this course.

§7.5.1. Principal Stresses and Maximum Shear

As discussed in §7.2.2, the stress matrix at a point $P(x, \rho, \theta)$ referred to a cylindrical coordinate system with axes reordered as (x, θ, ρ) takes the simple form

$$\begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tau = \tau_{\theta x} = \tau_{x\theta} = \tau_{max} \frac{\rho}{R}, \quad \tau_{max} = \frac{T R}{J}. \quad (7.16)$$

We can observe that all points are in *plate stress* because all stresses with a ρ subscript vanish, although the shaft is not a thin plate. It is also in *plane strain* because the transverse strains (that is, strain components with a ρ subscript): $\epsilon_{\rho\rho}$, $\gamma_{x\rho}$ and $\gamma_{\theta\rho}$, are zero.

Since the shear stress is proportional to ρ , we consider a point on the outer surface $\rho = R$, where $\tau = \tau_{max}$. The Mohr's circle for stresses in the (x, θ) plane passing through $P(x, R, \theta)$ is shown in Figure 7.10(a). By inspection we see that

- (a) The principal stresses are $\sigma_1 = \tau_{max}$ (tension) and $\sigma_2 = -\tau_{max}$ (compression). These occur at $\pm 45^\circ$ from the longitudinal x direction.
- (b) The maximum inplane shear is τ_{max} , which occurs on the shaft cross sections.

Consideration of 3D effects does not change these findings because the zero principal stress is the *intermediate* one. Thus the circle plotted in Figure 7.10(a) is the *outer* one, and the maximum overall shear stress $\tau_{max}^{overall}$ is the same as the maximum inplane shear stress, that is, $\tau_{max}^{inplane}$.

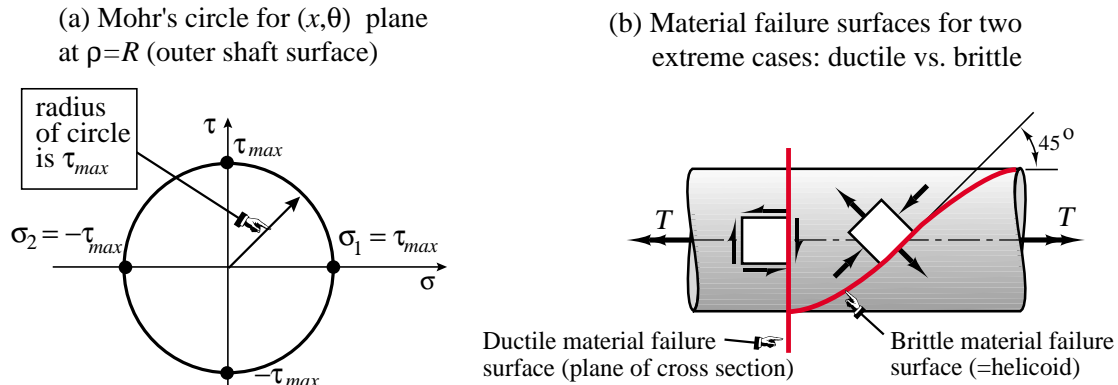


FIGURE 7.10. By inspection of the Mohr circle, maximum normal and shear stresses at a point on the outer shaft surface determine the kind of material failure.

§7.5.2. Material Failure: Ductile Versus Brittle

We are now in a position to explain the two material failure modes shown in Figure 7.9(a,b).

If the material is *ductile*, it fails by *maximum shear* τ_{max} reaching yield, followed by post-yield plastic flow. The failure surface is the cross section, as sketched in Figure 7.9(a). The picture in 7.11(a) shows a mild steel specimen taken to failure by torque.

If the material is *brittle*, it fails by *maximum tensile stress*, which is $\sigma_1 = \tau_{max}$. Since this occurs at 45° from the longitudinal direction, the fracture is more complicated and will look (at least near the shaft outer surface) like a helicoid, as sketched in Figure 7.9(b). The photographs in Figure 7.11 show two fractured specimen fabricated of brittle materials: cast iron and chalk, respectively.



FIGURE 7.11. Actual material failure observed in torqued specimen.

If the material is not isotropic, the failure process can be more involved than the two extreme cases discussed above. For example, wood and laminate composites may fail by delamination across layers or fibers.