Review of Continuum Mechanics: Kinematics
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§7.1. Introduction

Chapters 3 through 6 have covered general properties of the governing force residual equations of geometrically nonlinear structural systems with finite number of degrees of freedom (DOF). The DOFs are collected in the state vector, and driven by control parameters. The residual equations, being algebraic, are well suited for numerical computations.

Continuum models of actual structures, however, are expressed as ordinary or partial differential equations in space or space-time. As such, those models possess an infinite number of DOFs. Except for simple (typically linear) models, they cannot be directly solved analytically. The reduction to a finite number is accomplished by discretization methods. It was observed in Chapter 1 that for nonlinear problems in solid and structural mechanics the finite element method (FEM) is the most widely used discretization method.

This Chapter and the next one provide theoretical background for deriving geometrically nonlinear finite elements from continuum models. Collectively they give an overview of kinematic, kinetic and constitutive relations for a three-dimensional elastic deformable body, as needed in subsequent Chapters. Readers familiar with continuum mechanics may skim this and the next Chapter to get familiar with the notation.

§7.2. Notational Systems

Continuum mechanics deals with vector and tensor fields such as displacements, strains and stresses. Four notational systems are in common use.

Indicial Notation. Also called component notation. The key concept is that of an index. Indices identify components of vectors and tensors. It has convenient abbreviation rules, such as commas for partial derivatives and Einstein’s summation convention. The notation is general and powerful, and as such is preferred in analytic developments as well as publication in theory journals and monographs. It readily handles arbitrary tensors of any order, curvilinear coordinate systems and nonlinear expressions. When used in non-Cartesian coordinates, it sharply distinguishes between covariant and contravariant quantities. The main disadvantages are: (i) physics is concealed behind the index jungle, and (ii) highly inefficient for expressing numeric computations. Because of (i), it is not suitable for first-level (undergraduate) instruction.

Direct Notation. Sometimes called algebraic notation. Vectors and tensors are represented by single symbols, usually bold letters. These are linked by the well known operators of mathematical physics, such as . for dot product, × for cross products, and ∇ for gradient (or divergence). Has the advantage of compactness and quick visualization of intrinsic properties. Certain operations, however, become undefined beyond a certain range. Some of them overlap with matrix algebra while others do not. This fuzzyness can lead to confusion in computational work.

Matrix Notation. This is similar to the previous one, but entities are appropriately recast so that only matrix operations are used. This form can be directly mapped to discrete equations as well as high-level, matrix-oriented programming languages such as Matlab. It has the disadvantage of losing contact with the original physical entities along the way. For example, stress is a symmetric second-order tensor, but is recast as a 6-vector for FEM developments. This “covnience” change may forgo essential properties. For instance, it makes sense to say that principal stresses are
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Eigenvalues of the stress tensor. But the statement becomes nonsense when stresses are recast as vectors.\(^1\)

**Full Notation.** In full form notation every term is spelled out. No ambiguities of interpretation can arise. This works well as a notation of last resort, and also as a “comparison template” against which one can check out the meaning of more compact expressions. It is also useful for programming in low-order languages.

In this and following Chapters the direct, matrix and full notations are preferred. The indicial notation is used if either complicated tensor forms are needed, or nonlinear expressions not amenable to other notations appear. Often the expression is first given in direct form and confirmed by full form if feasible. It is then mapped to matrix notation for use in FEM developments. The decision chain leads to possible ambiguities against reuse of vector symbols in two contexts: continuum mechanics and FEM discretizations. Such ambiguities are resolved in favor of keeping FEM notation simple.

**Example 7.1.** Consider the well known dot product between two physical vectors in 3D space, \(a = (a_1, a_2, a_3)\) and \(b = (b_1, b_2, b_3)\) written in the four different notations:

\[
a_i b_j = a \cdot b = a^T b = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3.
\] (7.1)

**Example 7.2.** Take the internal static equilibrium equations of a continuum body, expressed in terms of Cauchy stresses and body forces per unit volume:

\[
\sigma_{ij,j} + b_i = 0, \quad \nabla \sigma + b = 0, \quad D^T \sigma + b = 0, \quad \begin{pmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + b_1 = 0, \text{ plus 2 more.} \end{pmatrix}
\]  (7.2)

In the matrix version, \(\sigma\) denotes the stress tensor cast as a 6-vector (in the sequel this is simply called \(\sigma\)).

**Example 7.3.** The strain-to-stress constitutive equations of a linearly elastic solid, written in a RCC frame and excluding initial strain effects, can be written as the generalized Hooke’s law

\[
\sigma_{ij} = E_{ijkl} e_{kl} \begin{pmatrix} \sigma_{11} = E_{1111} e_{11} + E_{1112} e_{12} + \ldots & \text{9 equations, 81 terms.} \end{pmatrix}
\]  (7.3)

Here the indicial and matrix forms are compact. The latter, however, requires that stresses \(\sigma_{ij}\) and strains \(e_{kl}\) be cast as 6-vectors while the elastic moduli tensor \(E_{ijkl}\) is cast to a 6 \(\times\) 6 matrix. The direct form requires the introduction of an ad-hoc operator that effects the product of a 4-dimensional tensor times a second order one; this varies from author to author. The full form will take up at least half a page.

**Example 7.4.** If a discrete mechanical system is conservative, it was shown in Chapter 6 that the total force residual is the gradient of a total potential energy function with respect to the state:

\[
r_i = \frac{\partial \Pi}{\partial u_i} = \Pi_{,i}, \quad \nabla \Pi = \begin{pmatrix} \frac{\partial \Pi}{\partial u_1} \\ \frac{\partial \Pi}{\partial u_2} \end{pmatrix}, \quad r_i = \frac{\partial \Pi}{\partial u_i}, r_1 = \frac{\partial \Pi}{\partial u_1}, r_2 = \frac{\partial \Pi}{\partial u_2}, \ldots
\]  (7.4)

The indicial form requires defining \(\Pi_{,i}\) as abbreviation of partial derivative with respect to \(u_i\) rather than \(x_i\). The direct form depends on the gradient operator symbol chosen. For example, \(\mathbf{r} = \mathbf{grad} \Pi\) is also used.

---

\(^1\) Stilted authors make a big deal about distinguishing between tensors, which are entities independent of any coordinate system, and their matrix and indicial representations, which are not. This is largely an exercise in peripatetic angelology, à la “How many angels can dance on the head of a pin?” Such scholastic preoccupations are eschewed here.
§7.3. The Continuum Model

In this and the next Chapter the structure is mathematically treated as a continuum body $B$. In this model, the body is considered as being formed by a set of points $P$ called particles, which are endowed with certain mechanical properties. For FEM analysis the body is divided into elements, which inherit the properties of the continuum model.

§7.3.1. Particle Motion

Particles displace or move in response to external actions, such as forces, characterized by the control parameters $\Lambda_i$ introduced in Chapter 3. Following the reduction process discussed there, in each stage the body responds to the single stage parameter $\lambda$.

A one-parameter series of positions occupied by the particles as they move in space is called a motion. The motion may be described by the displacement $\mathbf{u}(P) \equiv \mathbf{u}(\mathbf{x})$ of the particles\(^2\) with respect to a base or reference state in which particle $P$ is labeled $P_0$. The displacements of all particles $\mathbf{u}(\mathbf{x})$ such that $\mathbf{x} \equiv \{x, y, z\} \in B$, constitutes the displacement field.

The motion is said to be kinematically admissible if:

1. Continuity of particle positions is preserved so that no gaps or interpenetration occurs.\(^3\)
2. Kinematic constraints on the motion (for example, support conditions) are preserved.

A kinematically admissible motion along a stage will be called a stage motion. For one such motion the displacements $\mathbf{u}(\mathbf{x})$ characterize the state and the stage control parameter $\lambda$ characterizes the control. Both will be generally parametrized by the pseudo-time $t$ introduced in Chapter 3. Thus a stage motion can be generally represented by

$$
\lambda = \lambda(t), \quad \mathbf{u} = \mathbf{u}(\mathbf{x}, t), \quad \mathbf{x} \in B.
$$

§7.3.2. Configurations

If in (7.5) we freeze $t$, we have a configuration of the structure. Thus a configuration is formally the union of state and control. It may be informally viewed as a “snapshot” taken of the structure and its environment when the pseudotime is frozen. If the configuration satisfies the equilibrium equations, it is called an equilibrium configuration. In general, however, a randomly given configuration is not in equilibrium unless artificial body and surface forces are applied to it.

A staged response, or simply response, can be now mathematically defined as a series of equilibrium configurations obtained as $\lambda$ is continuously varied, starting from zero.

---

\(^2\) The underlining in $\mathbf{u}$ is used to distinguish the physical (particle level) displacement vector from the finite element node displacement array, which is a computational vector.

\(^3\) The mathematical statement of these compatibility conditions is quite complicated for finite displacements and will not be given here. See for example, [828, §34]. The displacement-based finite element formulations worked out in subsequent Chapters will automatically satisfy the requirement.
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<table>
<thead>
<tr>
<th>Name</th>
<th>Alias</th>
<th>Definition</th>
<th>Equilibrium Required?</th>
<th>Identification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Admissible</td>
<td>A kinematically admissible configuration</td>
<td>No</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>Perturbed</td>
<td>Kinematically admissible variation of an admissible configuration</td>
<td>No</td>
<td>C + δC</td>
<td></td>
</tr>
<tr>
<td>Current</td>
<td>Deformed Spatial</td>
<td>Any admissible configuration taken during the analysis process. Contains all others as special cases</td>
<td>No</td>
<td>C or C(t)</td>
</tr>
<tr>
<td>Base</td>
<td>Initial Undeformed Material</td>
<td>The configuration defined as the origin of displacements. Strain free but not necessarily stress free</td>
<td>Yes</td>
<td>C₀, C₀ or C(0)</td>
</tr>
<tr>
<td>Reference</td>
<td>Configuration to which stepping computations in an incremental solution process are referred</td>
<td>TL,UL: yes, CR: C₀ no, C₀ yes</td>
<td>TL: C₀, UL: C₀-1 CR: C₀ and C₀</td>
<td></td>
</tr>
<tr>
<td>Iterated</td>
<td>Configuration taken at the kth iteration of the nth increment step</td>
<td>No</td>
<td>Cₙ,k</td>
<td></td>
</tr>
<tr>
<td>Target</td>
<td>Equilibrium configuration accepted after completing the nth increment step</td>
<td>Yes</td>
<td>Cₙ</td>
<td></td>
</tr>
<tr>
<td>Corotated</td>
<td>Shadow Ghost</td>
<td>Body- or element-attached configuration obtained from C₀ through a RBM (CR description only)</td>
<td>No</td>
<td>Cₙ</td>
</tr>
<tr>
<td>Aligned</td>
<td>Preferred Directed</td>
<td>A fictitious body or element configuration aligned with a particular set of axes (usually global axes)</td>
<td>No</td>
<td>Cₐ</td>
</tr>
</tbody>
</table>

Definitions with blue background are used only in theoretical and applied mechanics for analytical formulations. Definitions with yellow background are only used in computational mechanics. Definitions with green background are used in both, hence the color choice. The meaning of reference configuration, however, may differ. The one stated above is for computational mechanics.

The base configuration C₀ is often the same as the natural state in which body (or element) is undeformed and stress free.

In dynamic analysis using the CR kinematic description, C₀ and Cₙ are called the inertial and dynamic reference configurations, respectively, when applied to an entire structure such as an airplane (e.g., autopilot simulations).

Figure 7.1. Distinguished configurations in geometrically nonlinear analysis.

§7.3.3. Distinguished Configurations

A distinctive feature of geometrically nonlinear analysis is the need to carefully distinguish among different configurations of the structure. As defined above, a set of kinematically admissible displacements \( \mathbf{u}(\mathbf{x}) \) plus a staged control parameter \( \lambda \) at a frozen \( t \) defines a configuration.

This is not necessarily an equilibrium configuration. In fact it will not usually be one. It is important to realize that an equilibrium configuration is not necessarily a physical configuration assumed by the actual structure.\(^4\) Configurations that are important in geometrically nonlinear analysis receive special qualifiers:

*admissible, perturbed, deformed, base, reference, iterated, target, corotated, aligned*

This terminology is collected in Figure 7.1 in tabular format. Of the nine instances listed there,

\(^4\) Recall the suspension bridge under zero gravity of Chapter 3.
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7.3 THE CONTINUUM MODEL

Current Configuration  
C or C(0)

Reference Configuration  
(identifier depends on kinematic description chosen)

Base Configuration  
C₀ or C(0)

Figure 7.2. The three most important configurations for geometrically nonlinear analysis. Specialization to the kinematic descriptions tabulated in Figure 7.3 are pictured in Figures 7.4, 7.5, and 7.6.

The first two are used extensively in theoretical and applied mechanics, the last four exclusively in computational mechanics, and three (current, base, and reference) used in both, although sometimes with different meanings.

The three most important configurations insofar as a FEM implementation is concerned, are: base, reference and current. These are pictured in Figure 7.2. An examination of this figure indicates that the choice of the reference configuration depends primarily on the kinematic description chosen. Such a choice is examined in the next subsection.

Remark 7.1. Many names can be found for the configurations pictured in Figure 7.1 in the literature dealing with finite elements and continuum mechanics. Here are some of those alternative names.

Perturbed configuration: adjacent, deviated, disturbed, incremented, neighboring, varied, virtual.
Current configuration: deformed, distorted, pull-forward, moving, present, space, spatial, varying.
Base configuration: baseline, body, initial, global, laboratory, material, natural, original, overall, undeformed, undistorted.
Reference configuration: fixed, frozen, known, pull-back.
Iterated configuration: corrected, intermediate, stepped, transient, transitory.
Target configuration: converged, equilibrated spatial, unknown.
Corotated configuration: attached, convected, ghost, phantom, shadow.
Aligned configuration: directed, body-matched, body-fitted, preferred.

§7.3.4. Kinematic Descriptions

Three kinematic descriptions of geometrically nonlinear finite element analysis are in current use in programs that solve nonlinear structural problems: Total Lagrangian or TL, Updated Lagrangian or UL, and Corotational or CR. They are described in Figure 7.3. They can be distinguished by the choice of reference configuration. The important configurations for these three descriptions are pictured in Figures 7.4, 7.5, and 7.6, respectively.

The TL formulation remains the most widely used in continuum-based FEM codes.⁵ The CR

⁵ A key reason is historic: three of the original nonlinear FEM codes: MARC, ABAQUS and ANSYS, originally implemented elements based on that description.
<table>
<thead>
<tr>
<th><strong>Name</strong></th>
<th><strong>Acronym</strong></th>
<th><strong>Definition</strong></th>
<th><strong>Primary applications</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Lagrangian</td>
<td>TL</td>
<td>Base and reference configurations coalesce and remain fixed throughout the solution process</td>
<td>Solid and structural mechanics with finite but moderate displacements and strains. Primarily used for elastic materials. Unreliable for flow-like behavior or topology changes</td>
</tr>
<tr>
<td>Updated Lagrangian</td>
<td>UL</td>
<td>Base configuration remains fixed but reference configuration is periodically. updated. Common update strategies: either set reference configuration to current solution, or to last converged solution</td>
<td>Solid and structural mechanics with finite displacements and possibly large strains. Handles material flow-like behavior well, (e.g., forming processes) as well as topology changes (fracture)</td>
</tr>
<tr>
<td>Corotational</td>
<td>CR</td>
<td>Reference configuration is split into base and corotated. Strains and stresses are measured from corotated to current, while base configuration is maintained as reference to measure rigid body motions</td>
<td>Solid and structural mechanics with arbitrarily large finite motions, but small strains and elastic material behavior. Extendible to nonlinear materials if inelasticity is localized so most of structure stays elastic.</td>
</tr>
</tbody>
</table>

TL and CR are Lagrangian: computations are always referred to a previous configuration (base and/or reference). UL is either Eulerian or a Lagrangian-Eulerian hybrid. Eulerian descriptions are popular in fluid mechanics, but less so in solid and structural mechanics unless the behavior is fluid-like (e.g., metal forming, nonlinear viscoelasticity).

**Figure 7.3.** Kinematic descriptions used in FEM programs that handle geometrically nonlinear problems in solid and structural mechanics.

Formulation is gaining in popularity for structural elements such as beams, plates, and shells, especially in Aerospace.\(^6\) The UL formulation is useful in treatments of vary large strains and flow-like behavior, as well as in processes involving topology and/or phase changes; e.g., metal forming.

§7.3.5. **Coordinate Systems**

Configurations taken by a body or element during the response analysis are linked by a Cartesian **global frame**, to which all computations are ultimately referred.\(^7\) There are actually two such frames:

(i) The **material global frame** with axes \{X_i\} or \{X, Y, Z\}.

(ii) The **spatial global frame** with axes \{x_i\} or \{x, y, z\}.\(^8\)

The material frame tracks the base configuration whereas the spatial frame tracks all others. This distinction agrees with the usual conventions of classical continuum mechanics. In this book both frames are taken to be identical, as nothing is gained by separating them. Thus only one set of global axes, with dual labels, is drawn in Figure 7.7

---

\(^6\) Geometrically nonlinear problems in Aerospace Engineering tend to involve large motions, in particular large rotations, but small strains. Reason: structures are comparatively thin to save weight.

\(^7\) In dynamic analysis the global frame may be moving in time as a Galilean or inertial frame. This is convenient to track the trajectory motion of objects such as aircraft or satellites.

\(^8\) The choice between \{X_1, X_2, X_3\} versus \{X, Y, Z\} and likewise \{x_1, x_2, x_3\} versus \{x, y, z\} is a matter of notational convenience. For example, when developing specific finite elements it is preferable to use \{X, Y, Z\} or \{x, y, z\} so as to reserve coordinate subscripts for node numbers. On the other hand, in derivations that make heavy use of indicial notation, the \(X_i\) and \(x_i\) notation is more appropriate.
§7.3 THE CONTINUUM MODEL

**Figure 7.4.** Important configurations in Total Lagrangian (TL) kinematic description.

**Figure 7.5.** Important configurations in Updated Lagrangian (UL) kinematic description.

**Figure 7.6.** Important configurations in Corotational (CR) kinematic description. The Note: the corotated and current configuration are shown highly offset for visualization convenience. In practical use they highly overlap; for example, the centroids coincide.

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\[ u = x - X \]

\[ \mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} x - X \\ y - Y \\ z - Z \end{bmatrix} = \mathbf{x} - \mathbf{X}. \]

§7.3.6. Finite Element Frames

In stark contrast to the uniqueness of global frames, the presence of finite elements means there are many local frames to keep track of. More precisely, each element is endowed with two local Cartesian frames:

(iii) The element base frame with axes \{\hat{X}_i\} or \{\hat{X}, \hat{Y}, \hat{Z}\}.
(iv) The element reference frame with axes \{\bar{x}_i\} or \{\bar{x}, \bar{y}, \bar{z}\}.

The base frame is attached to the base configuration. It remains fixed if the base is fixed. It is chosen according to usual FEM practices. For example, in a 2-node spatial beam element, \(\hat{X}_1\) is defined by the two end nodes whereas \(\hat{X}_2\) and \(\hat{X}_3\) lie along principal inertia directions. The origin is typically placed at the element centroid.

The meaning of the reference frame depends on the description chosen:

Total Lagrangian (TL). The reference and base frames coalesce, whence the \(X_i\) and \(x_i\) coalesce.

Updated Lagrangian (UL). This is described differently by different authors. Some specify that the reference configuration is set to be the current configuration, so it is effectively updated in each increment or iteration. Others authors relax this policy and specify that the reference configuration is updated once the process converges.

Corotational description (CR). The reference frame is renamed corotated frame or CR frame. It remains attached to the element and continuously moves with it.

The transformation

\[ \mathbf{x} = \mathbf{X} + \mathbf{u}, \]

maps the position of base particle \(P(X, Y, Z)\) to \(P(x, y, z)\). See Figure 7.7. Consequently the particle displacement vector is defined as

\[ \mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} x - X \\ y - Y \\ z - Z \end{bmatrix} = \mathbf{x} - \mathbf{X}. \]
§7.4 DEFORMATION MEASURES

in which \((X, Y, Z)\) and \((x, y, z)\) pertain to the same particle.

**Remark 7.2.** Variations of this notation scheme are employed as appropriate to the subject under consideration. For example, the coordinates of \(P\) in a target configuration \(C_n\) may be called \((x_n, y_n, z_n)\).

**Remark 7.3.** In continuum mechanics, \((X, Y, Z)\) and \((x, y, z)\) are called *material* and *spatial* coordinates, respectively. In general treatments both systems are curvilinear and need not coalesce. The foregoing relations are restrictive in two ways: the base coordinate systems for the reference and current configurations coincide, and that system is Cartesian. These assumptions are sufficient, however, for the problems addressed here.

**Remark 7.4.** The dual notation \((X, Y, Z) \equiv (x_0, y_0, z_0)\) is introduced on two accounts: (1) the use of \((x_0, y_0, z_0)\) sometimes introduces a profusion of additional subscripts, and (2) the notation agrees with that traditionally adopted in continuum mechanics for the material coordinates, as noted in the previous remark. The identification \(X \equiv x_0, Y \equiv y_0, Z \equiv z_0\) will be employed when it is convenient to consider the reference configuration as the initial target configuration; cf. Remark 7.1.

**Remark 7.5.** The TL and CR descriptions may be unified for a restricted class of elements: those containing only translational freedoms. The unification framework is presented in Chapters 14ee.

§7.3.7. Configurations and Staged Analysis

The meaning of some special configurations can be made more precise if the nonlinear analysis process is viewed as a sequence of *analysis stages*, as discussed in Chapter 3. We restrict attention to the Total Lagrangian (TL) and Corotational (CR) kinematic descriptions, which are the only ones covered in this book. In a staged TL nonlinear analysis, two common choices for the reference configuration are:

1. **Reference \(\equiv\) base.** The base configuration is kept as reference configuration for *all* stages.
2. **Reference \(\equiv\) stage start.** The configuration at the start of an analysis stage, *i.e.* at \(\lambda = 0\), is chosen as reference configuration.

A combination of these two strategies can be of course adopted. In a staged CR analysis the reference is split between base and corotated. The same update choices are available for the base. This may be necessary when rotations exceed \(2\pi\); for example in aircraft maneuvers.

The *admissible* configuration is a “catch all” concept that embodies all others as particular cases. The *perturbed* configuration is an admissible variation from an admissible configuration. An ensemble of perturbed configurations is used to establish *incremental* or *rate* equations.

The *iterated* and *target* configurations are introduced in the context of incremental-iterative solution procedures for numerically tracing equilibrium paths. The target configuration is the “next solution”. More precisely, an equilibrium solution (assumed to exist) which satisfies the total residual equations for a given value of the stage control parameter \(\lambda\). While working to reach the target, a typical solution process goes through a sequence of *iterated* configurations that are not in equilibrium.

The *corotated* configuration is a rigid-body rotation of the reference configuration that “follows” the current configuration like a “shadow”. It is used in the corotational (CR) kinematic description of nonlinear finite elements. Strains measured with respect to the corotated configuration may be considered “small” in many applications, a circumstance that allows linearization of several relations and efficient treatment of stability conditions.

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§7.4. Deformation Measures

Characterizing the deformation of the material with respect to the reference configuration requires the introduction of gradients of the motion.

§7.4.1. Deformation Gradients

The derivatives of \((x, y, z)\) with respect to \((X, Y, Z)\), arranged in Jacobian format, constitute the so-called deformation gradient matrix:

\[
F = \frac{\partial(x, y, z)}{\partial(X, Y, Z)} = \begin{bmatrix}
\frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\
\frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\
\frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z}
\end{bmatrix}.
\] (7.8)

which represents a second order tensor. The inverse relation gives the derivatives of \((X, Y, Z)\) with respect to \((x, y, z)\) as

\[
F^{-1} = \frac{\partial(X, Y, Z)}{\partial(x, y, z)} = \begin{bmatrix}
\frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} & \frac{\partial X}{\partial z} \\
\frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} & \frac{\partial Y}{\partial z} \\
\frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z}
\end{bmatrix}.
\] (7.9)

The indicial versions of (7.8) and (7.9) are more compact: \(F_{i,j} = \partial x_i / \partial X_j\), \(F^{-1}_{i,j} = \partial X_i / \partial x_j\). This is convenient when \(F\) and \(F^{-1}\) are interpreted as second order tensors. As illustrated in Figure 7.8, \(F\) and \(F^{-1}\) can be used to relate the coordinate differentials

\[
dx = \begin{bmatrix}
dx \\
dy \\
dz
\end{bmatrix} = F \begin{bmatrix}
dX \\
dY \\
dZ
\end{bmatrix} = F dx, \quad dX = F^{-1} dx.
\] (7.10)

§7.4.2. Displacement Gradients

Since \(u = x - X\), the displacement gradients with respect to \(C_0\) can be presented as the 3 \(\times\) 3 matrix

\[
G = F - I = \begin{bmatrix}
\frac{\partial x}{\partial X} - 1 & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\
\frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} - 1 & \frac{\partial y}{\partial Z} \\
\frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} - 1
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u_x}{\partial X} & \frac{\partial u_x}{\partial Y} & \frac{\partial u_x}{\partial Z} \\
\frac{\partial u_y}{\partial X} & \frac{\partial u_y}{\partial Y} & \frac{\partial u_y}{\partial Z} \\
\frac{\partial u_z}{\partial X} & \frac{\partial u_z}{\partial Y} & \frac{\partial u_z}{\partial Z}
\end{bmatrix} = \nabla u.
\] (7.11)
Likewise, displacement gradients with respect to \( C \) are given by

\[
\hat{G} = I - F^{-1} = \begin{bmatrix}
1 - \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} & \frac{\partial Z}{\partial x} \\
\frac{\partial Y}{\partial x} & 1 - \frac{\partial Y}{\partial y} & \frac{\partial Z}{\partial y} \\
\frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & 1 - \frac{\partial Z}{\partial z}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\
\frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\
\frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z}
\end{bmatrix}.
\] (7.12)

**Remark 7.6.** For some FEM formulations it will be found convenient to arrange the displacement gradients of (7.11) as a 9-component vector, which is written below as row vector to save space:

\[
g^T = \begin{bmatrix} g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 & g_8 & g_9 \end{bmatrix} = \begin{bmatrix}
\frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} & \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z}
\end{bmatrix}.
\] (7.13)

**Remark 7.7.** Displacement gradient matrices are connected by the relations

\[
G = (I - \hat{G})^{-1} - I, \quad \hat{G} = I - (I + G)^{-1}.
\] (7.14)

For small deformations \( G \approx \hat{G}^{-1} \) and \( \hat{G} \approx G^{-1} \). For finite deformations \( \hat{G} G \neq I \).

**Remark 7.8.** The ratio between infinitesimal volume elements \( dV = dx \, dy \, dz \) and \( dV_0 = dx \, dy \, dz \) in the current and reference configuration appears in many relations. Because of (7.10) this ratio may be expressed as

\[
\frac{dV}{dV_0} = \frac{\rho_0}{\rho} = \text{det} F = J,
\] (7.15)

where \( \rho \) and \( \rho_0 \) denote the mass densities in the current and reference configuration, respectively. The abbreviation \( J = \text{det} F \) is frequently used later. This equation expresses the law of conservation of mass.

### §7.4.3. Stretch Tensors

The combinations

\[
C_R = F^T F, \quad C_L = FF^T.
\] (7.16)

are two symmetric, positive definite matrices extensively used in following Chapters. When viewed as second-order tensors, they are called the *left* and *right stretch* tensors, respectively. Alternative names and notations in the literature are mentioned in §8.1. The matrices (7.16) naturally emerge from quadratic forms involving \( dX \) and \( d\mathbf{x} \) as follows:

\[
dX^T d\mathbf{x} = dX^T F^T F d\mathbf{x} = dX^T C_R d\mathbf{x},
\]

\[
dX^T d\mathbf{x} = dX^T F^{-T} F^{-1} d\mathbf{x} = dX^T C_L^T d\mathbf{x}.
\] (7.17)

From (7.17), finite strain measures can be constructed as described in the next Chapter. But an important property should be prenoted: if the body motion is rigid, \( dX^T d\mathbf{x} = dX^T d\mathbf{x} \) identically, whence both \( C_R \) and \( C_L \) must reduce to the identity \( I \).

Because of (7.17), deformation and strain measures that involve \( C_R \) are called *Lagrangian* or *material*, whereas those involving \( C_L \) are called *Eulerian* or *spatial*. The linkage may be direct or indirect; e.g., if \( \mathbf{U} = \sqrt{C_R} \), a measure that is a function of \( \mathbf{U} \) is classified as Lagrangian.
The foregoing definitions are illustrated next with simple motion examples. Examples 7.5 through 7.7 below illustrate motions associated with homogeneous deformations, whereas Example 7.8 deals with the simple nonhomogeneous extension of a bar member.

**Example 7.5. Simple extension.** The prototype case for this motion is a prismatic, homogeneous bar of reference length $L_0$ stretched to a current length $L$. See Figure 7.9. Select $X \equiv x$ along the bar axis, with $Y \equiv y$ and $Z \equiv z$ normal to it. The origin is at the left cross-section of the bar, which is $X$-fixed in the motion. Call $\lambda_1 = L/L_0$ the axial stretch ratio along $X$, while $\lambda_2$ and $\lambda_3$ denote stretches along $Y$ and $Z$, respectively. The reference-to-current motion is

$$x = \lambda_1 X, \quad y = \lambda_2 Y, \quad z = \lambda_3 Z.$$  \hfill (7.18)

The displacements are $u_X = x - X = (\lambda_1 - 1)X$, $u_Y = y - Y = (\lambda_2 - 1)Y$, and $u_Z = z - Z = (\lambda_3 - 1)Z$. The deformation and displacement gradients are

$$F = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad \hat{G} = \begin{bmatrix} \lambda_1 - 1 & 0 & 0 \\ 0 & \lambda_2 - 1 & 0 \\ 0 & 0 & \lambda_3 - 1 \end{bmatrix}. \hfill (7.19)$$

Since these are diagonal matrices, $F^{-1}$ and $\hat{G}$ are easily obtained.

If the material is transversally isotropic, $\lambda_2 = \lambda_3$. If so, for small deformations $-(\lambda_2 - 1)/(\lambda_1 - 1)$ becomes the classical Poisson’s ratio $\nu$. If the material is incompressible, the motion is isochoric (bar does not change volume), whence $\lambda_1 \lambda_2 \lambda_3 = 1$. If the material is incompressible and transversally isotropic, $\lambda_2 = \lambda_3 = 1/\sqrt{\lambda_1}$. The case of pure dilatation is obtained by making $\lambda_1 = \lambda_2 = \lambda_3$ in the foregoing equations.

This analysis is continued in Example 8.1 of next Chapter.

**Example 7.6. Simple shear.** Here a block of material is sheared by an angle $\theta$ as shown in Figure 7.10. Material fibers aligned with $X$ move horizontally and do not change length. As a result the block undergoes the motion

$$x = X + \gamma Y, \quad y = Y, \quad z = Z.$$  \hfill (7.20)

in which $\gamma = \tan \theta$ is called the amount of shear. Material fibers aligned with $X$ translate horizontally and do not change length, so the motion is isochoric. The displacements are $u_X = \gamma Y$, and $u_Y = u_Z = 0$. 

**Figure 7.10.** Simple shear of a square block. Points $A$ and $B$ in $C$ actually coincide with $A_0$ and $B_0$ in $C_0$; drawn offset for visualization convenience. Z axis normal to paper.
The deformation and displacement gradients are
\[ F = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G = F - I = \begin{bmatrix} 0 & \gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \] (7.21)

Since these are sparse the polar decomposition is easily found. It is worked out in Example 8.2 of next Chapter.

**Example 7.7. Combined stretch, rotation and rigid translation.** This example shows how to combine homogeneous motions. A rectangular, bar-like block of material is subject to the three motions pictured in Figure 7.11. They are applied in the following order:

1. A translational rigid motion that displaces the axes origin \( C_0 \) to \( C \) by \( u_{XC} \) and \( u_{YC} \) along \( X \) and \( Y \), respectively. No motion along \( Z \).
2. Stretchings of \( \lambda_1 = L/L_0 \), \( \lambda_2 = H_Y/H_{Y0} \) and \( \lambda_3 = 1 \) along \( X \), \( Y \), and \( Z \), respectively. (\( Z \) lengths are not pictured in the figure.)
3. A rigid rotation \( \psi \) (positive CCW) in the \( X, Y \) plane about \( C \).

The combined motion can be expressed in the matrix form
\[
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X - u_{XC} \\ Y - u_{YC} \\ Z \end{bmatrix}. \] (7.22)

in which \( c = \cos \psi \) and \( s = \sin \psi \). The combination of stretching and rigid rotation by a matrix product is called a multiplicative composition. Expanding gives
\[
x = \lambda_1 c (X - u_{XC}) - \lambda_2 s (Y - u_{YC}), \quad y = \lambda_1 s (X - u_{XC}) + \lambda_2 c (Y - u_{YC}), \quad z = Z. \] (7.23)

Since matrix products do not necessarily commute, the order in which stretch and rotation are applied is important. If the sequence were reversed: rotate-then-stretch, the motion would be different unless \( \lambda_1 = \lambda_2 \). By inspection, the deformation and displacement gradients are
\[ F = \begin{bmatrix} \lambda_1 c & -\lambda_2 s & 0 \\ \lambda_1 s & \lambda_2 c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G = F - I = \begin{bmatrix} \lambda_1 c - 1 & -\lambda_2 s & 0 \\ \lambda_1 s & \lambda_2 c - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \] (7.24)

Note that the translational rigid motion does not affect these tensors, but the rigid rotation does. The polar decomposition and strains are worked out in Example 8.4 of next Chapter.
Most convenient are those at which The displacement gradient $g$ is linearly 
This varies the isoparametric (natural) coordinate. The

$f_i$ The displacement 

$N_1$ are shape functions, and $\xi = 2X/L_0$ is 

$g = \frac{1}{L_0}(u_1 (2\xi - 1) + u_2 (2\xi + 1) - u_3 4\xi)$. 

This varies linearly in $\xi$, whence $g$ is fully specified if we know its value at two distinct points on the bar axis. Most convenient are those at which $g$ can be readily evaluated in terms of the lengths $L_1$, $L_2$ and $L_0$. Those are called gage points, which were introduced in the 1966 thesis [221] under the name strain nodes. A simple calculation show that the quarter points in $C_0$, located at $\xi = -\frac{1}{2}$ and $\xi = \frac{1}{2}$, are gage points labeled $G_1$ and $G_2$, respectively,\footnote{In $C_0$, $G_1$ and $G_2$ are midway between 1–3 and 3–2, respectively: $X_{G1} = -L_0/4$ and $X_{G2} = L_0/4$, whence the “quarter point” label. In $C$ their position may be obtained through the mapping $x_{G1} = X_{G1} + u(-\frac{1}{2}) = -L_0/4 + (3u_1 - u_2 + 6u_3)/8$ and $x_{G2} = X_{G2} + u(\frac{1}{2}) = L_0/4 + (-u_1 + 3u_2 + 6u_3)/8$. Those are no longer quarter points in the deformed bar unless the displacement is linear so $u_3 = (u_1 + u_2)/2.$} whence 

$$g_{G1} = \frac{2(u_3 - u_1)}{L_0} = \frac{L_1 - L_0}{L_0}, \quad g_{G2} = \frac{2(u_2 - u_3)}{L_0} = \frac{L_2 - L_0}{L_0}. \quad (7.27)$$ 

At any other location, $g(\xi) = \frac{1}{2}(g_{G1} + g_{G2}) + (g_{G2} - g_{G1})\xi$, which in terms of lengths gives 

$$g(\xi) = \frac{L - L_0}{L_0} + \frac{L_2 - L_1}{L_0}\xi. \quad (7.28)$$ 

Why is it so important to express the axial displacement gradient in terms of original and deformed lengths? Because those are invariant with respect to arbitrary rigid rotations in 2D or 3D. In practical FEM analysis the

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure7.12.png}
\caption{Rectangular block undergoing Nonhomogeneous axial motion.}
\end{figure}
reference and current configuration of an element are not global-frame-aligned as in Figure 7.12. But if the internal energy is expressed in terms of lengths, misalignment is irrelevant.

The next chapter continues the review of continuum mechanics by focusing on field equations used in geometrically nonlinear FEM analysis.

§7.5. *Useful Vector and Matrix Formulas

This section collects vector and matrix identities used in subsequent chapters. It also provides a quick “refresher” on matrix calculus operations.

§7.5.1. *Vector Identities

Below are some well known vector identities. Here $a$ and $b$ denote scalars, whereas $\mathbf{u}$ and $\mathbf{v}$ are generic $3 \times 1$ column vectors. All of these are assumed to be fields, that is, functions of position coordinates $\{x, y, z\}$; these arguments are usually omitted for brevity. Functions are assumed to be sufficiently smooth for differentiation operations to make sense. Direct notation is used; translation to matrix and indicial form is immediate. For full form expressions in terms of components, see [727].

\[
\nabla (a \ b) = a \nabla b + b \nabla a. \tag{7.29}
\]

\[
\nabla (\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}). \tag{7.30}
\]

\[
\nabla \times (\nabla a) = 0. \tag{7.31}
\]

\[
\nabla \cdot (\nabla a) = 0. \tag{7.32}
\]

\[
\nabla \times (\nabla \times \mathbf{u}) = \nabla (\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}. \tag{7.33}
\]

\[
\nabla \cdot (a \mathbf{u}) = a (\nabla \cdot \mathbf{u}) + (\mathbf{u} \cdot \nabla) a. \tag{7.34}
\]

\[
\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v}). \tag{7.35}
\]

\[
\nabla \times (a \mathbf{u}) = a(\nabla \times \mathbf{u}) + (\nabla a) \times \mathbf{u}. \tag{7.36}
\]

\[
\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u} (\nabla \cdot \mathbf{v}) - \mathbf{v} (\nabla \cdot \mathbf{u}) + (\mathbf{v} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{v}. \tag{7.37}
\]

Note that $\nabla \cdot \mathbf{u} = u_{i,j}$ is the divergence of $\mathbf{u}$ (a scalar). On the other hand, $\nabla \cdot a$ is the gradient of $a(x, y, z)$ stored as a column vector whereas $a \cdot \nabla$ is the same gradient stored as a row vector.

§7.5.2. *Integral Identities

In the following integral identities, $V$ denotes the volume of a body, $S$ is its surface, $\mathbf{n}$ the exterior unit normal, $C$ is a closed curve (circuit) on the surface, and $\mathbf{t}$ the unit tangent to $C$. The differentials of $V$, $S$ and $C$ are denoted as $dV$, $dS$ and $ds$, respectively. Appropriate smoothness of $S$ and $C$ is assumed.

Divergence theorem for scalar field $a = a(x, y, z)$:

\[
\int_S a \mathbf{n} \, dS = \int_V \nabla a \, dV. \tag{7.38}
\]

Divergence theorem for vector field $\mathbf{u}(x, y, z)$:

\[
\int_S \mathbf{u} \cdot \mathbf{n} \, dS = \int_V \nabla \cdot \mathbf{u} \, dS. \tag{7.39}
\]

Divergence theorem for vector curl $\mathbf{n}(x, y, z) \times \mathbf{u}(x, y, z)$:

\[
\int_S (\mathbf{n} \times \mathbf{u}) \, dS = \int_V (\nabla \times \mathbf{u}) \, dV. \tag{7.40}
\]
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Stokes' theorem for vector field \( \mathbf{u}(x, y, z) \):

\[
\int_C \mathbf{u} \cdot d\mathbf{s} = \int_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} \, dS. \tag{7.41}
\]

Stokes' theorem for dot product \( \mathbf{u}(x, y, z) \cdot \mathbf{v}(x, y, z) \):

\[
\oint_C (\mathbf{u} \cdot \mathbf{v}) \, d\mathbf{s} = \int_S (\nabla(\mathbf{u} \cdot \mathbf{v})) \times \mathbf{n} \, dS. \tag{7.42}
\]

§7.5.3. *Quick Refresher on Matrix Calculus

The material in this subsection is intended to help readers unfamiliar (or rusty) with matrix calculus as regards some finite element derivations. These involve taking derivatives of a scalar or vector with respect to a vector. The exposition is done using full-form examples. In what follows \( a, b, \) and \( c \) denote scalars, \( x \) and \( y \) are \( 2 \times 1 \) column vectors, \( \mathbf{A} \) is an arbitrary \( 2 \times 2 \) matrix, and \( \mathbf{S} \) is a symmetric \( 2 \times 2 \) matrix:

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix}. \tag{7.43}
\]

Matrix, full and indicial notations are used as appropriate; for the latter indices \( i, j \) run over 1,2.

Derivative of a scalar with respect to a vector:

\[
\frac{\partial c}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial c}{\partial x_1} \\ \frac{\partial c}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial c}{\partial x_1} \\ \frac{\partial c}{\partial x_2} \end{bmatrix} \tag{7.44}
\]

Derivative of a vector with respect to a vector:

\[
\frac{\partial \mathbf{x}}{\partial \mathbf{y}} = \begin{bmatrix} \frac{\partial x_i}{\partial y_1} \\ \frac{\partial x_i}{\partial y_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix}. \tag{7.45}
\]

Second derivative of a scalar with respect to two vectors:

\[
\frac{\partial^2 c}{\partial \mathbf{x} \partial \mathbf{y}} = \begin{bmatrix} \frac{\partial^2 c}{\partial x_1 \partial y_1} & \frac{\partial^2 c}{\partial x_1 \partial y_2} \\ \frac{\partial^2 c}{\partial x_2 \partial y_1} & \frac{\partial^2 c}{\partial x_2 \partial y_2} \end{bmatrix} \tag{7.46}
\]

This yields a symmetric matrix if \( \mathbf{x} = \mathbf{y} \), as in the second derivative of the GL strain given in (9.14)

Dyadic product (a.k.a. outer product) of two scalar-with-respect-to-vector derivatives:

\[
\frac{\partial a}{\partial \mathbf{x}} \otimes \frac{\partial b}{\partial \mathbf{y}} = \begin{bmatrix} \frac{\partial a}{\partial x_1} \frac{\partial b}{\partial y_1} \\ \frac{\partial a}{\partial x_1} \frac{\partial b}{\partial y_2} \\ \frac{\partial a}{\partial x_2} \frac{\partial b}{\partial y_1} \\ \frac{\partial a}{\partial x_2} \frac{\partial b}{\partial y_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial a}{\partial x_1} \\ \frac{\partial a}{\partial x_2} \\ \frac{\partial b}{\partial y_1} \\ \frac{\partial b}{\partial y_2} \end{bmatrix} = \frac{\partial a}{\partial \mathbf{x}} \left( \frac{\partial b}{\partial \mathbf{y}} \right)^T. \tag{7.47}
\]

In particular, if \( \mathbf{y} = \mathbf{x} \) and \( a = b = c \), the resulting matrix is symmetric:

\[
\frac{\partial c}{\partial \mathbf{x}} \otimes \frac{\partial c}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial c}{\partial x_1} \frac{\partial c}{\partial x_1} \\ \frac{\partial c}{\partial x_1} \frac{\partial c}{\partial x_2} \\ \frac{\partial c}{\partial x_2} \frac{\partial c}{\partial x_1} \\ \frac{\partial c}{\partial x_2} \frac{\partial c}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial c}{\partial x_1} \\ \frac{\partial c}{\partial x_2} \\ \frac{\partial c}{\partial x_1} \\ \frac{\partial c}{\partial x_2} \end{bmatrix} = \frac{\partial c}{\partial \mathbf{x}} \left( \frac{\partial c}{\partial \mathbf{x}} \right)^T. \tag{7.48}
\]

Some consequences of above formulas for dot product forms:

\[
\frac{\partial (\mathbf{x}^T \mathbf{y})}{\partial \mathbf{x}} = \mathbf{y}, \quad \frac{\partial (\mathbf{x}^T \mathbf{y})}{\partial \mathbf{y}} = \mathbf{x}, \quad \frac{\partial (\mathbf{x}^T \mathbf{x})}{\partial \mathbf{x}} = 2\mathbf{x}, \quad \frac{\partial \sqrt{\mathbf{x}^T \mathbf{x}}}{\partial \mathbf{x}} = \frac{\mathbf{x}}{\sqrt{\mathbf{x}^T \mathbf{x}}}. \tag{7.49}
\]

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And for quadratic forms:

\[
\frac{\partial (x^T A y)}{\partial x} = Ay, \quad \frac{\partial (x^T A y)}{\partial y} = A^T x, \quad \frac{\partial (x^T A x)}{\partial x} = (A + A^T) x, \quad \frac{\partial (x^T S x)}{\partial x} = 2 S x. \quad (7.50)
\]

Some textbooks in continuum mechanics use directional derivatives to derive these expressions. This approach introduces an additional parameter and requires taking limits as the parameter approaches zero. For the foregoing expressions, the extra work does not yield new results. But it can pay off for more complicated cases; e.g., if curvilinear coordinates are used.

**Notes and Bibliography**

Few textbooks collectively cover the kinematics of classical continuum mechanics as well as the extensions required to formulate geometrically nonlinear finite elements in three different descriptions (TL, UL and CR). In this regard the two volumes by Crisfield [181,182] may be cited. See Notes and Bibliography in next Chapter for additional references on books treating continuum mechanics proper.
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**Homework Exercises for Chapter 7**

**Review of Continuum Mechanics: Kinematics**

**EXERCISE 7.1** [A:15] The classical Lagrangian finite strain measure (“classical” in the sense of being the first one published and most popular) is denoted as Green-Lagrange (GL). It was introduced by (George) Green in 1841. To derive it, consider Figure 7.8. From a material particle in \( C_0 \) emanates an arbitrary differential vector: \( d\mathbf{X} \), which is mapped to \( d\mathbf{x} \) via \( \mathbf{F} \) as per (7.10). The GL strain tensor, denoted as \( \mathbf{e}^G \), is defined as

\[
d\mathbf{X}^T \mathbf{e}^G d\mathbf{X} \overset{\text{def}}{=} \frac{1}{2} \left( ||d\mathbf{x}||^2 - ||d\mathbf{X}||^2 \right) = \frac{1}{2} \left( d\mathbf{x}^T d\mathbf{x} - d\mathbf{X}^T d\mathbf{X} \right)
\]

Show that

\[
\mathbf{e}^G = \frac{1}{2} (\mathbf{F}^T \mathbf{F} - \mathbf{I}) = \frac{1}{2} (\mathbf{C}_R - \mathbf{I}). \tag{E7.2}
\]

**EXERCISE 7.2** [A:15] The classical Eulerian finite strain measure is denoted as Almansi-Euler (AE) — also often called Almansi-Hamel. To derive it, consider Figure 7.8. From a spatial particle in \( \mathcal{C} \) emanates an arbitrary differential vector: \( d\mathbf{x} \), which is mapped to \( d\mathbf{X} \) via \( \mathbf{F}^{-1} \) as per (7.10). The AE strain tensor, denoted by \( \mathbf{e}^A \), is defined as

\[
d\mathbf{x}^T \mathbf{e}^A d\mathbf{x} \overset{\text{def}}{=} \frac{1}{2} \left( ||d\mathbf{x}||^2 - ||d\mathbf{X}||^2 \right) = \frac{1}{2} \left( d\mathbf{x}^T d\mathbf{x} - d\mathbf{X}^T d\mathbf{X} \right)
\]

Show that

\[
\mathbf{e}^A = \frac{1}{2} (\mathbf{I} - \mathbf{F}^{-T} \mathbf{F}^{-1}) = \frac{1}{2} (\mathbf{I} - \mathbf{C}_L^{-1}). \tag{E7.4}
\]

**EXERCISE 7.3** [A:20] A bar of length \( L_0 \) lies originally along the \( X \) axis in the reference configuration \( C_0 \). The bar is rigidly rotated 90° about \( Z \equiv z \) to lie along the \( Y \equiv y \) axis while retaining the same length at the current configuration \( \mathcal{C} \). See Figure E7.1.

(a) Verify that the motion from \( C_0 \) to \( \mathcal{C} \) is given by

\[
x = -Y, \quad y = X, \quad z = Z.
\]

(b) Compute the displacement field \( \mathbf{u} \), the deformation gradient matrix \( \mathbf{F} \), the right stretch tensor \( \mathbf{C}_R \), the displacement gradient matrix \( \mathbf{G} \) and the Green-Lagrange (GL) strain tensor given in (E7.2) above. Show that the GL measure correctly predicts zero axial strain \( \epsilon_{XX}^G \) whereas the infinitesimal strain measure \( \epsilon_{XX} = \partial u_x / \partial x \) predicts the absurd value of \(-100\%\) strain.

**EXERCISE 7.4** [A:25] Repeat the previous Exercise for the Biot, Hencky, Swainger, Almansi, midpoint and Bazant strains listed in Figure 8.3 of the next Chapter. Verify that \( \epsilon_{XX} = 0 \) for all of those measures.

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