5

Critical Points
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§5.1. Introduction

This Chapter provides additional material on properties of the one-parameter force residual equations. It begins with a study of critical points, which are classified into limit and bifurcation points. Limit point “sensors” and turning points are briefly described. Two worked out examples have been added in the present revision.

§5.2. Critical Points

This section deals with the classification and characterization of critical points. The determination of such points is a key application of geometrically nonlinear analysis on account of the following property:

Along a static equilibrium path of a conservative system, transition from stability to instability can only occur at critical points.

This property does not extend to nonconservative systems, which generally require a dynamic treatment, as discussed in latter Chapters. In addition, it does not apply to conservative systems away from equilibrium.

§5.2.1. Behavioral Assumptions

We shall restrict the class of systems considered here to those that satisfy the following assumptions:

- There is only one control parameter: the staging parameter $\lambda$.
- The system is conservative: the total residual is the gradient of a real total-energy function $\Pi_1$:

$$r(u, \lambda) = \frac{\partial \Pi_1(u, \lambda)}{\partial u}.$$  (5.2)

Since $\Pi$ and the state vector entries are real, the entries of the residual vector are also real.

§5.2.2. Stiffness Matrix Properties

A consequence of the conservativeness assumption (5.2) is that the tangent stiffness matrix is the Hessian of the total energy function:

$$K = \frac{\partial r(u, \lambda)}{\partial u} = \frac{\partial \Pi_1(u, \lambda)}{\partial u \partial u}.$$  (5.3)

Transposing both sides of (5.3) gives $K = K^T$. Thus $K$ is symmetric real. This guarantees two important spectral properties:

- All eigenvalues of $K$ are real.
- $K$ has a full set of independent real eigenvectors that can be orthonormalized. Furthermore left and right eigenvectors coalesce.
To state these properties more precisely, let the eigensystem of the \( N \times N \) tangent stiffness matrix be
\[
K z_i = \kappa_i z_i, \quad i = 1, 2, \ldots N.
\] (5.4)
The eigenvalues \( \kappa_i \) of \( K = K^T \) are real, and the orthonormalized eigenvectors satisfy \( z_i^T z_j = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker delta.

**Remark 5.1.** If the system is nonconservative, \( K \) is generally unsymmetric, and the foregoing spectral properties are lost. The major consequence is that purely static stability analysis is no longer possible because of the possible occurrence of growing oscillations in real time (flutter). Investigation of that possibility requires a dynamic analysis, which brings in inertial and possibly damping effects. This substantially complicates the model as well as the analysis process. This case is relegated to the final Chapters.

§5.2.3. Regular Versus Critical Points

Each point of an equilibrium path represents a (static) equilibrium state. These are classified as follows according to whether the tangent stiffness matrix evaluated at that point is singular or not:

Regular point: \( K \) is nonsingular.

Critical point: \( K \) is singular. Also called singular or nonregular points.

Recall the incremental velocity vector defined in §4.2.3 is
\[
v = u' = K^{-1} q.
\] (5.5)
At a critical point \( v \) becomes undefined according to (5.5), since \( K^{-1} \) does not exist. Physically this means that the structural behavior cannot be controlled by the parameter \( \lambda \).

Since the determinant of a singular matrix is zero, the foregoing classification can be stated as

\[
\text{The determinant of } K \text{ vanishes at a critical point}
\] (5.6)

This rule provides a practical mean for locating critical points analytically in simple problems with closed form solutions for the response. The procedure is illustrated in §5.6.

§5.2.4. Isolated Versus Multiple Critical Points

We shall denote by
\[
u_{cr}, \quad \lambda_{cr}, \quad K_{cr}, \quad q_{cr},
\] (5.7)
the value of the state vector, control parameter, tangent stiffness matrix, and incremental load vector, respectively, evaluated at a critical point. Since \( K_{cr} \) is singular, at least one eigenvalue of \( K \) is zero.

The following subclassification takes into account the number of zero eigenvalues:

Isolated critical point: \( K_{cr} \) has only one zero eigenvalue. Its rank deficiency is one.

Multiple critical point: \( K_{cr} \) has two or more zero eigenvalues. Its rank deficiency is two or more.

This distinction has importance from both computational and engineering viewpoints. A multiple critical point is more difficult to “capture” and traverse numerically in a response computation process. Physically, a structure with a multiple critical point is more sensitive to imperfections in the vicinity of that critical state. It might be thought that critical point coalescence has a low probability of happening in a typical structure. However, such occurrence may be the unfortunate side effect of a design optimization process.
§5.3 LIMIT OR BIFURCATION POINT?

Figure 5.1. Critical points for a two degree of freedom system \((u_1, u_2)\) shown on the \(u_1\) versus \(\lambda\) plane: (a) Limit point (“snap through” behavior) \(L_1\) occurs before bifurcation \(B_1\); (b) Bifurcation point \(B_1\) occurs before limit point \(L_1\), in which case \(L_1\) is physically unreachable. Full lines represent physically “preferred” paths. A more realistic three-dimensional view of this case is shown in Figure 5.2.

§5.2.5. Limit Versus Bifurcation Points

For simplicity we restrict attention to isolated critical points, at which \(K_{cr}\) has a single zero eigenvalue and a rank deficiency of one. It is convenient to distinguish two types of critical points

Limit points, at which the tangent (4.35) to the equilibrium path is unique but normal to the \(\lambda\) axis so \(v\) becomes infinitely large.\(^1\) In geometric terms, we have a maximum, minimum or inflexion point of the response curve. (For the inflexion case, see Remark 5.6.)

Bifurcation point, also called branch point or branching point, from which two equilibrium path branches emanate and so there is no unique tangent.

Since the tangent at a limit point is normal to \(\lambda\), it must correspond to a maximum, minimum or inflexion point with respect to \(\lambda\). In the case of a maximum or a minimum, the occurrence of a limit point is informally called snap through or snap buckling by structural engineers for reasons explained in a Remark below.

The type classification a for multiple critical point is more complicated, and is discussed in the Chapters dealing specifically with stability.

Figures 5.1 illustrates two possible configurations of limit and bifurcation points (all of them isolated) for a two DOF system. It is assumed that the fundamental path occurs with \(u_2 = 0\), and so the response is shown on the \(\{\lambda, u_1\}\) plane for clarity. Limit points are identified as \(L_1, L_2, \ldots\) whereas bifurcation points are marked as \(B_1, B_2, \ldots\).

§5.3. Limit or Bifurcation Point?

We shall focus here on an isolated critical point, at which we have available the quantities listed in (5.7). How can we mathematically characterize its type? Consider the eigensystem (5.4), and let \(z_{cr}\) be the right eigenvector of \(K_{cr}\) associated with the single zero eigenvalue:

\[
K_{cr} z_{cr} = 0 = z_{cr}.
\]  

\(^{1}\) Some authors, e.g. Seydel [736], call limit points “turning points.” That term is here used for a different type cf. §5.5.
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This eigenvector will be called the null eigenvector. It spans the null space of \( K_{cr} \). Transposing both sides of (5.8) gives

\[
z_{cr}^T K_{cr}^T = z_{cr}^T K_{cr} = 0^T.
\]  

That is, \( z_{cr} \) is also a left null eigenvector. Recall the first-order rate equation of the equilibrium path, stated in (4.7) as \( K \dot{u} = q \dot{\lambda} \). Evaluate this equation at the critical point: \( K_{cr} \dot{u}_{cr} = q_{cr} \dot{\lambda}_{cr} \).

Multiply through by \( dt \) to express it in terms of differentials:

\[
K_{cr} du = q_{cr} d\lambda.
\]  

in which \( du \) and \( d\lambda \) denote the differentials at \( (u_{cr}, \lambda_{cr}) \). Premultiply both sides of (5.10) by \( z_{cr}^T \) and use (5.9) to get

\[
z_{cr}^T q_{cr} d\lambda = 0.
\]  

This states that the product of two scalars: \( z_{cr}^T q_{cr} \), and \( d\lambda \), must vanish. So one of them must be zero. If

\[
z_{cr}^T q_{cr} \neq 0,
\]  

then \( d\lambda \) must vanish, and we have a limit point. On the other hand, if

\[
z_{cr}^T q_{cr} = 0,
\]  

then \( d\lambda \) is not necessarily zero, and we have a bifurcation or branching point. The quantity \( z_{cr}^T q_{cr} \) is called an critical point type indicator.

The key physical characteristic of a bifurcation point is an abrupt transition from one deformation mode to another mode; the latter having been previously “concealed” by virtue of being orthogonal to the incremental load vector. This is plainly shown by (5.13).

Example 5.1. Consider the scalar residual equation for the single DOF \( u \):

\[
r = u - 2u^2 + u^3 - \lambda.
\]  

Both the tangent stiffness and the incremental load are scalars: \( K = \partial r / \partial u = 1 - 2u + 3u^2 = (1 - 3u)(1 - u) \) and \( q = -\partial r / \partial \lambda = 1 \), respectively. Since \( \det(K) = K \), the critical points are the roots of \( K = 0 \): \( u_1 = \frac{1}{3} \) and \( u_2 = 1 \), at which \( \lambda_1 = 4/27 \) and \( \lambda_2 = 0 \). The only normalized “eigenvector” is \( z = 1 \) everywhere so \( z_{cr} = z = 1 \). Since \( q z_{cr} = 1 \) throughout, both critical points are of limit type. It is not difficult to show that bifurcation points cannot appear for a scalar residual equation if \( \lambda \) appears linearly; cf. Exercise 5.4.

Remark 5.2. If \( K \) is not symmetric, several changes must be made in the previous assumptions and derivations. These are explained in the Chapters that deal with nonconservative systems. In such systems the possible loss of stability may occur on account of growing dynamic oscillations, a phenomenon called flutter when the nonconservative loading is due to aerodynamic effects.

Remark 5.3. If \( \lambda \) is an applied load multiplier, a limit point associated with a maximum or a minimum, such as \( L_1 \) and \( L_2 \) in Figure 5.1(a) is called a snap-through point by structural engineers. The reason is that, if the load is kept constant, the structure “snaps” dynamically to another equilibrium position. The term collapse applies to critical points beyond which the structure becomes useless.
§5.4 LIMIT POINT SENSORS

Remark 5.4. As an isolated limit point is approached, the incremental velocity vector \( \mathbf{v} \) tends to become parallel to \( \mathbf{z}_{cr} \) whereas its magnitude goes to \( \infty \). If \( \mathbf{v} \) is normalized (for example, to length one), then

\[
\frac{\mathbf{v}}{|\mathbf{v}|} \rightarrow \mathbf{z}_{cr}.
\]

Therefore the normalized \( \mathbf{v} \) may be a good null eigenvector estimate if \( \mathbf{K} \) has been factored near the limit point. (This is just a restatement of the well known inverse iteration process \([875]\) for finding eigenvectors.)

Remark 5.5. The set of control parameters for which \( \det \mathbf{K} = 0 \) while \( \mathbf{r}(\mathbf{u}, \lambda) = \mathbf{0} \) is sometimes called the bifurcation set in the applied mathematics literature. The name is misleading, however, in that the set may include limit points; the name critical set would be more appropriate.

Remark 5.6. An inflexion point in a load-deflection response curve is called an inflexion limit point if no other pass crosses at that point. At such a point no snapping occurs, so the term no-snap limit point is also appropriate. Such occurrences are comparatively rare.

Remark 5.7. Showing bifurcation points of a 2-DOF system on the \( \lambda \) versus \( u_1 \) plane as in Figure 5.1 may be misleading, as it conceals the phenomenon of transition from one mode of deformation to another. Figure 5.2 provides a more realistic picture. This diagram shows the classical bifurcation behavior for a symmetrically loaded shallow arch. Here \( u_1 \) and \( u_2 \) measure amplitude of symmetric and antisymmetric displacement shapes, respectively. At \( B_1 \) the arch, which had been deforming symmetrically, takes off along an antisymmetric deformation mode; at \( B_2 \) the latter disappears and the arch rejoins the symmetric path.

Remark 5.8. Physically the distinction between the two types of critical points is not so marked, inasmuch as imperfect structures display limit-point behavior. A bifurcation point may be viewed as the limit of a sequence of critical points of limit type, realized as the structure strives towards mathematical perfection. The example worked out in §5.6.3 clearly illustrates that point; see Figure 5.8.

§5.4. Limit Point Sensors

Scalar estimates of the overall stiffness of the structure as the control parameter varies are useful as limit points sensors. The following estimator is based on the Rayleigh quotient approximation to the fundamental eigenvalue of \( \mathbf{K} \):

\[
k_{x} = \frac{x^T \mathbf{K} x}{x^T x},
\]

\(5–7\)
where \( \mathbf{x} \) is an arbitrary nonnull vector, and \( \mathbf{K} \) is evaluated at an equilibrium position \( \mathbf{u}(\lambda) \). An “equilibrium-path stiffness” estimator is obtained by taking \( \mathbf{x} \) to be \( \mathbf{v} = \mathbf{K}^{-1}\mathbf{q} \), in which case

\[
k = k_{\mathbf{v}} = \frac{\mathbf{q}^T \mathbf{v}}{\mathbf{v}^T \mathbf{v}}.
\]

(5.17)

This value may depend on \( \lambda \). It is convenient in practice to work with the dimensionless ratio

\[
\kappa = \frac{k(\lambda)}{k(0)},
\]

(5.18)

This ratio takes the value 1 at the start of an analysis stage, and goes to zero as a limit point is approached. Therefore it is often called limit point sensor. A sensor with this behavior (although computed in a different way) was introduced by Bergan and coworkers \[87,89\] under the name current stiffness parameter. In practice (5.18) is often expressed in terms of the state.

It should be noted that no estimator of this type can reliably predict the occurrence of a bifurcation point. Sensors for such points are described later in the context of augmented equations.

§ 5.5. *Turning Points

Turning points are regular points at which the tangent is parallel to the \( \lambda \) axis so that \( \mathbf{v} = \mathbf{0} \). The unit tangent takes the form

\[
\mathbf{t}_u = [\mathbf{0} \pm 1]^T.
\]

(5.19)

Although turning points generally do not have any physical meaning, they can cause special problems in path-following solution procedures because of “turnback” effects.

To detect the vicinity of a turning point one can check the two mathematical conditions: \( \mathbf{v} \) becomes orthogonal to \( \mathbf{q} \) and \( \mathbf{u} \) tends to zero faster than \( \mathbf{q} \). For example:

\[
|\cos\langle \mathbf{v}, \mathbf{q} \rangle| < \delta, \quad |\kappa| > \kappa_{\text{min}},
\]

(5.20)

where \( \kappa \) is the current stiffness parameter. Typical values may be \( \delta = 0.01 \), \( \kappa_{\text{min}} = 100 \).

§ 5.6. Critical Point Computation Problems

This section goes over the computation of critical points and associated attributes, such as stability transitions. Two problems with one degree of freedom (DOF) are used. The first one is artificial, only used to simultaneously illustrate all types of special points (except fracture). The second one pertains to a real but highly idealized structure, and brings attention to the effect of imperfections. All calculations are carried out in closed form. No FEM discretization is required.

§ 5.6.1. The Circle Game

The first problem, dubbed The Circle Game,\(^2\) assumes the following total residual function

\[
r(\mu, \lambda) = (\lambda - \mu)(\lambda^2 + \mu^2 - 1) = 0,
\]

(5.21)

in which \( \mu \) is a dimensionless state parameter and \( \lambda \) a dimensionless control parameter. The non-separable residual (5.21) is wholly artificial: no real structure produces it.\(^3\) It is useful, however,

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\(^2\) After a well known song by Joni Mitchell; see http://www.youtube.com/watch?v=emMbmxExQv. In the song, each turn around the circle stands for a year of a young person’s life.

\(^3\) To get a roughly similar response associated with a real structural system, at least two DOF are required.
in illustrating several types of special points using simple diagrams. Plainly the equilibrium paths associated with (5.21) are: (i) the fundamental path $\lambda = \mu$ that passes through the origin reference point, and (ii) the unit-radius circle $\lambda^2 + \mu^2 = 1$, which forms a secondary path. See Figure 5.3(a).

The following interesting points are marked there:

- **One reference point** $R$, at $\lambda = \mu = 0$.
- **Two bifurcation points** $B_1$ and $B_2$, at $\lambda = \pm 1/\sqrt{2}$, $\mu = \pm 1/\sqrt{2}$.
- **Two limit points** $L_1$ and $L_2$, at $\lambda = \pm 1$, $\mu = 0$.
- **Two turning points** $T_1$ and $T_2$, at $\lambda = 0$, $\mu = \pm 1$.
- **Two non-equilibrium vortex points** $V_1$ and $V_2$, at $\lambda = \pm 1/\sqrt{6}$, $\mu = \mp 1/\sqrt{6}$.

The last type (vortex point) has not been introduced before. Those points are characterized below. Unlike the other seven points, $V_1$ and $V_2$ do not lie on an equilibrium path, but may be reached via perturbed residuals $r_c = \pm 4/(3\sqrt{6}) = \pm 0.544331$.

Figure 5.3(b) shows the incremental flow $r = r_c$ drawn for several sample values of $r_c$. The equation $r(\mu, \lambda) = r_c$ is cubic in both $\mu$ and $\lambda$, and its closed form solution for either $\mu$ or $\lambda$ gives three roots. For example, the solution $\lambda = \lambda(\mu, r_c)$ obtained by Mathematica can be expressed as

\[
\begin{align*}
A_1 &= 2\mu^2 - 3, \quad A_2 = 2\sqrt{2}A_1, \quad A_3 = 27r_c - 18\mu + 20\mu^3, \\
A_4 &= \sqrt{4A_1^3 + (27r_c - 18\mu + 20\mu^3)^2}, \quad A_5 = \frac{3}{2}A_3 + A_4, \\
B_1 &= \frac{3}{4}A_5, \quad B_2 = A_2/A_5, \quad C_1 = 1 + j\sqrt{3}, \quad C_2 = 1 - j\sqrt{3}, \quad \text{with } j = \sqrt{-1}, \\
\lambda_1 &= \frac{1}{6}(2\mu - B_2 + B_1), \quad \lambda_2 = \frac{1}{12}(4\mu + C_1B_2 - C_2B_1), \quad \lambda_3 = \frac{1}{12}(4\mu + C_2B_2 - C_1B_1).
\end{align*}
\]  

(5.22)

For specified $\mu$ and $r_c$, (5.22) gives 1 or 3 real roots, as can be graphically observed in Figure 5.3(b). The vortex points $V_1$ and $V_2$ are recognized by neighboring closed orbits of constant $r_c$.  

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4 Vortex points are also called centers or center points in the literature — they are important in nonlinear dynamics.
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The tangent stiffness matrix $K$, incremental load vector $q$ and incremental velocity vector $v$ reduce to scalars $K$, $q$ and $v$, respectively, which are obtained as

$$K = \frac{\partial r}{\partial \mu} = 1 - \lambda^2 + 2\lambda \mu - 3\mu^2, \quad q = -\frac{\partial r}{\partial \lambda} = 1 - 3\lambda^2 + 2\lambda \mu - \mu^2, \quad v = \frac{q}{K}.$$ 

Signs taken by $K$, $q$, and $v$ over the $\{\lambda, \mu\}$ plane are shown in Figure 5.4(a,b,c), respectively.

The $\{\lambda, \mu\}$ region where $K > 0$ is called stable, whereas that where $K < 0$ is unstable. The transition locus $K = 0$, which lies on the ellipse $\lambda^2 - 2\lambda \mu + \mu^2 = 1$ highlighted in Figure 5.4(a), is labelled neutrally stable or simply neutral. Note that transition from stability to instability along an equilibrium path occurs at the critical points $B_1$, $B_2$, $L_1$ and $L_2$. As noted previously in (5.1), this is a general property of a conservative system in static equilibrium.

Figure 5.4(b) displays sign regions for the incremental load $q$. This value vanishes over the ellipse $3\lambda^2 - 2\lambda \mu + \mu^2 = 1$. This ellipse is tilted with respect to the zero-stiffness one. It passes through the bifurcation and turning points, but not through the limit points.

The ellipses of Figures 5.4(a,b) intersect at four points: the two bifurcation points $B_1$ and $B_2$, and the two vortex points $V_1$ and $V_2$, as pictured in Figure 5.4(c). The bifurcation points lie on equilibrium paths but the vortex points do not. At those four points both $K$ and $q$ vanish, whence the incremental velocity $v = q/K$ takes on the indeterminate form $0/0$.

The conditions stated in §5.3 to distinguish limit and bifurcation points can be easily checked in this example. Since $K$ is a scalar, stiffness singularity means $K = 0$. The null eigenvector, normalized to unit length, has only one entry: $z_{cr} = 1$. Thus the indicator $z_{cr}^T q_c$ reduces to $q_{cr}$, which is $q$ evaluated at a critical point. From inspection of Figure 5.4(b) it is plain that $q_{cr} = 0$ at $B_1$ and $B_2$ whereas $q_{cr} \neq 0$ at $L_1$ and $L_2$. This corroborates the rules stated in (5.12) and (5.13).

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5 Those conclusions are justified in a later chapter that specifically deals with stability.
§5.6 CRITICAL POINT COMPUTATION PROBLEMS

![Diagram of a perfect propped rigid cantilever column with extensional spring remaining horizontal.](image)

**Figure 5.5.** Geometrically exact analysis of a perfect propped rigid cantilever (PRC) column with extensional spring remaining horizontal: (a) untilted, unloaded column; (b) tilted column after buckling.

### §5.6.2. Perfect Propped Rigid Cantilever Column

The second example considers the configuration shown in Figure 5.5(a). A rigid strut $AB$ of length $L$ is hinged at $B$ and supports a downward vertical load $P$ at tip $A$. The load remains vertical as the column tilts. The column is propped by an extensional spring of stiffness $k$ attached to $A$. This configuration will be called a propped cantilevered rigid column; or PCR column for short.

Since the column is rigid, there is only one DOF. Two convenient choices for it are: the tilt angle $\theta$, or the tip horizontal displacement: $u_A = L \cos \theta$. We select the latter as state parameter, but render it dimensionless on dividing by $L$: $\mu = u_A / L = \sin \theta$. For convenience, the single control parameter $\lambda$ is defined from $P = \lambda kL$, which also makes $\lambda$ dimensionless.

For a geometrically exact analysis it is important to know what happens to the extensional spring as the column tilts. The simplest assumption is that it remains horizontal, as pictured in Figure 5.5(b). The spring force $k u_A$ is then horizontal and points to the left if $u_A > 0$. Doing a FBD at the displaced tip position $A'$ and taking moments with respect to $B$ yields the equilibrium condition

$$ P L \sin \theta = k u_A L(1 - \cos \theta), \quad (5.23) $$

which can be transformed to $\lambda k L^2 \sin \theta = k L^2 \sin \theta (1 - \cos \theta)$. Cancelling out $k L^2$, replacing $\mu = \sin \theta$, and converting to total residual yields

$$ r(\lambda, \mu) = \mu - \lambda \frac{\mu}{\sqrt{1 - \mu^2}}. \quad (5.24) $$

This can be also be derived as the gradient $r = \partial \Pi / \partial \mu$ of the total potential energy function

$$ \Pi(\lambda, \mu) = \frac{1}{2} \mu^2 + \lambda \sqrt{1 - \mu^2}. \quad (5.25) $$

The two equilibrium solutions provided by $r = 0$ are

$$ \mu = 0 \quad \text{for any} \ \lambda, \quad \lambda = \sqrt{1 - \mu^2}. \quad (5.26) $$

---

6 It is important not to cancel out the $\sin \theta$ on both sides, as otherwise the primary path $\mu = \sin \theta = 0$ would be lost.
These solutions yield the vertical (untilted) and tilted column equilibrium paths, respectively. These are the primary and secondary paths shown in Figure 5.6(a). The secondary path falls on the unit circle \( \lambda^2 + \mu^2 = 1 \). For \( \lambda > 0 \) the two paths intersect at \( \mu = 0 \) and \( \lambda = \lambda_{cr} = 1 \), which is a bifurcation point. The secondary path exhibits two turning points at \( \lambda = 0, \mu = \pm 1 \).

As in the Circle Game example, the tangent stiffness matrix \( K \), incremental load vector \( q \) and incremental velocity vector \( v \) reduce to scalars \( K, q \) and \( v \), respectively, which are given by

\[
K(\mu) = \frac{\partial r}{\partial \mu} = -\frac{\mu^2}{1 - \mu^2}, \quad q(\mu) = -\frac{\partial r}{\partial \lambda} = \frac{\mu}{\sqrt{1 - \mu^2}}, \quad v(\mu) = \frac{q}{K} = -\frac{\sqrt{1 - \mu^2}}{\mu}.
\]  

(5.27)

In terms of the tilt angle, \( K = -\tan^2 \theta, q = \tan \theta \) and \( v = -\cot \theta \). The \( K \) and \( v \) given above, however, are only valid for the secondary path. On the primary path \( \mu = 0, K \to \pm \infty \) and \( v \to 0/0 \). The stiffness coefficient for the secondary branch is plotted in Figure 5.6(b). Since \( K < 0 \) except at \( \mu = 0 \), the entire path is unstable. In fact, of the four branches that emanate from \( B \), only one (the \( \mu = 0 \) primary path for \( \lambda < \lambda_{cr} \)) is stable. After buckling the tilting column supports only a decreasing load, which vanishes at the turning points. Consequently this structural configuration is poor from the standpoint of post-buckling safety.

§5.6.3. Imperfect Propped Rigid Cantilever Column

We now consider a variation of the previous problem, in which the PRC column exhibits an initial geometric imperfection, as shown in Figure 5.7(a). When the column is unloaded, it tilts by an angle \( \theta_0 \) or, equivalent, a tip horizontal displacement \( L \epsilon = L \sin \theta \). We shall take \( \epsilon = \sin \theta \) as measure of initial imperfection. The state parameter \( \mu = u_A/L \) and control parameter \( \lambda = P/(kL) \) are defined as before. The total residual (5.24) changes to

\[
r(\lambda, \mu, \epsilon) = \mu - \epsilon - \lambda \frac{\mu}{\sqrt{1 - \mu^2}}.
\]  

(5.28)

This is the gradient of the total potential energy function

\[
\Pi(\lambda, \mu, \epsilon) = \frac{1}{2} (\mu - \epsilon)^2 + \lambda \sqrt{1 - \mu^2}.
\]  

(5.29)
which reduces to (5.25) if $\epsilon = 0$. The equilibrium path satisfying $r = 0$ is given by

$$\lambda = (1 - \frac{\epsilon}{\mu})\sqrt{1 - \mu^2}.$$  \hfill (5.30)

If $\epsilon \neq 0$, this solution gives two branches separated by the $\lambda$ axis, as pictured in Figure 5.8(a). (The formal limit $\epsilon \to 0$ gives only the secondary path $\lambda = \sqrt{1 - \mu^2}$ of the perfect column.) The previous formulas for $K$ and $v$ given in (5.27) change to

$$K(\mu, \epsilon) = \frac{\partial r}{\partial \mu} = \frac{\epsilon - \mu^3}{\mu(1 - \mu^2)}, \quad v(\mu, \epsilon) = \frac{q}{K} = -\frac{\mu^2 \sqrt{1 - \mu^2}}{\mu^3 - \epsilon},$$  \hfill (5.31)

whereas $q$ remains the same. Figure 5.8(a) plots the equilibrium paths given by (5.30) for selected values of $\epsilon$. The resemblance of this picture to an incremental flow is not accidental. If the flow is produced by setting the residual (5.24) of the perfect column to $r_c = \epsilon$, the residual (5.28) of the imperfect column is obtained.

Inspection of Figure 5.8(a) shows that if $\epsilon \neq 0$ the bifurcation point disappears. If so the response consists of two paths, which do not intersect:

- A primary path that takes off from the unloaded but imperfect configuration $\lambda = 0 \mu = \epsilon$. This path exhibits a limit point. Those points are marked by circles in Figure 5.8(a).
- A secondary path located in the opposite half-plane. This path has no critical points.

The two paths are separated by the perfect column equilibrium paths (5.26). The primary path limit points collectively represent a set of critical load values, which lie on the curve

$$\lambda_{cr}(\mu) = (\sqrt{1 - \mu^2})^3 = \cos^{3/2} \theta.$$  \hfill (5.32)

This is called a critical point locus, which separates the stable and unstable regions, as can be surmised from the stiffness coefficient plotted in Figure 5.8(b). This separation is shown in further detail in Figure 5.9(a). Eliminating $\mu$ in (5.32) in favor of $\epsilon$ yields the imperfection sensitivity diagram

$$\lambda_{cr}(\epsilon) = (1 - \epsilon^{2/3})^{3/2} = 1 - \frac{3}{2}\epsilon^{2/3} + \frac{3}{8}\epsilon^{4/3} + \frac{1}{16}\epsilon^2 + \ldots$$  \hfill (5.33)

This expression is plotted in Figure 5.9(b). From the Taylor series given above one can see that this curve has a vertical tangent as $\epsilon \to 0$. This feature highlights that this structural configuration exhibits high sensitivity of load capacity to small initial imperfections.
Chapter 5: CRITICAL POINTS

(a) \[ k \]
(b) \[ \theta \]

**Figure 5.7.** Geometrically exact analysis of an imperfect propped rigid cantilever (PRC) column with extensional spring remaining horizontal: (a) untilted, unloaded column; (b) tilted loaded column.

(a) (b)

**Figure 5.8.** Response of imperfect PRC column: (a) Load-deflection response for sample values of \( \epsilon \); (b) Stiffness coefficient versus state parameter for those values. Sample values of \( \epsilon \) are annotated near the corresponding curves.

(a) (b)

**Figure 5.9.** More detailed display of imperfection effects: (a) critical point locus separating stable and unstable regions; (b) imperfection sensitivity diagram, which displays decreasing load capacity as the imperfection parameter grows.
Notes and Bibliography

Overall there is a huge and nondescript literature on critical points, and their application to stability. That Babel pile is fractured according to applications as well as communities.

As regards structural stability, the classic reference is Timoshenko [810], originally published in 1936 (there is an expanded 1961 second edition by Dover). It collects most everything known on the subject, from Euler until the mid 1930s. Focus is on linearized bifurcation buckling. Nonlinear stability and limit points receive scant attention, as such calculations were considered far too demanding given the absence of computers. Timoshenko’s problem-by-problem, example-focused approach has influenced books on structural stability since. Among them the textbook by Bazant and Cedolin [68] stands out by its comprehensive coverage that includes plasticity, creep, dynamics, localization and fracture. The example-driven monograph by Panovko and Gubanova [604], translated from the Russian, is less ambitious but makes enjoyable reading.

Stronger from a computational viewpoint is the textbook by Brush and Almroth [120]. The monograph by Bushnell [124] has more physics (e.g., temperature, plasticity and creep effects) and a wider selection of practical problems. The book by Seydel [736] has a nice description of computational methods although its focus is on chemical engineering problems. The two volumes by Crisfield [181,182] are oriented to structural mechanics and brim with physical insight, but unfortunately use Fortran as expository implementation language.7 (Only PL-1 — also designed by committee — was uglier, fortunately long departed.)

Before computers came along, perturbation methods were popular for treating complicated problems in elastic stability. Two monographs by Thompson and Hunt [804,806] delve on the topic. The 1975 survey by Gallagher [312] is more FEM oriented.

The connection between potential-based structural stability and “catastrophe theory” is presented in a highly readable manner by Poston and Steward [657] and Thompson [805]. That theory began as a serious effort to systematize singular behavior of potential-driven systems and ended as a joke after outlandish claims of application to the natural and social sciences. But, like chaos, it had its 15 minutes of fame. Another contemporary newcomer: fractals, has displayed more sustaining power thanks to use in computer graphics.

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7 A tongue in cheek comment by Ken Thompson (co-inventor of the C language and the original Unix) on accepting the 1983 ACM Turing Award seems appropriate: “Since this is an Exercise divorced from reality, the usual vehicle was Fortran. Actually Fortran was the language of choice for the same reason three-legged races are popular.”
Chapter 5: CRITICAL POINTS

Homework Exercise for Chapter 5
Critical Points and Related Properties

EXERCISE 5.1 [A:20] Consider the one-parameter, two-degree-of-freedom (DOF) residual-force system

\[ r(u_1, u_2, \lambda) = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 6u_1 - 2u_2 - u_1^2 - 12\lambda \\ -2u_1 + 4u_2 - u_2^2 + 2\lambda \end{bmatrix} \] (E5.1)

Choose the point \( P(u_1, u_2, \lambda) \) located at

\[ u_1 = 2, \quad u_2 = 1, \quad \lambda = \frac{1}{2}, \] (E5.2)

(a) Show that \( P \) is on an equilibrium path,
(b) Show that \( P \) is a critical point,
(c) Determine whether it is a limit or a bifurcation point. Hint: compute the null eigenvector \( z \) of \( K \) at that point and apply the criterion given in §5.3.
(d) Verify whether the limit point sensor \( \kappa \) defined by (5.17)–(5.18) vanishes at \( P \).

EXERCISE 5.2 [A:15] Show that all limit points of (E5.1) satisfy either of the equations

\[ 63 - u_1 - 36u_2 = 0, \quad 5 - 2u_1 - 3u_2 + u_1u_2 = 0 \] (E5.3)

called limit point surfaces (or LP surfaces for short), and that the only intersection of these surfaces and the primary equilibrium path is at (E5.2).

Hint: compute the LP sensor \( \kappa = (q^Tv)/(v^Tv) \) — in which \( v = K^{-1}q \) — and note that the numerator splits into 2 factors, which immediately gives the LP surfaces if \( \kappa = 0 \). But only one intersects with the equilibrium path \( r_1 = 0, r_2 = 0 \). Quick way to show that: set \( \det(K) = 0 \) and observe that it gives only one of the two LP surfaces; the other is spurious.

EXERCISE 5.3 [A:20] Show that the critical point surface defined by \( \det(K) = 0 \) is independent of \( \lambda \) if the residual force system is separable.

EXERCISE 5.4 [A:15] Show that \( q^Tz \) is independent of \( \lambda \) if the residual force system is separable and the load is proportional. (This used in a later Chapter to show that bifurcation points do not exist in that case, except under very special circumstances.)

EXERCISE 5.5 [A:15] Does The Circle Game residual (5.21) have a potential? If so, find it.

Hint: the alleged total-potential function \( \Pi(u, \lambda) \), exists if the stiffness matrix is symmetric: \( K = K^T \), or \( K_{ij} = \frac{\partial r_i}{\partial u_j} = K_{ji} = \frac{\partial r_j}{\partial u_i} \), and the indicated derivatives exist. If this condition holds, \( \Pi \) can be obtained by partial integration of \( r \) with respect to \( u \), plus an arbitrary function of \( \lambda \). Given all that, what can you say about the 1-DOF case such as (5.21)?

EXERCISE 5.6 [A:15] Generalization of above: a system governed by a 1-DOF residual has always a potential if it is integrable in the state \( u \).

EXERCISE 5.7 [A:30] (Advanced, requires knowledge of matrix eigensystem spectral theory). If \( K \) is not symmetric, the critical point classification argument based on \( q^Tz \) and stated in §5.3 fails. Explain why.
Homework Exercises for Chapter 5

Solutions

EXERCISE 5.1 Substituting \( u_1 = 2, u_2 = 1 \) and \( \lambda = 1/2 \) into the residual equations gives \( r_1 = r_2 = 0 \). Consequently those values correspond to an equilibrium state.

The tangent stiffness matrix evaluated at that point is

\[
K = \frac{\partial r}{\partial u} = \begin{bmatrix}
6 - 2u_1 & -2 \\
-2 & 4 - 2u_2
\end{bmatrix} = \begin{bmatrix}
2 & -2 \\
-2 & 2
\end{bmatrix}, \tag{E5.4}
\]

which is obviously singular. Thus those values \((u_1 = 2, u_2 = 1\) and \(\lambda = 1/2\)) pertain to a critical point. The null eigenvector is \( z^T = [1 \ 1] \). The incremental load vector is \( q = -\frac{\partial r}{\partial u} = [12 \ -2]^T \). Hence

\[
q^Tz = 12 - 2 = 10 \neq 0, \tag{E5.5}
\]

which classifies that critical point as a limit point. The incremental velocity is

\[
v = K^{-1}q = \begin{bmatrix}
6 - 2u_1 & -2 \\
-2 & 4 - 2u_2
\end{bmatrix}^{-1} \begin{bmatrix}
12 \\
-2
\end{bmatrix} = \frac{1}{5 - 2u_1 - 3u_2 + u_1u_2} \begin{bmatrix}
11 - 6u_2 \\
3 + u_1
\end{bmatrix} \tag{E5.6}
\]

The limit point sensor is

\[
\kappa = \frac{1}{\kappa_0} \frac{q^Tv}{v^Tv} = \frac{26(63 - u_1 - 36u_2)(5 - 2u_1 - 3u_2 + u_1u_2)}{63(130 + 6u_1 + u_1^2 - 132u_2 + 36u_2^2)} \tag{E5.7}
\]

This sensor vanishes at \( u_1 = 2, u_2 = 1 \), since \( 5 - 2 \times 2 - 3 \times 1 + 2 \times 1 = 0 \).

EXERCISE 5.2

This follows from the result (E5.7) of the previous exercise, since the two equations in (E5.3) make \( \kappa = 0 \). Substituting those equations into (E5.1) it is found that the only equilibrium solution is (E5.2). Can also be done graphically, plotting all equations on the \((u_1, u_2)\) plane.

EXERCISE 5.3 For a separable system \( p \) is independent of \( \lambda \); so is \( K = \frac{\partial p}{\partial u} \) as well as its determinant.

EXERCISE 5.4 From the previous exercise \( z \), which is an eigenvector of \( \det K = 0 \), is independent of \( \lambda \) if the residual is separable. If the load is proportional, \( q \) is constant. Thus \( q^Tz \) is independent of \( \lambda \).

EXERCISE 5.5 Yes, since a \( 1 \times 1 \) tangent stiffness matrix is always symmetric. Integrating the residual with respect to the state gives the total potential

\[
\Pi(\mu, \lambda) = \int r(\mu, \lambda) d\mu = \frac{\mu}{12} \left( 12\lambda^3 + 6\mu - 6\lambda^2\mu - 3\mu^3 + 4(\mu^2 - 3) \right) + g(\lambda), \tag{E5.8}
\]

in which \( g(\lambda) \) is an arbitrary function of \( \lambda \) alone.

EXERCISE 5.6 Plainly follows from the Hint of the previous Exercise and the definition of residual.

EXERCISE 5.7 If \( K \) is unsymmetric the eigensystem \( \det K = 0 \) has both right and left eigenvectors, which are generally different. The criterion \( q^Tz = 0 \) only considers the right eigenvector \( z \), and thus may fail.