F

Fitting Element Fields

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§F.1. Introduction

This Appendix studies the problem of Polynomial Field Fitting over a finite element domain. This is a subset of the more general curve- and surface-fitting problem, as covered for example in [476]. This special case is defined as follows.

(I) A polynomial interpolation of a scalar function f, called the *base function*, is defined over a finite element domain using n nodal values of f and n shape functions expressed in natural coordinates.

(II) A polynomial interpolation of a scalar function g, called the *fitted function*, is defined over the same element domain in terms of $m \leq n$ nodal values and m shape functions. This function approximates the base function f in a least square sense.

The interpolations involved in (I) and (II) will be called the *base* and *fitted* interpolants of f and g, respectively.

The case $m = n$ is not excluded, as it covers situations in which the same polynomial order is defined from different nodal values. A well know example is the linear triangle with $n = 3$ corner values (the Turner triangle) versus that with $m = 3$ midpoint values (the Veubeke triangle).

Passing from f to g involves a *least square fitting*. This is an operation that has been extensively studied in the applied mathematics literature. However, most of the work available there does not consider the use of natural coordinates as well as integration over the element domain. These two features are essential for finite element applications.

Passing from g to f is a *collocation* operation, which simply involves evaluating g at the f nodes.

§F.2. Least-Square Fitting

Let the node values for the base and reduced interpolant be arranged in column vectors denoted by $\mathbf{f}$ and $\mathbf{g}$ of lengths n and m, respectively. The associated shape functions are arranged in row vectors $\mathbf{N}_f$ and $\mathbf{N}_g$, respectively. The shape functions of both f and g will be assumed to be linearly independent. The interpolations may be expressed as the dot products

$$f = \mathbf{N}_f\,\mathbf{f}, \qquad g = \mathbf{N}_g\,\mathbf{g}. \tag{F.1}$$

The distance between g and f is $d = g - f = \mathbf{N}_g\,\mathbf{g} - \mathbf{N}_f\,\mathbf{f}$. Its square is

$$d^2 = (\mathbf{N}_g\,\mathbf{g} - \mathbf{N}_f\,\mathbf{f})^T\,(\mathbf{N}_g\,\mathbf{g} - \mathbf{N}_f\,\mathbf{f}) = \mathbf{g}^T\,\mathbf{N}_g^T\,\mathbf{N}_g\mathbf{g} - 2\,\mathbf{g}^T\,\mathbf{N}_g^T\,\mathbf{N}_f\,\mathbf{f} + \mathbf{f}^T\,\mathbf{N}_f^T\,\mathbf{N}_f\mathbf{f}. \tag{F.2}$$

As the distance norm we take

$$s = \tfrac{1}{2}\int_\Omega d^2\,d\Omega = \tfrac{1}{2}\left(\mathbf{f}^T\,\mathbf{A}\,\mathbf{f} - 2\,\mathbf{g}^T\,\mathbf{B}\,\mathbf{f} + \mathbf{g}^T\,\mathbf{C}\mathbf{g}\right) = \tfrac{1}{2}\,[\,\mathbf{f}^T \quad \mathbf{g}^T\,]\begin{bmatrix}\mathbf{A} & -\mathbf{B}^T\\ -\mathbf{B} & \mathbf{C}\end{bmatrix}\begin{bmatrix}\mathbf{f}\\ \mathbf{g}\end{bmatrix}. \tag{F.3}$$

in which Ω denotes the element domain, and

$$\mathbf{A} = \int_\Omega \mathbf{N}_f^T\,\mathbf{N}_f\,d\Omega, \quad \mathbf{B} = \int_\Omega \mathbf{N}_g^T\,\mathbf{N}_f\,d\Omega, \quad \mathbf{C} = \int_\Omega \mathbf{N}_g^T\,\mathbf{N}_g\,d\Omega. \tag{F.4}$$

Matrices $\mathbf{A}$ and $\mathbf{C}$ are called the *self kernels* for f and g, respectively. They are symetric and positive definite (because the shape functions are assumed to be liearly independent). Matrix $\mathbf{B}$ of order $m \times n$ is the *coupled kernel*, which is rectangular unless $n = m$.[1]

Both $\mathbf{A}$ and $\mathbf{C}$ are symmetric and positive definite, of orders $n \times n$ and $m \times m$, respectively, whereas the $m \times n$ matrix $\mathbf{B}$ is generally rectangular. Minimizing s with respect to $\mathbf{g}$ yields

$$\frac{\partial s}{\partial \mathbf{g}} = \mathbf{C}\,\mathbf{g} - \mathbf{B}\,\mathbf{f} = \mathbf{0}, \quad \text{whence} \quad \mathbf{g} = \mathbf{C}^{-1}\,\mathbf{B}\,\mathbf{f} = \mathbf{G}\,\mathbf{f}. \tag{F.5}$$

$\mathbf{G} = \mathbf{C}^{-1}\,\mathbf{B}$ will be called the $f \to g$ *fitting matrix*. When identification of orders m and n are convenient, the matrix will be denoted $\mathbf{G}^n_m$.

Minimizing s with respect to $\mathbf{f}$ yields

$$\frac{\partial s}{\partial \mathbf{f}} = \mathbf{A}\,\mathbf{f} - \mathbf{B}^T\,\mathbf{g} = \mathbf{0}, \quad \Leftarrow \quad \mathbf{f} = \mathbf{A}^{-1}\,\mathbf{B}^T\,\mathbf{g} = \mathbf{F}\,\mathbf{f}. \tag{F.6}$$

$\mathbf{F} = \mathbf{A}^{-1}\,\mathbf{B}$ will be called the $g \to f$ *collocation matrix*. It is a generalized inverse of $\mathbf{G}$ since

$$\mathbf{G}\,\mathbf{F} = \mathbf{I}, \qquad \mathbf{F}\,\mathbf{G} = \mathbf{H}, \tag{F.7}$$

in which $\mathbf{I}$ is the $m \times m$ identity matrix whereas $\mathbf{H}$ is an $n \times n$ projector (idempotent) matrix satisfying $\mathbf{H} = \mathbf{H}^2$. Matrix $\mathbf{H}$, which has rank m, is not generally symmetric.

Replacing the expressions of $\mathbf{G}$ and $\mathbf{F}$ into $\mathbf{G}\,\mathbf{F} = \mathbf{I}$ yields the property

$$\mathbf{C}^{-1}\mathbf{B}\,\mathbf{A}^{-1}\,\mathbf{B}^T = \mathbf{I}, \quad \text{or} \quad \mathbf{B}^T\,\mathbf{A}^{-1}\,\mathbf{B} = \mathbf{C}, \tag{F.8}$$

whereas replacing the expressions of $\mathbf{G}$ and $\mathbf{F}$ into $\mathbf{F}\,\mathbf{G} = \mathbf{H}$ yields the property

$$\mathbf{A}^{-1}\mathbf{B}^T\,\mathbf{C}^{-1}\,\mathbf{B} = \mathbf{H}, \quad \text{or} \quad \mathbf{B}^T\,\mathbf{C}^{-1}\,\mathbf{B} = \mathbf{A}\,\mathbf{H}. \tag{F.9}$$

Notice that $\mathbf{B}^T\,\mathbf{A}^{-1}\,\mathbf{B}$ and $\mathbf{B}^T\,\mathbf{C}^{-1}\,\mathbf{B}$ are the Schur complements of $\mathbf{C}$ and $\mathbf{A}$, respectively, in the supermatrix shown in (F.3). Since $\mathbf{B}^T\,\mathbf{C}^{-1}\,\mathbf{B}$ is symmetric, so is $\mathbf{A}\,\mathbf{H}$. It can be readily shown that the reverse product $\mathbf{H}\,\mathbf{A}$ is antisymmetric.

Inserting (F.5) into (F.3) gives the minimum distance as

$$s_{min} = \tfrac{1}{2}\,\mathbf{f}^T\,(\mathbf{A} - \mathbf{B}^T\,\mathbf{C}^{-1}\,\mathbf{B})\,\mathbf{f} = \tfrac{1}{2}\,\mathbf{f}^T\,(\mathbf{A} - \mathbf{A}\,\mathbf{H})\,\mathbf{f} = \tfrac{1}{2}\,\mathbf{f}^T\,\mathbf{A}\,\hat{\mathbf{H}}\,\mathbf{f}, \tag{F.10}$$

in which $\hat{\mathbf{H}} = \mathbf{I} - \mathbf{H}$. Since $\mathbf{H}$ is a projection matrix, so is $\hat{\mathbf{H}}$. As a quick check, take $\mathbf{N}_f \equiv \mathbf{N}_g$, whence $m = n$ and $\mathbf{C} = \mathbf{B} = \mathbf{A}$; if so $\mathbf{H} = \mathbf{I}$ and $s_{min} = 0$, as can be expected.

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[1] Readers familiar with computational dynamics may recognize $\mathbf{A}$ and $\mathbf{C}$ as the numerical part of the consistent mass matrices pertaining to the base and fitted elements, respectively.

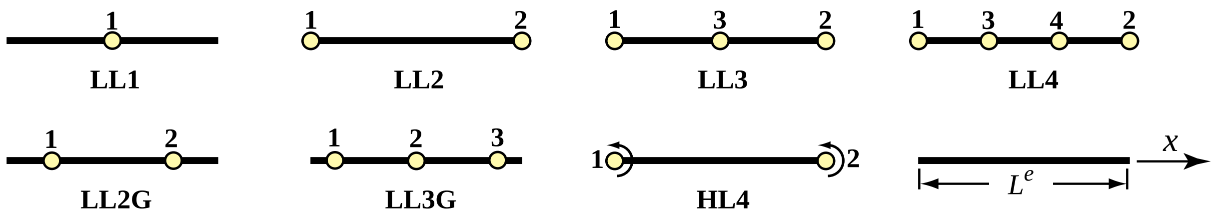

FIGURE F.1. Line element notation. LL and HL denote Lagrangian and Hermitian interpolation, respectively. Nodes shown as circles have only one DOF: the function value there. Nodes pictured with a rotation ("harpoon") symbol have two DOF there: the function value and its x-derivative scaled by the line length L^e.

```
LineShapeFunctions[typ_,ξ_,Le_]:=Module[{Nsf=Null,m=Null,ξnc=Null},
   If [typ=="LL1", Nsf={1}; m=0; ξnc={{0}}];
   If [typ=="LL2", Nsf={(1-ξ)/2,(1+ξ)/2}; m=1; ξnc={{-1},{1}}];
   If [typ=="LL3", Nsf={-(1-ξ)*ξ/2,(1+ξ)*ξ/2,1-ξ^2}; m=2;
          ξnc={{-1},{1},{0}}];
   If [typ=="LL4", Nsf={(1-ξ)*(9*ξ^2-1),(1+ξ)*(9*ξ^2-1),
          9*(3*ξ-1)*(ξ^2-1),9*(3*ξ+1)*(1-ξ^2)}/16; m=3;
          ξnc={{-1},{1},{-1/3},{1/3}}];
   If [typ=="LL2G", Nsf={1-Sqrt[3]*ξ,1+Sqrt[3]*ξ}/2;
          m=1; ξnc={{-Sqrt[3]/2},{Sqrt[3]/2}}];
   If [typ=="LL3G", Nsf={ξ*(-Sqrt[15]+5*ξ),6-10*ξ^2,ξ*(Sqrt[15]+5*ξ)}/6;
          m=2; ξnc={{-Sqrt[3/5]},{0},{Sqrt[3/5]}}];
   If [typ=="HL4", Nsf={2*(1-ξ)^2*(2+ξ),(1-ξ)^2*(1+ξ)*Le,2*(1+ξ)^2*(2-ξ),
        -(1+ξ)^2*(1-ξ)*Le}/8; m=3; ξnc={{-1},{-1},{1},{1}}];
   Return[{Nsf,m,ξnc}]];
```

FIGURE F.2. Line element shape function module.

```
IntegrateOverLine[f_,ξ_,Le_]:=Module[{fint},
     fint=Integrate[(Le/2)*f,{ξ,-1,1}]; Return[fint]];

LineFit[ftyp_,gtyp_,ξ_,Le_]:=Module[{Nf={},Ng={},ζf,ζg,mf,mg,
  Am,Bm,Cm,Ainv,Cinv,F,G,GF,FG},
  {Nf,mf,ζf}=LineShapeFunctions[ftyp,ξ,Le];
  {Ng,mg,ζg}=LineShapeFunctions[gtyp,ξ,Le];
  If [Nf==Null, Print["Unknown ftyp=",ftyp]; Return[Null]];
  If [Ng==Null, Print["Unknown gtyp=",gtyp]; Return[Null]];
  Am=IntegrateOverLine[Transpose[{Nf}].{Nf},ξ,Le]; Ainv=Inverse[Am];
  Bm=IntegrateOverLine[Transpose[{Ng}].{Nf},ξ,Le]; BmT=Transpose[Bm];
  Cm=IntegrateOverLine[Transpose[{Ng}].{Ng},ξ,Le]; Cinv=Inverse[Cm];
  G=Cinv.Bm;  F=Ainv.BmT; FG=F.G; GF=G.F;
  Return[{Am,Bm,Cm,G,F,GF,FG}]];
```

FIGURE F.3. Line element fitting module.

§F.3. Line Elements

Line elements are used for one-dimensional FEM models. In structural mechanics: bars, beams, shafts, etc. This Section tabulates results for the seven line elements shown in Figure F.1. The line segment is assumed straight with constant metric and lenth L^e along x. As a consequence, the integrals over the line can be evaluated exactly, without resource to numerical integration.

Elements labeled `"LLn"` are the standard Lagrangian ones, with function values at equally spaced `n` nodes, which include the end points if $n \geq 2$. Elements labeled `"LLnG"` have internal function values given at `n>=2` Gauss points.[2] That labeled `"HL4"` is the standard Hermite cubic-interpolated element with 4 DOF: two end-node values and two length-scaled[3] x-derivatives at the end nodes: $[\, f_1,\, (df/dx)_1\, L^e,\; f_2,\, (df/dx)_2\, L^e]$.

§F.3.1. Line Element Processing Modules

The shape functions for the line elements shown in Figure F.1 are returned by the *Mathematica* module `LineShapeFunctions` listed in Figure F.2. The module is invoked as

$$\texttt{\{Nsf,m,}\zeta\texttt{nc\}=LineShapeFunctions[typ,}\xi\texttt{,Le]} \tag{F.11}$$

in which

`typ` One of the identifiers shown in Figure **?** supplied as a character string (e.g., `"LL3"`)

ξ The isoparametric natural coordinate.

`Le` The element length. Only used for type `"HL4"`.

The module returns

`Nfs` A list of shape functions.

`m` The polynomial order of the shape functions.

ξ`nc` A list of isoparametric coordinates of the nodes.

If argument `typ` is not implemented, `Nsf` returns `Null`.

The polynomial fit operation is carried out by the *Mathematica* module `LTFit` listed in Figure F.6. The module is invoked as

$$\texttt{\{A,B,C,G,F,FG,GF\}=LineFit[ftyp,gtyp,}\xi\texttt{,Le]} \tag{F.12}$$

in which

`ftyp,gtyp` Identifiers of the base and fitted line element, respectively.

ξ The isoparametric natural coordinate.

`Le` The line segment length. Only relevant if `ftyp` or `gtyp` is `"HL4"`, but typically is set to 1 to reduce clutter in the fitting matrices.

The module returns a list of seven matrices: **A**, **B**, **C**, **G**, **F**, **F G** (which should be the identity), and $\mathbf{GF} = \mathbf{H}$ (which should be a projector). These matrices were introduced in §F.2.

---

[2] The one-point line element is denoted simply by `"LL1"` rather than ‘`"LL1G"` since no confusion can arise. These “Gauss elements” have specialized uses.

[3] The scaling by L^e is to reduce clutter in the fitting formulas. Scaling is supressed when node rotations are used.

§F.3.2. Line Self-Kernel Matrices

Self kernels are the matrices **A** and **C** defined in (F.4). Since $\mathbf{C} = \mathbf{A}$ if $\mathbf{N}_f \equiv \mathbf{N}_g$, only **A** is listed. Its subscript identifies the line element.

$$\mathbf{A}_{LL1} = [\,1\,], \quad \mathbf{A}_{LL2} = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{A}_{LL3} = \frac{1}{30}\begin{bmatrix} 4 & -1 & 2 \\ -1 & 4 & 2 \\ 2 & 2 & 16 \end{bmatrix},$$

$$\mathbf{A}_{LL4} = \frac{1}{1680}\begin{bmatrix} 128 & 19 & 99 & -36 \\ 19 & 128 & -36 & 99 \\ 99 & -36 & 648 & -81 \\ -36 & 99 & -81 & 648 \end{bmatrix}, \quad \mathbf{A}_{LL2G} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{F.13}$$

$$\mathbf{A}_{LL3G} = \frac{1}{18}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad \mathbf{A}_{HL4} = \frac{L_e}{420}\begin{bmatrix} 156 & 22\,L_e & 54 & -13\,L_e \\ 22\,L_e & 4\,L_e^2 & 13\,L_e & -3\,L_e^2 \\ 54 & 13\,L_e & 156 & -22\,L_e \\ -13\,L_e & -3\,L_e^2 & -22\,L_e & 4\,L_e^2 \end{bmatrix},$$

§F.3.3. Line Coupled Matrices

LL2 to LL1:

$$\mathbf{B} = \frac{1}{2}\,[\,1 \quad 1\,], \quad \mathbf{G} = \frac{1}{2}\,[\,1 \quad 1\,], \quad \mathbf{F} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{F.14}$$

LL2 to LL2G:

$$\mathbf{B} = \frac{1}{12}\begin{bmatrix} a_1 & a_2 \\ a_2 & a_1 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{6}\begin{bmatrix} a_1 & a_2 \\ a_2 & a_1 \end{bmatrix}, \quad \mathbf{F} = \frac{1}{2}\begin{bmatrix} a_3 & a_4 \\ a_4 & a_3 \end{bmatrix}, \tag{F.15}$$

in which $a_1 = 3 + \sqrt{3}$, $a_2 = 3 - \sqrt{3}$, $a_3 = 1 + \sqrt{3}$, and $a_4 = 1 + \sqrt{3}$.

LL3 to LL2:

$$\mathbf{B} = \frac{1}{6}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}, \quad \mathbf{F} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}. \tag{F.16}$$

LL3 to LL1:

$$\mathbf{B} = \frac{1}{6}\,[\,1 \quad 1 \quad 4\,], \quad \mathbf{G} = \frac{1}{6}\,[\,1 \quad 1 \quad 4\,], \quad \mathbf{F} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{F.17}$$

LL3 to LL3G:

$$\mathbf{B} = \frac{1}{36}\begin{bmatrix} b_1 & b_1 & 4 \\ 0 & 0 & 16 \\ b_1 & b_1 & 4 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{10}\begin{bmatrix} b_1 & b_1 & 4 \\ 0 & 0 & 10 \\ b_1 & b_1 & 4 \end{bmatrix}, \quad \mathbf{F} = \frac{1}{6}\begin{bmatrix} b_3 & -4 & b_4 \\ b_4 & -4 & b_3 \\ 0 & 6 & 0 \end{bmatrix}, \tag{F.18}$$

in which $b_1 = 3 + \sqrt{15}$, $b_2 = 3 - \sqrt{15}$, $b_3 = 5 + \sqrt{15}$, and $b_4 = 5 - \sqrt{15}$.

LL3 to LL2G:

$$\mathbf{B} = \frac{1}{12}\begin{bmatrix} a_3 & a_4 & 4 \\ a_4 & a_3 & 4 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{6}\begin{bmatrix} a_3 & a_4 & 4 \\ a_4 & a_3 & 4 \end{bmatrix}, \quad \mathbf{F} = \frac{1}{2}\begin{bmatrix} a_3 & a_4 \\ a_4 & a_3 \\ 1 & 1 \end{bmatrix}, \tag{F.19}$$

in which $a_1 = 3 + \sqrt{3}$, $a_2 = 3 - \sqrt{3}$, $a_3 = 1 + \sqrt{3}$, and $a_4 = 1 + \sqrt{3}$.

LL4 to LL3:

$$\mathbf{B} = \frac{1}{120}\begin{bmatrix} 11 & 0 & 18 & -9 \\ 0 & 11 & -9 & 18 \\ 4 & 4 & 36 & 36 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{80}\begin{bmatrix} 62 & 18 & 54 & -54 \\ 18 & 62 & -54 & 54 \\ -5 & -5 & 45 & 45 \end{bmatrix}, \quad \mathbf{F} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 2 & -1 & 8 \\ -1 & 2 & 8 \end{bmatrix}. \tag{F.20}$$

LL4 to LL2:

$$\mathbf{B} = \frac{1}{120}\begin{bmatrix} 13 & 2 & 36 & 9 \\ 2 & 13 & 9 & 36 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{20}\begin{bmatrix} 8 & -3 & 21 & -6 \\ -3 & 8 & -6 & 21 \end{bmatrix}, \quad \mathbf{F} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{F.21}$$

LL4 to LL1:

$$\mathbf{B} = \frac{1}{8}\,[\,1 \quad 1 \quad 3 \quad 3\,], \quad \mathbf{G} = \frac{1}{8}\,[\,1 \quad 1 \quad 3 \quad 3\,], \quad \mathbf{F} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \tag{F.22}$$

LL4 to LL3G:

$$\mathbf{B} = \frac{1}{720}\begin{bmatrix} h_1 & h_2 & h_3 & h_4 \\ -20 & -20 & 180 & 180 \\ h_2 & h_1 & h_4 & h_3 \end{bmatrix}, \mathbf{G} = \frac{1}{400}\begin{bmatrix} h_5 & h_6 & h_7 & h_8 \\ -25 & -25 & 225 & 225 \\ h_6 & h_5 & h_8 & h_7 \end{bmatrix}, \mathbf{F} = \frac{1}{54}\begin{bmatrix} h_9 & -36 & h_{10} \\ h_{10} & -36 & h_9 \\ h_{11} & 44 & h_{12} \\ h_{12} & 44 & h_{11} \end{bmatrix}. \tag{F.23}$$

in which $h_1 = 55 + 11\sqrt{15}$, $h_2 = 55 - 11\sqrt{15}$, $h_3 = 45 + 27\sqrt{15}$, $h_4 = 45 - 27\sqrt{15}$, $h_5 = 110 + 22\sqrt{15}$, $h_6 = 110 - 22\sqrt{15}$, $h_7 = 90 + 54\sqrt{15}$, $h_8 = 90 - 54\sqrt{15}$, $h_9 = 45 + 9\sqrt{15}$, $h_{10} = 45 - 9\sqrt{15}$, $h_{11} = 5 + 3\sqrt{15}$, and $h_{12} = 5 - 3\sqrt{15}$.

LL4 to LL2G:

$$\mathbf{B} = \frac{1}{240}\begin{bmatrix} g_1 & g_2 & g_3 & g_4 \\ g_2 & g_1 & g_4 & g_3 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{120}\begin{bmatrix} g_1 & g_2 & g_3 & g_4 \\ g_2 & g_1 & g_4 & g_3 \end{bmatrix}, \quad \mathbf{F} = \frac{1}{2}\begin{bmatrix} g_5 & g_6 \\ g_6 & g_5 \\ g_5 & g_6 \\ g_6 & g_5 \end{bmatrix} \tag{F.24}$$

in which $g_1 = 15 + 11\sqrt{3}$, $g_2 = 15 - 11\sqrt{3}$, $g_3 = 45 + 27\sqrt{3}$, $g_4 = 45 - 27\sqrt{3}$, $g_5 = 1 + \sqrt{3}$, and $g_6 = 1 - \sqrt{3}$.

(HL4 coupled matrices to be inserted here)

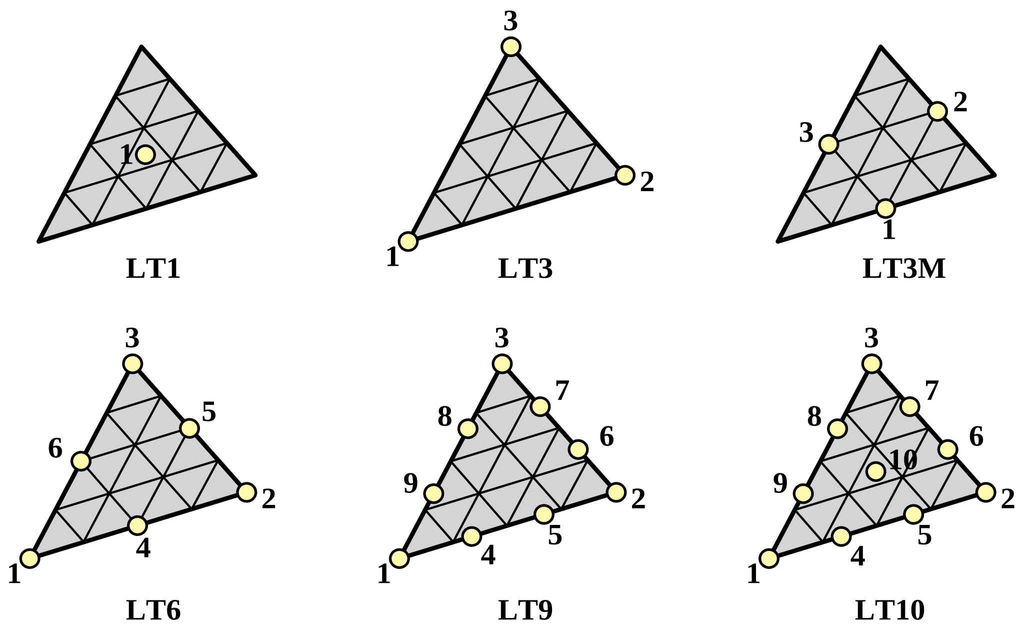

FIGURE F.4. Notation for Lagrangian triangles considered in §F.2. Figures taken from [221].

```
LTShapeFunctions[typ_,tcoor_]:=Module[{ζ1,ζ2,ζ3,Nsf=Null,m=Null,
  ζnc=Null}, {ζ1,ζ2,ζ3}=tcoor;
  If [typ=="LT1",  Nsf={1}; m=0; ζnc={{0,0,0}}];
  If [typ=="LT3",  Nsf={ζ1,ζ2,ζ3}; m=1; ζnc={{1,0,0},{0,1,0},{0,1,0}}];
  If [typ=="LT3M", Nsf={ζ1+ζ2-ζ3,-ζ1+ζ2+ζ3,ζ1-ζ2+ζ3}; m=1;
      ζnc={{1/2,1/2,0},{0,1/2,1/2},{1/2,0,1/2}}];
  If [typ=="LT6",  Nsf={ζ1*(2*ζ1-1),ζ2*(2*ζ2-1),ζ3*(2*ζ3-1),
      4*ζ1*ζ2,4*ζ2*ζ3,4*ζ3*ζ1}; m=2; ζnc={{1,0,0},{0,1,0},{0,0,1},
           {1/2,1/2,0},{0,1/2,1/2},{1/2,0,1/2}}];
  If [typ=="LT9", Nsf={ζ1*(3*ζ1-1)*(3*ζ1-2),ζ2*(3*ζ2-1)*(3*ζ2-2),
      ζ3*(3*ζ3-1)*(3*ζ3-2),9*ζ1*ζ2*(3*ζ1-1),9*ζ1*ζ2*(3*ζ2-1),
      9*ζ2*ζ3*(3*ζ2-1),9*ζ2*ζ3*(3*ζ3-1),9*ζ3*ζ1*(3*ζ3-1),
      9*ζ3*ζ1*(3*ζ1-1)}/2+9*{-2,-2,-2,3,3,3,3,3,3}*ζ1*ζ2*ζ3/4; m=3;
      ζnc={{1,0,0},{0,1,0},{0,0,1},{2/3,1/3,0},{1/3,2/3,0},
           {0,2/3,1/3},{0,1/3,2/3},{2/3,0,1/3},{1/3,0,2/3}}];
  If [typ=="LT10", Nsf={ζ1*(3*ζ1-1)*(3*ζ1-2),ζ2*(3*ζ2-1)*(3*ζ2-2),
      ζ3*(3*ζ3-1)*(3*ζ3-2),9*ζ1*ζ2*(3*ζ1-1),9*ζ1*ζ2*(3*ζ2-1),
      9*ζ2*ζ3*(3*ζ2-1),9*ζ2*ζ3*(3*ζ3-1),9*ζ3*ζ1*(3*ζ3-1),
      9*ζ3*ζ1*(3*ζ1-1),54*ζ1*ζ2*ζ3}/2; m=3;
      ζnc={{1,0,0},{0,1,0},{0,0,1},{2/3,1/3,0},{1/3,2/3,0},{0,2/3,1/3},
           {0,1/3,2/3},{2/3,0,1/3},{1/3,0,2/3},{1/3,1/3,1/3}}];
  Return[{Nsf,m,ζnc}]];
```

FIGURE F.5. Module `LTShapeFunctions` that returns shape function information for the Lagrange triangles of Figure F.4.

§F.4. Lagrangian Triangles

A *Lagrangian triangle* (LT) is one interpolated from node values only. They are primarily used for 2D problems of variational index 1, such as plane stress and heat conduction. This Section tabulates results for the six triangular elements shown in Figure F.4. It assumes flat triangles with constant

```
LTFit[ftyp_,gtyp_,tcoor_]:=Module[{Nf={},Ng={},mf,mg,ζf,ζg,
  Am,Bm,Cm,Ainv,Cinv,F,G,GF,FG},
  {Nf,mf,ζf}=LTShapeFunctions[ftyp,tcoor];
  {Ng,mg,ζg}=LTShapeFunctions[gtyp,tcoor];
  If [Nf==Null, Print["Unknown ftyp=",ftyp]; Return[Null]];
  If [Ng==Null, Print["Unknown gtyp=",gtyp]; Return[Null]];
  Am=IntegrateOverTriangle[Transpose[{Nf}].{Nf},1,mf+mf]; Ainv=Inverse[Am];
  Bm=IntegrateOverTriangle[Transpose[{Ng}].{Nf},1,mg+mf]; BmT=Transpose[Bm];
  Cm=IntegrateOverTriangle[Transpose[{Ng}].{Ng},1,mg+mg]; Cinv=Inverse[Cm];
  G=Cinv.Bm; F=Ainv.BmT; FG=F.G; GF=G.F;
  Return[{Am,Bm,Cm,G,F,GF,FG}]];
```

FIGURE F.6. Module `LTFit` that return the fitting matrices introduced in §F.2.

metric, in which the area integrations can be performed exactly, without resource to numerical integration.

§F.4.1. Lagrangian Triangle Processing Modules

The shape functions for the LT shown in Figure F.4 are returned by the *Mathematica* module `LTShapeFunctions` listed in Figure F.5. The module is invoked as

`{Nsf,m,ζnc}=LTShapeFunctions[typ,tcoor]` (F.25)

in which

- `typ` One of the LT identifiers shown in Figure F.4 supplied as a character string (e.g., `"LT3"`)
- `tcoor` A 3-item list of triangular coordinate symbols, usually `{ζ1,ζ2,ζ3}`.

The module returns

- `Nfs` A list of shape functions.
- `m` The polynomial order of the shape functions.
- `ζnc` A list of triangular coordinates of the nodes.

If argument `typ` is not implemented, `Nsf` returns `Null`.

The polynomial fit operation is carried out by the *Mathematica* module `LTFit` listed in Figure F.6. The module is invoked as

`{A,B,C,G,F,FG,GF}=LTFit[ftyp,gtyp,tcoor]` (F.26)

in which

- `ftyp,gtyp` Identifiers of the base and fitted triangle, respectively.
- `tcoor` A 3-item list of triangular coordinate symbols. The module returns a list of seven matrices: $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{G}$, $\mathbf{F}$, $\mathbf{F\,G}$ (which should be the identity), and $\mathbf{GF} = \mathbf{H}$ (which should be a projector). These matrices were introduced in §F.2.

Remark F.1. The shape functions for LT9 and LT10 were derived in the author's 1966 thesis [221]. The four others, due to Argyris and Fraeijs de Veubeke, were published one year earlier.

Remark F.2. Module `LTFit` carries out triangle integrals through module `IntegrateOverTriangle`, which is described in Exercise 15.11 of Chapter 15. The area integration is based on the formula (**?**), which is valid for a constant metric triangle.

§F.4.2. Lagrangian Triangle Self Kernels

Self kernels are the matrices $\mathbf{A}$ and $\mathbf{C}$ defined in (F.4). Since $\mathbf{C} = \mathbf{A}$ if $\mathbf{N}_f \equiv \mathbf{N}_g$, only $\mathbf{A}$ is listed.

1-Node Linear Triangle with Centroid Value. Labeled LT1 in Figure F.4.

$$\mathbf{A} = [\,1\,]. \tag{F.27}$$

3-Node Linear Triangle With Corner Values. Labeled LT3 in Figure F.4.

$$\mathbf{A} = \frac{1}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}. \tag{F.28}$$

3-Node Linear Triangle With Midpoint Values. Labeled LT3M in Figure F.4.

$$\mathbf{A} = \frac{1}{3}\,\mathbf{I}_3. \tag{F.29}$$

6-Node Quadratic Triangle. Labeled LT6 in Figure F.4.

$$\mathbf{A} = \frac{1}{180}\begin{bmatrix} 6 & -1 & -1 & 0 & -4 & 0 \\ -1 & 6 & -1 & 0 & 0 & -4 \\ -1 & -1 & 6 & -4 & 0 & 0 \\ 0 & 0 & -4 & 32 & 16 & 16 \\ -4 & 0 & 0 & 16 & 32 & 16 \\ 0 & -4 & 0 & 16 & 16 & 32 \end{bmatrix}, \tag{F.30}$$

9-Node Cubic Triangle. Labeled LT9 in Figure F.4.

$$\mathbf{A} = \frac{1}{13440}\begin{bmatrix} 236 & 106 & 106 & -162 & -198 & -144 & -144 & -198 & -162 \\ 106 & 236 & 106 & -198 & -162 & -162 & -198 & -144 & -144 \\ 106 & 106 & 236 & -144 & -144 & -198 & -162 & -162 & -198 \\ -162 & -198 & -144 & 1485 & 27 & 135 & 297 & 135 & 945 \\ -198 & -162 & -144 & 27 & 1485 & 945 & 135 & 297 & 135 \\ -144 & -162 & -198 & 135 & 945 & 1485 & 27 & 135 & 297 \\ -144 & -198 & -162 & 297 & 135 & 27 & 1485 & 945 & 135 \\ -198 & -144 & -162 & 135 & 297 & 135 & 945 & 1485 & 27 \\ -162 & -144 & -198 & 945 & 135 & 297 & 135 & 27 & 1485 \end{bmatrix}. \tag{F.31}$$

10-Node Cubic Triangle. Labeled LT10 in Figure F.4.

$$\mathbf{A} = \frac{1}{6720}\begin{bmatrix} 76 & 11 & 11 & 18 & 0 & 27 & 27 & 0 & 18 & 36 \\ 11 & 76 & 11 & 0 & 18 & 18 & 0 & 27 & 27 & 36 \\ 11 & 11 & 76 & 27 & 27 & 0 & 18 & 18 & 0 & 36 \\ 18 & 0 & 27 & 540 & -189 & -135 & -54 & -135 & 270 & 162 \\ 0 & 18 & 27 & -189 & 540 & 270 & -135 & -54 & -135 & 162 \\ 27 & 18 & 0 & -135 & 270 & 540 & -189 & -135 & -54 & 162 \\ 27 & 0 & 18 & -54 & -135 & -189 & 540 & 270 & -135 & 162 \\ 0 & 27 & 18 & -135 & -54 & -135 & 270 & 540 & -189 & 162 \\ 18 & 27 & 0 & 270 & -135 & -54 & -135 & -189 & 540 & 162 \\ 36 & 36 & 36 & 162 & 162 & 162 & 162 & 162 & 162 & 1944 \end{bmatrix}. \tag{F.32}$$

§F.4.3. Lagrangian Triangle Fitting Matrices

Base: LT3, Fitted: LT1. Matrices **A** and **C** are given by (F.28) and (F.27), respectively.

$$\mathbf{B} = \mathbf{G} = \frac{1}{3}\,[\,1 \quad 1 \quad 1\,], \qquad \mathbf{F} = [\,1 \quad 1 \quad 1\,]^T. \tag{F.33}$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_1$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 1.

Base: LT3, Fitted: LT3M. Matrices **A** and **C** are given by (F.28) and (F.29), respectively.

$$\mathbf{B} = \frac{1}{6}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{G} = 3\,\mathbf{B}, \quad \mathbf{F} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}. \tag{F.34}$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_3$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 3.

Base: LT6, Fitted: LT3. Matrices **A** and **C** are given by (F.30) and (F.28), respectively.

$$\mathbf{B} = \frac{1}{60}\begin{bmatrix} 2 & -1 & -1 & 8 & 4 & 8 \\ -1 & 2 & -1 & 8 & 8 & 4 \\ -1 & -1 & 2 & 4 & 8 & 8 \end{bmatrix},$$
$$\mathbf{G} = \frac{1}{5}\begin{bmatrix} 2 & -1 & -1 & 3 & -1 & 3 \\ -1 & 2 & -1 & 3 & 3 & -1 \\ -1 & -1 & 2 & -1 & 3 & 3 \end{bmatrix}, \quad \mathbf{F} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}. \tag{F.35}$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_3$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 3.

Base: LT6, Fitted: LT3M. Matrices **A** and **C** are given by (F.30) and (F.29), respectively.

$$\mathbf{B} = \frac{1}{30}\begin{bmatrix} 1 & 1 & -2 & 6 & 2 & 2 \\ -2 & 1 & 1 & 2 & 6 & 2 \\ 1 & -2 & 1 & 2 & 2 & 6 \end{bmatrix}, \quad \mathbf{G} = 3\,\mathbf{B}, \quad \mathbf{F} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{F.36}$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_3$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 3.

Base: LT6, Fitted: LT1. Matrices **A** and **C** are given by (F.30) and (F.27), respectively.

$$\mathbf{B} = \mathbf{G} = \frac{1}{3}\,[\,0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1\,], \quad \mathbf{F} = [\,1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\,]^T. \tag{F.37}$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_1$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 1.

Base: LT9, Fitted: LT6. Matrices **A** and **C** are given by (F.31) and (F.30), respectively.

$$\mathbf{B} = \frac{1}{1680}\begin{bmatrix} 26 & 8 & 8 & 45 & -45 & -21 & -21 & -45 & 45 \\ 8 & 26 & 8 & -45 & 45 & 45 & -45 & -21 & -21 \\ 8 & 8 & 26 & -21 & -21 & -45 & 45 & 45 & -45 \\ -40 & -40 & -32 & 168 & 168 & 120 & 48 & 48 & 120 \\ -32 & -40 & -40 & 48 & 120 & 168 & 168 & 120 & 48 \\ -40 & -32 & -40 & 120 & 48 & 48 & 120 & 168 & 168 \end{bmatrix},$$

$$\mathbf{G} = \frac{1}{112}\begin{bmatrix} 64 & 24 & 24 & 72 & -72 & 0 & 0 & -72 & 72 \\ 24 & 64 & 24 & -72 & 72 & 72 & -72 & 0 & 0 \\ 24 & 24 & 64 & 0 & 0 & -72 & 72 & 72 & -72 \\ -10 & -10 & 6 & 63 & 63 & 9 & -9 & -9 & 9 \\ 6 & -10 & -10 & -9 & 9 & 63 & 63 & 9 & -9 \\ -10 & 6 & -10 & 9 & -9 & -9 & 9 & 63 & 63 \end{bmatrix}, \tag{F.38}$$

$$\mathbf{F} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 \\ 2 & -1 & 0 & 8 & 0 & 0 \\ -1 & 2 & 0 & 8 & 0 & 0 \\ 0 & 2 & -1 & 0 & 8 & 0 \\ 0 & -1 & 2 & 0 & 8 & 0 \\ -1 & 0 & 2 & 0 & 0 & 8 \\ 2 & 0 & -1 & 0 & 0 & 8 \end{bmatrix}.$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_6$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 6.

Base: LT9, Fitted: LT3. Matrices **A** and **C** are given by (F.31) and (F.30), respectively.

$$\mathbf{B} = \frac{1}{240}\begin{bmatrix} -2 & -4 & -4 & 27 & 9 & 9 & 9 & 9 & 27 \\ -4 & -2 & -4 & 9 & 27 & 27 & 9 & 9 & 9 \\ -4 & -4 & -2 & 9 & 9 & 9 & 27 & 27 & 9 \end{bmatrix},$$

$$\mathbf{G} = \frac{1}{80}\begin{bmatrix} 2 & -6 & -6 & 63 & -9 & -9 & -9 & -9 & 63 \\ -6 & 2 & -6 & -9 & 63 & 63 & -9 & -9 & -9 \\ -6 & -6 & 2 & -9 & -9 & -9 & 63 & 63 & -9 \end{bmatrix}, \tag{F.39}$$

$$\mathbf{F} = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 & 2 & 1 & 0 & 0 & 1 & 2 \\ 0 & 3 & 0 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix}^T$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_3$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 3.

Base: LT9, Fitted: LT1. Matrices **A** and **C** are given by (F.31) and (F.27), respectively.

$$\mathbf{B} = \mathbf{G} = \frac{1}{48}\,[\,-2 \quad -2 \quad -2 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9\,],$$
$$\mathbf{F} = [\,1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\,]^T. \tag{F.40}$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_1$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 1.

Base: LT10, Fitted: LT9. Matrices **A** and **C** are given by (F.32) and (F.31), respectively.

$$\mathbf{B} = \frac{1}{13440}\begin{bmatrix} 140 & 10 & 10 & -18 & -54 & 0 & 0 & -54 & -18 & -576 \\ 10 & 140 & 10 & -54 & -18 & -18 & -54 & 0 & 0 & -576 \\ 10 & 10 & 140 & 0 & 0 & -54 & -18 & -18 & -54 & -576 \\ 54 & 18 & 72 & 1161 & -297 & -189 & -27 & -189 & 621 & 1296 \\ 18 & 54 & 72 & -297 & 1161 & 621 & -189 & -27 & -189 & 1296 \\ 72 & 54 & 18 & -189 & 621 & 1161 & -297 & -189 & -27 & 1296 \\ 72 & 18 & 54 & -27 & -189 & -297 & 1161 & 621 & -189 & 1296 \\ 18 & 72 & 54 & -189 & -27 & -189 & 621 & 1161 & -297 & 1296 \\ 54 & 72 & 18 & 621 & -189 & -27 & -189 & -297 & 1161 & 1296 \end{bmatrix},$$

$$\mathbf{G} = \frac{1}{70}\begin{bmatrix} 64 & -6 & -6 & 9 & 9 & 9 & 9 & 9 & 9 & -36 \\ -6 & 64 & -6 & 9 & 9 & 9 & 9 & 9 & 9 & -36 \\ -6 & -6 & 64 & 9 & 9 & 9 & 9 & 9 & 9 & -36 \\ 4 & 4 & 4 & 64 & -6 & -6 & -6 & -6 & -6 & 24 \\ 4 & 4 & 4 & -6 & 64 & -6 & -6 & -6 & -6 & 24 \\ 4 & 4 & 4 & -6 & -6 & 64 & -6 & -6 & -6 & 24 \\ 4 & 4 & 4 & -6 & -6 & -6 & 64 & -6 & -6 & 24 \\ 4 & 4 & 4 & -6 & -6 & -6 & -6 & 64 & -6 & 24 \\ 4 & 4 & 4 & -6 & -6 & -6 & -6 & -6 & 64 & 24 \end{bmatrix}, \tag{F.41}$$

$$\mathbf{F} = \frac{1}{12}\begin{bmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 \\ -2 & -2 & -2 & 3 & 3 & 3 & 3 & 3 & 3 \end{bmatrix}.$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_9$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 9.

Base: LT10, Fitted: LT6. Matrices **A** and **C** are given by (F.32) and (F.30), respectively.

$$\mathbf{B} = \frac{1}{840} \begin{bmatrix} 10 & 1 & 1 & 27 & -18 & -6 & -6 & -18 & 27 & -18 \\ 1 & 10 & 1 & -18 & 27 & 27 & -18 & -6 & -6 & -18 \\ 1 & 1 & 10 & -6 & -6 & -18 & 27 & 27 & -18 & -18 \\ 4 & 4 & 8 & 48 & 48 & 24 & -12 & -12 & 24 & 144 \\ 8 & 4 & 4 & -12 & 24 & 48 & 48 & 24 & -12 & 144 \\ 4 & 8 & 4 & 24 & -12 & -12 & 24 & 48 & 48 & 144 \end{bmatrix},$$

$$\mathbf{G} = \frac{1}{280} \begin{bmatrix} 136 & 36 & 36 & 216 & -144 & 36 & 36 & -144 & 216 & -144 \\ 36 & 136 & 36 & -144 & 216 & 216 & -144 & 36 & 36 & -144 \\ 36 & 36 & 136 & 36 & 36 & -144 & 216 & 216 & -144 & -144 \\ -4 & -4 & 36 & 126 & 126 & -9 & -54 & -54 & -9 & 126 \\ 36 & -4 & -4 & -54 & -9 & 126 & 126 & -9 & -54 & 126 \\ -4 & 36 & -4 & -9 & -54 & -54 & -9 & 126 & 126 & 126 \end{bmatrix}, \tag{F.42}$$

$$\mathbf{F} = \frac{1}{9} \begin{bmatrix} 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 \\ 2 & -1 & 0 & 8 & 0 & 0 \\ -1 & 2 & 0 & 8 & 0 & 0 \\ 0 & 2 & -1 & 0 & 8 & 0 \\ 0 & -1 & 2 & 0 & 8 & 0 \\ -1 & 0 & 2 & 0 & 0 & 8 \\ 2 & 0 & -1 & 0 & 0 & 8 \\ -1 & -1 & -1 & 4 & 4 & 4 \end{bmatrix}.$$

$\mathbf{G\,F} = \mathbf{I}_6$, $\mathbf{H} = \mathbf{F\,G}$ = projection matrix of rank 6.

Base: LT10, Fitted: LT3. Matrices **A** and **C** are given by (F.32) and (F.28), respectively.

$$\mathbf{B} = \frac{1}{120} \begin{bmatrix} 2 & 1 & 1 & 9 & 0 & 0 & 0 & 0 & 9 & 18 \\ 1 & 2 & 1 & 0 & 9 & 9 & 0 & 0 & 0 & 18 \\ 1 & 1 & 2 & 0 & 0 & 0 & 9 & 9 & 0 & 18 \end{bmatrix},$$

$$\mathbf{G} = \frac{1}{40} \begin{bmatrix} 4 & 0 & 0 & 27 & -9 & -9 & -9 & -9 & 27 & 18 \\ 0 & 4 & 0 & -9 & 27 & 27 & -9 & -9 & -9 & 18 \\ 0 & 0 & 4 & -9 & -9 & -9 & 27 & 27 & -9 & 18 \end{bmatrix}, \tag{F.43}$$

$$\mathbf{F} = \frac{1}{3} \begin{bmatrix} 3 & 0 & 0 & 2 & 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 3 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 & 1 & 2 & 2 & 1 & 1 \end{bmatrix}^T.$$

$\mathbf{G\,F} = \mathbf{I}_3$, $\mathbf{H} = \mathbf{F\,G}$ = projection matrix of rank 3.

Base: LT10, Fitted: LT1. Matrices **A** and **C** are given by (F.32) and (F.27), respectively.

$$\mathbf{B} = \mathbf{G} = \frac{1}{120} [\,4 \quad 4 \quad 4 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 54\,], \qquad \mathbf{F} = [\,1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\,]^T. \tag{F.44}$$

$\mathbf{G\,F} = \mathbf{I}_1$, $\mathbf{H} = \mathbf{F\,G}$ = projection matrix of rank 1..