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I on the Prize: Inquiry approaches in undergraduate mathematics

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Abstract

In the United States (US) and elsewhere across the world, undergraduate mathematics instructors are increasingly aware of the value of inquiry-based instruction. In this research commentary, we describe the intellectual origins and development of two major strands of inquiry in US higher education, offer an explanation for apparent differences in these strands, and argue that they be united under a common vision of Inquiry-Based Mathematics Education (IBME). Central to this common vision are four pillars of IBME: student engagement in meaningful mathematics, student collaboration for sensemaking, instructor inquiry into student thinking, and equitable instructional practice to include all in rigorous mathematical learning and mathematical identity-building. We conclude this commentary with a call for a four-pronged agenda for research and practice focused on learning trajectories, transferable skills, equity, and a systems approach.

Keywords

Active learning, inquiry-based learning, inquiry-based mathematics education, inquiry-oriented instruction, undergraduate mathematics education

I on the Prize: Inquiry approaches in undergraduate mathematics

In the United States (US), a growing chorus of voices is calling for post-secondary mathematics teaching to provide students with learning experiences that are rich and meaningful: centered on students' ideas, requiring their mental engagement in and out of class, and accountable to their prior understandings. These calls are grounded in evidence from education research that such research-based, student-centered teaching practices benefit student learning, attitudes, success and persistence in mathematics and related fields (see e.g., Freeman, et al., 2014; Kober, 2015). They also offer students access to a range of rewarding and well-paid career paths. And, because success in mathematics courses is essential for many other education and career paths, these experiences and outcomes also support students to pursue interests in science, engineering, technology, business, health care, social science, teaching, and many other fields. While research-based instructional practices are not yet the norm in North American classrooms, they are becoming more mainstream (Stains et al., 2018)—as, indeed, they must in order to have widespread benefit.

Such calls for reformed instruction—in the US and elsewhere—are often motivated by national or regional concerns for economic competitiveness—for education that prepares STEM workers to fuel the innovation economy (e.g., President's Council of Advisors on Science & Technology (PCAST), 2012; Rocard, et al., 2007; West, 2012). As Artigue and Blomhøj (2013) note, such sociopolitical justifications merit critical consideration of the intellectual origins and pedagogical practices that are endorsed. Within the discipline of mathematics, research-based instructional practices have also been endorsed by leaders of North American professional societies across the mathematical sciences (CBMS, 2016; MAA, 2017; Saxe & Braddy, 2015). These statements are noteworthy in emphasizing how students benefit in ways that in turn strengthen the discipline, such as increased student interest and persistence in mathematics and better inclusion of diverse students. These statements have fostered mathematics' educators curiosity and attention to research-based active teaching and learning; they are both responses to and drivers of the growing visibility of active learning within mathematics.

As scholars who have studied active learning and teaching in postsecondary mathematics education, especially approaches known as inquiry-based learning (IBL, Laursen) and inquiry-oriented instruction (IOI, Rasmussen), we are encouraged to see this growing interest in educational practices we know to be effective for students. We have also observed growing concern for defining and differentiating particular strategies (e.g., Cook, Murphy & Fukawa-Connelly, 2016; Kuster, Johnson, Keene, & Andrews-Larson, 2017). In this commentary, we propose some key principles of mathematical inquiry in the undergraduate classroom, describe the history and development of two major strands of inquiry in US higher education, offer an explanation for apparent differences in these strands, and argue that they be united under a common vision of Inquiry-Based Mathematics Education (IBME). Our project here is to delineate and connect these two landscapes of IBME, to illustrate that the commonalities are more essential than the differences, and to encourage researchers and practitioners to keep their eye on the inquiry prize by focusing on how inquiry experiences matter for students, instructors, mathematics departments, and the profession.

This commentary draws on the authors' extensive research and experience with undergraduate mathematics education in the US. Our search of the literature, and conversations with international colleagues, suggests that inquiry-based mathematics education is more strongly developed in the US, both in research and in classroom practice. For US readers, we hope this commentary will clarify and unify what some see as different inquiry traditions. For non-US readers, we hope it will make visible unrecognized similarities or differences in trajectories of change in higher education, and perhaps lay groundwork for future development of post-secondary inquiry approaches in their own countries. All readers will benefit from the agenda we lay out for future research and practice. Thus, while this commentary is focused on two US traditions of inquiry, its value and contribution extend well beyond US borders.

What is Inquiry in Mathematics?

We begin by situating inquiry within the broader landscape of active learning and teaching. Decades ago, Bonwell and Eison (1991) defined active teaching strategies as those that "involve students in doing things and thinking about what they are doing" (p. 19). Students may "do" and "think" by reading, writing, discussing, or solving problems, but they must take part in higher-order thinking tasks such as analysis, synthesis and evaluation. We add to this definition the explicit expectation that students talk to each other about what they are doing and thinking, as conversations are powerful in clarifying, solidifying, and elaborating learners' ideas. They also take advantage of the inherently social nature of classrooms and provide the instructor with the feedback needed to identify fruitful next steps toward her learning goals for students.

The instructor's role is to orchestrate this doing, thinking, and talking—to choose the important mathematical ideas and to develop tasks that enable students to meet and grapple with them; to structure opportunities in advance for students to reflect, analyze, synthesize, and communicate, and to make use of student ideas to structure these opportunities in the moment; to ensure that all students have an equal chance to participate and grow. This does not mean there is no instructor talk, but rather that such talk is well timed and well targeted to surface and explore students' prior knowledge, to help students organize or connect important ideas, and to support students' changing views (Neumann, 2014). As Campbell and coauthors (2017) show in their multi-institution observation study, these cognitively responsive practices are often missing, even in classes that feature interactive and hands-on activities. Instructor skill and thought are required to make active learning truly active.

We consider inquiry a branch of active learning. As with active learning more generally, students in inquiry classrooms are engaged in doing mathematics, and the instructor is orchestrating and structuring student learning opportunities. Inquiry, however, has several additional distinguishing characteristics. First, inquiry curricula exhibit a longer-term trajectory that sequences daily tasks to build toward big ideas. These coherent task sequences scaffold students' mathematical work on challenging problems over weeks of instruction and may lead to proving a major theorem or (re)inventing a mathematical idea, definition, or procedure. To support such task sequences, instructors must deeply understand the mathematics so they can capitalize on students' mathematical ideas, thus recognizing and nurturing the seeds of student ideas that have the potential to grow and develop, without getting lost in the weeds. A good task sequence of course helps to provide the framework.

A second distinguishing characteristic of inquiry is the nature of students' mathematical work. In inquiry classrooms students reinvent or create mathematics that is new to them. They do so by engaging in mathematical practices similar to those of practicing mathematicians: conjecturing and proving, defining, creating and using algorithms, and modeling (Moschkovitz, 2002; Rasmussen, Zandieh, King, & Teppo, 2005). As such, students not only develop deep mathematical understanding, but they also develop a sense of ownership through creation and reinvention. Instructors, for their part, allow students intellectual space to be creative, while at the same time they seek ways to extend student ideas and connect these to formal or conventional mathematics. This requires adaptive and responsive facilitation skills, not just expertise in exposition and delivery of content.

A third distinguishing characteristic of inquiry is a consequence of the previous two: it offers students and instructors greater opportunity to develop a critical stance toward previous, perhaps unquestioned learning and teaching routines. A critical stance is "an attitude or disposition towards oneself, others and the object of inquiry that challenges and impels learners to reflect, understand and act in the milieu of potentiality" (Curzon-Hobson, 2003, p. 201). For example, inquiry provides occasions for students to reconsider their past experiences and think anew about what mathematics is, and about what it means to know math, to do math, and to teach math. For instructors, listening to and making sense of student thinking may challenge how they think about the process of learning something new—how ideas may develop, what it means to "cover" material, and how tentative ideas and errors contribute to the learning-teaching process. A necessary part of developing a critical stance is to have learning experiences that differ from past experiences, and the opportunity to reflect on those experiences. Inquiry classrooms can offer such experiences.

Inquiry learning in mathematics may seem distinct from how this term has long been used in science education (see Bybee, 2011, for a brief history and key references). Yet at the core, these approaches are the same in seeking to involve students in the behaviors and practices of expert scientists or mathematicians. In science, these practices center on evidence: designing and carrying out investigations, evaluating and interpreting evidence, and making and critiquing arguments from evidence (NRC, 1996, 2000, 2012). In mathematics, the working material differs: students may explore patterns, generate conjectures, prove theorems, (re)invent definitions or procedures or compare solutions. But they are still engaging in the practices of experts and, through first-hand experience, coming to understand disciplinary ways of knowing. Artigue and Blomhøj (2013) link these broad notions of inquiry to American philosopher John

Dewey's notion that education should generate both particular knowledge and general knowledge-building capacities or habits of mind useful for making sense of the world.

U.S. Traditions of Inquiry

We focus on two main traditions of inquiry in U.S. post-secondary mathematics, known as inquiry-oriented (IO) and inquiry-based learning (IBL). We argue that the similarities are more important than the (apparent) differences; to do so, we first trace their intellectual origins and practical reach in the United States.

IOI: Inquiry-oriented Instruction

Several different IO curricula cover a variety of content areas for post-secondary mathematics, including abstract algebra, differential equations, linear algebra, and mathematics for future elementary school teachers. Additional materials are currently being developed in combinatorics and advanced calculus. A major intellectual source of inspiration and influence for this work (especially in differential equations and linear algebra) comes from the pioneering research of Paul Cobb, Erna Yackel and colleagues in elementary school classrooms (e.g., Cobb et al., 1991; Cobb & Yackel, 1996; Yackel & Cobb, 1996; Yackel, Cobb, & Wood, 1991). Their innovative, classroom-based work was grounded in both cognitive and social theories of learning. Their use of the term "inquiry" came from Richards (1991), who characterized inquiry classrooms as those where students learn to speak and act mathematically by discussing and solving new or unfamiliar problems. The classrooms Cobb and Yackel studied were characterized by students routinely explaining their own thinking, listening to and attempting to make sense of others' thinking, asking questions if they didn't understand someone's work, offering different solution strategies, and indicating their agreement or disagreement, with reasons. Such patterns of classroom talk represent *social norms* and could aptly apply as well to a science class or a history class (Yackel & Cobb, 1996).

Cobb and Yackel also identified classroom talk that was specific to mathematics. For example, when students routinely offer different solution strategies, a relevant mathematical issue is what constitutes a *different* solution. Is Angie's solution different from Juan's? If yes, how so and why? When someone explains their reasoning, what makes for a mathematically *acceptable* solution, or what constitutes an *elegant* solution? Difference, acceptability, and elegance are all criteria that fall under the realm of mathematics and are thus referred to as *sociomathematical norms* (Yackel & Cobb, 1996). While this work originated in second and third grade classrooms, the constructs of social and sociomathematical norms provide powerful and useful tools for researchers and practitioners in IO approaches at the university level. For example, two IO goals for instruction incorporate social norms:(1) students share their thinking, and (2) students orient to and engage in others' thinking. In IO classes, researchers and practitioners are working together to identify how instructors can realize these goals. For example, Rasmussen, Yackel, and King (2003, p. 153-154) identified a number of concrete things that instructors can say to promote student explanation and justification, such as, "Tell us how you thought about it, that is what we are interested in", "Did anyone think about that in a different way?", and "What do the rest of you think about what Jason just said?" Yackel, Rasmussen, and King (2000) showed the applicability and usefulness of norms in a differential equations class. In particular, they found that acceptable mathematical explanations sought to interpret rate of change, not just recount a procedure. Such research can be useful for all practitioners in raising their awareness of what kinds of explanation and justification they value and want to promote among their students.

Also inspired by the work of Cobb and Yackel, current IO researchers use research methods that takes place in actual classrooms where teachers function as partners in the research or a member of the research team is the classroom instructor. This research approach (sometimes referred to as developmental research or design-based research) cycles between designing instructional material, implementing it day-to-day in the classroom, and analyzing what results (Cobb, 2000; Gravemeijer, 1994). Data sources may include video-recordings of class sessions, problem solving interviews with students, records of team meetings with the teacher/co-researcher, and copies of student work, over multiple weeks to an entire semester. Such classroom-based research seeks to investigate how students build particular ideas, what teaching strategies promote students' mathematical progress, how social aspects of classroom interaction relate to student identity and mathematical growth, as well as creating research-based, shareable curricular materials. As these curricula spread beyond the original research teams, a new cadre of mathematicians and mathematics educators are investigating productive ways to support others in using these materials and adapting them to their local context and circumstances (e.g., the National Science Foundation-supported project Teaching Inquiry-Oriented Mathematics: Establishing Supports (TIMES, n.d.)).

Another cornerstone of IO curricula is their grounding in the instructional design theory of Realistic Mathematics Education (RME). Traditional curricula are typically designed based on expert understanding of the mathematics, but RME takes a bottom-up approach where curricula are designed based on how learners might reinvent important mathematical ideas and procedures (Freudenthal, 1991; Gravemeijer, 1999). That is, rather than seeing mathematics as a collection of pre-established truths and procedures that learners must assimilate, RME offers a set of design heuristics where students can, with the support of their instructor, reinvent mathematics at successively higher levels. The classroom, design-based research approach is an ideal method for revealing and generating such routines and practices as well as the kinds of knowledge and dispositions that instructors need (Andrews-Larson, Wawro, & Zandieh, 2017; Johnson, 2013; Johnson & Larson, 2012; Kuster et al., 2017; Marrongelle & Rasmussen, 2008; Rasmussen, Zandieh, & Wawro, 2009; Wagner, Speer, & Rossa, 2007).

Visitors to IO classrooms would see students working in small groups on unfamiliar and challenging problems, students presenting and sharing their work, even if tentative, and whole-class discussions where students question and refine their classmates' reasoning. The students' intellectual work lies in creating and revising definitions, making and justifying conjectures and justifying them, developing their own representations, and creating their own algorithms and methods for solving problems—and, in this work, following the two social norms described earlier as goals for IO classrooms: (1) for students to share their thinking, and (2) for students to orient to and engage in others' thinking. Carefully designed, sequenced, and classroom-tested instructional materials are key here, as is the instructor's role in listening to and interpreting their students' thinking, connecting it to conventional or formal mathematics, and using student ideas to move forward the joint mathematical agenda (Kuster et al., 2018; Rasmussen & Marrongelle, 2006; Rasmussen, Marrongelle, Kwon, & Hodge, 2017). Clearly the role of an IO instructor is multi-faceted with practices and routines that go well beyond those required of lecturing and the dissemination of knowledge. As research is revealing how instructors realize these goals, we can add two additional goals for IO instructors: (3) helping students deepen their thinking, and; (4) building on and extending student ideas. Rasmussen et al. (2018) give concrete, actionable talk moves that instructors can use tomorrow to help realize these four goals. Taken together, they highlight that, in IO classrooms, inquiry applies to both students and instructors (Rasmussen & Kwon, 2007).

IBL: Inquiry-based learning

In contrast to the research-based history of IO instruction, IBL emerges from practical work by educators and the collegial community they formed. Among the key supports for this community have been activities fostered by the Educational Advancement Foundation (EAF). Former students of UT Austin topologist R. L. Moore, aided by the EAF, initially sought to commemorate and share Moore's distinctive teaching style, known as the "Moore method" (W. S. Mahavier 1999; W. T. Mahavier 1997; Parker 2005). Although student-centered pedagogies had appeared in the US and Europe well before the 1990s (Artigue & Blomhøj 2013), this Moore-derived movement developed largely independently of those concepts and practices, especially through collegial exchange and a bootstrapping approach to professional development. Moore did not refer to his method as inquiry-based learning, but early leaders of the movement saw similarities between Moore's teaching and the general principles of inquiry-based teaching that were gaining momentum in higher education at the time (NSF 1996; Brint 2011); the term inquiry-based learning and the initialism IBL came into currency within this community at this time. As the movement has grown in size and vitality, it has broadened its conception of IBL teaching practices to what is known as the "big tent" (Hayward, Kogan & Laursen, 2016; also Ernst, Hodge & Yoshinobu, 2017). By this we mean that, within the IBL approach, instructors may choose varied and multiple instructional strategies to engage students and facilitate learning: one size does not fit all. Haberler, Laursen and Hayward (2018; also Haberler, forthcoming) have traced aspects of the history and sociology of this particular IBL movement and identified some of the drivers toward its evolution from "modified Moore method" to "IBL" terminology and an inclusive, "big tent" understanding and enactment of IBL.

Whereas IO continues to develop through design-based research on different courses, the IBL community continues to grow as a lively place for practitioners to exchange ideas and deepen their practice—a network of people and events. The Academy of Inquiry Based Learning offers many resources for instructors, including workshops that have been backed by National Science

Foundation funding (http://www.inquirybasedlearning.org/). Earlier workshop series have also supported many new practitioners to develop their skills (Hayward, Kogan & Laursen, 2016; Hayward & Laursen, 2016, 2018). The IBL SIGMAA, a Special Interest Group of the Mathematical Association of America, was formed in 2017 (http://sigmaa.maa.org/ibl/); it hosts mini-workshops and organizes symposia at professional meetings where practitioners can share experiences, strategies, and findings from scholarly examination of their own practice. For example, at the 2018 Joint Mathematics Meeting, the sessions on IBL filled five half-days with over 50 talks. The Educational Advancement Foundation and its successor, Mathematics Learning by Inquiry, have hosted periodic conferences on IBL teaching and learning. Regional consortia are beginning to mobilize in some parts of the country. A solicitation for a special issue of PRIMUS on IBL drew so many contributions that it was expanded to a double volume (Katz & Thoren, 2017a,b, and references therein), reflecting growing practitioner interest in documenting their methods and observations.

Typically, IBL courses are based on a carefully scaffolded sequence of problems or proofs, set up so that as students work through these problems they jointly build up the big ideas of the course through discovering and explaining the mathematical arguments. Commonly, the problem sequences or 'scripts' are based in instructors' mathematical knowledge and classroom experience with how students may productively develop ideas. But they may not be grounded in instructional design principles from education research; they are shared colleague to colleague through informal networks or a course repository, the Journal of Inquiry Based Learning, and increasingly, through practitioner-oriented journals or scholarship of teaching and learning outlets such as PRIMUS. While traditionally Moore method courses emphasized upper division topics such as real analysis and abstract algebra, today IBL approaches have been adapted to nearly all courses in the mathematics curriculum, from first-year to advanced courses for mathematics-focused students, for general education of non-STEM students, and for preservice teachers.

Visitors to IBL courses would see class work that is highly interactive, emphasizing student communication and critique of these ideas, whether through student presentations at the board or small group discussions. Whole-class discussion and debriefs are used to aid collective sense-making, and instructors may provide mini-lectures to provide closure and signposting. Instructors' classroom role is thus shifted from telling and demonstrating to guiding, managing, coaching and monitoring student inquiry. There is a long tradition of practical literature from reflective educators describing IBL teaching practices and curricula (see, e.g., Coppin, Mahavier, May & Parker 2009; Ernst, Hodge & Yoshinobu 2017; Hotchkiss, Ecke, Fleron & Renesse 2014, and references therein; Katz & Thoren 2017a,b, and references therein; W. S. Mahavier, 1997; W. T. Mahavier, 1999; Yoshinobu & Jones 2013). More recently, IBL practices have been characterized by a team of researchers who sought to understand student outcomes emerging from a variety of IBL courses taught at four institutions (Laursen, 2013; Laursen, Hassi & Hough, 2016; Laursen, Hassi, Kogan, Hunter & Weston, 2011; Laursen, Hassi, Kogan & Weston, 2014). This research has in turn increased the visibility of IBL methods within US mathematics education and has provided language and foundations for deeper practitioner inquiry. Thus, we do not describe IBL as "research-based" practice but rather as consistent with and supported by education research (Laursen, et al., 2014).

Differences in the Research Bases for Inquiry Traditions

It is in the research studies of IBL and IOI where apparent differences arise between these approaches. This is largely due to different emphases in what are still small literatures. For example, the most frequently cited publications about IBL courses all stem from a single major study of IBL that examined aggregate outcomes for large numbers of students in different IBL courses. Because the courses were not jointly planned or developed by instructors, the study samples included, and findings describe, substantial, natural variation: students were taking a range of mathematics courses taught by multiple instructors in different institutions and using a wide range of curricular materials (Laursen, et al., 2011; 2014). Classroom observations served to characterize courses in the aggregate, identifying patterns and documenting differences between courses using IBL and those using lecture-based, non-IBL methods. Observations also made it possible to distinguish some subtleties, such as differences in the IBL approaches instructors used with pre-service teachers from those chosen for courses aimed at STEM majors (Laursen, Hassi & Hough, 2016). However, to protect instructors' anonymity, the team did not examine any one instructor's practice in detail. This contrasts with studies of IO, which have more often focused on a small number of classrooms to examine instructor moves and student discourse in greater detail. In IO studies, typically instructors taught from one of several carefully designed research-based curricula, another source of contrast with the high heterogeneity of IBL course materials.

Published studies of student outcomes from IO and IBL also differ in focus and specificity. Some IO studies have examined outcomes of content-based assessments for specific courses, such as differential equations (Kwon, Rasmussen, Allen, 2005; Rasmussen, Kwon, Allen, Marrongelle, & Burtch, 2006), linear algebra (Bouhjar, Andrews-Larson, Haider, & Zandieh, 2018), and abstract algebra (Johnson et al, 2018; Larson, Johnson, & Bartlo, 2013). In general, such comparative studies show that IO students outperform non-IO students. Taking a different tack, IBL researchers selected student outcome measures that could be generalized across different topical courses—for example, student grades, and self-reported outcomes from surveys and interviews, rather than course-specific measures (Hassi & Laursen, 2015; Kogan & Laursen, 2015; Laursen, 2013). These measures accommodated their large, multi-institution sample and varied course contexts. Yet, similar to IOI studies, these studies broadly show greater benefits to IBL students than to their non-IBL peers across cognitive and non-cognitive domains. Some outcome measures show no difference; importantly, there is no evidence of harm done to IBL students despite reduced content "coverage."

In addition to these differences in focus and methods of the existing research studies, there are differences in the researchers' stance with respect to the teaching tradition. As mathematics-trained researchers, IO researchers were interested in investigating student learning of particular mathematical ideas and in developing and studying instructors' practices and the knowledge they find useful in IO teaching. As described, they drew on social and sociomathematical norms from earlier theoretical work of Cobb and Yackel, the instructional design theory of RME, and the K-12 literature on mathematical knowledge for teaching. Instructors who participated in these studies tended to be part of the extended research team, typical of design-based research. More recently, IO researchers have been leading professional development and investigating the teaching practices of mathematicians as they implement IO curricula (e.g., Andrews-Larson & McCrackin, 2018; Keene, Fortune, & Hall, 2018). In contrast, Laursen and colleagues have brought an external perspective to IBL; while the team included people trained in mathematics as well as in other areas of natural and social science, they were not IBL instructors themselves. This group began their work with a very practical orientation as evaluators commissioned to study courses taught in four university IBL Centers, embedding themselves in the IBL community but also attentive to its relationship to the broader national interest in active learning in STEM higher education.

We describe these differences not to value one approach over another, but to point out some differences in the bodies of RUME scholarship emerging from these two inquiry traditions. These differences in the research questions, methods and perspectives may lead RUME researchers and mathematics educators to focus on the differences between IBL and IOI methods, rather than on their commonalities. But we argue that the commonalities are more significant for improving practice and for generating fruitful and impactful research. While these two inquiry traditions are based in the United States, we suggest that they raise interesting questions for scholars worldwide to explore in different higher education contexts, and suggest different ways that research may contribute to practice.

The Four Pillars of Inquiry-Based Mathematics Education

Because these descriptions make clear that IBL and IOI mathematics share common foundational practices, we discuss them jointly under the term Inquiry-Based Mathematics Education, or IBME (Artigue & Blomhøj, 2013). In their study of student outcomes, Laursen and coauthors (2014) identified "twin pillars" (p. 413) that support student learning, deep engagement with meaningful mathematics and collaborative processing of mathematical ideas. Deep engagement occurs as students encounter, grapple with, and revisit important ideas over time, in and out of class. And, as students discuss, elaborate and critique these ideas together, they deepen their understanding and build communication skills, collaborative skills, and appreciation for diverse paths to solutions. These pillars of learning emphasize what students do that leads to the good outcomes; they imply, but do not make explicit instructors' roles in selecting and staging meaningful tasks and orchestrating students' conversation about them to build up the big ideas of the discipline in an intellectually coherent way. Rasmussen and Kwon (2007) characterized inquiry using two similar pillars and a third, instructor inquiry into student thinking. This pillar emphasizes the instructor's role to strengthen the student pillars by eliciting student ideas and making them public, building a classroom community where students can fruitfully engage with and refine those ideas together, and elaborating and extending student ideas—a role that requires that instructors value and attend to students' ideas.

We add to these three a fourth pillar, equitable instructional practice. The research base in undergraduate mathematics education does not reveal just how to accomplish this in inquiry-based college classrooms: current studies show that inquiry classrooms can level the playing field for women (Laursen, et al., 2014) and argue why this may occur (Hassi & Laursen, 2015; Tang, Savic, El Turkey, Karakok, Cilli-Turner, & Plaxco, 2017) but also show that this is not automatic (Andrews, Can & Angstadt, 2018; Brown, 2018; Ellis, 2018; Johnson et al. 2018). Research on high school classrooms offers useful lessons, however: Boaler (2006) describes seven teaching practices that yielded higher and more equitable educational attainment and fostered students' respect and felt responsibility for each other. It is striking, yet no coincidence, that these practices overlap well with the first three pillars of inquiry. For example, asking students to justify their answers and share their reasoning is a form of inquiry into students' mathematical thinking—but as Boaler's study showed, this also contributed to equity and respect, instilling a norm that students explain their own ideas and ask for others' explanations and help. However, equity-oriented practices such as assigning competence—publicly raising the status of a student's intellectual contribution—require instructor attention to classroom dynamics as well as mathematics. Instructors must consider not just what students think but what they may feel and experience; they must notice whose thoughts are heard, acknowledged and valued and actively shape those experiences in ways that foster respect and responsibility.

To recap, four pillars of IBME support student learning. Two emphasize student behaviors and two emphasize instructor behaviors:

- Students engage deeply with coherent and meaningful mathematical tasks
- Students collaboratively process mathematical ideas
- Instructors inquire into student thinking
- Instructors foster equity in their design and facilitation choices.

Research Agendas for Inquiry-Based Mathematics Education

As core IBME principles, these four pillars are the foundations of effective IBME practice; they account for student learning and thus offer guidance to instructors seeking to develop their teaching practice. The four pillars also offer guidance to researchers interested in IBME about fruitful and important questions to pursue. We identify four agendas as important for researchers and practitioners to explore in higher education settings where inquiry approaches hold promise.

The Learning Trajectory Agenda

At the elementary and secondary school levels, research and development on learning trajectories holds great promise to make significant impact on learning and teaching (Daro, Mosher, & Corcoran, 2011; National Research Council, 2007). Comparable work at the post-secondary level, however, is relatively sparse, both in general and in particular to inquiry curricula. IBME classrooms offer ideal settings for surfacing ideas and explicating learning trajectories. In their recent review of the learning trajectory literature, Lobato and Walters (2017) call out seven approaches to research in this field, with foci ranging from individual students' cognitive levels to disciplinary logic; similar approaches may apply in studying undergraduate learning or developing inquiry problem sequences.

Another promising approach focuses on various aspects of instruction. For example, Sztajn, Confrey, Wilson, and Edgington (2012) offer a comprehensive model by which to coordinate research on teaching with research on learning, which they call learning trajectory based instruction (LTBI). Their analysis examines how learning trajectory research can inform research on mathematical knowledge for teaching, discourse facilitation, task analysis, and formative assessment. Post-secondary mathematics education will benefit from embracing a more explicit agenda focused on learning trajectories and instruction based on them. Doing so is likely to result in more generalizable and useful products for practitioners and will tap into the strong interest of researchers worldwide to unpack and investigate basic processes of learning and teaching. LTBI approaches to collaborative curriculum development also have potential to amplify practitioners' contributions and increase their knowledge and classroom use of student learning trajectories.

The Transferable Skills Agenda

An untapped potential of inquiry instruction and research is explicit attention to skills that are transferable to other disciplines and to work settings— student competencies such as communication to experts and non-experts, writing, working in teams, critical thinking, metacognition, and thinking ethically. Because of their emphasis on collaboration, communication, and problem solving, inquiry classrooms offer ample opportunities for IBME students to develop these skills. Yet students may not perceive these as areas where they have made gains applicable outside mathematics. Indeed, King, Varsavsky, Belward, and Matthews (2017) surveyed graduating university mathematics majors at four Australian universities about their perceptions of the opportunities they had had to develop mathematical knowledge and transferable skills. They found a startling difference in what students reported about content skills compared to transferable skills. Students valued content skills and said these were taught and assessed in their curricula; their content-related confidence and knowledge increased—but for transferable skills, students reported less confidence and knowledge and perceived these skills to be neither taught nor assessed in their coursework. Yet these are precisely the kinds of skills that employers report as being highly valued but missing in prospective employees, and that are widely seen as essential for good global citizenship in the 21st century (AAC&U, 2007; Jungic & Lovric, 2017; Prinsley & Baranyai, 2015; Wake & Burkhardt, 2013).

Clearly these skills must be explicitly valued and called out as instructors plan and facilitate daily interactions and set tasks and assessments. IBME classrooms are well suited to explicitly teach and assess transferable skills, so we call for researchers and practitioners to take up this agenda. Challenges for researchers include whether inquiry curricula do indeed generate such skills, and how to measure them, how to design curricula and identify teacher knowledge and practices that support students to develop transferable skills. Practitioners may wish to emphasize particular transferable skills in their syllabi and planning, and help students reflect on their broader gains. In turn emphasis on these skills may be useful in helping instructors to justify their choices to teach with IBME to colleagues and to demonstrate accreditation and assessment outcomes at the unit level.

The Equity Agenda

Many questions for instructors and researchers in higher education are prompted by recent studies that suggest that making inquiry work well for all students may not be as easy as it seems (Andrews, Can & Angstadt, 2017; Johnson et al., 2018). For instructors, self-reflection, peer observation, professional development and open-minded reading may be tools for understanding how their own and students' behaviors can shape classroom climates (e.g., Burgstahler, 2017; DiAngelo, 2016; Marquez Kiyama & Rios-Aguilar, 2018; Quaye & Harper, 2007). They may wish to investigate and apply strategies for promoting an equitable environment (e.g., Montgomery County Public Schools, 2010; Tanner, 2013). For researchers, attention to equity in IBME classrooms may mean designing studies that have the statistical power needed to unpack average gains or outcomes in more intersectional ways, or developing measures to probe particular phenomena classroom more deeply (e.g., Reinholz & Shah, 2018). There are opportunities to explore new theoretical perspectives (see Adiredja & Andrews-Larson, 2017) and build theory across multiple instantiations of IBME when examining topics such as teaching practice, classroom discourse and power, epistemological ownership, intersectionality and student identity. Scholars and educators in different countries will face different specific concerns about whose voices are privileged or excluded in mathematics but will recognize similar issues of identity, agency and power in their own higher education settings.

The Systems Agenda

Many studies of IBME so far have focused on students and teachers, but taking a departmental or institutional perspective can shed different light. Most teaching happens behind closed doors, and this may fool us into considering only the individual student, instructor, or classroom environment as the focus of a study. Yet what goes on in a classroom is inseparable from the culture, norms and practices of a department, discipline, or institution. For instance, Austin (2011) describes some of the ways these contexts shape instructors' choices of teaching practice. For researchers, attention to systems may give rise to fruitful questions about whether and how instruction is changing within departments or in networked communities to align with recommended practices in the discipline (e.g., Apkarian, 2018; Apkarian, Bowers, O'Sullivan & Rasmussen, 2018; Haberler, Laursen & Hayward, 2018; Kezar, Gehrke & Elrod, 2015; Laursen, 2016; Smith, Webb, Bowers & Voigt, 2017). They may elucidate the features of departments and institutions that promote or hinder equitable student experiences and learning outcomes (Reinholz & Apkarian, 2018). Fine-grained studies in multiple settings may reveal interesting variations in student experiences or outcomes that depend on classroom dynamics or instructors' facilitation skills; they may demonstrate ways to adapt IBME for different student audiences, or to organize inquiry for online courses. Studies in different international contexts would add much to our understanding about how the organization and national culture of higher education shapes instructors' choices and students' responses. Systems-focused studies must necessarily attend to variability, recognizing that one size does not fit all and accommodating that variability as a feature—not a bug—of the research design. For practitioners, systems-oriented thinking is essential to address the broad challenges we have already raised: designing course sequences to align with typical student learning trajectories and to thoughtfully build and assess transferable skills; preparing new instructors and teaching assistants to implement IBME across a multi-section course; developing holistic approaches to recruiting and retaining diverse students in the mathematics program.

Closing Thoughts

We recognize that these are challenging, higher-order problems. Yet investigation of these agendas will benefit both research and practice. For research, these agendas will generate greater coherence of the body of knowledge across all IBME traditions, and will focus scholars' attention on challenging educational problems of wide interest, with potential for significant impact. Practitioners will likewise benefit from greater commonality and coherence in the body of research-based advice for improving their practice. And their attention to these higher-order issues will help them decide where their efforts may be best placed—in pedagogy, curriculum, program structure—in order to enhance the learning and success of all students.

These questions are rooted in the four pillars of IBME: student engagement in meaningful mathematics, student collaboration for sensemaking, instructor inquiry into student thinking, and equitable instructional practice to include all in rigorous mathematical learning and mathematical identity-building. The shared agenda is reflected in the shared terminology of inquiry-based mathematics education. We encourage educators and scholars alike to invest their effort on these challenging agendas and to create and promote opportunities for these communities to interact fruitfully.

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