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Abstract

Applying Johansen cointegration test to U.S. annual data constructed from the EU KLEMS database, the paper documents that the productivities of consumption-goods and equipment-goods sector are cointegrated. It confirms further, using the non-linear cointegration test framework developed by Kapetanios *et al.* (2006), that the cointegrating relation is non-linear. The cointegration of sectoral productivities is also documented in the empirical findings of Schmitt-Grohé and Uribe (2011). I successfully derive a theoretical proposition that implies that sectoral productivities of the consumption-goods and equipment-goods sectors are cointegrated if and only if the aggregate neutral productivity and the investment-specific technology are cointegrated. Plus, I consider the non-linear cointegration of sectoral productivities to examine the role of the common stochastic trend of sectoral productivities in explaining the movements of investment-specific technology as well as those of interesting macroeconomic aggregates such as output, consumption, investment and hours worked. For this end, I construct a two-sector dynamic stochastic general equilibrium (DSGE) model where the productivities of the consumption and equipment sectors feature a non-linear error correction (NEC) in the vector error correction model (VECM). The maximum likelihood estimation successfully estimates most of structural parameters, including the sectoral capital shares, and it identifies all structural shocks. The paper finds that the innovations of common stochastic trends of sectoral productivities account for half of consumption, 79 percent of investment, and only 6 percent of hours worked variabilities in long-run.

Keywords: Two-sector model; Business cycles; Investment-specific technology; Productivity; Cointegrateion; Non-linear error correction

JEL Classification Numbers: E32

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1 Introduction

Since the seminal work of Greenwood, Hercowitz and Krusell (1997, 2000), investment-specific technology (IST) has become a leading candidate as a main source of economic growth and fluctuation rather than total factor productivity (TFP). They suggest also that IST can be expressed by the ratio of the productivity in the equipment sector to that in the consumption-goods sector. There is a hardship, however, in interpreting the progress of IST as technological progress of the capital-goods (or equipment) sector.[1] Oulton (2007) suggests that IST may change without a change in the difference of sectoral productivities between consumption-goods and equipment.[2] Furthermore, Whelan (2003) insists that a two-sector approach incorporating relatively high technological progress of durable goods better explain the long-run behavior of the U.S. economy. As another modification to the IST literatures, Schmitt-Grhoé and Uribe (2011) introduce a cointegrated relationship between TFP and IST, which is supported by an empirical analysis that shows a common stochastic trend in TFP and IST. They insist that the innovation in the common stochastic trend explains a sizeable fraction of volatilities of output, consumption, investment, and hours.

To investigate business cycles features in the U.S. economy, this paper considers the two ways of modification exhibited above. Ireland and Schuh (2008) establish a two-sector economy model incorporating both level and growth-rate shocks of sectoral productivities, inspired by Whelan (2003), to study the U.S. business cycles. Their study, however, does not reflect the fact that the sectoral productivities are cointegrated. Therefore, one key feature of this study is the cointegrated relationship between sectoral productivities.

What makes the cointegrated sectoral productivities so important in business cycles studies? Sectoral production performance is affected by the amount of factor inputs, such as labor and capital, and sector-specific production knowledge as well as some countrywide environments such

[1] Recent empirical studies show that the relative price of capital goods does not correctly measure the relative productivity changes. Basu *et al.* (2010) estimate technological changes at a disaggregated industry level and aggregate them by using the U.S. input-output tables. Their finding suggests that relative price does not properly measure the relative technological change. Adopting the two-sector model calibrated on the U.S. input-output tables, Guerrieri *et al.* (2010) conclude that the effect of TFP in the machinery sector is qualitatively different from that of IST. They argue that the shock of TFP in machinery boosts consumption at all succeeding periods while investment-specific technological shock reduce consumption in initial periods.

[2] In the case of different factor intensity in the two sectors, Oulton (2007) points out that the relative price may change without a change in sectoral productivities.

as infrastructure, education, politics, culture, and so on. In the neoclassical growth accounting framework, we can derive a sectoral TFP as a residual measure, called Solow residuals. In turn, the schedule of sectoral TFPs depends on sector-specific production knowledge as well as country-wide economic environments that affect the production of all sectors simultaneously. Accordingly, there may exist a common stochastic trend among sectoral TFPs, which implies the cointegrated relationship in sectoral productivities.

To shed light on the cointegrated relationship, two independent analyses are performed. First, I conduct the Johansen cointegration test on two sectoral productivities of consumption-goods and equipment sectors, which are reconstructed from the EU KLEMS database[3]. The test statistics confirm the cointegration between sectoral productivities[4]. As the second way to illuminate sectoral cointegration in productivity, I establish theoretical propositions based on the findings of Schmitt-Grohé and Uribe (2011) that the aggregate neutral productivity and IST are cointegrated. The propositions imply that the sectoral productivities are cointegrated if and only if the aggregate neutral productivity and IST are cointegrated. Thereby the sectoral cointegrated relationship is supported by the empirical findings of Schmitt-Grohé and Uribe (2011).

Applying the cointegration of sectoral productivities into a dynamic stochastic general equilibrium (DSGE) model, the present paper examines the effects and roles of each structural shock, such as the shocks of preference and productivities, in the U.S. business cycles. As in Ireland and Schuh (2008), the level and growth-rate shocks of preference, and those of the productivities of consumption-goods and equipment sectors are employed. To incorporate the cointegrated relationship of sectoral productivities into the DSGE model, we have to consider the fact that the cointegrated relationship of sectoral productivities may possess a dynamic instability, if the long-run equilibrium between the sectoral productivities is not linear. To resolve this problem and ensure globally-stationary error correction dynamics, I introduce a smooth transition non-linear error correction (STR NEC) featured by exponential function into the vector error correction model

[3]For more details about the EU KLEMS database, refer to O'Mahony and Timmer(2009). The data is available at www.euklems.net.

[4]Marquis and Trehan (2008) capture the idea that the productivities of consumption-goods and equipment shares common shocks. They fail to estimate, however, the cointegrated relationship between sectoral productivities, and just incorporate the correlation between the growth rate of the equipment productivity and that of consumption-goods productivity.

(VECM) framework for sectoral productivities. Using the established stationary model, I perform the maximum likelihood estimation to estimate the deep parameters including sectoral capital shares without symmetric assumption. The model estimation successfully identifies all parameters. The estimated sectoral capital shares confirm the conventional wisdom that consumption-goods sector is relatively labor-intensive, whereas equipment sector is capital-intensive. More importantly, different to Ireland and Schuh (2008) which fail to identify the growth rate shock of equipment sector, this paper successfully identifies all structural shocks.

As results, I find a sizeable effect of common stochastic trends in sectoral productivities to business cycles with persistence. Innovations in the common stochastic trends, which mostly rely on the equipment sector, increase consumption and investment almost permanently, and explains the long-run variabilities of about 48 percent and 79 percent in consumption and investment, respectively, and account for only 6 percent of hours-worked variability. Similarly to Ireland and Schuh (2008), the innovation of preference gives highly persistent and sizeable effects on hours-worked. Also, the preference shocks account for half of consumption variability and most of hours-worked variability. The level shocks of productivities explain only short-run fluctuations; there is no persistence in these shocks.

The remainder of the paper is organized as follows. Section 2 illuminates the cointegrated relationship in the U.S. sectoral productivities both in empirical and theoretical ways. Section 3 establishes a model economy incorporating the cointegrated sectoral productivities. Section 4 estimates the model with the maximum likelihood and discusses the estimates. Section 5 examines the impulse responses and the contributions of structural shocks to forecast error variance. Lastly, Section 6 concludes this paper.

2 Cointegrated productivities

This section examines whether sectoral productivities are cointegrated both theoretically and empirically. Schmitt-Grohé and Uribe (2011) have found that the aggregate TFP and IST are cointegrated by using the U.S. quarterly data. Keeping the empirical finding of Schmitt-Grohé and Uribe

(2011) in mind, for the empirical analysis, I construct annual sectoral productivities, aggregate TFP, and IST from the EU KLEMS database and test the cointegrated relationship among the productivities. Furthermore, I derive theoretical propositions to support the cointegrated relationship between two sectoral productivities.

2.1 Empirical evidence

To construct sectoral productivities as well as aggregate neutral and investment-specific productivities, I use the annual U.S. growth accounting data of EU KLEMS database ranging 1970-2005. This data selection is different to that of Schmitt-Grohé and Uribe (2011), which adopts the U.S. quarterly data. Because of sectoral productivity analysis, the U.S. quarterly data, which is available only on the aggregate level, cannot be used. Although the sample size of the annual data is much smaller than that of the quarterly data, 36 observations are sufficient to produce an auxiliary result, which supports the findings from the quarterly data of Schmitt-Grohé and Uribe (2011).

Data

EU KLEMS includes the 72 sectoral definitions. For use in this analysis, 72 industrial levels have to be aggregated into two sectors; consumption-goods and equipment sectors. For aggregation, I define the equipment sector as the aggregation of Electrical and optical equipment (30t33), Machinery (29) and Transport equipment (34t35)[5], and the rest are aggregated for consumption goods sector. The Törnqvist index (or Divisia index) is applied for the aggregation. For example, log-difference capital service input of a higher sector i is the weighted average of the log-differenced capital service of its sub-sectors; the applied formula is

$$\Delta \ln K_t^i = \sum_j \bar{\omega}_{K,j,t}^i \Delta \ln K_{j,t}^i,$$

where K_t^i is the capital service of sector i, $K_j^{i,t}$ exhibits the capital demand in sub-sector j of sector i, $j \in i$, and $\bar{\omega}_{K,j,t}^i$ is the two-period moving average of the capital input share demanded by sub-

[5]The number inside of parentheses indicates the industry code in the EU KLEMS database (the version of additional industry aggregation).

sector j out of the total demand of sector i, which satisfies $\sum_j \bar{\omega}^i_{K,j,t} = 1$, $\forall t$. The aggregations for sectoral output, intermediate input, and labor services adopt the same method of capital service. Under the growth accounting framework suggested by Jorgenson and Griliches (1967), I construct the sectoral productivity measures for the two sectors by using the aggregated input and output series through the following formula:

$$\Delta \ln A^i_t = \Delta \ln Y^i_t - \bar{v}^i_{X,t} \Delta \ln X^i_t - \bar{v}^i_{K,t} \Delta \ln K^i_t - \bar{v}^i_{L,t} \Delta \ln L^i_t,$$

where A^i_t represents the Solow residual (or TFP) of sector i for $i \in \{tot, cons, equip\}$.[6] Y^i_t, X^i_t, K^i_t and L^i_t respectively denote the output, intermediate input, capital service and labor service of sector i. $\bar{v}^i_{l,t}$ indicates the two-period moving average of the share of input factor l, which satisfies $\sum_l \bar{v}^i_{l,t} = 1$, $\forall i, t$.

Price movements can be captured by the implicit GDP deflators. Log-difference GDP deflator of sector i is formulated as

$$\Delta \ln P^i_t = \Delta \ln N.VA^i_t - \Delta \ln R.VA^i_t,$$

where $N.VA^i_t$ and $R.VA^i_t$ represent the nominal value added and real value added in sector i, $i \in \{cons, equip\}$, respectively. Then, I can construct the log-difference relative price of equipment in terms of consumption ($\Delta \ln RP$) from the following:

$$\Delta \ln RP = \Delta \ln P^{equip}_t - \Delta \ln P^{cons}_t.$$

Using the log-difference variables constructed above, I derive the index series of those variables with base year 1995. First, I set the year before the starting year of each series to 100, and then apply the following formula forwardly:

$$x_{t+1} = x_t \times \exp\left(\Delta \ln x_{t+1}\right),$$

where x_t is a time-series variable, which starts with 100 and has a known $\Delta \ln x_{t+1}$, $\forall t$. Finally, I

[6] *tot*, *cons* and *equip* stand for aggregate economy, consumption goods sector and equipment sector, respectively.

normalize indices with the value of base year 1995.

Table 1: Unit-root tests for the logarithms of productivities and relative price of equipment

Data	Test	Trend	Lags (AIC)	Test-stats.	Critical values (5%)	Null hypothesis
TFP.cons	ADF	No	1	1.15	-1.95	Accept
	ADF	Yes	1	-2.12	-3.5	Accept
	DF-GLS	No	1	-0.319	-1.95	Accept
	DF-GLS	Yes	1	-2.38	-3.19	Accept
TFP.equip	ADF	No	1	2.72	-1.95	Accept
	ADF	Yes	1	-0.46	-3.5	Accept
	DF-GLS	No	1	1.48	-1.95	Accept
	DF-GLS	Yes	1	-0.976	-3.19	Accept
TFP.tot	ADF	No	1	1.83	-1.95	Accept
	ADF	Yes	1	-1.44	-3.5	Accept
	DF-GLS	No	1	0.901	-1.95	Accept
	DF-GLS	Yes	1	-1.93	-3.19	Accept
RP	ADF	No	1	-3.07	-1.95	Reject
	ADF	Yes	1	-0.357	-3.5	Accept
	DF-GLS	No	1	1.48	-1.95	Accept
	DF-GLS	Yes	1	-0.772	-3.19	Accept

Notes: All unit-root tests fail to reject except the ADF test for RP without trend. Tests are conducted using the R program with the "urca" package. ADF stands for Augmented Dickey-Fuller, and DF-GLS stands for Dickey-Fuller Generalized Least Squares. TFP.cons, TFP.equip, TFP.tot, and RP denote the productivity of consumption goods sector, the productivity of equipment sector, the productivity of aggregate economy, and the relative price of equipment, respectively.

Empirical findings

Unit-root and cointegration tests are conducted for the logarithms of aggregated TFP, sectoral productivities, and relative price of equipment by using the data constructed above. As first, augmented Dickey-Fuller (ADF) and Dickey-Fuller GLS (DF-GLS) tests are performed to test the unit root. Table 1 presents the results. The ADF test fails to reject the unit-root hypothesis except for the relative price of equipment without trend. DF-GLS can be considered as the increased power of the test, but it cannot reject the null hypothesis of unit root in all tested variables in both with and without trend. I also conduct the unit-root tests for the first-differenced logged variables, which are not reported here, and all test statistics reject the null hypothesis. Based on the results so far, I can therefore conclude that logged aggregate TFP, TFP in consumption-goods, TFP in equipment

Table 2: The Johansen trace test for cointegration

Database	Cointegration rank	Lags (AIC)	Test-stats.	Critical values (5%)	Null hypothesis
db1	r <= 2	3	0.103	8.18	-
	r <= 1		13.524	17.95	Accept
	r = 0		40.328	31.52	Reject
db2	r <= 2	3	0.35	8.18	-
	r <= 1		7.31	17.95	Accept
	r = 0		37.00	31.52	Reject
db3	r <= 2	3	0.0765	8.18	-
	r <= 1		7.4565	17.95	Accept
	r = 0		37.0785	31.52	Reject
db4	r <= 2	3	0.433	8.18	-
	r <= 1		7.375	17.95	Accept
	r = 0		36.863	31.52	Reject
db5	r <= 1	3	1.62	8.18	Accept
	r = 0		21.13	17.95	Reject
db6	r <= 1	3	0.324	8.18	Accept
	r = 0		20.898	17.95	Reject

Notes: The Johansen trace tests confirm cointegrated relation for all specified datasets with one cointegrating vector. Tests are conducted using the R program with the "urca" package. Test models don't include both constant and trend. The dataset used for the Johansen cointegration test are defined as follows:
db1: TFP.tot, TFP.cons, TFP.equip
db2: RP, TFP.cons, TFP.equip
db3: TFP.tot, RP, TFP.cons
db4: TFP.tot, RP, TFP.equip
db5: TFP.tot, RP
db6: TFP.cons, TFP.equip

and relative price of equipment are integrated by order one.

Schmitt-Grohé and Uribe (2011) find the cointegration of TFP and relative price of equipment with the U.S. quarterly data. To confirm the consistency of their result, I conduct Johansen cointegration tests with various sets of variables including the dataset of TFP and the relative price of equipment with the U.S. data from the EU KLEMS database. The test results of the Johansen trace and maximum eigenvalue tests are exhibited in Table 2 and 3, respectively.

Both Johansen tests, trace and maximum eigenvalue, confirm that the system of logged aggregate TFP and sectoral productivities (db1) have one cointegrating vector, which implies logged TFP can be expressed as a linear combination of two sectoral productivities and one anonymous stationary series. Conventional wisdom on growth accounting also supports this result. The system

Table 3: The Johansen maximum eigenvalue test for cointegration

Database	Cointegration rank	Lags (AIC)	Test-stats.	Critical values (5%)	Null hypothesis
db1	r = 2	3	0.103	8.18	-
	r = 1		13.421	14.9	Accept
	r = 0		26.804	21.07	Reject
db2	r = 2	3	0.35	8.18	-
	r = 1		6.96	14.9	Accept
	r = 0		29.68	21.07	Reject
db3	r = 2	3	0.0765	8.18	-
	r = 1		7.3799	14.9	Accept
	r = 0		29.6221	21.07	Reject
db4	r = 2	3	0.433	8.18	-
	r = 1		6.941	14.9	Accept
	r = 0		29.489	21.07	Reject
db5	r = 1	3	1.62	8.18	Accept
	r = 0		19.50	14.9	Reject
db6	r = 1	3	0.324	8.18	Accept
	r = 0		20.574	14.9	Reject

Notes: The Johansen maximum eigenvalue tests confirm the cointegrated relation for all specified datasets with one conintegrating vector. Tests are conducted using the R program with the "urca" package. Test models don't include both constant and trend. The dataset used for the Johansen cointegration test are defined as follows:
db1: TFP.tot, TFP.cons, TFP.equip
db2: RP, TFP.cons, TFP.equip
db3: TFP.tot, RP, TFP.cons
db4: TFP.tot, RP, TFP.equip
db5: TFP.tot, RP
db6: TFP.cons, TFP.equip

of the logged relative price of equipment and sectoral productivities (db2) have cointegrated with one cointegrating vector.[7] The cointegration of TFP and the relative price of equipment (db5) is tested and confirms the result of Schmitt-Grohé and Uribe (2011). Adding each sectoral productivity on "db5", two three-variable systems (db3 and db4) are also examined for cointegration. Interestingly, both systems accept cointegration with one cointegrating vector. The simultaneous cointegration of the two systems of variables (TFP and IST, and TFP, IST, and an augmented sectoral productivity) gives us an important implication; the result allows us to infer a possible

[7] According to Greenwood *et al.* (1997), the logged relative price of equipment equals the difference of logged productivity of equipment and that of consumption; the implied cointegrating vector is $(1, 1, -1)$ for the system of $(\ln \mathrm{RP}, \ln \mathrm{TFP.equip}, \ln \mathrm{TFP.cons})$. The estimated cointegrating vector from the Johansen test, however, fails to produce the implied sign of the cointegrating vector.

cointegrated relation of sectoral productivities. The cointegration test for sectoral productivities (db6) confirms that the inference is right.

The cointegrated relation among sectoral productivities indicates the possibility that the comovements of aggregate variables and sectoral comovements can arise not only from structural linkages but also from common stochastic trends. Most of the literature in multi-sector business cycles has investigated the sectoral comovements with sectoral structural linkages: Hornstein and Praschink (1997), and Horvath (2002) incorporate intermediate inputs into their model economy to foster sectoral linkages and find positive sectoral comovement in output and employment. However, the empirical findings in Tables 2 and 3, which exhibit the existence of a common stochastic trend in sectoral productivities, suggest that the common stochastic trend of sectoral productivities is another key to solving the sectoral comovement puzzle.

2.2 Theoretical approach

Schmitt-Grohé and Uribe (2011) exhibit that the U.S. quarterly data indicate that the neutral productivity and IST share common stochastic trends. Then, where do the stochastic trends come from? To address this question, I first ignore the empirical results of the previous subsection except for the findings of Schmitt-Grohé and Uribe (2011). There are two reasons. First, the cointegration test with annual data is sensitive to lag selection due to the small sample property. Hence, the findings of quarterly data ranging 1948-2006 are much more reliable compared to the annual data. Secondly, I show that the existence of the common trends in sectoral productivity can be proven without using the sophisticatedly disaggregated high-quality database.

Since Greenwood *et al.* (1997), many studies with two-sector models identify IST as the ratio of the productivity of equipment to the productivity of consumption goods. As such, the behavior of IST reflects the change of sectoral productivities. To examine the relation formally, let us consider a simplified neoclassical two-sector model, as in Oulton (2007); one sector is for producing consumption goods and the other produces equipment. A benevolent social planner would maximize aggregate social utility, $U(C_t, N_t)$, in an infinite time horizon with the given resource constraint,

$$C_t + J_t = Y_t, \tag{1}$$

where C_t is an aggregate consumption, J_t is a forgone consumption or savings for investment spending, and Y_t is a composite output consisting of consumption goods and equipment. The investment spending is used for purchasing equipment and eventually contributes to capital accumulation as follows:

$$K_{t+1} = (1 - \delta)K_t + I_t, \tag{2}$$

where K_t is a capital stock at the beginning of period t, δ implies depreciation rate of capital stocks, and I_t stands for the amount of newly produced equipment used for gross investment during period t. Note that the gross investment, I_t, is measured in the unit of equipment, whereas the investment spending, J_t, takes the unit of consumption. In capital accumulation the investment spending must be therefore transformed into the unit of equipment. Suppose that Q_t governs the linear transformation of the forgone consumption, then we can rewrite Eq.(2) as [8]

$$K_{t+1} = (1 - \delta)K_t + J_t Q_t. \tag{3}$$

Since the nominal investment spending, $P_{c,t}J_t$, should equal the market value of investment, $P_{e,t}I_t$, Eq.(2) and Eq.(3) imply

$$Q_t \equiv \frac{P_{c,t}}{P_{e,t}}, \tag{4}$$

where $P_{c,t}$ is the market price of consumption goods, $P_{e,t}$ is the price for newly produced equipment and Q_t is known as IST from Greenwood *et al.* (1997).

Each representative producer of both sectors uses capital and labor in its constant return to

[8]Schmitt-Grhoé and Uribe (2011) estimate the power of transformation as unity, which implies a linear transformation from consumption to investment.

scale production function with its own neutral technological progress as follows:

$$Y_{c,t} = Z_{c,t} F^{c}\left(K_{c,t}, N_{c,t}\right), \tag{5}$$

$$Y_{e,t} = Z_{e,t} F^{e}\left(K_{e,t}, N_{e,t}\right), \tag{6}$$

where $Y_{c,t}$ and $Y_{e,t}$ are the outputs of consumption goods and equipment sector, respectively. $K_{j,t}$ and $N_{j,t}$ stand for capital and labor inputs, respectively, of sector $j \in \{c, e\}$. The sum of each input across sectors satisfies the feasibility conditions: $N_t \geq N_{c,t} + N_{e,t}$ and $K_t \geq K_{c,t} + K_{e,t}$. Suppose that $Z_{j,t}$ represents the neutral productivity of sector j and has a random walk process as follows:

$$\ln Z_{c,t} = \ln Z_{c,t-1} + \epsilon_{c,t}, \tag{7}$$

$$\ln Z_{e,t} = \ln Z_{e,t-1} + \epsilon_{e,t}, \tag{8}$$

where both $\epsilon_{c,t}$ and $\epsilon_{e,t}$ are independent white noises. Note that both sectoral productivities follow uncorrelated random walk processes due to the independently distributed disturbances, $\epsilon_{c,t}$ and $\epsilon_{e,t}$.

Suppose both sectors are in perfect competition, then the representative firms would set their prices at marginal costs, which implies

$$\frac{P_{c,t}}{P_{e,t}} = \frac{Z_{e,t} F_1^{e}\left(K_{e,t}, N_{e,t}\right)}{Z_{c,t} F_1^{c}\left(K_{c,t}, N_{c,t}\right)}, \tag{9}$$

where $F^{j}(\cdot,\cdot)$ is a constant-returns production function of sector j and $F_1^{j}(\cdot,\cdot)$ is the partial derivative with respect to the first argument. By considering the equivalence for IST, the inverse relative price of equipment given by Eq.(4), and the constant returns of production function, we can rewrite Eq.(9) as

$$Q_t = \frac{Z_{e,t} f^{e'}\left(k_{e,t}\right)}{Z_{c,t} f^{c'}\left(k_{c,t}\right)}, \tag{10}$$

where $k_{j,t}$ exhibits a capital per worker in sector j and $f^{j}(k_{j,t}) = F^{j}\left(K_{j,t}/N_{j,t}, 1\right)$. Suppose further that the production function is Cobb-Douglas such that $f^{j}(k_{j,t}) = k_{j,t}^{\alpha_j}$, then Eq.(10) is extended by logged variables as

$$\ln Q_t = \ln Z_{e,t} - \ln Z_{c,t} + S_{q,t}, \tag{11}$$

where $S_{q,t} = \ln \alpha_e - \ln \alpha_c - (1-\alpha_e)\ln k_{e,t} + (1-\alpha_c)\ln k_{c,t}$, and α_j indicates the capital share of sector j. Without loss of generality, we can assume that the capital/worker ratios of both sectors change with a deterministic trend, which implies a trend-stationary stochastic process. Thus, $S_{q,t}$ is stationary. Since logged Q_t is composed of two uncorrelated random walk processes and a stationary process, the investment-specific productivity, Q_t, also has a random walk process.

On the other hand, the composite output Y_t consists of $Y_{c,t}$ and $Y_{e,t}$ with an aggregator $\Phi(\cdot)$. To make things more precise, suppose that the aggregator is Cobb-Douglas as

$$Y_t = \Phi\left(Y_{c,t}, Y_{e,t}\right) = {Y_{c,t}}^{\phi} {Y_{e,t}}^{1-\phi}, \tag{12}$$

where $\phi \in [0,1]$ indicates the share of output for consumption goods to the total output. Using the production functions given in Eq.(5) and Eq.(6), the composite output can be extended by logged variables as

$$\begin{aligned} \ln Y_t &= \phi \ln Z_{c,t} + (1-\phi)\ln Z_{e,t} \\ &\quad + \alpha_c \phi \ln K_{c,t} + \alpha_e (1-\phi)\ln K_{e,t} \\ &\quad + (1-\alpha_c)\phi \ln N_{c,t} + (1-\alpha_e)(1-\phi)\ln N_{e,t}, \end{aligned}$$

which implies that the Solow residuals of the aggregate output from a typical growth accounting method is a linear combination of $\ln Z_{c,t}$ and $\ln Z_{e,t}$:

$$\ln A_t \equiv \phi \ln Z_{c,t} + (1-\phi)\ln Z_{e,t}, \tag{13}$$

where A_t represents Solow residuals or the aggregate TFP.

Then, logged A_t has to be a random walk because logged $Z_{c,t}$ and $Z_{e,t}$ are uncorrelated $I(1)$ processes by construction. Normalizing Eq.(13) with respect to $\ln Z_{e,t}$ and substituting for Eq.(11)

yields

$$\ln Q_t - (1-\phi)^{-1} \ln A_t + (1-\phi)^{-1} \ln Z_{c,t} = S_{q,t}. \tag{14}$$

According to Eq.(14), a linear combination of three $I(1)$ processes gives a stationary process, which means the cointegration system of $\ln Q_t$, $\ln A_t$ and $\ln Z_{c,t}$ with the cointegrating vector of $(1, -(1-\phi)^{-1}, (1-\phi)^{-1})$. Another cointegrated relation is derived by substituting Eq.(13) for Eq.(11) with respect to $\ln Z_{c,t}$:

$$\ln Q_t + \phi^{-1} \ln A_t - \phi^{-1} \ln Z_{e,t} = S_{q,t}. \tag{15}$$

Eq.(15) implies that $\ln Q_t$, $\ln A_t$ and $\ln Z_{e,t}$ are cointegrated with the cointegration vector of $(1, \phi^{-1}, -\phi^{-1})$. These results can be summarized in the following Proposition 1:

Proposition 1. *Suppose that sectoral productivities,* $\ln Z_{c,t}$ *and* $\ln Z_{e,t}$*, follow uncointegrated* $I(1)$ *processes, then there exists a cointegrating vector that makes the system of three* $I(1)$ *processes* $(\ln Q_t, \ln A_t, \ln Z_{c,t})$ *(or* $(\ln Q_t, \ln A_t, \ln Z_{e,t})$*) stationary.*

Independent sectoral shocks are broadly assumed in most of the literature on multi-sector business cycles, including two-sector specification.[9] According to Proposition 2, however, Proposition 1 contradicts the empirical findings, indicating a cointegrated relation between TFP and IST, which is supported by Schmitt-Grohé and Uribe (2011).

Proposition 2. *Under the assumption of uncointegrated sectoral productivities,* $\ln Z_{c,t}$ *and* $\ln Z_{e,t}$*, following* $I(1)$ *processes, if TFP* $(\ln A_t)$ *and IST* $(\ln Q_t)$ *are cointegrated, there is no such cointegrating vector that makes three* $I(1)$ *processes of* $(\ln A_t, \ln Q_t, \ln Z_{c,t})$ *(or* $(\ln A_t, \ln Q_t, \ln Z_{e,t})$*) stationary.*

Proof: refer to Appendix A

To reconcile Proposition 1 with the empirical findings of Schmiit-Grhoé and Uribe (2011), I reconsider the underlying assumptions on Proposition 1. First, I consider relaxing the random

[9] Consistent with Proposition 1, Ireland and Schuh (2008) introduce growth stationary (or log difference stationary implies $I(1)$) sectoral productivities in their two-sector model but two productivities are uncorrelated.

walk assumption from both sectoral productivities to either one of the two. This modification does not hurt the non-stationary property of the aggregate neutral and investment-specific productivities, while ensuring cointegration between them; at least one non-stationary process is enough to make any linear combination of productivities non-stationary. However, this has not been supported by data. According to Table 1, U.S. sectoral productivities constructed from the EU KLEMS database reveal that the sectoral productivities have $I(1)$ processes in both sectors.

Another possible modification is introducing a cointegrated relation of both sectoral productivities, which is also supported by the empirical results for "db6" in Tables 2 and 3. To derive a formal theoretical result, first of all, we have to check if this additional assumption grants the property of $I(1)$ process to TFP and IST. For the validity, the cointegrating vector has to satisfy a specific condition. It is helpful to refer to IST given in Eq.(11) and aggregate TFP in Eq.(13). Both logged TFP and IST are a special linear combination of logged sectoral productivities, $\ln Z_{c,t}$ and $\ln Z_{e,t}$, with different scale vectors; respectively, $(\phi, 1-\phi)$ and $(-1, 1)$. Now suppose that the uncovered cointegrating vector of $(\ln Z_{c,t}, \ln Z_{e,t})$ is $(1, \kappa)$. To ensure the non-stationary property of TFP and IST, κ should not be equal to $(1-\phi)/\phi$ or -1. Accordingly, if the cointegrating vector of sectoral productivities satisfies the conditions mentioned above, the non-stationarity of TFP and IST are preserved and Proposition 3 follows:

Proposition 3. *Suppose $\ln A_t$, $\ln Q_t$, $\ln Z_{c,t}$ and $\ln Z_{e,t}$ follow $I(1)$ processes. Then, $\ln A_t$ and $\ln Q_t$ are cointegrated if and only if $\ln Z_{c,t}$ and $\ln Z_{e,t}$ are cointegrated.*

Proof: refer to Appendix A

As we have already seen in Tables 2 and 3, Proposition 3 stands on the support of empirical findings. Consequently, an appropriate model for a two-sector economy is better to introduce the cointegrated relation of sectoral productivities. In the following section, the cointegrated sectoral productivities are incorporated into a two-sector DSGE model and are used to estimate deep parameters and analyze the role of the stochastic common trend of sectoral productivities.

3 Model

Throughout SECTION 2, I have explained why we consider the cointegrated relationship of sectoral productivities in a two-sector economy model. Considering PROPOSITION 3, this section develops a two-sector business cycle model extended from Ireland and Schuh (2008); their model is established for two-sector economy of consumption goods and equipment with both level and growth rate shocks of preference and productivities. The main difference of this model is the cointegrated relationship of sectoral productivities. Additionally, to ensure fully mobile capital across sector, capital accumulation is allowed only at the aggregate level. Also, as real rigidities, capital adjustment cost and habit persistence in consumption are employed. Solving the competitive equilibrium, I introduce IST explicitly into the model; Ireland and Schuh (2008) regard IST as a shadow price.

3.1 The Household

Consider that the infinitely lived representative household has the preference, described over the habit persistent consumption, C_t, and hours worked, H_t, which is given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \ln\left(C_t - \xi C_{t-1}\right) - H_t/X_t \right\}, \tag{16}$$

where β and $\xi \in [0,1)$, respectively, denote the subjective discount factor and the degree of habit persistence. X_t stands for the preference shock. The preference shock consists of two stochastic components: level-stationary cyclical part, $X_{l,t}$, and growth-stationary trend part, $X_{g,t}$. The functional form of preference shocks are given by

$$X_t = X_{l,t} X_{g,t}, \tag{17}$$

$$\ln X_{l,t} = \rho_{xl} \ln X_{l,t-1} + \epsilon_{xl,t}, \tag{18}$$

$$\ln\left(\frac{X_{g,t}/X_{g,t-1}}{\eta^{xg}}\right) = \rho_{xg} \ln\left(\frac{X_{g,t-1}/X_{g,t-2}}{\eta^{xg}}\right) + \epsilon_{xg,t}, \tag{19}$$

where $\rho_j \in [0,1)$ and ϵ_j, respectively, indicate the autoregressive coefficients and disturbance of stochastic process which is *iid* normal with mean zero and variance σ_j^2 for $j \in \{xl, xg\}$. η^{xg} stands for the long-run steady-state growth rate of preference shock.

In this model economy, the household earns income by supplying labor force and renting capital to the firms, and spends its fortune for consumption and investment purposes. Hence, the household faces the budget constraint of

$$C_t + I_t/Q_t \leq \tilde{W}_t H_t + \tilde{R}_t K_t, \tag{20}$$

where $\tilde{W}_t$ and $\tilde{R}_t$ stand for the wage and rent rate in terms of the unit of consumption goods. As we have seen from Eqs.(1)-(4), investment expenditure, J_t, is equal to the gross investment in terms of consumption goods, I_t/Q_t. Capital, K_{t+1}, accumulates through investment, I_t, with capital adjustment cost and constantly depreciated previous capital stock, K_t, as follows:

$$K_{t+1} \leq (1-\delta) K_t + I_t \left[1 - \frac{\psi}{2}\left(\frac{I_t}{I_{t-1}} - \tau^I\right)^2\right], \tag{21}$$

where $\psi > 0$ is the parameter for capital adjustment cost, and τ^I denotes the steady state level of investment growth.

The representative household would maximizes its life-time utility, Eq.(16), subject to the budget constraint, Eq.(20), including the capital accumulation process, Eq.(21). The first-order conditions from solving the household's problem are derived as follows:

$$\begin{aligned}
\Lambda_{1,t} &= \frac{1}{C_t - \xi C_{t-1}} - \beta\xi\mathbb{E}_t \frac{1}{C_{t+1} - \xi C_t}, && (22)\\
\frac{1}{X_t} &= \Lambda_{1,t}\tilde{W}_t, && (23)\\
\frac{\Lambda_{1,t}}{Q_t} &= \Lambda_{2,t}\left[1 - \frac{\psi}{2}\left(\frac{I_t}{I_{t-1}} - \tau^I\right)^2 - \psi\frac{I_t}{I_{t-1}}\left(\frac{I_t}{I_{t-1}} - \tau^I\right)\right] + \beta\mathbb{E}_t\Lambda_{2,t+1}\psi\left(\frac{I_{t+1}}{I_t}\right)^2\left(\frac{I_{t+1}}{I_t} - \tau^I\right), && (24)\\
\Lambda_{2,t} &= \beta\mathbb{E}_t\left[\Lambda_{1,t+1}\tilde{R}_{t+1} + \Lambda_{2,t+1}(1-\delta)\right], && (25)
\end{aligned}$$

Eq.(20), and Eq.(21) with equality, in which $\Lambda_{1,t}$ and $\Lambda_{2,t}$ stand for the Lagrange multipliers on the budget constraint, Eq.(20), and capital accumulation process, Eq.(21), respectively.

3.2 Firms

Two producing firms represent this model economy; one produces consumption goods and the other produces equipment. For the sake of clarity, I assume that all consumption goods are non-durables and all equipment are durables. This assumption is consistent with the definition that I used to construct the data of two-sector productivity in Section 2.1. Equipment is usually demanded for the two purposes: durable consumption and investment. By assuming all consumption goods are non-durable, however, I justify that all products of the equipment sector are used for investment without being spent for consumption. This assumption is by no means at odds; if we consider a household production, the durable consumptions can be regarded as an investment for the household's production. This assumption is also applied to the construction of observed data for consumption and investment.

Each firm $i \in \{c, e\}$, uses physical capital, $K_{i,t}$, and hours worked, $H_{i,t}$, as inputs to produce its output, $Y_{i,t}$, through a Cobb-Douglas type production function of homogeneous-degree-one as

$$Y_{c,t} = A_{c,t} {K_{c,t}}^{\alpha_c} (Z_{c,t} H_{c,t})^{1-\alpha_c}, \tag{26}$$

$$Y_{e,t} = A_{e,t} {K_{e,t}}^{\alpha_e} (Z_{e,t} H_{e,t})^{1-\alpha_e}, \tag{27}$$

where α_i denotes the substitute elasticity of physical capital for the production in sector i. $A_{i,t}$ indicates a Hicks-neutral productivity level shock of sector i and is assumed independent across sectors; these productivity level shocks are supposed to have mutually uncorrelated $AR(1)$ processes as follows:

$$\ln A_{c,t} = \rho_{ac} \ln A_{c,t-1} + \epsilon_{ac,t} \tag{28}$$

$$\ln A_{e,t} = \rho_{ae} \ln A_{e,t-1} + \epsilon_{ac,t}, \tag{29}$$

where $\rho_j \in [0, 1)$ and $\epsilon_{j,t}$ denotes the autoregressive coefficient and disturbance term which is *iid* normal with mean zero and variance σ_j^2, for $j \in \{ac, ae\}$, respectively.

$Z_{i,t}$ is the productivity growth rate shock and exhibited as labor-augmented type. Following

PROPOSITION 3, I assume that $Z_{c,t}$ and $Z_{e,t}$ are cointegrated and incorporated into the system through the vector error correction model (VECM) including the smooth transition non-linear error correction (STR NEC) as

$$\begin{bmatrix} \ln\left(\dfrac{Z_{c,t}/Z_{c,t-1}}{\eta^{zc}}\right) \\ \ln\left(\dfrac{Z_{e,t}/Z_{e,t-1}}{\eta^{ze}}\right) \end{bmatrix} = \begin{bmatrix} \rho_{ce} & \rho_{cc} \\ \rho_{ee} & \rho_{ec} \end{bmatrix} \begin{bmatrix} \ln\left(\dfrac{Z_{c,t-1}/Z_{c,t-2}}{\eta^{zc}}\right) \\ \ln\left(\dfrac{Z_{e,t-1}/Z_{e,t-2}}{\eta^{ze}}\right) \end{bmatrix} + \begin{bmatrix} f_c\left(ect_{t-1}\right) \\ f_e\left(ect_{t-1}\right) \end{bmatrix} + \begin{bmatrix} D_{ce} & D_{cc} \\ D_{ee} & D_{ec} \end{bmatrix} \begin{bmatrix} \epsilon_{zc,t} \\ \epsilon_{ze,t} \end{bmatrix}, \quad (30)$$

where $\epsilon_{zc,t}$ and $\epsilon_{ze,t}$ are *iid* normal with mean zero and variance σ^2_{zc} and σ^2_{ze}, respectively, and ect indicates the error correction term defined as

$$ect_t = \ln Z_{c,t} - \kappa \ln Z_{e,t}, \quad (31)$$

which implies $Z_{c,t}$ and $Z_{e,t}$ are cointegrated with cointegrating vector $(1, -\kappa)$. The functional forms of $f_i(\cdot)$ include both linear and non-linear for $i \in \{c, e\}$; if linear, it is a typical VECM. Here I assume $f_i(\cdot)$ follows the exponential smooth transition (ESTR) functional form as

$$f_i(ect_{t-1}) = \gamma_i ect_{t-1}\left(1 - \exp^{-\theta(ect_{t-1}-\nu)^2}\right), \quad (32)$$

for $i \in \{c, e\}$, where $\theta \geq 0$ and ν is a transition parameter. Furthermore, according to Kapetanio, Shin and Snell (2003), ect_t is geometrically ergodic or globally stationary as long as $\theta > 0$, $0 < \gamma_e < 2$ and $-2 < \gamma_c < 0$.

Since firms would maximize profits in competitive markets subject to their production technology given in Eq.(26) and (27), their profit maximization should satisfy the following conditions:

$$\tilde{R}_t = \alpha_c Y_{c,t}/K_{c,t}, \quad (33)$$

$$\tilde{W} = (1 - \alpha_c) Y_{c,t}/H_{c,t}, \quad (34)$$

$$Q_t \tilde{R}_t = \alpha_e Y_{e,t}/K_{e,t}, \quad (35)$$

$$Q_t \tilde{W} = (1 - \alpha_e) Y_{e,t}/H_{e,t}, \quad (36)$$

Eq.(26), and Eq.(27). Accordingly, these firms' profit-maximizing conditions imply that IST is the ratio of the marginal product of capital in equipment to the marginal product of capital in consumption-goods sector, which is given as follows:

$$Q_t = \frac{\alpha_e Y_{e,t}/K_{e,t}}{\alpha_c Y_{c,t}/K_{c,t}}. \tag{37}$$

3.3 Market Clearing

On the equilibrium, the four markets, consumption goods, equipment, capital and labor, of the model economy have to be cleared. Hence, the following market clearing conditions should be satisfied:

$$C_t = Y_{c,t}, \tag{38}$$

$$I_t = Y_{e,t}, \tag{39}$$

$$K_t = K_{c,t} + K_{e,t}, \tag{40}$$

$$H_t = H_{c,t} + H_{c,t}. \tag{41}$$

In addition, the aggregate output measured by unit of consumption goods is defined as

$$Y_t = Y_{c,t} + Y_{e,t}/Q_t. \tag{42}$$

3.4 Solution

The variables of this model economy possess non-stationary properties granted by Z_c, Z_e and X_g of $I(1)$ stochastic processes. Consequently, I need to transform each non-stationary variable into a stationary one on the balanced growth path. Since each variable grows with different rates along the balanced growth path, the functional form of the transformation depends on each of them. Through the following transformation equations, each non-stationary variable, denoted in upper-case, is replaced by its stationary form, denoted in lower-case, : $Y_t = y_t \mathrm{T}^c_{t-1}$; $C_t = c_t \mathrm{T}^c_{t-1}$; $H_t = h_t \mathrm{T}^h_{t-1}$; $\Lambda_{1,t} = \lambda_{1,t}/\mathrm{T}^c_{t-1}$; $\Lambda_{2,t} = \lambda_{2,t}/\mathrm{T}^i_{t-1}$; $\tilde{R}_t = \tilde{r}_t \mathrm{T}^c_{t-1}/\mathrm{T}^i_{t-1}$; $\tilde{W}_t = \tilde{w}_t \mathrm{T}^c_{t-1}/\mathrm{T}^h_{t-1}$; $Q_t = q_t \mathrm{T}^i_{t-1}/\mathrm{T}^c_{t-1}$;

$K_t = k_t \mathrm{T}^i_{t-1}$; $I_t = i_t \mathrm{T}^i_{t-1}$; $Y_{c,t} = y_{c,t}\mathrm{T}^c_{t-1}$; $Y_{e,t} = y_{e,t}\mathrm{T}^i_{t-1}$; $K_{c,t} = k_{c,t}\mathrm{T}^i_{t-1}$; $K_{e,t} = k_{e,t}\mathrm{T}^i_{t-1}$; $H_{c,t} = h_{c,t}\mathrm{T}^h_{t-1}$; $H_{e,t} = h_{e,t}\mathrm{T}^h_{t-1}$; $X_{l,t} = x_{l,t}$; $A_{c,t} = a_{c,t}$; $A_{e,t} = a_{e,t}$, where $\mathrm{T}^c_t = {Z_{c,t}}^{1-\alpha_c}{Z_{e,t}}^{\alpha_c}X_{g,t}$, $\mathrm{T}^i_t = Z_{e,t}X_{g,t}$ and $\mathrm{T}^h_t = X_{g,t}$.

Applying the above transformation to the non-stationary system of equations, Eqs.(17)-(42) except the redundant Eqs.(35) and (36), we obtain the stationary system of equations: the equations are presented in Appendix B.1. In the substitution process, I define the exogenous fundamental growth rates, denoted ηs, and the growth rates of endogenous variables, denoted τs, as follows: $\eta^{zc}_t = Z_{c,t}/Z_{c,t-1}$, $\eta^{ze}_t = Z_{e,t}/Z_{e,t-1}$ and $\eta^{xg}_t = X_{g,t}/X_{g,t-1}$; $\tau^c_t = \mathrm{T}^c_t/\mathrm{T}^c_{t-1}$, $\tau^i_t = \mathrm{T}^i_t/\mathrm{T}^i_{t-1}$ and $\tau^h_t = \mathrm{T}^h_t/\mathrm{T}^h_{t-1}$.

To solve the stationary non-linear system, I employ the method of Klein (2000). Since this solution method requires a linearized system, I log-linearize the stationary non-linear system on the steady-state values.[10]

3.5 Non-linear Error Correction

Before moving to the next section, we need to address one question: Why is the non-linear error correction considered in the model economy? A linear error correction is dominantly applied in cointegration models; Schmitt-Grohé and Uribe (2011) incorporate VECM into their model with a linear error correction. The estimated adjustment-speed coefficient with linear assumption, such as Johansen test statistics, however, does not guarantee the dynamic global stationary process of the cointegration system. Therefore, what we need now for the structural model is ensuring the dynamic stability of the system.

Table 4 exhibits the estimated cointegration parameters from the Johansen test for the dataset "db6" represented in Tables 2 and 3. From Table 4, we can see that the estimated cointegrating vector, $(1, \kappa)$, is $(1, -0.087)$ and the adjustment-speed, $(\gamma_{zc}, \gamma_{ze})$, is revealed $(-0.653, -0.613)$. The fastest adjustment-speed vector is necessarily orthogonal to the estimated cointegrating vector but the estimated adjustment-speed vector is far from the orthogonal vector. Figure 1 illustrates the estimated cointegrating vector and adjustment-speed vector and implies that the signs of the

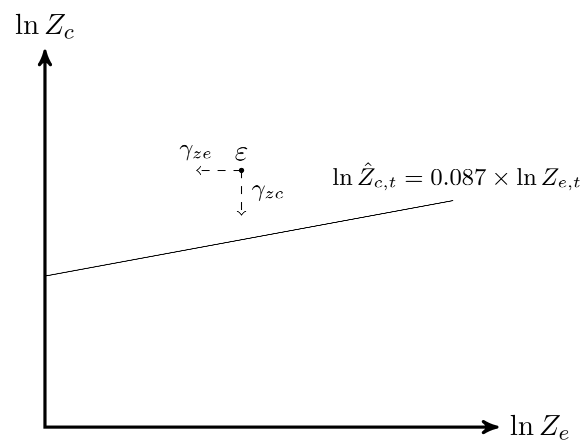
[10]The steady-state values are explicitly derived and presented in Appendix B.2. Also, the log-linearization method applied is explained in Appendix B.3.

Table 4: Cointegrated relation of sectoral productivities

	TFP.cons	TFP.equip
Cointegration Vector	1	-0.087
Adjustment parameter	-0.653	-0.613

Notes: The estimated cointegrating vector and adjustment parameters are obtained by Johansen test for the dataset named 'db6' represented in TABLE 2 and 3. The cointegrating vector is normalized by TFP.cons. TFP.cons and TFP.equip stand for the productivity of consumption goods and equipment, respectively.

Figure 1: Linear adjustment of the cointegrated sectoral productivities



estimated adjustment-speed vector is different to that of the fastest adjustment-speed vector. We can readily notice from FIGURE 1 that the linear adjustment from the deviation may not lead it back on the long-run equilibrium, if the deviation point, ε, is far enough from the long-run equilibrium path.

How can we then ensure the global stability of the system of equations? One possible answer is suggested by Kapetanio, Shin and Snell (2006), who develop a method of testing non-linear cointegration using non-linear error correction. To check the applicability of their model, I test the non-linear cointegrated relationship of the annual sectoral productivities constructed from the EU KLEMS database using the methods of Kapetanio, Shin and Snell (2006). The statistic of F_{nec} tests the null hypothesis of no cointegration with no underlying assumptions. The statistic of F^*_{nec} tests the null hypothesis of no cointegration with the assumption that the switching point is zero.

Table 5: Cointegration test under non-linear error correction assumptions

	Case	Lags(AIC)	Test statistic	Critical value(95%)	Null hypothesis
F_{nec}	Constant	3	0.908	13.73	Accept
	Trend	3	1.112	16.13	Accept
F^*_{nec}	Constant	3	1.459	12.17	Accept
	Trend	3	1.873	15.07	Accept
t_{nec}	Constant	3	-3.224	-3.22	Reject
	Trend	3	-4.477	-3.59	Reject

Notes: The statistics of F_{nec} tests the null hypothesis of no cointegration with no under-lying assumptions. The statistics of F^*_{nec} tests the null hypothesis of no cointegration with the assumption that the switching point is zero. The statistic of t_{nec} tests the null hypothesis of no cointegration with the assumption that the switching point is zero and the error correction term follow unit roots process in the middle regime.

The statistic of t_{nec} tests the null hypothesis of no cointegration with the assumption that the switching point is zero and the error correction term follows the unit roots process in the middle regime. Table 5 shows the test statistics. The test statistics without underlying assumption (F_{nec}) and with the assumption of zero switching point (F^*_{nec}) fail to reject the null hypothesis of no cointegration. The test statistics with the assumption of zero switching point and the unit roots process in the middle regime (t_{nec}), however, significantly reject the null hypothesis of no cointegration.

As such, the non-linear cointegrated relationship between sectoral productivities is confirmed from the cointegration test with non-linear error correction. Accordingly, if we push the assumption of linear error correction, dynamic instability is likely to occur in the maximum-likelihood estimation discussed in the next section. To ensure the dynamic stationary process on the DSGE model with VECM of the sectoral productivities, I assume non-linear error correction featuring exponential adjustment function.

4 Estimation

One goal of this paper is to identify the common stochastic trends of sectoral productivities, which requires estimation of the structural parameters, especially those in shock processes, such as autoregressive coefficients and the variance of disturbances. As in Ireland and Schuh (2008), and Schmitt-Grohé and Uribe (2011), I adopt the maximum likelihood estimation to estimate the deep parameters that lie on the structural model economy. The linear solution method of Klein (2000) provides the approximated solution of the non-linear system, which is defined on a state-space. Accordingly, we can employ the Kalman filter with given observable variables and construct a likelihood function. In estimation, the growth rate of consumption, investment, and hours worked are adopted for the observable variables. I construct the series of consumption and investment from the U.S. quarterly data of national income and product accounts (NIPAs) available on the BEA website.[11]

To be consistent with the model economy, consumption data is constructed by aggregating non-durable and service consumption. Also, investment is constructed by aggregating "durable consumption," and "equipment and software" in NIPAs. For aggregation, as in SECTION 2.1, the Törnqvist index is applied. Hours worked is obtained from the Federal Reserve Bank of St. Louis' FRED website, under "hours of all persons for nonfarm business sector." All data, ranging 1948:Q2-2011:Q4, are seasonally adjusted and reconstructed in per capita terms by applying "the civilian non-institutional population age 16 and over," which is available on the BLS website.

A subset of the structural parameters is calibrated. It is quite well known that the maximum likelihood estimates of the discount factor, β, and the capital depreciation rate, δ, are extremely difficult to get. Hence, as in Ireland and Schuh (2008), I impose $\beta = 0.99$ and $\delta = 0.025$. The diagonal elements of innovation coefficients, D_{cc} and D_{ee}, of VECM, without loss of generality, are normalized to unity. The steady-state quarterly growth rates of consumption, investment, and hours worked are calibrated as 1.0042, 1.0092, and 0.9995, respectively, from the average growth rate of the quarterly data constructed above. The cointegrating parameter (κ) and the steady-state growth rate of sectoral productivities, and preference (η^{zc}, η^{ze} and η^{xg}) are calculated from the steady-state conditions of the model.

The rest of the structural parameters are estimated via maximum likelihood. TABLE 6 presents

[11] Table 1.1.4 (Price index for GDP) and Table 1.1.5 (Nominal GDP) of NIPAs are used to construct real consumption for non-durables and services, and real investment, which is redefined as the aggregate of "durable consumption," and "equipment and software" in NIPAs.

Table 6: The maximum likelihood estimates and standard errors of the structural parameters

Parameter	Estimate	Standard error
ξ	0.2028	0.0327
ψ	0.3148	0.0410
θ	0.9349	0.0186
ν	-0.0551	0.0900
α_c	0.3307	0.0310
α_e	0.4009	0.0723
ρ_{cc}	0.2986	0.1450
ρ_{ce}	0.0000	0.0429
ρ_{ec}	0.0000	0.0686
ρ_{ee}	0.0000	0.0352
γ_c	-0.1825	0.4684
γ_e	1.7946	0.0671
D_{ce}	0.3000	0.0778
D_{ec}	0.0236	0.0949
ρ_{xl}	0.8911	0.1324
ρ_{xg}	0.5493	0.1156
ρ_{ac}	0.0000	0.1141
ρ_{ae}	0.0000	0.0702
σ_{xl}	0.0033	0.0014
σ_{xg}	0.0046	0.0009
σ_{ac}	0.0029	0.0005
σ_{ae}	0.0086	0.0020
σ_c	0.0042	0.0011
σ_e	0.0200	0.0052
μ_c	0.0004	0.0003
μ_i	0.0078	0.0000
μ_h	0.0023	0.0002

Notes: Sample period is 1948:Q2 to 2011:Q4. The observables are the growth rates of consumption, investment, and hours worked. Each of the observables is assumed to possess measurement error. During estimation $\beta = 0.99$ and $\delta = 0.025$ are imposed. The diagonal elements of VECM innovations, D_{cc} and D_{ee}, are normalized to unity.

the estimated 27 parameters estimated with standard errors, which come from a parametric bootstrapping procedure as in Ireland and Schuh (2008). I generate $1,000$ sets of artificial data from the estimated model by assigning random disturbances for each period having the same length of actual data. The artificially generated $1,000$ sets of data are used to estimate $1,000$ sample parameters. The reported standard errors in TABLE 6 are the standard deviations of the samples. The model estimates a significant habit-persistence parameter, ξ, of 0.2028; it is much higher than the estimate of Ireland and Schuh (2008) but a little bit lower than that of Schmitt-Grohé and Uribe (2011). The capital adjustment-cost parameter is estimated as 0.3148, which is even lower than reported in existing literature; however, the estimate is significant. The estimation allows the existence of measurement errors in consumption, investment, and hours worked series: denoted μ_c, μ_i, and μ_h, respectively. I curb the estimates of these measurement errors not to exceed 25% of the standard error of each series.

In the estimation, I estimate the capital share of each sector without assuming symmetry across sectoral production functions; most of the two-sector models, including Ireland and Schuh (2008), employ symmetric capital shares. The symmetry assumption, however, does not reflect the reality, but is done for convenience. The maximum likelihood method estimates the capital share of consumption goods, α_c, as 0.3307 and that of equipment, α_e, as 0.4009: the estimate of capital share in equipment production, however, has a twice as large standard deviation than that for consumption. The estimated sectoral capital shares are worth comparing with others: Ireland and Schuh (2008) estimate the capital share of 0.39 with s.e. 0.06, and Schmitt-Grhoé and Uribe (2011) estimate 0.37 with s.e. 0.03. Therefore, we can see the estimate is not much different to the estimates of existing studies but rather lie within their two-standard error confidence intervals both in consumption goods and equipment. Additionally, the estimates correspond to the conventional wisdom, which says consumption goods production is relatively labor-intensive, meanwhile equipment production is capital-intensive.

The most interesting features of the estimation is the parameters of cointegration, volatility, and persistence of the shocks. The existence of cointegration can be tested by evaluating the estimate of θ.[12] If $\theta = 0$, the error-correction term of non-linear VECM will vanish; it implies a regular VAR model. Applying the standard deviation of estimated θ, we can easily test the null hypothesis of $\theta = 0$: we can reject the null because the estimated θ of 0.9349 lies far outside the two-standard deviation from the null. Accordingly, the cointegration of sectoral productivities is

[12] The maximum likelihood estimates have asymptotically normal distributions. Therefore, for hypothesis tests, we can apply t-test. See Canova (2007), pp. 225-228, for details.

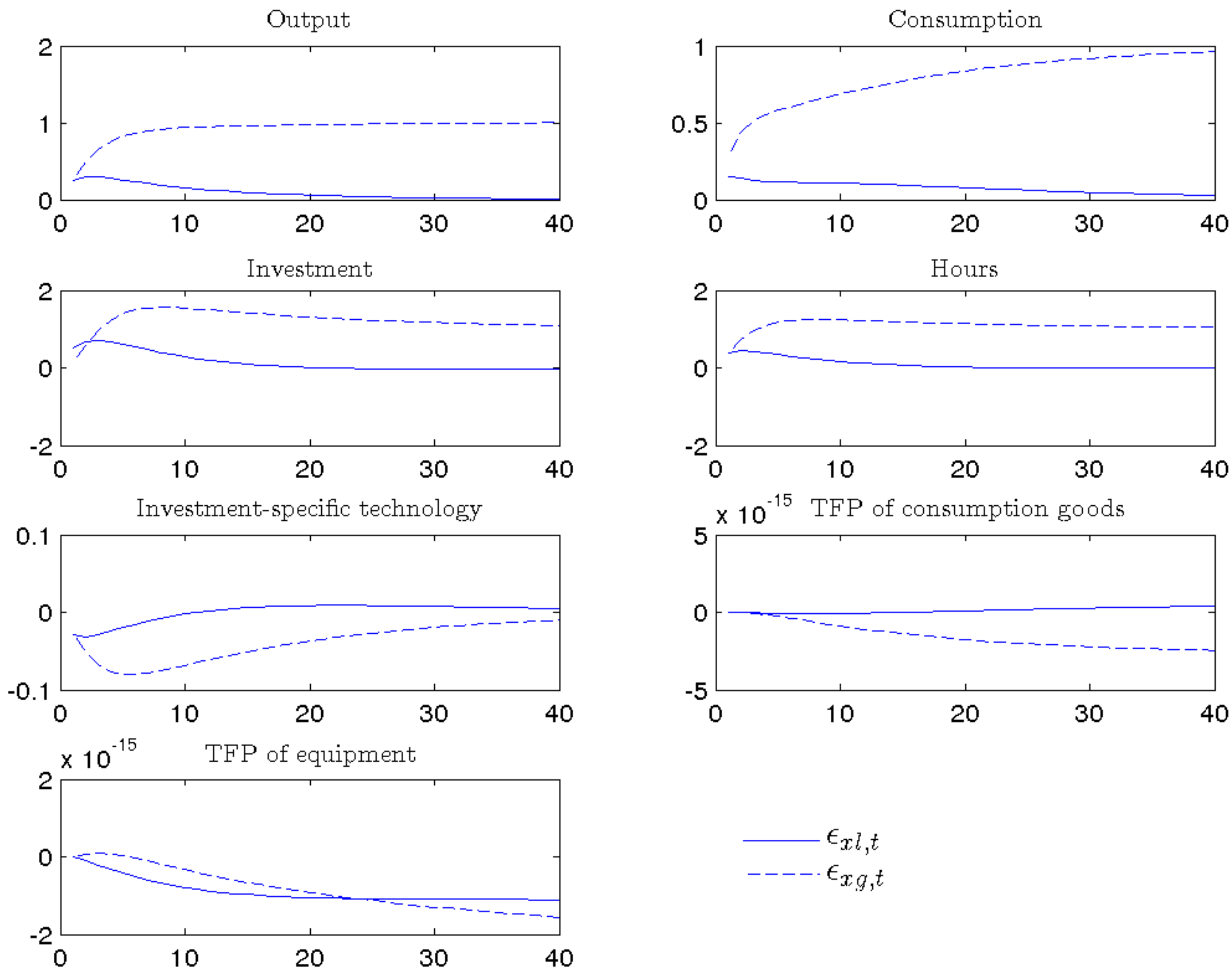
confirmed. The persistence parameters of common trend shocks (ρ_{cc}, ρ_{ce}, ρ_{ec}, and ρ_{ee}) are estimated as 0.2986 and zeros, respectively, which mean the persistence of common trend shocks is delivered to the next period only through the consumption goods channel. The correlation parameters of the innovation of common trend, D_{ce} and D_{ec} indicate that the innovations of common trend shocks are significantly correlated: about 30% of growth rate innovation of equipment, $\epsilon_{e,t}$, is correlated to that of consumption goods. As we will see in the next section, these features bring out impressive results in impulse response analysis. The estimates of adjustment-speed parameters (γ_c and γ_e) are -0.1825 and 1.7946, which mean that most of the adjustment occurs in the equipment sector: that is, the productivity of consumption goods is weakly exogenous.

One of the important features of the estimates is that all shocks are identified. Ireland and Schuh (2008) identify shocks except the growth rate component of investment goods sector and they conclude that no equipment-sector-specific technology has had permanent effects on the postwar U.S. economy. The results of this paper suggest, however, that their failure in identifying growth rate shock of equipment is due to the misspecified their structural model. The volatility of the common trend shocks, σ_c and σ_e, are estimated as 0.0042 and 0.0200, respectively. Consequently, the largest shocks among estimates returns to σ_e, which identifies the stochastic trend of equipment production in the postwar U.S. data. The estimated volatilities of the remained shocks, σ_{xl}, σ_{xg}, σ_{ac}, and σ_{ae}, are 0.0033, 0.0046, 0.0029, and 0.0086, respectively. The level and growth rate shocks of preference are estimated with high persistence: the autoregressive coefficients of the level and growth rate shocks, ρ_{xl} and ρ_{xg}, are estimated as 0.8911 and 0.5493, respectively. However, the persistence of the level shocks of sectoral productivities, ρ_{ac} and ρ_{ae}, are estimated as zero; that is, no persistence is estimated.

5 Results

The estimated structural disturbances from Section 4 have different implications on the model economy. This section investigates the effects of each shock and discusses its roles. The impulse responses in Figures 2-4 depict the responses of output, consumption, investment, hours, IST and

Figure 2: Impulse responses on preference shocks both in level and growth rate

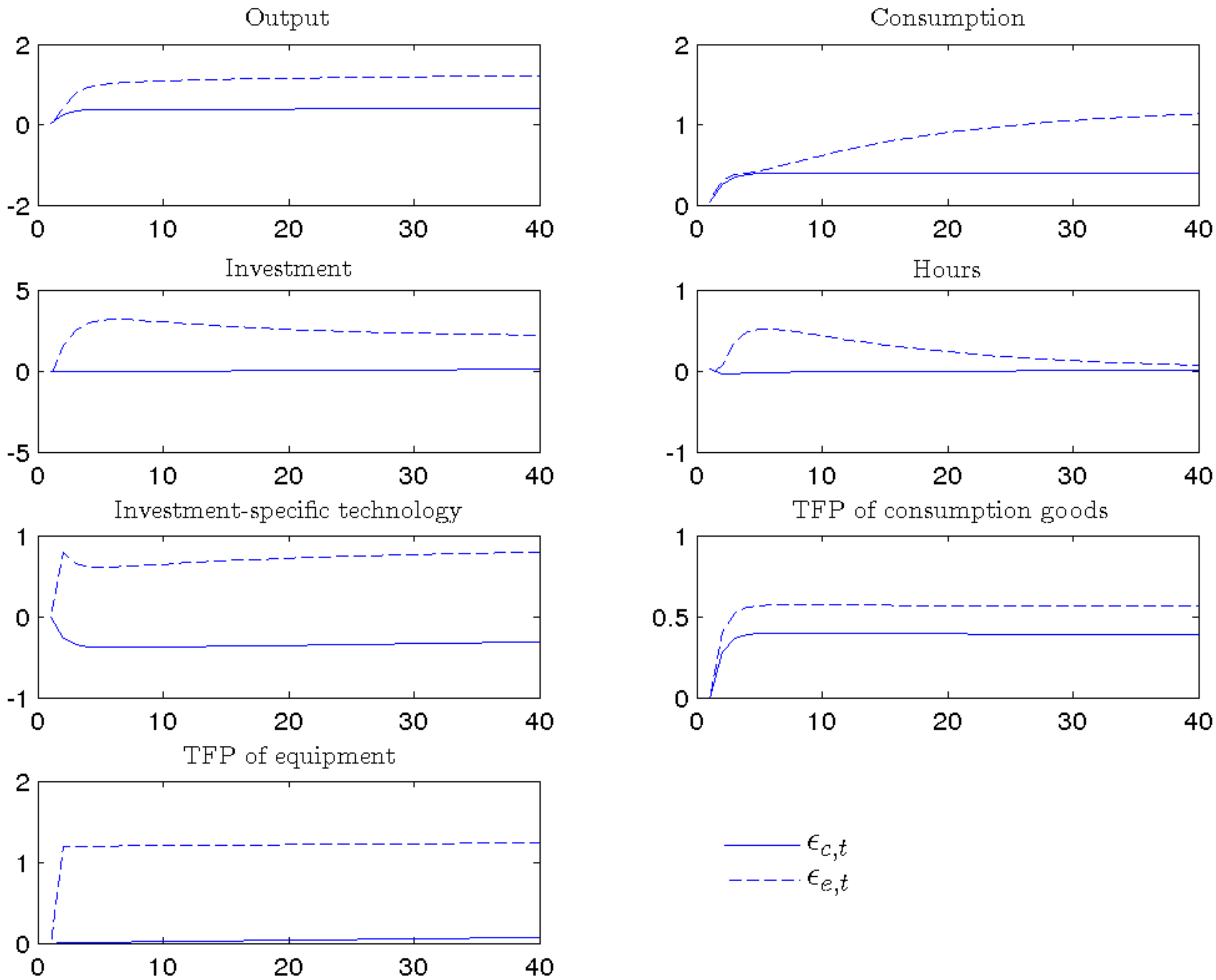


Notes: Each panel shows the percentage deviation of output, consumption, investment, hours worked, IST, TFP of consumption goods sector, and TFP of equipment sector to a one-standard-deviation shock to level and growth rate of preference.

sectoral TFPs to a one-standard-error innovation of each shock. Figure 2 displays the impulse responses to the level and growth rate shocks in preference. Figures 3 and 4 exhibit the impulse responses to the shocks of sectoral productivities, respectively, in common trends and in level.

Figure 2 indicates that both level and growth rate shocks have positive effects on all four

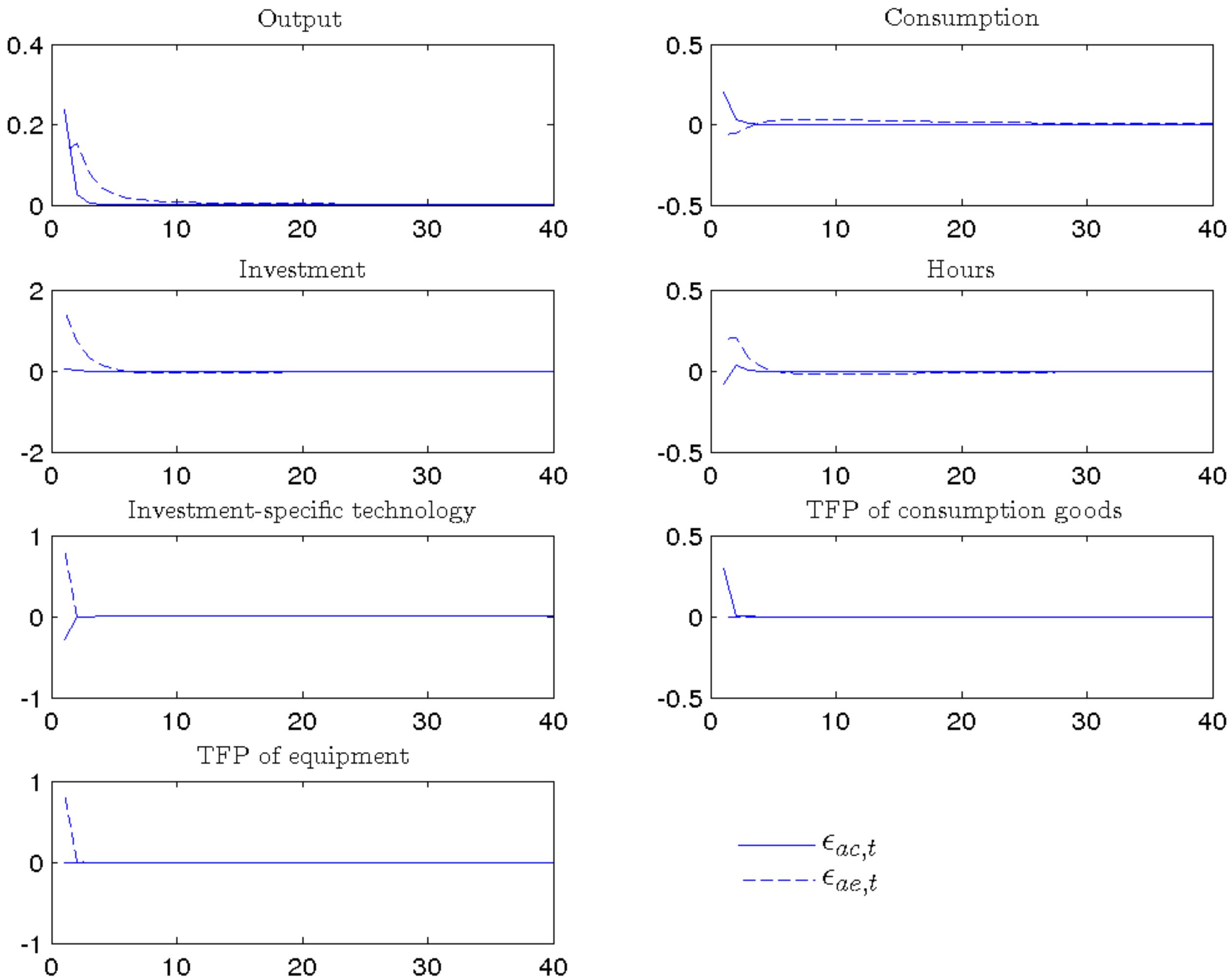
Figure 3: Impulse responses on productivity shocks in common stochastic trend



Notes: Each panel shows the percentage deviation of output, consumption, investment, hours worked, IST, TFP of consumption goods sector, and TFP of equipment sector to a one-standard-deviation shock to common stochastic trend in productivities.

macroeconomic aggregates: output, consumption, investment, and hours worked. In particular, the growth rate shock in preference has nearly equally sizeable effects on the macroeconomic aggregates with high persistence. This result is consistent with Ireland and Schuh (2008); they find that only the growth rate shock in preference has a highly persistent sizeable effect on hours worked. Since

Figure 4: Impulse responses on productivity shocks in level



Notes: Each panel shows the percentage deviation of output, consumption, investment, hours worked, IST, TFP of consumption goods sector, and TFP of equipment sector to a one-standard-deviation shock to the productivity level of each sector.

the preference shocks are not related to the changes in productivities, they have no effect on sectoral TFPs.

Another notable fact in FIGURE 2 is the decrease of IST in the short run, which recovers its original state in the long run. This fact confirms Oulton (2007)'s argument: The relative price of

Table 7: Forecast-error variance decomposition

Quaters ahead	ϵ_{xl}	ϵ_{xg}	ϵ_{ac}	ϵ_{ae}	ϵ_c	ϵ_e
			Consumption			
1	14.0	52.5	26.2	2.2	1.3	3.8
4	4.2	49.5	2.5	0.4	20.0	23.4
8	2.5	48.6	0.9	0.2	20.1	27.6
12	1.9	47.3	0.5	0.2	17.7	32.5
20	1.1	44.9	0.2	0.1	13.7	40.0
40	0.5	41.7	0.1	0.0	9.4	48.3
			Investment			
1	10.1	1.0	0.1	84.0	0.2	4.6
4	6.8	10.9	0.0	11.3	0.0	70.9
8	3.7	15.8	0.0	3.8	0.0	76.7
12	2.5	17.6	0.0	2.4	0.0	77.4
20	1.6	18.9	0.0	1.5	0.0	78.0
40	0.9	19.4	0.0	0.9	0.0	78.8
			Hours worked			
1	41.9	43.8	2.0	11.1	0.2	1.0
4	16.5	71.4	0.2	2.2	0.1	9.5
8	8.6	78.1	0.1	0.8	0.0	12.5
12	5.8	81.9	0.0	0.5	0.0	11.7
20	3.6	86.5	0.0	0.3	0.0	9.5
40	2.0	91.8	0.0	0.2	0.0	6.0

Notes: The decomposed forecast error variances in consumption, investment, and hours worked are exhibited. The decomposition consists of the contribution of all 6 shocks to the forecast error variances.

equipment can change without the relative change of sectoral productivities. In the model economy, equipment production is capital-intensive, meanwhile consumption production is labor-intensive; these are estimated rather than assumed. The positive preference shocks increase labor supply and subsequently push down equilibrium wage. Accordingly, the production of consumption, which is labor-intensive, rise and it is accompanied by a decrease in consumption prices. Therefore, IST is decreasing in the short run. As we can see, however, the magnitude of the effect is very limited. Consequently, we can say that Oulton's argument is right but not likely to be a dominant effect in a real economy.

According to Figure 3, the shocks to common stochastic trend generally have persistent effects on the model but the propagation paths differ for each source of shocks. The shock due to $\epsilon_{e,t}$ has a very sizeable effect on output, consumption, and investment. In particular, the effect on investment is much larger than that on consumption and remains for a long period of time. $\epsilon_{e,t}$ also increases the hours worked in the short run and shrink rapidly to its original level. The shock due to $\epsilon_{c,t}$ mostly affects the productivity of consumption goods. The effect of $\epsilon_{c,t}$ on the productivity of the equipment is negligibly small; subsequently, IST decreases almost permanently. However, investment does not shrink from that; instead it remains almost unchanged. The consequent effect of $\epsilon_{c,t}$ on consumption is returned as very persistent.

The impulse responses to level shocks of sectoral productivities have effects for short periods of time. Since these shocks are not mutually correlated, there is no cross-over effect. As we can see in FIGURE 4, the consumption productivity shock, $\epsilon_{ac,t}$ has an effect only on TFP in consumption goods and consumption, while a positive shock in equipment productivity, $\epsilon_{ae,t}$, leads to an increase in TFP in equipment and investment.

TABLE 7 exhibits the decomposed forecast error variances of consumption, investment, and hours worked; decomposition consists of the contribution of all 6 shocks to the forecast error variances. About half of consumption variability depends on the growth rate of preference innovations almost equally both in the short and long runs. The level shock of consumption productivity to consumption takes a small portion of the variability only in the short run. Interestingly, about half of consumption variability in the long run is explained by the common stochastic shocks. The variability of investment mostly explained by the common stochastic trend, especially, $\epsilon_{e,t}$ takes about 71-79% of the variability after the fourth quarter predicted time horizon. The productivity shock in equipment accounts for most of the one-period-ahead forecast variance for investment; however, its explanation power shrinks radically with the increase of forecast period. Around 20% of investment variability is due to preference shocks. As I have pointed out before, most of the volatility of hours worked, about 90%, is associated with preference shocks: long-run variability of 10% in hours worked is explained by common stochastic trend of productivities.

6 Conclusion

This paper theoretically and empirically presents the existence of a cointegrated relationship in sectoral productivities, which is motivated by the findings of Schmitt-Grohé and Uribe (2011). Furthermore, I incorporate the cointegrated relationship of sectoral productivities into the two-sector model of Ireland and Schuh (2008). By introducing non-linear error correction into the model economy, I conduct maximum likelihood estimation and successfully identify all structural shocks. The subsequent impulse-response investigation finds that the innovation of common stochastic trends in sectoral productivities increases consumption and investment simultaneously and permanently in two ways. First the innovation of $\epsilon_{c,t}$, which accounts for most of growth rate of consumption productivity, increases the TFP of consumption goods with a negligible effect on the TFP of equipment. Consequently, the effect mostly causes consumption increase without changing investment. Secondly, the innovation of $\epsilon_{e,t}$, which is correlated to consumption productivity, increases the sectoral TFPs simultaneously. Since the magnitude of the effect on equipment TFP is twice as large compared to that on consumption TFP, IST suddenly increases. Because of the positive effect on both TFPs and IST, consumption and investment increase simultaneously with persistence.

The knowledge included in the paper can be applied to disentangle the sectoral comovement puzzle. The existing studies on this issue, such as Hornstein and Praschnik (1997) and Horvath (2000), address the problem by introducing intermediate inputs, which foster the sectoral linkages. The impulse response results from common stochastic trends, however, indicate that the sectoral outputs, consumption and investment, increase at the same time: that is, sectoral comovement is explained without introducing intermediate inputs. The assumptions of the paper, however, are too restrictive to be applied in the general sense: the model assumes perfect segregation between consumption goods and investment goods. Therefore, by relaxing the restrictive assumption, we may extend the idea of this study to the future study on addressing the sectoral comovement puzzle.

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Appendix

A Proofs

A.1 Proof for Proposition 2

Suppose $\ln A_t$ and $\ln Q_t$ are cointegrated, then there exist $(1, \psi)$ such that $\ln A_t + \psi \ln Q_t = S_t^1$, where S_t^1 is a stationary stochastic process. Suppose a negation that there exist a cointegrating vector $(1, \mu_1, \mu_2)$ in the system of $(\ln A_t, \ln Q_t, \ln Z_{c,t})$ and assume that S_t^2 is another stationary process which is independent of S_t^1, then

$$
\begin{aligned}
& \ln A_t + \mu_1 \ln Q_t + \mu_2 \ln Z_{c,t} = S_t^2 \\
\rightarrow \quad & S_t^1 - \psi \ln Q_t + \mu_1 \ln Q_t + \mu_2 \ln Z_{c,t} = S_t^2 \\
\rightarrow \quad & (\mu_1 - \psi) \ln Q_t + \mu_2 \ln Z_{c,t} = S_t^2 - S_t^1 \\
\rightarrow \quad & (\mu_1 - \psi)(\ln Z_{e,t} - \ln Z_{c,t}) + \mu_2 \ln Z_{c,t} = S_t^2 - S_t^1 \\
\rightarrow \quad & (\mu_1 - \psi) \ln Z_{e,t} + (\mu_2 - \mu_1 + \psi) \ln Z_{c,t} = S_t^2 - S_t^1
\end{aligned}
$$

Since *RHS* of the above equation is stationary, *LHS* has to be stationary either. Since $\ln Z_e$ and $\ln Z_c$ are not coitegrated, to make *LHS* stationary the following condition has to be satisfied:

$$
\begin{aligned}
& \mu_1 - \psi = 0, \ \ and \\
& \mu_2 - \mu_1 + \psi = 0,
\end{aligned}
$$

which implies $\mu_2 = 0$. However, $\mu_2 = 0$ contradicts the assumption that $(\ln A_t, \ln Q_t, Z_{c,t})$ is a cointegrated system. Therefore, $(\ln A_t, \ln Q_t, \ln Z_{c,t})$ is not cointegrated.

The proof for $(\ln A_t, \ln Q_t, \ln Z_{e,t})$ is omitted because of the similarity to the above. $\square$

A.2 Proof for Proposition 3

Case1: $\ln A_t$ and $\ln Q_t$ are cointegrated $\Longrightarrow$ $\ln Z_{c,t}$ and $\ln Z_{e,t}$ are cointegrated.

Suppose $\ln Z_{c,t}$ and $\ln Z_{e,t}$ consist of random walk, $\mu_{c,t}$ and $\mu_{e,t}$, and stationary parts, $e_{c,t}$ and $e_{e,t}$, as follows:

$$\ln Z_{c,t} = \mu_{c,t} + e_{c,t}$$
$$\ln Z_{e,t} = \mu_{e,t} + e_{e,t},$$

then $\ln A_t$ and $\ln Q_t$ are represented as follows:

$$\begin{aligned}
\ln A_t &= \phi \ln Z_{c,t} + (1-\phi)\ln Z_{e,t} \\
&= \phi\mu_{c,t} + (1-\phi)\mu_{e,t} + \phi e_{c,t} + (1-\phi)e_{e,t} \\
\ln Q_t &= \ln Z_{e,t} - \ln Z_{c,t} \\
&= \mu_{e,t} - \mu_{c,t} + e_{e,t} - e_{c,t}.
\end{aligned}$$

Since $\ln A_t$ and $\ln Q_t$ are cointegrated, there exists $(1, \psi)$ such that $\ln A_t + \psi \ln Q_t = S_t$ where S_t is a stationary process. $\ln A_t + \psi \ln Q_t$ can be rewritten as

$$\begin{aligned}
\ln A_t + \psi \ln Q_t &= \phi\mu_{c,t} + (1-\phi)\mu_{e,t} + \psi\mu_{e,t} - \psi\mu_{c,t} + D \\
&= (\phi-\psi)\mu_{c,t} + (1-\phi+\psi)\mu_{e,t} + D,
\end{aligned}$$

where D is a stationary process, defined as $\phi e_{c,t} + (1-\phi)e_{e,t} + \psi e_{e,t} - \psi e_{c,t}$. Suppose further that $\mu_{c,t}$ and $\mu_{e,t}$ are not cointegrated, then the cointegrated $\ln A_t$ and $\ln Q_t$ requires the following conditions:

$$\phi - \psi = 0, \ \ and$$
$$1 - \phi + \psi = 0.$$

The two equations, however, cannot be solved simultaneously. Therefore, $\mu_{c,t}$ and $\mu_{e,t}$ have to be cointegrated, which further implies the cointegration of $\ln Z_{c,t}$ and $\ln Z_{e,t}$.

Case2: $\ln Z_{c,t}$ and $\ln Z_{e,t}$ are cointegrated $\Longrightarrow$ $\ln A_t$ and $\ln Q_t$ are cointegrated.

Since $\ln Z_{c,t}$ and $\ln Z_{e,t}$ are cointegrated, there exists a cointegrating vector $(1,\kappa)$ such that $\ln Z_{c,t}+\kappa\ln Z_{e,t}=S_t$ with a stationary S_t. $\ln A_t$ and $\ln Q_t$ can be rewritten as

$$\ln A_t = (1-\phi-\kappa\phi)\ln Z_{e,t}+\phi S_t$$
$$\ln Q_t = (1+\kappa)\ln Z_{e,t}-S_t.$$

Then there exists a linear combination for $\ln A_t$ and $\ln Q_t$ such that

$$\begin{aligned}\ln A_t-\frac{1-\phi-\phi\kappa}{1+\kappa}\ln Q_t &= (1-\phi-\kappa\phi)\ln Z_{e,t}+\phi S_t-(1-\phi-\kappa\phi)\ln Z_{e,t}+\frac{1-\phi-\kappa\phi}{1+\kappa}S_t\\ &= \frac{1}{1+\kappa}S_t.\end{aligned}$$

Therefore, $\ln A_t$ and $\ln Q_t$ are cointegrated with the cointegrating vector $\left(1,-\frac{1-\phi-\phi\kappa}{1+\kappa}\right)$. $\square$

B Model solution

B.1 Stationary system

The Household's Conditions

$$\lambda_{1,t}=\frac{1}{c_t-\xi c_{t-1}/\tau^c_{t-1}}-\beta\xi\mathbb{E}_t\frac{1}{c_{t+1}\tau^c_t-\xi c_t} \tag{B.1.1}$$

$$\frac{1}{x_{l,t}\eta^{xg}_t}=\lambda_{1,t}\tilde{w}_t \tag{B.1.2}$$

$$\begin{aligned}\lambda_{1,t}/q_t &= \lambda_{2,t}\left[1-\frac{\psi}{2}\left(\frac{i_t}{i_{t-1}}\tau^i_{t-1}-\tau^I\right)^2-\psi\frac{i_t}{i_{t-1}}\tau^i_{t-1}\left(\frac{i_t}{i_{t-1}}\tau^i_{t-1}-\tau^I\right)\right]\\ &+\beta\psi\mathbb{E}_t\left(\frac{i_{t+1}}{i_t}\right)^2\tau^i_t\left(\frac{i_{t+1}}{i_t}\tau^i_t-\tau^I\right)\end{aligned} \tag{B.1.3}$$

$$\lambda_{2,t}\tau^i_t=\beta\mathbb{E}_t\left\{\lambda_{1,t+1}\tilde{r}_{t+1}+\lambda_{2,t+1}(1-\delta)\right\} \tag{B.1.4}$$

$$c_t+i_t/q_t=\tilde{w}_th_t+\tilde{r}_tk_t \tag{B.1.5}$$

$$k_{t+1}\tau^i_t=(1-\delta)k_t+i_t\left[1-\frac{\psi}{2}\left(\frac{i_t}{i_{t-1}}\tau^i_{t-1}-\tau^I\right)^2\right] \tag{B.1.6}$$

The Firms' Conditions

$$y_{c,t} = a_{c,t}(k_{c,t})^{\alpha_c}(\eta_t^{zc}h_{c,t})^{1-\alpha_c} \tag{B.1.7}$$

$$y_{e,t} = a_{e,t}(k_{e,t})^{\alpha_e}(\eta_t^{ze}h_{e,t})^{1-\alpha_e} \tag{B.1.8}$$

$$\tilde{r}_t = \alpha_c y_{c,t}/k_{c,t} \tag{B.1.9}$$

$$\tilde{w}_t = (1-\alpha_c)y_{c,t}/h_{c,t} \tag{B.1.10}$$

$$q_t = \frac{\alpha_e y_{e,t}/k_{e,t}}{\alpha_c y_{c,t}/k_{c,t}} \tag{B.1.11}$$

Market Clearing Conditions

$$k_t = k_{c,t} + k_{e,t} \tag{B.1.12}$$

$$h_t = h_{c,t} + h_{e,t} \tag{B.1.13}$$

$$c_t = y_{c,t} \tag{B.1.14}$$

$$i_t = y_{e,t} \tag{B.1.15}$$

$$y_t = y_{c,t} + y_{e,t}/q_t \tag{B.1.16}$$

Growth Rates

$$\tau_t^c = (\eta_t^{zc})^{1-\alpha_c}(\eta_t^{ze})^{\alpha_c}\eta_t^{xg} \tag{B.1.17}$$

$$\tau_t^i = \eta_t^{ze}\eta_t^{xg} \tag{B.1.18}$$

$$\tau_t^h = \eta_t^{xg} \tag{B.1.19}$$

Observable Variables

$$\tau_t^C = \tau_{t-1}^c \frac{c_t}{c_{t-1}} \tag{B.1.20}$$

$$\tau_t^I = \tau_{t-1}^i \frac{i_t}{i_{t-1}} \tag{B.1.21}$$

$$\tau_t^H = \tau_{t-1}^h \frac{h_t}{h_{t-1}} \tag{B.1.22}$$

Exogenous Stochastic Processes

$$ect_t - ect_{t-1} = \ln \eta_t^{zc} - \kappa \ln \eta_t^{ze} \tag{B.1.23}$$

$$\begin{bmatrix} \ln(\eta_t^{zc}/\eta^{zc}) \\ \ln(\eta_t^{ze}/\eta^{ze}) \end{bmatrix} = \begin{bmatrix} \rho_{cc} & \rho_{ce} \\ \rho_{ec} & \rho_{ee} \end{bmatrix} \begin{bmatrix} \ln\left(\eta_{t-1}^{zc}/\eta^{zc}\right) \\ \ln\left(\eta_{t-1}^{ze}/\eta^{ze}\right) \end{bmatrix} + \begin{bmatrix} f_c(ect_{t-1}) \\ f_e(ect_{t-1}) \end{bmatrix} + \begin{bmatrix} D_{cc} & D_{ce} \\ D_{ec} & D_{ee} \end{bmatrix} \begin{bmatrix} \epsilon_{zc,t} \\ \epsilon_{ze,t} \end{bmatrix} \tag{B.1.24}$$

$$\ln x_{l,t} = \rho_{x,l} \ln x_{l,t-1} + \epsilon_{xl,t} \tag{B.1.25}$$

$$\ln(\eta_t^{xg}/\eta^{xg}) = \rho_{xg} \ln(\eta_{t-1}^{xg}/\eta^{xg}) + \epsilon_{xg,t} \tag{B.1.26}$$

$$\ln a_{c,t} = \rho_{ac} \ln a_{c,t-1} + \epsilon_{ac,t} \tag{B.1.27}$$

$$\ln a_{e,t} = \rho_{ae} \ln a_{e,t-1} + \epsilon_{ae,t} \tag{B.1.28}$$

B.2 The steady states

The steady-state values of the variables in the model economy are determined by exogenously given parameter set, Θ, and the long-run average of the deterministic growth rates: η^{zc}, η^{ze} and η^{xg}. Substituting these parameters and growth rates into the Eqs.(B.1.17)-(B.1.19), we can get the steady-state of endogenous growth rates:

$$\tau^c = (\eta^{zc})^{1-\alpha_c}(\eta^{ze})^{\alpha_c}\eta^{xg} \tag{B.2.1}$$

$$\tau^i = \eta^{ze}\eta^{xg} \tag{B.2.2}$$

$$\tau^h = \eta^{xg}. \tag{B.2.3}$$

Using Eqs.(B.1.20)-(B.1.22), additionally, the long-run growth rate of the non-stationary variables are obtained as follows: $\tau^C = \tau^c$, $\tau^I = \tau^i$, and $\tau^H = \tau^h$.

The household's optimization conditions exhibited in Eqs.(B.1.1)-(B.1.6), respectively, implies the following conditions on steady states:

$$\lambda_1 c = \Phi_1, \tag{B.2.4}$$

$$1/\eta^{xg} = \lambda_1 \tilde{w}, \tag{B.2.5}$$

$$\lambda_1/q = \lambda_2, \tag{B.2.6}$$

$$\lambda_2 \tau^i = \beta \left\{ \lambda_1 \tilde{r} + \lambda_2 (1-\delta) \right\}, \tag{B.2.7}$$

$$c + i/q = \tilde{w}h + \tilde{r}k, \tag{B.2.8}$$

$$i = \Phi_i k, \tag{B.2.9}$$

where $\Phi_1 = \frac{\tau^c - \beta\xi}{\tau^c - \xi}$ and $\Phi_i = \tau^i - 1 + \delta$. Also, Eqs.(B.2.6) and (B.2.7) indicates

$$\tilde{r}q = \bar{rq}, \tag{B.2.10}$$

where $\bar{rq} = \tau^i/\beta - 1 + \delta$.

Market clearing conditions, Eqs.(B.1.12)-(B.1.16), give the important steady-state equalities, respectively, as follows:

$$k = k_c + k_e \tag{B.2.11}$$

$$h = h_c + h_e \tag{B.2.12}$$

$$c = y_c \tag{B.2.13}$$

$$i = y_e \tag{B.2.14}$$

$$y = y_c + y_e/q \tag{B.2.15}$$

By considering Eq.(35) with stationary transformation, Eqs.(B.2.9), (B.2.10), (B.2.11) and

(B.2.14), we can write the steady-state capital of each sector in terms of aggregate capital stock:

$$k_e = \Pi k, \tag{B.2.16}$$

$$k_c = (1 - \Pi)k, \tag{B.2.17}$$

where $\Pi = \alpha_e \Phi_i / \bar{r}q$. Eq.(B.1.8), with Eqs.(B.2.9), (B.2.14) and (B.2.16), implies the steady-state of h_e;

$$h_e = \Phi_{he} k, \tag{B.2.18}$$

where $\Phi_{he} = (\Phi_i / \Pi^{\alpha_e})^{1/(1-\alpha_e)} / \eta^{ze}$.

From (B.1.14), we can see that in steady-state $c = y_c$. Eqs.(B.2.4) and (B.2.5) gives

$$\frac{\tilde{w}}{c} = \frac{1}{\eta^{xg}\Phi_1}. \tag{B.2.19}$$

Applying Eq.(B.2.19) to Eq.(B.1.10), the steady-state level of h_c is obtained as the following:

$$h_c = \Phi_{hc}, \tag{B.2.20}$$

where $\Phi_{hc} = (1 - \alpha_c)\eta^{xg}\Phi_1$. With the implicit steady-state condition for q, $q = \frac{(1-\alpha_e)y_e/h_e}{(1-\alpha_c)y_c/h_c}$, Eqs.(B.1.11), (B.2.16), (B.2.17), (B.2.18) and (B.2.20), we can get the steady-state level of capital stock as follows:

$$\bar{k} = \frac{\alpha_c \Pi (1 - \alpha_e)\bar{h}_c}{\alpha_e (1 - \Pi)(1 - \alpha_c)\Phi_{he}}. \tag{B.2.21}$$

B.3 Log-Linearization

To calculate a numerical solution for the decision rules of this model economy, I do linearize the system of equations given in Section B.1 on its steady-state value of Section B.2. Instead of solving for the log-linearized system by hand, I have derived the linearized system on `Matlab` by applying the standard-method of log-linearization with the `Symbolic` toolbox in `Matlab`.

From here, I briefly describe the standard-method of log-linearization with a simple example.

Suppose that we have an equation given as follows:

$$f(X_t) + g(Y_t) = 0, \tag{B.3.1}$$

where X and Y are strictly positive variables. Using the identity $X = e^{\ln X}$, we can rewrite Eq.(B.3.1) as

$$f\left(e^{\ln X_t}\right) + g\left(e^{\ln Y_t}\right) = 0. \tag{B.3.2}$$

Taking the first-order Taylor expansion for Eq.(B.3.2) with respect to $\ln X$ and $\ln Y$ around the steady-state values, $\ln \bar{X}$ and $\ln \bar{Y}$, we can have

$$f(\bar{X}) + f'(\bar{X})(\ln X_t - \ln \bar{X}) + g(\bar{Y}) + g'(\bar{Y})(\ln Y_t - \ln \bar{Y}) = 0. \tag{B.3.3}$$

Using the identity of $f(\bar{X}) + g(\bar{Y}) = 0$ and letting $\hat{x} = \ln X - \ln \bar{X}$ and $\hat{y} = \ln Y - \ln \bar{Y}$, Eq.(B.3.3) is simplified as

$$f'(\bar{X})\hat{x} + g'(\bar{Y})\hat{y} = 0. \tag{B.3.4}$$

This standard-method of log-linearization can be coded on `Matlab` as follows:

```
ff_lv = subs(ff, {xx}, {exp(xx)});
grad = jacobian(ff_lv ,xx);,
```

where `ff` stands for the system of equation before log-linearized and `xx` indicates a set of variables in the system. In the first line, `Matlab`, using the identity of $X = e^{\ln X}$, substitutes `xx` to logged `xx`. And then, take derivatives with respect to logged `xx` on the second line. With the simple two-line code, we can linearize more complicated system of equations easily.

Through the above method, I linearize the non-linear system of equations, Eqs.(B.1.1)-(B.1.28) around their steady state values.

B.4 Solving the Model

This section explains the solution method of the model economy. I adopt the generalized Schur method (QZ Decomposition) of Klein (2000), and Gomme and Klein (2011); they develop solution algorithm in both first- and second-order approximation with tensor-free mechanism. In what follows, I describe the solution procedure of the model by using their way of explanation. Also, I would announce that for practical reason I employ the code, '`gx_hx.m`', written by Schmitt-Grohe and Uribe (2004), which is available on their web site. The algorithm is actually same to Klein's (2000) for the first-order approximation.

The system of log-linearized equations from the previous section consists of a state vector, $\tilde{x}$, and a non-state vector, $\tilde{y}$, for period t and $t+1$ as follows:

$$\mathbb{E}_t\left[f(\tilde{x}_{t+1}, \tilde{y}_{t+1}, \tilde{x}_t, \tilde{y}_t)\right] = 0 \tag{B.4.1}$$

where f maps $\mathbb{R}^{2n_{\tilde{x}}+2n_{\tilde{y}}}$ into $\mathbb{R}^{n_{\tilde{x}}+n_{\tilde{y}}}$. The state vector and the non-state vector for time t of the model are defined as

$$\tilde{x}_t = [\hat{c}_{t-1}, \hat{i}_{t-1}, \hat{h}_{t-1}, \hat{k}_{t-1}, \hat{ec}t_t, \hat{\eta}^{xg}_{t-1}, \hat{\eta}^{zc}_{t-1}, \hat{\eta}^{ze}_{t-1}, \hat{\eta}^{zc}_t, \hat{\eta}^{ze}_t, \hat{x}_{l,t}, \hat{\eta}^{xg}_t, \hat{a}_{c,t}, \hat{a}_{e,t}, \hat{\epsilon}_{zc,t}, \hat{\epsilon}_{ze,t}],$$

$$\tilde{y}_t = [\hat{\tau}^C_t, \hat{\tau}^I_t, \hat{\tau}^H_t, \hat{y}_t, \hat{c}_t, \hat{i}_t, \hat{h}_t, \hat{q}_t, \hat{x}_t, \hat{y}_{c,t}, \hat{y}_{e,t}, \hat{k}_{c,t}, \hat{k}_{e,t}, \hat{h}_{c,t}, \hat{h}_{e,t}, \hat{\tilde{w}}_t, \hat{\tilde{r}}_t, \hat{\tau}^c_t, \hat{\tau}^i_t, \hat{\tau}^h_t, \hat{\lambda}_{1,t}, \hat{\lambda}_{2,t}],$$

where $\hat{\cdot}$ indicates the percent deviation from steady-state value. The linearized system of equations, Eq.(B.4.1), can be written as

$$A\begin{bmatrix}\tilde{x}_{t+1}\\ \mathbb{E}_t\tilde{y}_{t+1}\end{bmatrix} = B\begin{bmatrix}\tilde{x}_t\\ \tilde{y}_t\end{bmatrix} + \begin{bmatrix}\varepsilon_{t+1}\\ 0\end{bmatrix}, \tag{B.4.2}$$

where A denotes the coefficient matrix of the time $t+1$ variables including both state and non-state, and B is the coefficient matrix of the time t variables. Note that both A and B are $(n_{\tilde{x}} + n_{\tilde{y}}) \times (n_{\tilde{x}} + n_{\tilde{y}})$ matrices. The theorem of generalized Schur form presented in Golub and van Loan (1996), and cited by Klein (2000) and Gomme and Klein (2011) is required here.

Theorem [Generalized Schur Form]. *Let A and B be $n \times n$ matrices. If there is a $z \in \mathbb{C}$ such that $|B - zA| \neq 0$, then there exist matrices Q, Z, S and T such that*

1. *Q **and** Z **are Hermitian**, i.e. $Q^H Q = QQ^H = I_n$ and similarly for Z, where H denotes the Hermitian transpose.*

2. *T and S upper triangular.*

3. *$QA = SZ^H$ and $QB = TZ^H$.*

4. *There is no i such that $s_{ii} = t_{ii} = 0$.*

Moreover, the matrices Q, Z, S and T can be chosen in such a way as to make the diagonal entries s_{ii} and t_{ii} appear in any desired order.

For ordering of i, the ones satisfying $|s_{ii}| > |t_{ii}|$ will be chosen to appear first; these s_{ii} and t_{ii} pairs are called stable generalized eigenvalues.

The following equation can be derived from Eq.(B.4.2) by taking conditional expectation.

$$A\mathbb{E}_t \begin{bmatrix} \tilde{x}_{t+1} \\ \tilde{y}_{t+1} \end{bmatrix} = B \begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \end{bmatrix} \tag{B.4.3}$$

According to the above THEOREM, there exist upper triangular matrices S and T satisfying $QA = SZ^H$ and $QB = TZ^H$. Consequently, by premultiplying Q in both sides of Eq.(B.4.3), we can rewrite Eq.(B.4.3) as

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \mathbb{E}_t \begin{bmatrix} s_{t+1} \\ u_{t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix}, \tag{B.4.4}$$

where

$$\begin{bmatrix} s_t \\ u_t \end{bmatrix} \equiv Z^H \begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \end{bmatrix}, \tag{B.4.5}$$

and s_t and u_t have same length as $\tilde{x}_t$ and $\tilde{y}_t$ respectively. The last block of Eq.(B.4.4) can be written out as

$$S_{22}\mathbb{E}_t[u_{t+1}] = T_{22}u_t.$$

If S_{22} and T_{22} constitute a (weakly) unstable matrix pair, $|s_{ii}| < |t_{ii}|$ (for weakly $|s_{ii}| \leq |t_{ii}|$), then any solution to Eq.(B.4.2) with bounded variance must satisfy $u_t = 0, \forall t$ (for weakly, unless $\Sigma = 0$). Given $u_t = 0, \forall t$, the first block of Eq.(B.4.4) should hold

$$S_{11}\mathbb{E}_t[s_{t+1}] = T_{11}s_t. \tag{B.4.6}$$

If S_{11} and T_{11} constitute a stable matrix pair, $|s_{ii}| > |t_{ii}|$, then S_{11} is invertible. Hence we may write

$$\mathbb{E}_t[s_{t+1}] = S_{11}^{-1}T_{11}s_t. \tag{B.4.7}$$

Rewrite Eq.(B.4.5) as

$$\begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \end{bmatrix} = Z \begin{bmatrix} s_t \\ u_t \end{bmatrix} \tag{B.4.8}$$

where $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$.

If Z_{11} is invertible, then we may found the first-order approximation of policy rules as follows:

$$\tilde{y}_t = \underbrace{Z_{21}Z_{11}^{-1}}_{F}\tilde{x}_t \tag{B.4.9}$$

$$\tilde{x}_{t+1} = \underbrace{Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}}_{P}\tilde{x}_t + \varepsilon_{t+1}. \tag{B.4.10}$$

C Estimating model parameters[13]

Applying Kalman filter, I construct a log-likelihood function and find a parameter set, Θ, such that maximizes the likelihood function. The log-likelihood function that I want to construct is given as

[13] Writing this section, I found great usefulness in Hamilton(1997), Canova(2007), and the technical appendix of Ireland and Schuh (2008).

follows[14];

$$\mathcal{L}(d|\Theta) = -\frac{Tl}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln|\Sigma_{t|t-1}| - \frac{1}{2}\sum_{t=1}^{T}\varepsilon_t \Sigma_{t|t-1}^{-1}\varepsilon_t, \tag{C.1.1}$$

where T shows the time-length of the observed-vector, d, and l is the number of element of vector d, and ε_t and $\Sigma_{t|t-1}$ indicate the one-period-ahead forecast error of the observed-vector and its mean-square error, respectively.

For the consistency purpose from the previous sections, I suppose the state-space of this model economy as follows:

$$x_{t+1} = Cx_t + v_{t+1},$$
$$d_t = Dx_t + w_t,$$

where x and d respectively represent the state-vector of $k \times 1$, and the observed-vector of $l \times 1$. v and w stand for the stochastic disturbance of the state-vector and the measurement error, respectively, with $\mathbb{E}(vv') = \Sigma_v$ and $\mathbb{E}(ww') = \Sigma_w$. Due to the recursive nature of the state-space, the task will start from determining the initial conditions, mean and mean-square error, for the one-period-ahead forecast of the state-vector:

$$x_{1|0} = \mathbb{E}(x_1),$$
$$\mathbb{E}(x_1 - x_{1|0})(x_1 - x_{1|0})' = \Omega_{1|0}$$
$$vec(\Omega_{1|0}) = [I_{k^2} - (C \otimes C)]^{-1}\, vec(\Sigma_v),$$

where $x_{1|0}$ implies the expected value of x_1 on the information available at time 0. With the predetermined state-vector, we can find the one-period-ahead forecast for observed-vector and its

[14] The log-likelihood function is derived by using prediction error decomposition for the computational purpose. For more detail, see Canova(2007), pp.221-225.

mean-square error:

$$d_{t|t-1} = Dx_{t|t-1}, \tag{C.1.2}$$

$$\Sigma_{t|t-1} = D\Omega_{t|t-1}D' + \Sigma_w, \tag{C.1.3}$$

and the forecast error, ε, is written as

$$\varepsilon_t = d_t - d_{t|t-1}, \tag{C.1.4}$$

where d_t is the observed-vector at time t. Substituting Eqs.(C.1.3) and (C.1.4) into Eq.(C.1.1) recursively, we can construct the log-likelihood function. To move next period's forecast, we have to update the state-vector with the information of time t.

Using the formula for updating a linear projection, we can update state equation estimates[15]:

$$\begin{aligned} x_{t|t} &= x_{t|t-1} + \mathbb{E}(x_t - x_{t|t-1})(d_t - d_{t|t-1})' \\ &\quad \times \mathbb{E}(d_t - d_{t|t-1)})(d_t - d_{t|t-1})' \times (d_t - d_{t|t-1}) \\ &= x_{t|t-1} + \Omega_{t|t-1}D'\Sigma_{t|t-1}^{-1}\varepsilon_t, \end{aligned} \tag{C.1.5}$$

$$\Omega_{t|t} = \Omega_{t|t-1} - \Omega_{t|t-1}D'\Sigma_{t|t-1}^{-1}D\Omega_{t|t-1}. \tag{C.1.6}$$

The next period's forecast of the state-vector are then given as follows:

$$\begin{aligned} x_{t+1|t} &= Cx_{t|t} \\ &= Cx_{t|t-1} + K_t\varepsilon_t, \end{aligned} \tag{C.1.7}$$

$$\begin{aligned} \Omega_{t+1|t} &= \mathbb{E}(x_{t+1} - x_{t+1|t})(x_{t+1} - x_{t+1|t})' \\ &= \mathbb{E}(C(x_t - x_{t|t}) + v_{t+1})(C(x_t - x_{t|t}) + v_{t+1})' \\ &= C(x_t - x_{t|t})(x_t - x_{t|t})'C' + \Sigma_v \\ &= C\Omega_{t|t}C' + \Sigma_v, \end{aligned} \tag{C.1.8}$$

[15]see Hamilton(1997), pp.92-100

where K_t implies the Kalman-gain given by

$$K_t = C\Omega_{t|t-1} D' \Sigma_{t|t-1}^{-1}. \tag{C.1.9}$$

D Evaluating the model: Variance decomposition[16]

This section ascertains how to decompose the forecast error variance for the observable variables, such as consumption, investment, and hours worked into percentage due to each of the model shocks.

We can rewrite the state space equation and decision rule as follows:

$$x_{t+1} = Px_t + v_{t+1}, \tag{D.1.1}$$

$$y_t = Fx_t. \tag{D.1.2}$$

Eq.(D.1.1) can be rewritten as MA representation:

$$(1 - PL)x_t = v_t$$

$$x_t = \textstyle\sum_{j=0}^{\infty} P^j v_{t-j} \tag{D.1.3}$$

The s-period-ahead forecast error of state vector on the information of time t is

$$x_{t+s} - x_{t+s|t} = \sum_{j=0}^{s-1} P^j v_{t+s-j}, \tag{D.1.4}$$

and MSE of the forecast is exhibited as

$$\mathbb{E}[(x_{t+s} - x_{t+s|t})(x_{t+s} - x_{t+s|t})'] \equiv \Sigma_{x,s} = \Sigma_v + P\Sigma_v P' + P^2 \Sigma_v P'^2 + \ldots + P^{s-1}\Sigma_v P'^{s-1}. \tag{D.1.5}$$

[16] This section mostly comes from the technical appendix of Ireland and Schuh (2008). I just redefine some variables to fit to the model economy and try to increase the readability.

Next we can get the forecast error of the non-state vector of Eqs.(D.1.2) as

$$y_{t+s} - y_{t+s|t} = F(x_{t+s} - x_{t+s|t}). \tag{D.1.6}$$

Then MSE of the forecast for non-state vector is

$$\mathbb{E}[(y_{t+s} - y_{t+s|t})(y_{t+s} - y_{t+s|t})'] \equiv \Sigma_{y,s} = F\Sigma_{x,s}F'. \tag{D.1.7}$$

What we are interested in this analysis is mainly on the behavior of non-stationary aggregate variable such as consumption, investment, and hours worked per worker. Accordingly, we would get the variance decomposition for these non-stationary variables. In what follows, I describe the procedure for the variance decomposition of consumption as an example.

From the model solution given above we can rewrite the decision rule for consumption growth rate as follows:

$$\ln C_t - \ln C_{t-1} - \ln g^c = F_{gc} x_t, \tag{D.1.8}$$

where F_{gc} indicate the row for the consumption growth (g_t^c) in matrix F. Then we can derives the following s-period-ahead forecasts from Eq.(D.1.8):

$$\ln C_{t+s} - \ln C_t - s \ln g^c = F_{gc} \sum_{j=1}^{s} x_{t+j}, \tag{D.1.9}$$

$$\ln C_{t+s|t} - \ln C_t - s \ln g^c = F_{gc} \sum_{j=1}^{s} x_{t+j|t}. \tag{D.1.10}$$

Then the forecast error and MSE of forecast are derived as

$$\ln C_{t+s} - \ln C_{t+s|s} = F_{gc} \sum_{l=1}^{s} \left(x_{t+l} - x_{t+l|t} \right) = F_{gc} \sum_{l=1}^{s} \sum_{j=0}^{l-1} P^j v_{t+l-j} \tag{D.1.11}$$

$$\mathbb{E}\left[\ln C_{t+s} - \ln C_{t+s|t}\right]\left[\ln C_{t+s} - \ln C_{t+s|t}\right]' = F_{gc} \mathbb{E}\left[\sum_{l=1}^{s} \sum_{j=0}^{l-1} P^j v_{t+l-j}\right]\left[\sum_{l=1}^{s} \sum_{j=0}^{l-1} P^j v_{t+l-j}\right]' F'_{gc}, \tag{D.1.12}$$

where $\sum_{l=1}^{s}\sum_{j=0}^{l-1} P^j v_{t+l-j}$ is extended as

$$\begin{aligned}
\sum_{l=1}^{s}\sum_{j=0}^{l-1} P^j v_{t+l-j} &= \sum_{l=1}^{s}\left\{v_{t+l} + P v_{t+l-1} + \ldots + P^{l-1} v_{t+1}\right\} \\
&= [\{v_{t+1}\} \\
&\quad + \{v_{t+2} + P v_{t+1}\} + \ldots \\
&\quad + \left\{v_{t+s} + P v_{t+s-1} + \ldots + P^{s-1} v_{t+1}\right\}] \\
&= v_{t+s} + (I+P) v_{t+s-1} + \ldots + (I + P + \ldots + P^{s-1}) v_{t+1}.
\end{aligned}$$

Then the middle term of Eq.(D.1.12) is represented as

$$\mathbb{E}\left[\sum_{l=1}^{s}\sum_{j=0}^{l-1} P^j v_{t+l-j}\right]\left[\sum_{l=1}^{s}\sum_{j=0}^{l-1} P^j v_{t+l-j}\right]' = \Sigma_v + (I+P)\Sigma_v(I+P)' + \ldots + (I+P+\ldots+P^{s-1})\Sigma_v(I+P+\ldots+P^{s-1})'. \tag{D.1.13}$$