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Land Taxation in a Dualcentric City

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Abstract

While there is a large literature both on the spatial impacts of taxation and competing land uses in a dualcentric city, the two topics have yet to be analyzed under a common framework. This paper describes the optimal choice of a differential land tax across competing land uses in the framework of a dualcentric city model. The key results of the analysis indicate that higher levels of spatial competition lead to smaller tax differentials. These differentials are also sensitive to the relative size of the tax bases and the slopes of the bid rent functions at the respective land-use boundaries.

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1 Introduction

Differential taxes are a powerful policy instrument for state and local governments, as tax levels contribute to the composition as well as land use outcomes within jurisdictions. These taxes may vary across residential uses or according to residential/commercial designation. In the most common formulations, residents receive a discount either through a lower tax rate or assessment ratio. An example of this may be seen in Michigan where owner-occupied dwellings of residents are taxed at a rate of up to 18 mills less than vacation homes, rental properties, or commercial usage types. In Arizona, assessment ratios are based on property type - 25% for commercial and industrial and 10% for owner-occupied residential property. While discounting strategies may differ, many states and localities take advantage of these tools, the impacts of which may result in land market distortions. In spite of the prevalence of these within jurisdiction tax rate differences, the subject has received little attention in the literature.

The dualcentric city model provides a convenient framework to examine this issue. Traditionally, analysis of the dualcentric city has focused on the decisions of firms and households who locate endogenously as a result of spatially defined productivity or consumption considerations. Whatever the agent motivation, the incorporation of a land market over a set of possible locational choices defines the resulting spatial equilibrium. However, the analysis of the impact and choice of tax rates within this framework has, to the author's knowledge, been left unexplored. Enabling the jurisdiction to set differential taxes in this framework captures the desired composition and land market distortions which affect the optimal choice of tax instruments. The key results of this paper are that higher levels of spatial competition lead to smaller tax differentials - the exact levels of which are determined by the relative size of the tax bases and the slopes of the bid rent curves at the usage boundaries. While the focus of this paper is on the rent revenue maximizing jurisdiction, the results also serve to illustrate the impact of state imposed differentials on different jurisdiction types.

2 Background

The literature on the spatial decisions of firms and households in a dualcentric city has focused on two main areas; one in which households purchase goods from firms, and the other where there is a link between workers and employment centers. The endogenous formation of this city type was first examined in the framework of Fujita and Thisse (1986), where the Hotelling (1929) linear market model was combined with land consumption in the tradition of VonThunen (1826) and Alonso (1964). In this model, household location decisions are based on firm locations, and spatial competition is examined when the households simultaneously choose their consumption levels of land as well as the firm's output. In this early work, it was assumed households have constant population densities over locations. In a similar line of research, Fujita and Thisse (1991) relax this constant population density assumption to enable the identification of differing types of land use outcomes. Here two different firms choose their locations under the assumption of a spatial duopoly. After the firms choose their best locations, households are free to locate within the jurisdiction [1]. By relaxing this fixed population density constraint, the authors are able to characterize three different land use outcomes in this spatial duopoly model: i) a monocentric city in which firms locate at the center - occurs in small land areas, ii) a dualcentric city where firms locate separately and residents directly compete for land, and finally iii) a city where firms and residential patterns are completely separated - occurs in jurisdictions with large areas. These differing land use outcomes are used in the evaluation of the tax structure examined in this paper.

[1] A key assumption in these models is that households are making their choice in an open city framework. Therefore, the jurisdiction may be thought of as being located in a system of cities, which results in a fixed level of household utility, exogenously defined within the system. Therefore, at any chosen location, the household will be indifferent between the current choice and other choices which satisfy the utility constraint.

A second source of endogeneity in the locational choice of households is through spatially delineated labor markets. Examples of this are described in Smith (1997), Gabszewicz and Thisse (1986), and Fujita et al. (1997). Whatever the modeling strategy used, the existence of the endogenous formation of multiple centers in a city has been extensively explored in the literature. In the current paper, the location of the city centers are taken as given, and the choice of the optimal land tax under differing land use assumptions is examined.

The analysis of taxation in a spatial framework has been extensively studied in the monocentric city framework, with differing assumptions regarding the disposal of tax revenues and the closed vs. open nature of the city. These models are traditionally based on the spatial, monocentric city model of Wheaton (1974). This analysis shows how composition, the urban boundary, and lot sizes change in response to perturbations in exogenous variables such as income, reservation utility, location, and travel cost. Wheaton makes the assumption of a utility function which requires only the numerarie and housing goods to be normal, with positive income effects. The utility function lends itself to a comparative static analysis, generalizable to a large set of functional form assumptions. This generalized functional form is used in the current paper. Fujita (1989) describes a similar mechanism in the monocentric city with inclusion of property tax rates which are passed on to an absentee landowner. In these spatial taxation models, tax revenue may be either be freely disposed of, returned as income, or used for the provision of public goods. Grieson (1974), LeRoy (1976) and Carlton (1981) assume the free disposal of tax revenue and examine the effects of the spatial distribution of residents. Brueckner (2003) adopts this approach in his study of the relationship between property taxation and urban sprawl. Polinsky and Rubinfeld (1978) examine the long-run effects of residential property taxes and local public services in an open city urban spatial model. They allow for the adjustment of wages and land prices in response to changes in the local fiscal structure. In this model, the jurisdiction does not have to have a balanced budget, but faces a fixed public good expenditure level. Tax revenues above or below this level are simply given away to neighboring jurisdictions. The effects of sales and property taxes on land rents, city size, and housing consumption are examined by Pasha and Ghaus (1995) under the assumption of a closed city model. Following the assumption of Polinsky and Rubinfeld, they assume an exogenously fixed level of public good provision. Spatial analysis of the effects of the property tax is also considered in Haurin (1980), however the tax revenue is returned as income transfers. This is done in the context of an open city, which allows for population migration in response to fiscal changes. The current paper uses the assumptions of a fixed tax revenue constraint, and an open city model in which agent utilities are fixed. However, the assumption of a monocentric city is relaxed, and a dualcentric city is examined.

3 The Model

The framework for the model is an open, dualcentric city. The purpose of the model is to identify the optimal choice of ad valorem land tax rates under differing levels of spatial competition and tax revenue constraints. In this model, the Solow bid rent function is used to describe agents willingness to pay for land within a jurisdiction. In the simplest framework, freely mobile agents are homogeneous and share a common utility level. The components of the model are the bid rent function $\psi(r, u^h)$, the quantity of land consumed x^h, the amount of the numeraire z^h consumed by residents at each location, the distance from the city center r, increasing travel cost to location r, $T^h(r)$, and the fixed utility level u^h. For the purpose of the dual centric city model, the city centers are defined as city center one and city center two with differeing land uses associated with each center. It is assumed that the planner can differentiate between the land use associated with either of these types.

The bid rent function for agents associated with city center one is expressed as

$$\psi(r, u_h) = \max_{z_h, x_h} \left\{ \frac{y_h - T_h(r) - z_h}{x_h} | U(z_h, x_h) = u_h \right\} \tag{1}$$

This is the traditional bid rent construction for a monocentric city with a city center located at $r = 0$.[2] It is costly to travel to the city center, so the bid rent function is a decreasing function of the distance from $r = 0$.

$$\frac{\partial \psi}{\partial r} < 0 \tag{2}$$

The distance between city center one and city center two is normalized to one ($r \in [0, 1]$) without loss of generality.

The bid rent function for city center two agents may then be defined as

$$\phi(r, u_s) = max_{z_s, x_s} \left\{ \frac{y_s - T_s(1 - r) - z_s}{x_s} | U(z_s, x_s) = u_s \right\}. \tag{3}$$

The travel cost for a city center two agent at location r is $T(1 - r)$.

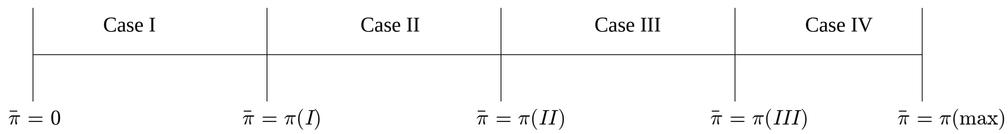
$$\frac{\partial \phi}{\partial r} > 0 \tag{4}$$

Rents are increasing as the distance to city center two is decreased. In the tradition of vonThunen, the price floor of land is the agricultural rent, denoted here by R^a. Agricultural land use may be present in the jurisdiction if all land is not occupied by either of the city center land use types.

It is assumed that the two agent types are homogeneous within group and heterogeneous across groups, so that the values of z, x, y and U for the two types are not necessarily identical. Heterogeneity across groups allows for differing bid rent curves between city center one and

[2] Following the assumptions of Wheaton (1974), utility is strictly quasiconcave and both goods x and z have positive income effects and are assumed to be normal goods

Figure 1: Tax Constraints and Land Use



city center two populations, a fact which will influence the choice of the optimal levels of the tax instruments.

In the model, absentee landowners bear the full incidence of the ad valorem land tax, given by τ^h for city center one and τ^s for city center two, where $(\tau^h, \tau^s) \geq 0$. [3] Therefore, the net rent received at any location for the city center one land use type is $(1 - \tau^h)\psi(r)$[4] , net rent from city center two is $(1 - \tau^s)\phi(r)$, and rent from agricultural land is R^a (agricultural taxes are set to 0). However, the landowner can only rent to one type at each location, and chooses to rent to the type with the highest bid rent net of taxes. This condition is described by the upper envelope of the bid rent functions given by:

$$R(r) = max\{(1 - \tau^h)\psi(r), R^a, (1 - \tau^s)\phi(r)\}$$

There are four different types of land use outcomes and corresponding optimal tax regimes in the model. These are a function of a fixed tax revenue constraint and are shown graphically in Figure 1.

For relatively low levels of fixed tax revenue, the jurisidtion is in the tax regime denoted as Case I, defined as the land use outcome where city center one and city center two usage types outbid agricultural use at all locations. In this land use outcome, there is direct competition for land between the two types at their borders. Therefore, the city center one/city center

[3] This is, of course, a substantial assumption, as changes in the tax incidence would have significant impacts on the results that follow. For a survey on the determinants of tax incidence see Wildasin (1986).

[4] For notational simplicity, $\phi(r, u^s)$ and $\psi(r, u^h)$ will be represented as $\phi(r)$ and $\psi(r)$.

two border net rents are described as

$$(1 - \tau^h)\psi(r^c) = (1 - \tau^s)\phi(r^c) \tag{5}$$

Where r^c is the common boundary shared by the two types. Additionally, at the boundary, the difference between the city center one and city center two bid rent functions is

$$\Delta R^{hs} = \frac{\tau^s - \tau^h}{1 - \tau^s}$$

This outcome is presented graphically in the left panel of Figure 2 where the boundary is denoted r^c.

At $\bar{\pi} = \pi(I)$, the increased level of tax revenue required by the jurisdiciton causes a discrete tax policy change to Case II, which describes a spatial structure where the usage types from city center one and city center two net boundary rents are equal to agricultural rent, but no agricultural land exists in the jurisdiction. The main difference associated with Case II is that the price floor for the two usage types at the boundary is constrained by agricultural land, which changes the optimal tax choice of the planner.

If the tax constraint is increased to $\pi(II)$, then the jurisidiction faces another regime switch into Case III. Here, agricultural use outbids land use one and land use two types over some portion of the jurisdiction, and therefore there is not direct competition between the respective city center boudaries. This results in two boundaries; city center one/agricultural (r^h) and agricultural/city center two (r^s). In order for the absentee landowner to be indifferent between land use types the border conditions for Case III requires that:

$$R^a = (1 - \tau^h)\psi(r^h) \tag{6}$$

$$R^a = (1 - \tau^s)\phi(r^s) \tag{7}$$

The gross rent differential between city center one and agricultural land is

$$\Delta R^{ha} = \tau^h \psi(r^h)$$

Similarly, the gross border rent differential between agricultural land and city center two is

$$\Delta R^{as} = \tau^s \phi(r^s)$$

This equilibrium type is presented graphically in the right panel of Figure 2. The City center one land use type occupies the land between $r = 0$ and $r = r^h$, agricultural land is located between r^h and r^s, and city center two occupies the land from r^s to 1. So, for Case III, it is necessary that $r^s > r^h$

These cases differ both in their level of land competition and tax revenue constraints which in turn define the optimal tax differential.

Finally, $\pi(III)$, occurs when one of the land use types reaches a corner solution, as a result of the Laffer effect [5]. Case IV then simply solves the case where tax revenues for one type have been maximized, and the second tax rate is simply set by substituting the corner solution tax rate into the tax revenue constraint. $\pi(max)$ describes the case where both types are at the corner solution. This is the maximum amount of tax revenue which may be raised by the jurisdiction. The following analysis will focus on Cases I-III, as the results for the Case IV restrict the planner's choice of tax instruments, and are simply corner solutions.

The goal of the central planner is to maximize net rent revenue (NR) subject to a tax revenue constraint, $\bar{\pi}$. The differing sets of the binding constraints on the tax revenue constraint affect the planner's choice of tax levels. It is assumed that the central planner has full control over the ad valorem land taxes levied on the city center one and city center

[5] This effect is discussed in Appendix C

two land use types. The maximization problem is described as

$$max_{\tau^h,\tau^s} \quad NR = (1-\tau^h)\int\limits_0^{r^h(\tau^h,\tau^s)} \psi(r)dr + \int\limits_{r^h(\tau^h,\tau^s)}^{r^s(\tau^h,\tau^s)} R^a dr + (1-\tau^s)\int\limits_{r^s(\tau^h,\tau^s)}^{1} \phi(r)dr$$

$$s.t. \quad \bar{\pi} = \tau^h \int\limits_0^{r^h(\tau^h,\tau^s)} \psi(r)dr + \tau^s \int\limits_{r^s(\tau^h,\tau^s)}^{1} \phi(r)dr \tag{8}$$

$$(1-\tau^h)\psi(r^h) = (1-\tau^s)\phi(r^s) \tag{9}$$

$$R^a \leq (1-\tau^h)\psi(r^h) \tag{10}$$

$$R^a \leq (1-\tau^s)\phi(r^s) \tag{11}$$

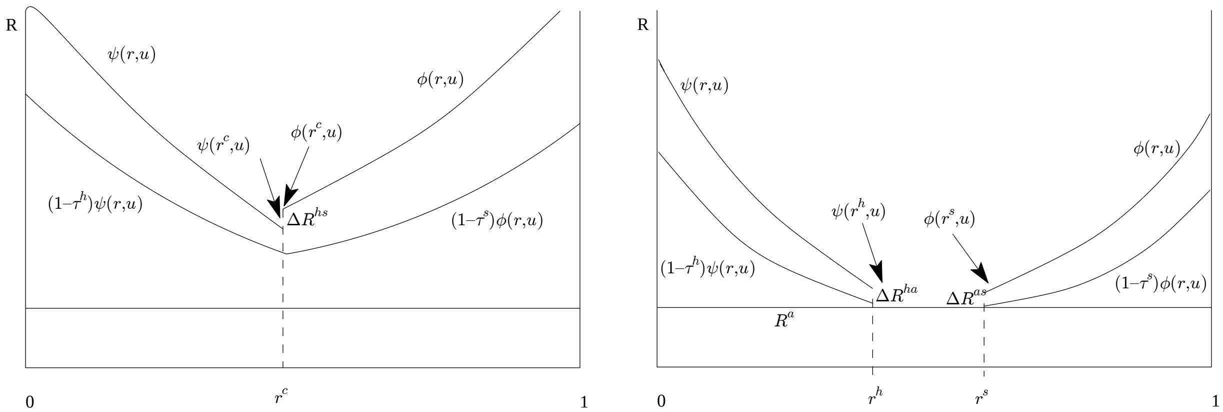
$$r^h \leq r^s \tag{12}$$

$$0 \leq \tau^h \leq 1 \tag{13}$$

$$0 \leq \tau^s \leq 1 \tag{14}$$

The set of binding constraints in the maximization problem define the possible tax and land use regimes. Case I describes a state in which there is not any agricultural land in the jurisdiction, and the city center populations directly compete for land at their respective boundaries, therefore, constraint (12) holds with equality, constraints (10) and (11) do not hold, and $r^h = r^s = r^c$. For the planner, a spatial structure with this outcome will be the best option, as net rents under these assumptions will dominate any other tax regime. However, it may be the case that the tax revenue constraint is too high and this outcome is unattainable, then the planner's next best option is to set taxes at the level which defines the land use type as Case II, where constraints (10), (11) and (12) are all binding. If this tax regime still does not fulfill the revenue requirement, then the tax regime which defines Case III occurs. Here, (12) is not binding as the two types do not share a common border, but (10) and (11) are binding, and net boundary rents for each type are equal to agricultural rents.

Figure 2: Land Use for Case I and Case III



In the three sections that follow, the optimal tax structure under each of these three cases is described. The relative net rent revenue maximizing levels of τ^h and τ^s are shown to be dependent upon the size of the relative tax bases, slopes of the bid rent functions at the boundary and the level of land competition.

4 Rent Maximization for Case I

If the tax revenue constraint can be filled in Case I, then this is the best tax regime[6]. In the following analysis, it is shown that the optimal taxation scheme under direct land competition between the two usage types incorporates identical tax rates for both. For the intuition behind this result, consider a case where tax rates are equal and then city center one tax rates are increased a marginal amount. In order for the tax revenue constraint to hold, city center two tax rates must be decreased. This change causes land use to switch from usage one to usage two at the border, an effect which will strictly decrease rent revenue. This is first shown analytically and then graphically below.

[6] This can be seen by examining the upper envelope of the bid rent functions over each of the three possible spatial formations

Net rent revenues for the Case I are defined as

$$NR = (1-\tau^h) \int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr + (1-\tau^s) \int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr \tag{15}$$

Where (12) is the only binding constraint. The net revenue maximizing condition is then:[7]

$$\left(\int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr \frac{\partial r^c}{\partial \tau^h} - \int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr \frac{\partial r^c}{\partial \tau^s} \right) \left(\tau^h \psi(r^c) - \tau^s \phi(r^c) \right) = 0 \tag{16}$$

Proposition 4.1. *In Case I, the optimal tax rates are such that* $\tau_h^* = \tau_s^*$.

Proof. Let the solution, (τ_h^*, τ_s^*), satisfy the tax revenue constraint. In order for (16) to hold, at least one of the two left hand terms must equal zero.

Step 1: Show that the first term in (16) $\left(\int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr \frac{\partial r^c}{\partial \tau^h} - \int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr \frac{\partial r^c}{\partial \tau^s} \right) \neq 0$.

First note that from the boundary constraint (9).

$$\frac{\partial r^c}{\partial \tau^h} = \frac{\psi(r^c)}{(1-\tau^h)\frac{\partial \psi}{\partial r} - (1-\tau^s)\frac{\partial \phi}{\partial r}} \tag{17}$$

$$\frac{\partial r^c}{\partial \tau^s} = -\frac{\phi(r^c)}{(1-\tau^h)\frac{\partial \psi}{\partial r} - (1-\tau^s)\frac{\partial \phi}{\partial r}} \tag{18}$$

Substituting (17) and (18) into the first term of (16) and setting this equal to 0 yields:

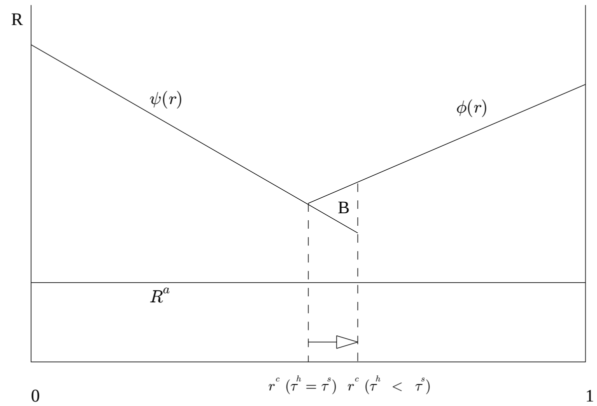
$$\frac{\int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr}{\int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr} = -\frac{\phi(r^c)}{\psi(r^c)}$$

Which cannot hold if $\{\int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr,\ \int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr,\ \psi(r^c),\ \phi(r^c)\} > 0$.

Thus, for the first order condition to hold, $\tau^h\psi(r^c) - \tau^s\phi(r^c)$ must equal 0. This condition is satisfied when $\tau_h^* = \tau_s^*$. Therefore, under Case I, the rent revenue maximizing tax pair

[7] Derivation is shown in Appendix A

Figure 3: Case I: Tax Pairs



(τ_h^*, τ_s^*) is such that $\tau_h^* = \tau_s^*$. □

The intuition of this result is clear from Figure 3. Any differential between the tax rates shifts the boundary relative to the case where $\tau^h = \tau^s$ and results in an unambiguous loss in rent revenue, shown as Area B in the figure (for the case $\tau^s > \tau^h$). The distortionary aspect of this differential tax on the spatial distribution of differing land use types causes a decrease in the rent envelope. This equilibrium type corresponds to a jurisdiction with relatively high rents and population densities, where land is valued significantly higher than agricultural use over all locations. In this case, the uniform tax rates used to generate the tax revenues do not cause any distortions in the land market. This is not the result in the Case II or Case III land distribution types, as interactions with agricultural land cause the planner to implement a differential tax between types which will distort land use outcomes.

The upper bound on tax revenue that can be raised in Case I is denoted as $\bar{\pi}_1$, and defines the transition between Cases I and II. This is described as:

$$\bar{\pi}_1 = max\left(\tau^* \int_{0}^{r^c(\tau^h,\tau^s)} \psi(r)dr + \tau^* \int_{r^c(\tau^h,\tau^s)}^{1} \phi(r)dr)\right)$$

Subject to

$$r^h = r^s = r^c$$

$$\psi(r^c) = \phi(r^c)$$

$$\tau^h = \tau^s = \tau^*$$

In cases where $\bar{\pi} > \bar{\pi}_1$, the planner is forced to set tax rates higher than is possible under equal tax rates. The analysis of these possible tax regimes is described next.

5 Rent Maximization for Case II

In Case II, the planner sets the relative tax rates so that there is still not any agricultural land in the jurisdiction, but the net rents for each usage type equal the agricultural rents at the unique boundary. The transition from Case I to Case II occurs as the tax revenue requirement increases to the point where the tax rates make the landowners indifferent between agricultural and other usage types at the boundary.

The landowners then receive total net rent revenue equal to

$$NR = (1-\tau^h) \int_{0}^{r^c(\tau^h,\tau^s)} \psi(r)dr + (1-\tau^s) \int_{r^c(\tau^h,\tau^s)}^{1} \phi(r)dr \qquad (19)$$

Where constraints (10), (11), and (12) are binding. The tax revenue constraint is described by:

$$\bar{\pi} = \tau^h \int_{0}^{r^c(\tau^h,\tau^s)} \psi(r)dr + \tau^s \int_{r^c(\tau^h,\tau^s)}^{1} \phi(r)dr \tag{20}$$

The solution to the planner's problem is then [8].

$$\frac{\int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr}{-\frac{\phi(r^c)}{(1-\tau^s)\frac{\partial\phi}{\partial r}}} = \frac{\int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr}{\frac{\psi(r^c)}{(1-\tau^h)\frac{\partial\psi}{\partial r}}} \tag{21}$$

At the optimal tax solution (τ_h^*, τ_s^*), the marginal cost in terms of rent revenue of increasing tax revenues from either usage must be equal. Here, the absolute loss in rent revenue from increasing τ^s is given by $-\frac{\phi(r^c)}{(1-\tau^s)\frac{\partial\phi}{\partial r}} < 0$, and the loss from increasing τ^h is $\frac{\phi(r^c)}{(1-\tau^h)\frac{\partial\psi}{\partial r}} < 0$. These loses represent the effect of land use switching between land use type one and two. Under most specifications of the total rents and bid rent slopes, this will lead to a case where the optimal tax differential is $\tau^h \neq \tau^s$.

In order to examine this result in more detail, the optimal tax pairs under differing assumptions on the relative bid rent slopes at the land use borders $(\frac{\partial\psi}{\partial r}, \frac{\partial\phi}{\partial r})$, as well as relative gross rents for each type $(\int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr, \int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr)$ are next presented analytically. The graphical interpretation and intuition are discussed in more detail for Case III.

Proposition 5.1. *In Case II when the slopes of the rent curves are equal at the boundaries under equal tax rates, but the size of the tax bases differ, the optimal tax differential is defined at some (τ_h^*, τ_s^*) pair where $\tau_s^* > \tau_h^*$ if $\int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr > \int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr$ and $\tau_h^* > \tau_s^*$ if $\int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr > \int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr$.*

Proof. Let gross rents be represented by $\int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr \neq \int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr$, the slopes of the rent curves at the boundary by $\frac{\partial\phi}{\partial r} = -\frac{\partial\psi}{\partial r} = \bar{m} > 0$, and the optimal tax pair (τ_h^*, τ_s^*) satisfies the tax revenue constraint, Substituting $\bar{m}$ into (21) yields:

[8] The derivation of this is shown in Appendix B

$$\frac{\int_{r^c(\tau^h,\tau^s)}^{1}\phi(r)dr}{\int_0^{r^c(\tau^h,\tau^s)}\psi(r)dr} = \frac{-\frac{\phi(r^c)}{(1-\tau^s)\bar{m}}}{\frac{\psi(r^c)}{(1-\tau^h)\bar{m}}} \tag{22}$$

Step 1: Prove by contradiction that $\tau_h^* \neq \tau_s^*$ at the optimum.

Assume $\tau^h = \tau^s$, then (22) becomes

Reducing this expression yields

$$\frac{\int_{r^c(\tau^h,\tau^s)}^{1}\phi(r)dr}{\int_0^{r^c(\tau^h,\tau^s)}\psi(r)dr} = 1$$

However, since $\frac{\int_{r^c(\tau^h,\tau^s)}^{1}\phi(r)dr}{\int_0^{r^c(\tau^h,\tau^s)}\psi(r)dr} \neq 1$, this equality does not hold.

Step 2: Show that if $\int_{r^c(\tau^h,\tau^s)}^{1}\phi(r)dr > \int_0^{r^c(\tau^h,\tau^s)}\psi(r)dr$ then $\tau_s^* > \tau_h^*$

Rewrite (22) as

$$\frac{(1-\tau^h)}{\psi(r^c)}\left(\int_0^{r^h(\tau^h)}\psi(r)dr\right) - \frac{(1-\tau^s)}{\phi(r^c)}\left(\int_{r^s(\tau^s)}^{1}\phi(r)dr\right) = 0 \tag{23}$$

However, if $\tau^h = \tau^s$,

$$\frac{(1-\tau^h)}{\psi(r^c)}\left(\int_0^{r^h(\tau^h)}\psi(r)dr\right) - \frac{(1-\tau^s)}{\phi(r^c)}\left(\int_{r^s(\tau^s)}^{1}\phi(r)dr\right) < 0 \tag{24}$$

Therefore, it is necessary to change the relative tax rates in order for (24) to hold. In order for a rise in τ^s to move toward the rent maximum, the first term must be increasing in τ^s and the second term must be decreasing.

Starting with the second term, let $g(\tau^s) = (1-\tau^s)$ then $\frac{\partial g(\tau^s)}{\partial \tau^s} < 0$. Additionally, $\frac{\partial \phi(r^c)^{-1}}{\partial \tau^s} < 0$ Since

$$\frac{\partial \phi(r^c)^{-1}}{\partial \tau^s} = -\phi(r^c)^{-2}\frac{\partial \phi}{\partial r}\frac{\partial r^c}{\partial \tau^s} < 0$$

as $\frac{\partial \phi}{\partial r} > 0$ and $\frac{\partial r^c}{\partial \tau^s} > 0$. Finally, $\frac{\partial \int_{r^s(\tau^s)}^1 \phi(r)dr}{\partial \tau^s} = -\phi(r^s)\left(\frac{\partial r^s}{\partial \tau^s}\right) < 0$. Therefore, the right side of this term in decreasing in τ^s.

Next, for the first term, let $g(\tau^h(\tau^s)) = (1-\tau^h)$ and $\frac{\partial g}{\partial \tau^s} > 0$ since $\frac{d\tau^h}{d\tau^s} < 0$ [9]. Additionally, $\frac{\partial \psi^{-1}}{\partial \tau^s} = -\psi(r^c)^{-2}\frac{\partial \psi}{\partial r}\frac{\partial r^c}{\partial \tau^h}\frac{d\tau^h}{d\tau^s} > 0$ since $\frac{\partial \psi}{\partial r} < 0$, $\frac{\partial r^c}{\partial \tau^h} < 0$, $\frac{d\tau^h}{d\tau^s} < 0$. Finally, the term $\frac{\partial \int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr}{\partial \tau^s} = \psi(r^h)\left(\frac{\partial r^c}{\partial \tau^h}\frac{d\tau^h}{d\tau^s}\right) > 0$, so the left hand side is increasing in τ^s, and the condition for the optimal pair of (τ_s^*, τ_h^*) is $\tau_s^* > \tau_h^*$.

Step 3: If $\int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr > \int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr$ then $\tau_h^* > \tau_s^*$

Follows from Step 2. □

Thus, in general under Case II, the optimal tax strategy involves shifting the tax burden toward the larger tax base. Next, I consider a case where, under equal tax rates, the size of the tax bases are equal but the slope of the rent curves at the boundary differ.

Proposition 5.2. *In Case II, when the slopes of the rent curves differ at the boundaries but the tax bases are the equal ($\int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr = \int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr$, $\frac{\partial \phi}{\partial r} \neq -\frac{\partial \psi}{\partial r}$) under equal tax rates, the optimal tax differential between types is dependent upon the relative magnitude of the slopes. The optimal tax pair is defined as (τ_h^*, τ_s^*) where $\tau_s^* > \tau_h^*$ if $\frac{\partial \phi}{\partial r} > -\frac{\partial \psi}{\partial r}$ and $\tau_h^* > \tau_s^*$ if $\frac{\partial \phi}{\partial r} < -\frac{\partial \psi}{\partial r}$.*

Let $\int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr = \int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr = \bar{k} > 0$

Proof. Step 1: Prove by contradiction that $\tau_h^* \neq \tau_s^*$.

Assume that $\tau^h = \tau^s$. Then (21) becomes

$$-\frac{\partial \psi}{\partial r} = \frac{\partial \phi}{\partial r}$$

However, since $\frac{\partial \psi}{\partial r} \neq -\frac{\partial \phi}{\partial r}$, $\tau_s^* = \tau_h^*$ cannot be the solution.

[9] The sign of $\frac{d\tau^h}{d\tau^s}$ is found by totally differentiating the tax revenue constraint, and showing that this sign may only be negative at the rent revenue maximizing levels of $(\tau_h^*, \tau_s^*$, see Appendix C for a proof

Step 2: Prove that if $\frac{\partial \phi}{\partial r} > -\frac{\partial \psi}{\partial r}$ then $\tau_s^* > \tau_h^*$.

Let $\bar{m} = \frac{\partial \phi}{\partial r} > -\frac{\partial \psi}{\partial r} = -\beta \bar{m}$ where $\beta \in (0,1)$.

Substituting these conditions into 21 yields:

$$\frac{(1-\tau^s)}{\phi(r^c)} - \frac{\beta(1-\tau^h)}{\psi(r^c)} = 0$$

Using the identities for Case II with differing total rents, the condition for that optimal pair, (τ_s^*, τ_h^*) is $\tau_s^* > \tau_h^*$.

Step 3: If $-\frac{\partial \psi}{\partial r} > \frac{\partial \phi}{\partial r}$ then $\tau_h^* > \tau_s^*$.

The proof of this result follows Step 2. □

Thus, in general under Case II, the optimal tax strategy involves shifting the tax burden toward the usage type with the steeper bid rent curve.

Again, the choice of (τ_h^*, τ_s^*) is dependent upon generating enough tax revenue to satisfy the tax revenue constraint. In Case II, the maximum amount of tax revenue which can be generated is given by $\bar{\pi}_2$.

$$\bar{\pi}_2 = max \left(\tau_h^* \int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr + \tau_s^* \int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr \right)$$

Subject to

$$r^h = r^s = r^c$$

$$(1-\tau_h^*)\psi(r^c) = (1-\tau_s^*)\phi(r^c)$$

$$(\tau_h^*, \tau_s^*) \text{ satisfies } \frac{\int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr}{\int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr} = \frac{\frac{\phi(r^c)}{(1-\tau^s)\bar{m}}}{\frac{\psi(r^c)}{(1-\tau^h)\bar{m}}}$$

6 Rent Maximization for Case III

If the planner is still not able to generate enough revenue, then tax rates must be set a such a level that induces a land use type defined by Case III. Here, there is not direct competition for land between the city center one and city center two land usage types. As a result, the only connection between the two types is through the tax revenue constraint. For a simple example of this, consider a case where, starting from identical tax rates that meet the tax revenue constraint, the city center one tax is increased a marginal amount. This increase in the tax rate causes a strict decrease in net rents at all locations covered by the city center one land-use type. This drop results in land use switching as agricultural outbids the city center one land use at the city center one/agricultural border. The increase in tax revenues allows for a corresponding decrease in city center two tax rates, thereby expanding city center two. This results in impacts on net rent revenues which are not obvious, as the gain in rents from the expansion of city center two may or may not equal the loss in revenues from the contraction of city center one. The analysis that follows highlights the effects of this linkage.

Landowners receive total net rent revenue equal to

$$NR = (1-\tau^h)\int_0^{r^h(\tau^h)} \psi(r)dr + \int_{r^h(\tau^h)}^{r^s(\tau^s)} R^a + (1-\tau^s)\int_{r^s(\tau^s)}^{1} \phi(r)dr \qquad (25)$$

subject to the boundary constraints $r^h < r^s$, $(1-\tau^h)\psi(r^h) = R^a$ and $(1-\tau^s)\phi(r^s) = R^a$. The tax revenue constraint is described by:

$$\bar{\pi} = \tau^h \int_0^{r^h(\tau^h)} \psi(r)dr + \tau^s \int_{r^s(\tau^s)}^{1} \phi(r)dr \qquad (26)$$

The solution to the planner's problem is then [10].

$$\frac{\int_{r^s(\tau^s)}^{1} \phi(r)dr}{-\tau^s \phi(r^s)\frac{\partial r^s}{\partial \tau^s}} = \frac{\int_0^{r^h(\tau^h)} \psi(r)dr}{\tau^h \psi(r^h)\frac{\partial r^h}{\partial \tau^h}} = \tag{27}$$

$\frac{\partial r^h}{\partial \tau^h}$ is found by differentiating Equation 10 and using the result that $\frac{\partial \psi}{\partial r} < 0$ from (2). So,

$$\frac{\partial r^h}{\partial \tau^h} = \frac{\psi(r^h)}{\frac{\partial \psi}{\partial r}(1-\tau^h)} < 0 \tag{28}$$

Therefore, an increase in τ^h will cause the city center one border to contract toward $r = 0$. From Equations (7) and (4) the relationship between the city center two boundary and τ^s is:

$$\frac{\partial r^s}{\partial \tau^s} = \frac{\phi(r^s)}{\frac{\partial \phi}{\partial r}(1-\tau^s)} > 0 \tag{29}$$

Here, increasing τ^s will contract the city center two border toward $r = 1$.

Substituting these identities into (27) yields the following optimization condition:

$$\frac{\int_{r^s(\tau^s)}^{1} \phi(r)dr}{-\frac{\tau^s \phi(r^s)^2}{(1-\tau^s)\frac{\partial \phi}{\partial r}}} = \frac{\int_0^{r^h(\tau^h)} \psi(r)dr}{\frac{\tau^h \psi(r^h)^2}{(1-\tau^h)\frac{\partial \psi}{\partial r}}} \tag{30}$$

In this case, just as in Case II, the marginal cost in terms of rent revenues of generating additional tax revenue from the two land uses must be equal under the optimal tax solution. However, the presence of agricultural land makes the cost effect even more pronounced, as the boundary effects are magnified by land use switching to agricultural land at the borders, rather than from the land use one to land use two types. Again, the optimal tax differential is non-zero, under most assumptions. The analytical results for Case III follow those of Case II quite closely and are included in Appendix D. Here, I focus on an intuitive discussion of the analytical results, shown graphically in the following analysis.

[10] The derivation of this result is shown in Appendix D

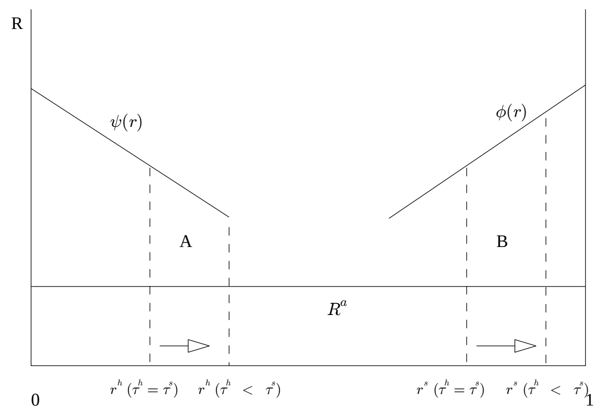


Figure 4: Optimal Tax Differential for Case III with Symmetric Bid Rent Functions

Under Case III, the optimal tax policy for symmetric bid rent functions when the tax rates are equal is $(\tau_h^* = \tau_s^*)$. This result is shown graphically in Figure 3[11]. Any movement away from this tax combination results in strictly lower rent revenues, since, from the figure, the gain from A is outweighed by the loss in B.

If the bid rent curves are not symmetric when the tax rates are equal, then the optimal tax differentials will, in general, be non zero. The case of having differing total rents rents under equal tax rates, but the same slope at the boundary is described next.

This case shows that it is in the interest of the planner to tax the usage type with the higher tax base under equal tax rates at a higher tax rate. The intuition for this result can be seen in Figure 5 for the case where $\int_{r^s(\tau^s)}^1 \phi(r)dr > \int_0^{r^h(\tau^h)} \psi(r)dr$, starting from the equal tax solution. The analytical results show that the gain in A from decreasing τ^h is larger than the loss in B from increasing τ^s. This result is driven by the fact that relatively small increase in τ^s will cause a larger relative decrease in τ^h. Therefore, the boundary of the city center one agents will shift out at a greater magnitude than the boundary of city center two contracts, which results in net rent revenue gains relative to a uniform tax rate.

[11] While the model describes outcomes for marginal changes in τ^h and τ^s, the Figures represent discrete changes. However, the intuitions from the Figures are consistent with the marginal results. Additionally, the use of the linear bid rent curves in the graphical analysis is assumed for its simplicity

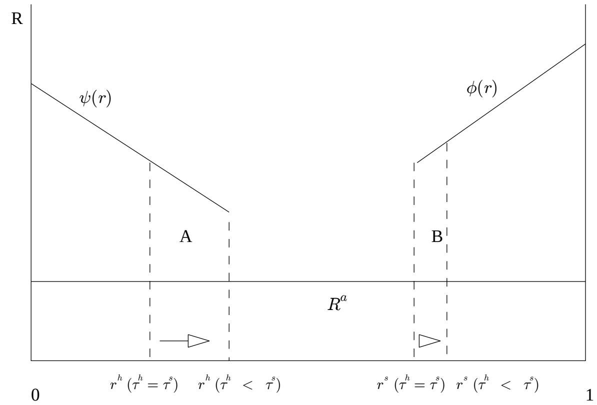


Figure 5: Case III: Tax Pairs for Non-Symmetric Tax Base $\left(\int_{r^s(\tau^s)}^1 \phi(r)dr > \int_0^{r^h(\tau^h)} \psi(r)dr\right)$

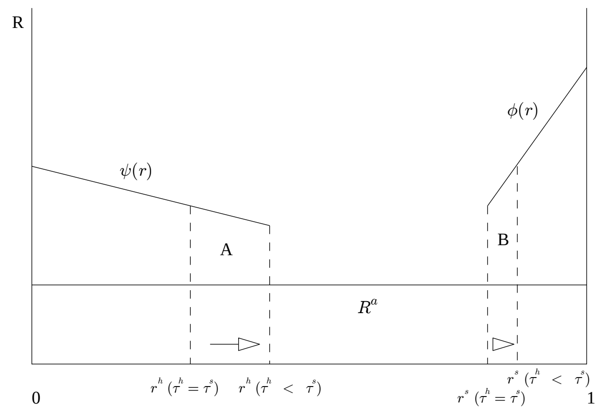


Figure 6: Case III: Tax Pairs in Non-Symmetric boundary bid-rent curves $(\frac{\partial\phi}{\partial r} > -\frac{\partial\psi}{\partial r})$

The final analysis for Case III is where, under equal tax rates, the slope of city center one and city center two rent curves differ at the boundary, but the total gross rents from each side are equal. Again, a differential tax is preferred by the planner. The intuition for the case where $\frac{\partial\phi}{\partial r} > -\frac{\partial\psi}{\partial r}$ under equal tax rates can be seen in Figure 6. Although the gross rents from both sides are the same at $\tau^h = \tau^s$, an increase in τ^s which results in a corresponding decrease in τ^h will raise net rents since the return to a decrease in τ^h (shown by Area A), is greater than the loss from the increase in τ^s (Area B). This condition is again, verified by the analytical results.

Results from Case III show that in general the only time it is efficient to set $\tau^h = \tau^s$ is under the assumption of completely symmetric bid rent curves under equal tax rates. Otherwise, the usage type with either the steeper bid-rent curve at the boundary or larger tax base under equal tax rates should be taxed at a higher rate. Therefore, the ability to set tax rates differentially will increase net rent revenues.

Under this final regime, the maximum amount of tax revenue which may be raised is:

$$\bar{\pi}_3 = max \left(\tau_h^* \int_0^{r^h(\tau^h)} \psi(r)dr + \tau_s^* \int_{r^s(\tau^s)}^{1} \phi(r)dr) \right)$$

Subject to

$$r^h < r^s$$

$$(1 - \tau_h^*)\psi(r^c) = (1 - \tau_s^*)\phi(r^c)$$

$$(\tau_h^*, \tau_s^*) \text{ satisfies } \frac{\int_{r^s(\tau^s)}^1 \phi(r)dr}{-\frac{\tau^s \phi(r^s)^2}{(1-\tau^s)\frac{\partial \phi}{\partial r}}} = \frac{\int_0^{r^h(\tau^h)} \psi(r)dr}{\frac{\tau^h \psi(r^h)^2}{(1-\tau^h)\frac{\partial \psi}{\partial r}}}$$

7 Conclusion

This analysis shows how spatial land competition in a dualcentric city affects the optimal choice of an advalorem land tax pair, a subject which up to this point has been left unexamined in the literature. Because there may be differing land use outcomes and tax revenue constraints in the dualcentric city, it is shown that there is heterogeneity in the optimal tax differential. The determination of the level of these tax rates depends upon the price floor of the jurisdiction - represented by agricultural land, the amount of tax revenue required in the jurisdiction, and the relative size and slope of the respective land use bid rent functions. The results have several policy implications. In highly dense, urban areas where the price floor is not a constraint, it is best to tax all land uses at an equal rate, since any movements away from this tax regime will cause an unambiguous loss in rent revenues. In less densely developed areas, where the price floor is binding, the use of differential tax will give the policy maker the ability to increase revenues. Therefore, if a local jurisdiction has complete control over the tax rates, land use must be a consideration in the decision of how much to tax. In cases where taxes are set at the state level, it is shown that the implementation of a forced tax differential will serve to lower total rents in urban areas, with ambiguous but identifiable results for less densely populated jurisdictions.

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A Case I: Revenue Maximization

Write the Lagrangian as

$$L = (1-\tau^h) \int\limits_0^{r^c(\tau^h,\tau^s)} \psi(r)dr + (1-\tau^s) \int\limits_{r^c(\tau^h,\tau^s)}^{1} \phi(r)dr + \lambda \left(\tau^h \int\limits_0^{r^c(\tau^h,\tau^s)} \psi(r)dr + \tau^s \int\limits_{r^c(\tau^h,\tau^s)}^{1} \phi(r)dr \right) \tag{31}$$

The first order conditions for the Lagrangian are

$$\begin{aligned} \frac{\partial L}{\partial \tau^h} &= - \int\limits_0^{r^c(\tau^h,\tau^s)} \psi(r)dr + (1-\tau^h)\psi(r^c)\frac{\partial r^c}{\partial \tau^h} - (1-\tau^s)\phi(r^c)\frac{\partial r^c}{\partial \tau^h} \\ &+ \lambda \left(\int\limits_0^{r^c(\tau^h,\tau^s)} \psi(r)dr + \tau^h \psi(r^c)\frac{dr^c}{d\tau^s} - \tau^s \phi(r^c)\frac{\partial r^c}{\partial \tau^h} \right) = 0 \\ \frac{\partial L}{\partial \tau^s} &= - \int\limits_{r^c(\tau^h,\tau^s)}^{1} \phi(r)dr + (1-\tau^h)\phi(r^c)\frac{\partial r^c}{\partial \tau^s} - (1-\tau^s)\phi(r^c)\frac{\partial r^c}{\partial \tau^s} \\ &+ \lambda \left(\tau^h \phi(r^c)\frac{\partial r^c}{\partial \tau^s} + \int\limits_{r^c(\tau^h,\tau^s)}^{1} \phi(r)dr - \tau^s \phi(r^c)\frac{\partial r^c}{\partial \tau^s} \right) = 0 \end{aligned}$$

Using the boundary condition identity that $(1-\tau^h)\psi(r^c) = (1-\tau^s)\phi(r^c)$, the FOCs can be reduced to

$$\begin{aligned} \frac{\partial L}{\partial \tau^h} &= - \int\limits_0^{r^c(\tau^h,\tau^s)} \psi(r)dr + \lambda \left(\int\limits_0^{r^c(\tau^h,\tau^s)} \psi(r)dr + \tau^h \psi(r^c)\frac{dr^c}{d\tau^s} - \tau^s \phi(r^c)\frac{\partial r^c}{\partial \tau^h} \right) = 0 \\ \frac{\partial L}{\partial \tau^s} &= - \int\limits_{r^c(\tau^h,\tau^s)}^{1} \phi(r)dr + \lambda \left(\tau^h \phi(r^c)\frac{\partial r^c}{\partial \tau^s} + \int\limits_{r^c(\tau^h,\tau^s)}^{1} \phi(r)dr - \tau^s \phi(r^c)\frac{\partial r^c}{\partial \tau^s} \right) = 0 \end{aligned}$$

Combining the FOCs yields:

$$\frac{\int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr + \left(\tau^h\psi(r^c) - \tau^s\phi(r^c)\right)\frac{\partial r^c}{\partial \tau^h}}{\int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr + \left(\tau^h\psi(r^c) - \tau^s\phi(r^c)\right)\frac{\partial r^c}{\partial \tau^s}} = \frac{\int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr}{\int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr}$$

Rearranging terms yields:

$$\left(\int\limits_{r^c(\tau^h,\tau^s)}^{1} \phi(r)dr\frac{\partial r^c}{\partial \tau^h} - \int\limits_{0}^{r^c(\tau^h,\tau^s)} \psi(r)dr\frac{\partial r^c}{\partial \tau^s}\right)\left(\tau^h\psi(r^c) - \tau^s\phi(r^c)\right) = 0$$

B Case II: Revenue Maximization

Write the maximization problem as

$$L = (1-\tau^h)\int\limits_{0}^{r^c(\tau^h,\tau^s)} \psi(r)dr + (1-\tau^s)\int\limits_{r^c(\tau^h,\tau^s)}^{1} \phi(r)dr + \lambda\left(\tau^h\int\limits_{0}^{r^c(\tau^h,\tau^s)} \psi(r)dr + \tau^s\int\limits_{r^c(\tau^h,\tau^s)}^{1} \phi(r)dr\right)$$

$$\frac{\partial L}{\partial \tau^h} = \psi(r^c)\frac{\partial r^c}{\partial \tau^h} - \phi(r^c)\frac{\partial r^c}{\partial \tau^h} + \lambda_1\left(\int\limits_{0}^{r^c(\tau^h,\tau^s)} \psi(r)dr + \tau^h\psi(r^c)\frac{\partial r^c}{\partial \tau^h} - \tau^s\phi(r^c)\frac{\partial r^c}{\partial \tau^h}\right) = 0$$

$$\frac{\partial L}{\partial \tau^s} = \psi(r^c)\frac{\partial r^c}{\partial \tau^s} - \phi(r^c)\frac{\partial r^c}{\partial \tau^s} + \lambda_1\left(\int\limits_{r^c(\tau^h,\tau^s)}^{1} \phi(r)dr + \tau^s\psi(r^c)\frac{\partial r^c}{\partial \tau^s} - \tau^s\phi(r^c)\frac{\partial r^c}{\partial \tau^s}\right) = 0$$

Solving these First Order conditions yields:

$$\frac{\int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr}{\int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr} = \frac{\frac{\partial r^c}{\partial \tau^s}}{\frac{\partial r^c}{\partial \tau^h}} \tag{32}$$

Since the constraints (10), (11) are binding, the right hand side of this is described as

$$\frac{\partial r^c}{\partial \tau^h} = \frac{\psi(r^c)}{-(1-\tau^h)\frac{\partial \psi}{\partial r}}$$
$$\frac{\partial r^c}{\partial \tau^s} = \frac{\phi(r^c)}{(1-\tau^s)\frac{\partial \phi}{\partial r}}$$

Substituting these conditions into (32) gives the condition for τ_h^*, τ_s^*.

$$\frac{\int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr}{\int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr} = \frac{\frac{\phi(r^c)}{(1-\tau^s)\frac{\partial \phi}{\partial r}}}{-\frac{\psi(r^c)}{(1-\tau^h)\frac{\partial \psi}{\partial r}}} \tag{33}$$

C Case II: $\frac{d\tau^h}{d\tau^s}$

The relationship between τ^h and τ^s can be seen by totally differentiating the tax revenue constraint, given as

$$\tau^h \int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr + \tau^s \int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr = \bar{\pi} \tag{34}$$

and

$$d\tau^h \int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr + \tau^h \frac{\partial r^c}{\partial \tau^h}\psi(r^c)d\tau^h + d\tau^s \int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr - \tau^s \frac{\partial r^c}{\partial \tau^s}\phi(r^h) = 0 \tag{35}$$

Solving for $\frac{d\tau^h}{d\tau^s}$ yields:

$$\frac{d\tau^h}{d\tau^s} = -\frac{-\tau^s \frac{\partial r^c}{\partial \tau^s}\phi(r^c) + \int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr}{\tau^h \frac{\partial r^c}{\partial \tau^h}\psi(r^c) + \int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr} \tag{36}$$

Proposition C.1. *At the rent revenue maximizing levels of* (τ_h^*, τ_s^*), $\frac{d\tau^h}{d\tau^s} < 0$

Proof. Proof by contradiction.

Step 1: Assume that $\frac{d\tau^h}{d\tau^s} > 0$. There are two cases where this may hold . The first case where the numerator of (36) is positive and the denominator is negative. The second case is simply the contrapositive of the first.

These conditions are defined as

$$-\tau^s \frac{\partial r^c}{\partial \tau^s}\phi(r^c) + \int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr > 0$$

$$\tau^h \frac{\partial r^c}{\partial \tau^h}\psi(r^c) + \int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr < 0$$

Rearranging terms and combining these conditions yields

$$\frac{\int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr}{-\tau^h \frac{\partial r^c}{\partial \tau^h}\psi(r^c)} < 1 < \frac{\int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr}{\tau^s \frac{\partial r^c}{\partial \tau^s}\phi(r^c)}$$

However, from the solution to the maximization problem,

$$\frac{\int_0^{r^c(\tau^h,\tau^s)} \psi(r)dr}{-\tau^h \frac{\partial r^c}{\partial \tau^h}\psi(r^c)} = \frac{\int_{r^c(\tau^h,\tau^s)}^1 \phi(r)dr}{\tau^s \frac{\partial r^c}{\partial \tau^s}\phi(r^c)}$$

The same result applies to the contrapositive, and therefore $\frac{d\tau^h}{d\tau^s} > 0$ does not exist at the maximum.

Step 2: Assume that $\frac{d\tau^h}{d\tau^s} = 0$ Then

$$\tau^s \frac{\partial r^c}{\partial \tau^s}\phi(r^c) = \int_{r^c(\tau^h,\tau^s)}^{1} \phi(r)dr$$

Then, from the maximization results, the following condition must hold:

$$-\tau^h \frac{\partial r^c}{\partial \tau^h}\psi(r^c) = \int_{0}^{r^c(\tau^h,\tau^s)} \psi(r)dr$$

However, if this is true, then the denominator of $\frac{d\tau^h}{d\tau^s} = 0$, and the solution is undefined. □

So, if $\frac{d\tau^h}{d\tau^s}$ must be less than zero, there still remain two possible (τ^h, τ^s) pairs which satisfy, which correspond to the the possible equilibrium points, that in which the both the denominator and numerator share the same sign, as a result of a Laffer-type curve. In the first case (τ_l^h, τ_l^s), the numerator and denominator are positive, then an increase in τ^h or τ^s will cause tax revenues to increase, which corresponds to being on the left hand side of the Laffer curve. However, if both are negative (τ_r^h, τ_r^s), then we are to the right of the Laffer curve. In the case when both are negative, the planner may increase both tax and rent revenues by decreasing the either of the tax rates. Therefore, it is clear that if $\tau_l^h < \tau_r^h$, and $\tau_l^s < \tau_r^s$ then

$$NR(\tau_l^h, \tau_l^s) > NR(\tau_r^h, \tau_r^s) \tag{37}$$

since

$$\frac{\partial NR}{\partial \tau^h} = -\int_{0}^{r^c(\tau^h,\tau^s)} \psi(r)dr + (1-\tau^h)\psi(r^c)\frac{\partial r^c}{\partial \tau^h} < 0$$

$$\frac{\partial NR}{\partial \tau^s} = -\int_{r^c(\tau^h,\tau^s)}^{1} \phi(r)dr - (1-\tau^s)\phi(r^c)\frac{\partial r^c}{\partial \tau^s} < 0$$

D Case III: Revenue Maximization

The planner's problem is

$$\max_{\tau^h,\tau^s} \quad (1-\tau^h)\int_{0}^{r^h(\tau^h)} \psi(r)dr + \int_{r^h(\tau^h)}^{r^s(\tau^s)} R^a dr + (1-\tau^s)\int_{r^s(\tau^s)}^{1} \phi(r)dr$$

$$\text{s.t.} \quad \tau^h \int_{0}^{r^h(\tau^h)} \psi(r)dr + \tau^s \int_{r^s(\tau^s)}^{1} \phi(r)dr = \bar{\pi}$$

The Lagrangian is then defined as

$$L = (1-\tau^h) \int_0^{r^h(\tau^h)} \psi(r)dr + \int_{r^h(\tau^h)}^{r^s(\tau^s)} R^a dr + (1-\tau^s) \int_{r^s(\tau^s)}^{1} \phi(r)dr + \lambda \left(\tau^h \int_0^{r^h(\tau^h)} \psi(r)dr + \tau^s \int_{r^s(\tau^s)}^{1} \phi(r)dr \right)$$

7 The first order conditions are then

$$\frac{\partial L}{\partial \tau^h} = (1-\tau^h)\psi(r^h)\frac{\partial r^h}{\partial \tau^h} - \int_0^{r^h(\tau^h)} \psi(r)dr - R^a \frac{\partial r^h}{\partial \tau^h} + \lambda \left(\tau^h \psi(r^h)\frac{\partial r^h}{\partial \tau^h} + \int_0^{r^h(\tau^h)} \psi(r)dr \right) = 0 \quad (38)$$

$$\frac{\partial L}{\partial \tau^s} = R^a \frac{\partial r^s}{\partial \tau^s} - (1-\tau^s)\phi(r^s)\frac{\partial r^s}{\partial \tau^s} - \int_{r^s(\tau^s)}^{1} \phi(r)dr + \lambda \left(-\tau^s \phi(r^s)\frac{\partial r^s}{\partial \tau^s} + \int_{r^s(\tau^s)}^{1} \phi(r)dr \right) = 0 \quad (39)$$

Combining (38) and (39) yields

$$\psi(r^h)\frac{\partial r^h}{\partial \tau^h}\tau^s\phi(r^s)\frac{\partial r^s}{\partial \tau^s} - \psi(r^h)\frac{\partial r^h}{\partial \tau^h}\int_{r^s(\tau^s)}^{1} \phi(r)dr - R^a\frac{\partial r^h}{\partial \tau^h}\tau^s\phi(r^s)\frac{\partial r^s}{\partial \tau^s}$$

$$+ R^a\frac{\partial r^h}{\partial \tau^h}\int_{r^s(\tau^s)}^{1} \phi(r)dr + R^a\tau^h\psi(r^h)\frac{\partial r^h}{\partial \tau^h}\frac{\partial r^s}{\partial \tau^s} + R^a\frac{\partial r^s}{\partial \tau^s}\int_0^{r^h(\tau^h)} \psi(r)dr$$

$$- \tau^h\psi(r^h)\frac{\partial r^h}{\partial \tau^h}\phi(r^s)\frac{\partial r^s}{\partial \tau^s} - \phi(r^s)\frac{\partial r^s}{\partial \tau^s}\int_0^{r^h(\tau^h)} \psi(r)dr = 0$$

Rearranging terms yields

$$\frac{\partial r^h}{\partial \tau^h}\frac{\partial r^s}{\partial \tau^s}\left(\psi(r^h)\tau^s\phi(r^s) - R^a\tau^s\phi(r^s) + R^a\tau^h\psi(r^h) - \tau^h\psi(r^h)\phi(r^s)\right) \quad (40)$$

$$+ \frac{\partial r^h}{\partial \tau^h}\left(R^a \int_{r^s(\tau^s)}^{1} \phi(r)dr - \psi(r^h)\int_{r^s(\tau^s)}^{1} \phi(r)dr \right) \quad (41)$$

$$+ \frac{\partial r^s}{\partial \tau^s}\left(R^a \int_0^{r^h(\tau^h)} \psi(r)dr - \phi(r^s)\int_0^{r^h(\tau^h)} \psi(r)dr \right) = 0 \quad (42)$$

Step 1: Show that $\psi(r^h)\tau^s\phi(r^s) - R^a\tau^s\phi(r^s) + R^a\tau^h\psi(r^h) - \tau^h\psi(r^h)\phi(r^s) = 0$
Rewrite (40) as

$$\tau^s\phi(r^s)(\psi(r^h) - R^a) + \tau^h\psi(r^h)(R^a - \phi(r^s)) \quad (43)$$

From the city center one boundary condition - $(1-\tau^h)\psi(r^h) = R^a$

$$\psi(r^h) - R^a = \frac{\tau^h R^a}{(1-\tau^h)}$$

and from the city center two boundary condition - $(1-\tau^s)\phi(r^s) = R^a$

$$R^a - \phi(r^s) = -\frac{\tau^s R^a}{(1-\tau^s)}$$

Substituting these identities into 43 yields

$$\frac{\tau^s \tau^h \phi(r^s) R^a}{(1-\tau^h)} - \frac{\tau^h \tau^s \psi R^a}{(1-\tau^s)} = \tau^h \tau^s R^a \left(\frac{\phi(r^s)}{(1-\tau^h)} - \frac{\psi(r^h)}{(1-\tau^s)} \right) \tag{44}$$

And, since $R^a = (1-\tau^h)\psi(r^h) = (1-\tau^s)\phi(r^s)$, The interior of this expression is zero, and term (40) is zero.

Step 2:

Now, from (41) and (42)

$$\frac{\partial r^h}{\partial \tau^h} \left(R^a \int\limits_{r^s(\tau^s)}^{1} \phi(r)dr - \psi(r^h) \int\limits_{r^s(\tau^s)}^{1} \phi(r)dr \right) + \frac{\partial r^s}{\partial \tau^s} \left(R^a \int\limits_{0}^{r^h(\tau^h)} \psi(r)dr - \phi(r^s) \int\limits_{0}^{r^h(\tau^h)} \psi(r)dr \right) = 0$$

And

$$\frac{\frac{\partial r^h}{\partial \tau^h} \int_{r^s(\tau^s)}^{1} \phi(r)dr \left(R^a - \psi(r^h)\right)}{\frac{\partial r^s}{\partial \tau^s} \int_{0}^{r^h(\tau^h)} \psi(r)dr \left(R^a - \phi(r^s)\right)} = -1 \tag{45}$$

Again, from the boundary conditions:

$$R^a - \psi(r^h) = -\tau^h \psi(r^h)$$
$$R^a - \phi(r^s) = -\tau^s \phi(r^s)$$

Substitute the results into (45) to get

$$\frac{\tau^h \psi(r^h) \frac{\partial r^h}{\partial \tau^h} \int_{r^s(\tau^s)}^{1} \phi(r)dr}{\tau^s \phi(r^s) \frac{\partial r^s}{\partial \tau^s} \int_{0}^{r^h(\tau^h)} \psi(r)dr} = -1$$

E Case III: Tax differentials

The following analysis describes the optimal tax pair under differing assumptions on the relative bid rent slopes at the land use borders $(\frac{\partial \psi}{\partial r}, \frac{\partial \phi}{\partial r})$, as well as relative gross rents for each type $(\int_0^{r^h(\tau^h)} \psi(r)dr, \int_{r^s(\tau^s)}^{1} \phi(r)dr)$.

Proposition E.1. *In Case III, when both the city center one and city center two bid rent curves are symmetric at the boundary and the tax bases are equal when the tax rates are equal, the optimal tax differential is defined at the* (τ_h^*, τ_s^*) *pair where* $\tau_h^* - \tau_s^* = 0$.

Proof. Let gross rents under equal tax rates to the tax revenue constraint be represented by $\int_{r^s(\tau^s)}^{1} \phi(r)dr = \int_0^{r^h(\tau^h)} \psi(r)dr = \bar{k} > 0$, the slopes of the rent curves at the boundary by $\frac{\partial \phi}{\partial r} = -\frac{\partial \psi}{\partial r} = \bar{m} > 0$, and the optimal tax pair (τ_h^*, τ_s^*) satisfy the tax revenue constraint.

Substituting in $\bar{m}$ and $\bar{k}$ into 30 yields

$$\frac{\frac{-(1-\tau^h)\bar{m}}{\tau^h \psi^2}\left[\bar{k}\right]}{\frac{(1-\tau^s)\bar{m}}{\tau^s \phi^2}\left[\bar{k}\right]} = -1$$

Rearranging terms yields

$$\bar{k}\bar{m}\left(\frac{(1-\tau^h)}{\tau^h \psi^2} - \frac{(1-\tau^s)}{\tau^s \phi^2}\right) = 0 \tag{46}$$

If $\tau^h > \tau^s$ and $\bar{k}\bar{m} > 0$, then (46) < 0 since $\tau^h \psi^2 > \tau^s \phi^2$ and $(1-\tau^h) < (1-\tau^s)$. Conversely, for $\tau^s > \tau^h$ (46) > 0. Therefore, the maximum can only occur when $\tau_h^* = \tau_s^*$, since this is the only pair which satisfies $\frac{(1-\tau^h)}{\tau^h \psi^2} = \frac{(1-\tau^s)}{\tau^s \phi^2}$. □

Proposition E.2. *In Case III, when the slopes of the rent curves are symmetric at the boundaries under equal tax rates, but the tax bases differ, the optimal tax differential is defined at some* (τ_h^*, τ_s^*) *pair where* $\tau_s^* > \tau_h^*$ *if* $\int_{r^s(\tau^s)}^{1} \phi(r)dr > \int_0^{r^h(\tau^h)} \psi(r)dr$ *and* $\tau_h^* > \tau_s^*$ *if* $\int_0^{r^h(\tau^h)} \psi(r)dr > \int_{r^s(\tau^s)}^{1} \phi(r)dr$.

Proof. Let gross rents be represented by $\int_{r^s(\tau^s)}^{1} \phi(r)dr \neq \int_0^{r^h(\tau^h)} \psi(r)dr$, the slopes of the rent curves at the boundary by $\frac{\partial \phi}{\partial r} = -\frac{\partial \psi}{\partial r} = \bar{m} > 0$, and the optimal tax pair (τ_h^*, τ_s^*) satisfies the tax revenue constraint,

Step 1: Prove by contradiction that $\tau_h^* \neq \tau_s^*$ at the optimum.

Assume $\tau^h = \tau^s$, then (30) becomes

$$\frac{-\bar{m}\left[\int_0^{r^h(\tau^h)} \psi(r)dr\right]}{\bar{m}\left[\int_{r^s(\tau^s)}^{1} \phi(r)dr\right]} = -1$$

Rearranging terms yields

$$\bar{m}\left(\int_0^{r^h(\tau^h)} \psi(r)dr - \int_{r^s(\tau^s)}^{1} \phi(r)dr\right) = 0 \tag{47}$$

However, since $\int_{r^s(\tau^s)}^{1} \phi(r)dr \neq \int_0^{r^h(\tau^h)} \psi(r)dr$ and $\bar{m} > 0$, this equality does not hold .

Step 2: If $\int_{r^s(\tau^s)}^{1} \phi(r)dr > \int_0^{r^h(\tau^h)} \psi(r)dr$ then $\tau_s^* > \tau_h^*$

Rewrite (30) as

$$\frac{(1-\tau^h)}{\tau^h \psi(r^h)^2}\left(\int_0^{r^h(\tau^h)} \psi(r)dr\right) - \frac{(1-\tau^s)}{\tau^s \phi(r^s)^2}\left(\int_{r^s(\tau^s)}^{1} \phi(r)dr\right) = 0 \tag{48}$$

If $\tau^h = \tau^s$,

$$\frac{(1-\tau^h)}{\tau^h\psi(r^h)^2}\left(\int_0^{r^h(\tau^h)}\psi(r)dr\right)-\frac{(1-\tau^s)}{\tau^s\phi(r^s)^2}\left(\int_{r^s(\tau^s)}^{1}\phi(r)dr\right)<0 \tag{49}$$

Therefore, it is necessary to change the relative tax rates in order for (48) to hold. In order for a rise in τ^s to move toward the rent maximum, the first term must be increasing in and the second term must be decreasing in τ^s.

Starting with the second term, let $g(\tau^s)=\frac{(1-\tau^s)}{\tau^s}$ then $\frac{\partial g(\tau^s)}{\partial\tau^s}<0$. Additionally, $\frac{\partial\phi(r^s)^{-2}}{\partial\tau^s}<0$ Since

$$\frac{\partial\phi(r^s)^{-2}}{\partial\tau^s}=-2\phi(r^s)\frac{\partial\phi}{\partial r}\frac{\partial r^s}{\partial\tau^s}<0$$

as $\frac{\partial\phi}{\partial r}>0$ and $\frac{\partial r^s}{\partial\tau^s}>0$. Finally, $\frac{\partial\int_{r^s(\tau^s)}^1\phi(r)dr}{\partial\tau^s}=-\phi(r^s)\left(\frac{\partial r^s}{\partial\tau^s}\right)<0$. Therefore, the right side of this term in decreasing in τ^s.

Next, for the first term, let $g(\tau^h(\tau^s))=\frac{(1-\tau^h)}{\tau^h}>0$ since $\frac{d\tau^h}{d\tau^s}<0$ [12].
Additionally, $\frac{\partial\psi^{-2}}{\partial\tau^s}=-2\psi(r^h)\left(\frac{\partial\psi}{\partial r}\left(\frac{\partial r^h}{\partial\tau^h}\frac{d\tau^h}{d\tau^s}+\frac{\partial r^h}{\partial\tau^s}\right)\right)>0$ since $\frac{\partial\psi}{\partial r}<0$, $\frac{\partial r^h}{\partial\tau^h}<0$, $\frac{d\tau^h}{d\tau^s}<0$, and $\frac{\partial r^h}{\partial\tau^s}>0$. Finally, the term $\frac{\partial\int_0^{r^h(\tau^h)}\psi(r)dr}{\partial\tau^s}=\psi(r^h)\left(\frac{\partial r^h}{\partial\tau^h}\frac{d\tau^h}{d\tau^s}\right)>0$, so the left hand side is increasing in τ^s, and the condition for the optimal pair of (τ_s^*,τ_h^*) is $\tau_s^*>\tau_h^*$.

Step 3: If $\int_0^{r^h(\tau^h)}\psi(r)dr>\int_{r^s(\tau^s)}^1\phi(r)dr$ then $\tau_h^*>\tau_s^*$ Follows from Step 2. □

Proposition E.3. *In Case III, when the slopes of the rent curves differ at the boundaries but the tax bases are equal* ($\int_{r^s(\tau^s)}^1\phi(r)dr=\int_0^{r^h(\tau^h)}\psi(r)dr$, $\frac{\partial\phi}{\partial r}\neq-\frac{\partial\psi}{\partial r}$) *under equal tax rates, the optimal tax differential between types is dependent upon the relative magnitude of the slopes. Then the optimal tax pair is defined as* (τ_h^*,τ_s^*) *where* $\tau_s^*>\tau_h^*$ *if* $\frac{\partial\phi}{\partial r}>-\frac{\partial\psi}{\partial r}$ *and* $\tau_h^*>\tau_s^*$ *if* $\frac{\partial\phi}{\partial r}<-\frac{\partial\psi}{\partial r}$.

Let $(\int_{r^s(\tau^s)}^1\phi(r)dr=\int_0^{r^h(\tau^h)}\psi(r)dr=\bar{k}>0)$

Proof. Step 1: Prove by contradiction that $\tau_h^*\neq\tau_s^*$.

Assume that $\tau^h=\tau^s$. Then (30) becomes

$$\frac{\frac{\partial\psi}{\partial r}(\bar{k})}{\frac{\partial\phi}{\partial r}(\bar{k})}=-1$$

Rearranging terms yields

$$\bar{k}\left(\frac{\partial\psi}{\partial r}+\frac{\partial\phi}{\partial r}\right)=0$$

However, since $\frac{\partial\psi}{\partial r}\neq-\frac{\partial\phi}{\partial r}$ and $\bar{k}>0$, $\tau_s^*=\tau_h^*$ cannot be the solution.

Step 2: Prove that if $\frac{\partial\phi}{\partial r}>-\frac{\partial\psi}{\partial r}$ then $\tau_s^*>\tau_h^*$.

Let $m=\frac{\partial\phi}{\partial r}>-\frac{\partial\psi}{\partial r}=-\beta m$ where $\beta\in(0,1)$.

[12] The proof of $\frac{d\tau^h}{d\tau^s}<0$ follows the same logic as from the touching case

Substituting these conditions into 30 yields:

$$\frac{-\frac{(1-\tau^h)\beta m}{\tau^h \psi (r^h)^2}\bar{k}}{\frac{(1-\tau^s)m}{\tau^s \phi (r^s)^2}\bar{k}} = -1$$

And therefore:

$$m\bar{k}\left(\frac{(1-\tau^s)}{\tau^s \phi^2} - \frac{(1-\tau^h)}{\tau^h \psi^2}\beta\right) = 0$$

Using the identities for the Case III with differing tax bases, the condition for that optimal pair, (τ_s^*, τ_h^*) is $\tau_s^* > \tau_h^*$.

Step 3: If $-\frac{\partial \psi}{\partial r} > \frac{\partial \phi}{\partial r}$ then $\tau_h^* > \tau_s^*$. The proof of this result follows Step 2. □