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Why is Pigou sometimes Wrong? Explaining how Distortionary Taxation can Cause Public Spending to Exceed the Efficiency Level

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WHY IS PIGOU SOMETIMES WRONG? EXPLAINING HOW DISTORTIONARY TAXATION CAN CAUSE PUBLIC SPENDING TO EXCEED THE EFFICIENT LEVEL

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ABSTRACT

When a public good is financed by a proportional tax, the price distortion increases the marginal cost of the public good above its resource cost. Pigou (1928) conjectured that the higher cost lowers the second-best public good level below the first-best level. I explain why this conjecture is sometimes false. In particular, if the public good is financed by a commodity tax, the price distortion normally raises the marginal benefit, and I provide an example in which the increased benefit dominates the increased cost; the second-best public good level lies *above* the first-best level. In contrast, if the public good is financed by a wage tax and leisure is a normal good, the price distortion lowers the marginal benefit; if the wage elasticity is positive, the second-best public good level lies below the first-best level.

Key words: proportional tax, second-best, public service level.

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1. INTRODUCTION

With lump-sum taxation, the first-best public good level is characterized by the "Samuelson Rule": expenditure on the public good should be increased until the marginal benefit equals the marginal resource cost.[1] The use of lump-sum taxation to finance a public good is of course unrealistic, but it is a useful benchmark against which to compare other tax structures. If the lump-sum tax is replaced by a proportional tax, there is a price distortion - the consumer price exceeds the marginal cost of the taxed good - which gives rise to a welfare cost; this welfare cost is henceforth termed the “distortion cost”. Pigou (1928) recognized that the distortion cost should be included in the cost of the public good, and conjectured that the increased cost of the public good would cause the second-best public good level to fall below the first-best level:

> "When there is indirect change, it ought to be added to the direct loss of satisfaction involved in the withdrawal of the marginal unit of resources by taxation, before this is balanced against the satisfaction yielded by the marginal expenditure. It follows that, in general, *expenditure ought not to be carried so far* as to make the real yield of the last unit of resources expended by the government equal to the real yield of the last unit left in the hands of the representative citizen." (Pigou, (1928, p. 53); italics added).

Formal analysis by Atkinson and Stern (1974), King (1986) and Wilson (1991a) confirms that the second-best public good level is indeed less than the first-best level when the household's utility function has either Cobb-Douglas or CES form.

Is Pigou's conjecture more generally correct? Pigou's intuition hinges on the increase in the marginal cost of funds. What is missing in his analysis is the effect of the distortion cost on the marginal benefit of the public good. In particular, if a proportional tax is placed on a commodity, the price distortion causes the household to substitute into the untaxed numeraire. The direct effect of the increased consumption of the numeraire is to raise the marginal benefit of the public good. I provide an example in which the increase in the marginal benefit exceeds the increase in the marginal cost, so that the net effect of the distortion is to increase the second-best public good level above the first-best level. In addition, my approach provides intuition for the result of Gaube (2000) that, if the second-best public good level exceeds the first-best level, the taxed commodity must be a gross complement; and it allows me to show that, if the second-best public good level exceeds the first-best level, the taxed commodity must be normal.[2]

In my structure, a commodity tax is equivalent to a labor tax. However, as noted by Atkinson and Stern (1974), there are descriptive differences. In particular, a labor tax can be considered as a lump-sum tax plus a leisure subsidy. In this case, my earlier result is reversed: if the subsidized commodity (leisure) is normal, the second-best public good level is less than the first-best level. Thus my analysis explains the result of Gaube (2000) that the second-best level is less than the first-best level when leisure is a normal good.[3]

Other authors note situations which seem to contradict Pigou's intuition, although their emphasis continues to be on the marginal cost of funds.[4] The marginal cost of funds is lowered whenever the increase in the public good interacts with the pre-existing tax structure to create additional tax revenue. In Diamond and Mirrlees (1971), this occurs because the public good level is complementary with the taxed commodities. In Stiglitz and Dasgupta (1971), it occurs because the higher tax rate required to finance the marginal expenditure induces the household to substitute into other taxed commodities. The lowering of the marginal cost of funds creates the possibility that the second-best level of the public good may exceed the first-best level. Since I want to use the simplest model to challenge Pigou's conjecture, I choose a model in which these complications are absent. In particular, I assume that the demand for the taxed commodity is independent of the level of the public good, and that all taxes are collected using a single proportional tax.

An alternative argument which stresses equity is put forward by Besley and Coate (1991): if the main beneficiaries of the public good are the poor, the public good is a useful instrument with which to effect redistribution; this causes the second-best level to possibly exceed the first-best level. Wilson (1991b) shows that the absence of a lump-sum tax is a crucial assumption of earlier models. If lump-sum taxes and indirect taxes are chosen optimally (reflecting redistribution concerns), the lump-sum tax may be used to finance an increase in the public good. The consequent reduction in the size of the distorted private sector is a favorable effect and may cause the second-best level to exceed the first-best level. Although these effects are important, I abstract from them: my results are obtained in a model in which all households are identical and in which, as noted earlier, all tax revenue is collected using a single proportional tax.

The paper is organized as follows. Section 2 characterizes the first-best and second-best allocations if (in the case of the second-best) tax revenue is collected using a commodity tax. Section 3 shows how the distortion effects the public good level. Section 4 discusses the case of labor taxation. Section 5 concludes.

2. THE MODELS

The population is comprised of a large number of households, each of which has identical tastes and income. A representative household consumes the numeraire commodity ℓ, another commodity x, and the public service[5] z. The utility achieved by the household is

$$U(\ell, x) + G(z).$$

$U(\cdot\,,\cdot)$ and $G(\cdot)$ are strictly concave functions. As noted in the Introduction, the utility function is chosen to be additively separable between private goods and the public service in order to prevent the public service *per se* influencing the marginal cost of funds. In this section, we consider ℓ to be a commodity - in Section 4 ℓ is reinterpreted to be leisure.

The units of the commodity x and of the public service z are chosen so that the resource cost of producing one unit of commodity x is p units of numeraire and the resource cost of producing one unit of the public service is k units of numeraire. The household has endowment M; its consumption of the numeraire commodity is therefore

$$\ell = M - px - kz. \tag{1}$$

2.1 First-best Allocation.

The benchmark analysis is the first-best case in which the public service is financed by a lump-sum tax. Using Equation (1) to substitute for ℓ, the allocation problem is written as a two-stage maximization problem

$$\max_{z}\ \max_{x}\quad U(M - px - kz\,,x)\ +\ G(z)\,.$$

In the first stage of the maximization, the optimal choice of x is found conditional on a given level of z. This conditional choice $x^F(z)$ is defined implicitly by the the first-order condition

$$\frac{U_x(M - px^F(z) - kz\,, x^F(z))}{U_\ell(M - px^F(z) - kz\,, x^F(z))}\ =\ p\ \ , \tag{2}$$

where the subscript identifies the position of differentiation. x should be increased until its marginal benefit equals its resource cost.

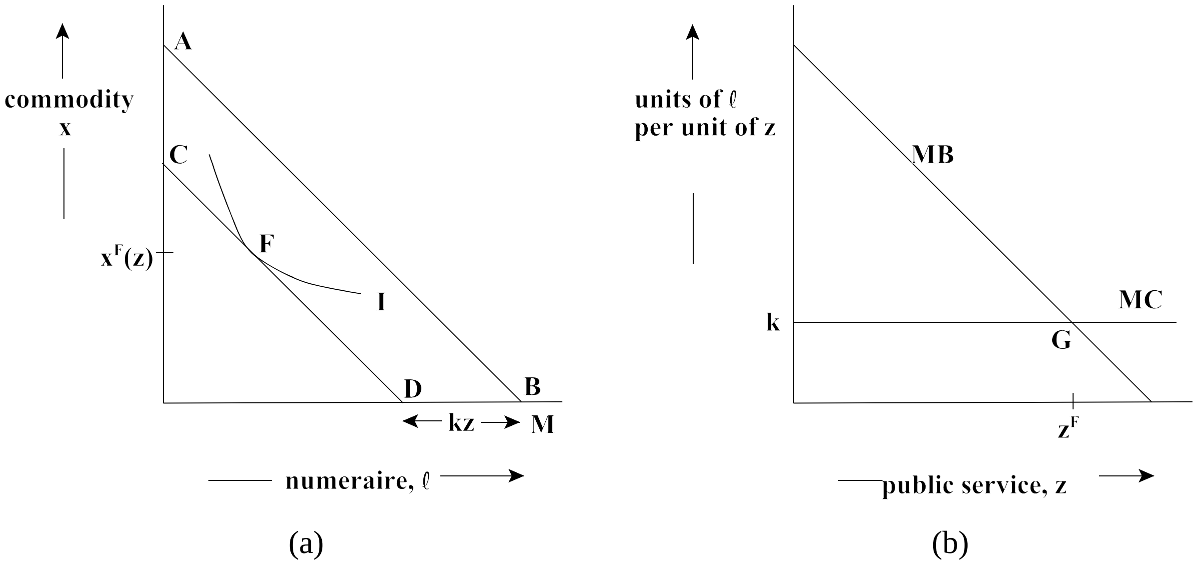

Figure 1: first-best allocation

In Figure 1(a), *AB* is the household's budget line if $z = 0$; *CD* is the budget line if $z > 0$ and if the public service is financed by a lump-sum tax kz. With lump-sum taxation, the optimal allocation $x^F(z)$ is at the point of tangency *F* of the indifference curve *I* and *CD*.

In the second stage of the maximization, z is chosen. Substituting $x^F(z)$ for x in the objective, differentiate to obtain the first-order condition

$$U_\ell \left(-p \frac{dx^F}{dz} - k \right) + U_x \frac{dx^F}{dz} + G_z = 0 \ .$$

Using Equation (2), this first-order condition is reduced to

$$\frac{G_z(z)}{U_\ell(M - px^F(z) - kz, x^F(z))} = k \ . \tag{3}$$

The value of z which solves Equation (3) is the first-best public service, z^F. The left-hand side of Equation (3) is the marginal benefit of the public service. It is shown in the Appendix A that the marginal benefit is a decreasing function of z and, in Figure 1(b), it is shown as the downward-sloping line *MB*. The right-hand side of Equation (3) is the marginal cost of the public service, shown in Figure 1(b) as the horizontal line *MC*. First-best efficiency requires that the marginal benefit equals the marginal cost, or z^F occurs at the intersection *G* of *MB* and *MC*.

2.2 Second-best allocation

The second-best model assumes that the public service must be financed by a tax on the commodity x. Commodity x is competitively produced, so that its producer price is the resource cost p. If the tax rate is t, the consumer price of a unit of commodity x is

$$q \equiv p(1 + t) \ .$$

Because p is exogenous to the model, the consumer price q is a representation of the commodity tax rate.

The second-best problem is modeled as a two-stage problem. In the first-stage the government sets the tax rate q and the public service level z, and in the second stage the household chooses its consumption of the taxed commodity. Working backwards, at the second stage, the household takes q and z as given and chooses x to maximize his utility:

$$\max_{x} \ U(M - qx, x) \ + \ G(z) \ .$$

The household's choice of x conditional on q, $x^S(q)$, is implicitly defined by the first-order condition

$$\frac{U_x(M - qx^S(q), x^S(q))}{U_\ell(M - qx^S(q), x^S(q))} = q \ . \tag{4}$$

Each household buys x until the marginal benefit equals the consumer price.

At the first stage, the government is restricted to choices of q and z which balance its budget in the second stage, or

$$(q - p)\, x^S(q) = kz . \tag{5}$$

Hence q is an implicit function of z, $q(z)$, and the household choice of x becomes an implicit function of z, $x^S(q(z))$. Figure 2 superimposes the second-best problem on the first-best problem. In Figure 2(a) the government sets the tax rate so that tax revenue kz is collected or so that the household's budget line *BE* induces the household to choose an allocation on *CD*: facing the budget line *BE* , the household achieves the indifference curve I' at *S*. $x^S(q(z))$ is the consumption of x at *S*.

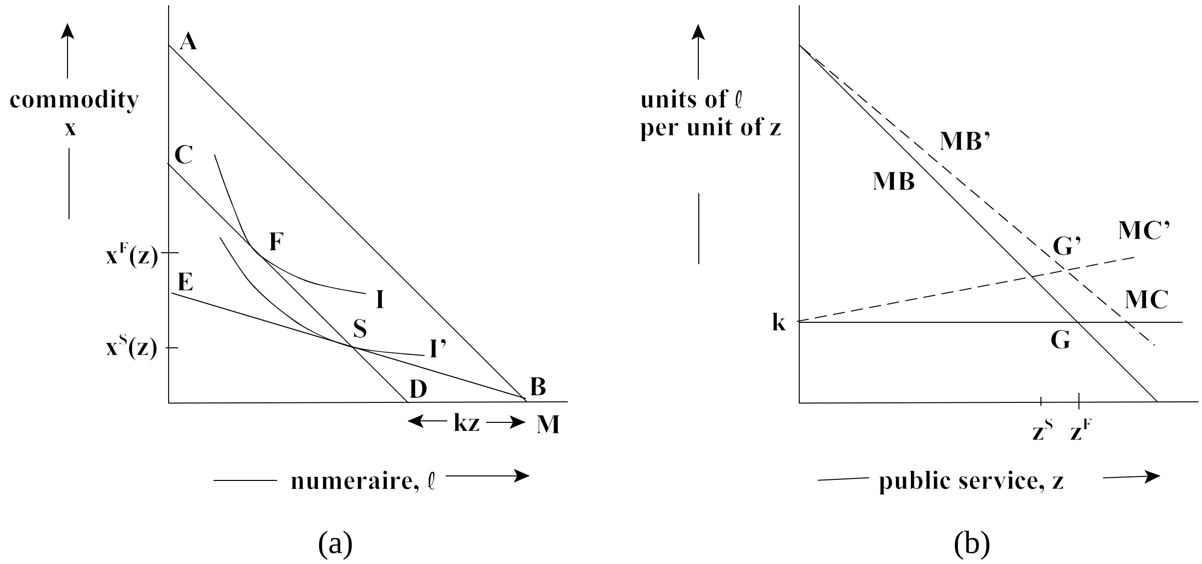

Figure 2: second-best efficiency with a commodity tax

The first stage of the maximization is to choose the public service level (and the implied tax rate) as

$$\max_z \; U(M - q(z)\, x^S(q(z)) \,,\; x^S(q(z))) \;+\; G(z) \,.$$

Differentiating to obtain the first-order condition,

$$U_\ell \left(-q \frac{dx^S}{dq}\frac{dq}{dz} - x^S \frac{dq}{dz} \right) \;+\; U_x \frac{dx^S}{dq}\frac{dq}{dz} \;+\; G_z \;=\; 0 \;.$$

Using Equation (4), the first-order condition is written as

$$\frac{G_z(z)}{U_\ell(M - q(z)x^S(q(z)), x^S(q(z)))} \;=\; x^S \frac{dq}{dz} \;. \tag{6}$$

Differentiating Equation (5) and rearranging,

$$x^S \frac{dq}{dz} = k \frac{1}{1 - \frac{q-p}{q}\varepsilon} \, ,$$

where $\varepsilon = -(q/x)\, dx^S/dq$ is the price elasticity of demand for commodity x.

Substituting for $x^S dq/dz$ in Equation (6), the second best public service z^S is the solution to

$$\frac{G_z(z)}{U_\ell(M - q(z)x^S(q(z)), x^S(q(z)))} = k \frac{1}{1 - \frac{q(z)-p}{q(z)}\varepsilon} \, . \tag{7}$$

The left-hand side of Equation (7) is the marginal benefit of an additional unit of the public service - it measures the amount of numeraire the individual is willing to give up to gain an extra unit of z. The right-hand side of Equation (7) is the marginal resource cost multiplied by the marginal cost of funds - it measures the amount of numeraire the individual needs as compensation if k units of revenue are raised using tax instrument q.

ASSUMPTION 1: the price elasticity of the taxed commodity is positive, or $\partial x/\partial q < 0$.

The assumption $\varepsilon > 0$ implies that the marginal cost of funds exceeds unity or that the marginal distortion cost is positive.

3. DISTORTION AND PUBLIC SERVICE LEVEL

3.1 Discussion

I now compare the first-best outcome z^F with the second-best outcome z^S by comparing Equation (3) and (7) using Figure 2. The marginal cost *MC'* of the public service in the second-best case includes the distortion cost of the proportional tax and, with $\varepsilon > 0$, exceeds the marginal cost *MC* in the first-best case. In Figure 2(b), MC' lies strictly above *MC*.

What is new in my analysis is how the distortion cost is likely to raise the marginal benefit of the public service. In Figure 2(a), if the public service is preset at z and financed by a proportional tax on x, the household's budget line is *BE* and the household chooses the allocation *S*. The commodity tax causes a consumption distortion - the household consumes less commodity x and therefore more numeraire than if a lump-sum tax is used.[6] The direct effect of the increased consumption of the numeraire is to lower the marginal utility associated with the numeraire. This in turn increases the marginal benefit of the public service.

Formally, the increase in the numeraire ℓ affects the marginal benefit schedule as

$$\frac{d}{d\ell}\,\frac{G_z(z)}{U_\ell\left(\ell\,,\,\dfrac{M-\ell-kz}{p}\right)} \;=\; -\,\frac{G_z}{U_\ell^2}\left[U_{\ell\ell}-\frac{U_{\ell x}}{p}\right]. \tag{8}$$

In the neighborhood of the first-best allocation $\ell^F(z) \;\equiv\; M - px^F(z) - kz$, normality of x implies that the expression $[\,\cdot\,]$ is negative.[7] Hence the proportional tax increases the consumption of numeraire and, at least in the neighborhood of the first-best allocation, raises the marginal benefit of z.

In Figure 2(b), *MB'* is the marginal benefit schedule when the commodity tax is used instead of the lump-sum tax. It is drawn above *MB*, the marginal benefit schedule in the first-best. The second-best public service level, z^S, lies at the intersection G' of MB' and MC'. Because the price distortion affects both the marginal benefit and the marginal cost schedules of the public service, it is potentially ambiguous whether the second-best public service level is less or greater than the first-best public service level. $z^S > z^F$ if the shift in the marginal benefit schedule is sufficiently large.

Equation (8) shows that the shift in the marginal benefit schedule depends on the direct effect of the increase in the numeraire (the $U_{\ell\ell}$ term) and the indirect effect associated with the decrease in the commodity x (the $U_{\ell x}$ term). If $U_{\ell x} > 0$, the decrease in the quantity of x also lowers the marginal utility of the numeraire; *ceteris paribus* this indirect effect reinforces the direct effect and increases the shift in the marginal benefit schedule. Descriptively, ℓ gives more utility when it is used with x, or x and ℓ are complements. This provides an intuitive explanation for Gaube's (2000) result that, if x and ℓ are normal, $z^S > z^F$ requires that x is a gross complement of ℓ.[8]

Considering the alternative case, a sufficient condition for $z^S < z^F$ is that the use of the porportional tax causes *MC* to shift up and *MB* to shift down. We have already seen that, with $\varepsilon > 0$, *MC'* lies above *MC*. *MB'* lies below *MB* if, as the allocation moves from *F* to *S* in Figure 2, U_ℓ increases or

$$\frac{d}{d\ell} U_\ell(\ell, \frac{M - \ell - kz}{p}) > 0$$

or

$$U_{\ell\ell} - \frac{1}{p} U_{\ell x} > 0$$

which is exactly the condition which ensures that $\partial x / \partial M < 0$.[9] This gives a new sufficient condition which ensures that $z^S < z^F$.

LEMMA 1: *if the taxed commodity has a positive price elasticity and is inferior, then* $z^S < z^F$

3.2 Example In Which $z^s > z^F$

I now provide an example in which $z^S > z^F$. This example is based on de Bartolome (1998). I restrict attention to utility functions $U(\ell, x)$ which give an iso-elastic demand for commodity x.[10] The demand for commodity x is

$$x = \frac{C M^b}{q^a}, \quad \text{with } 0 \leq a, \ \ 0 \leq C. \tag{9}$$

We restrict attention to $a, b < 1$. If $b > 0$, x is a normal good. If $b > a$, x is a gross complement with respect to the price of ℓ.[11] The budget constraint gives the demand for the numeraire

$$\ell = M - qx = M - CM^b q^{1-a}. \tag{10}$$

The indirect utility function which generates these demand functions is recovered in Appendix B, and is

$$V(q, M) = M^{1-b} - C \frac{1-b}{1-a} q^{1-a}.$$

Appendix C shows that this indirect utility function implies $U_{\ell x} > 0$. The acceptable set $\{q , M\}$ is restricted because (1) $\ell \geq 0, x \geq 0$, and because (2) the Slutsky Restriction (the own-derivative of the implied compensated demand must be non-positive or the implied utility function must be concave).

For ease of calculation, I set

$$G(z) \equiv \sqrt{z}.$$

The first-best problem is

$$z^F = \underset{z}{argmax} \; (M-z)^{1-b} - C\frac{1-b}{1-a}\, p^{1-a} + \sqrt{z} \;\; ;$$

The second-best problem is

$$\max_{q,z} \; M^{1-b} - C\frac{1-b}{1-a}\, q^{1-a} + \sqrt{z} \qquad \text{s.t.} \qquad (q-p)\frac{CM^b}{q^a} = z \;\; .$$

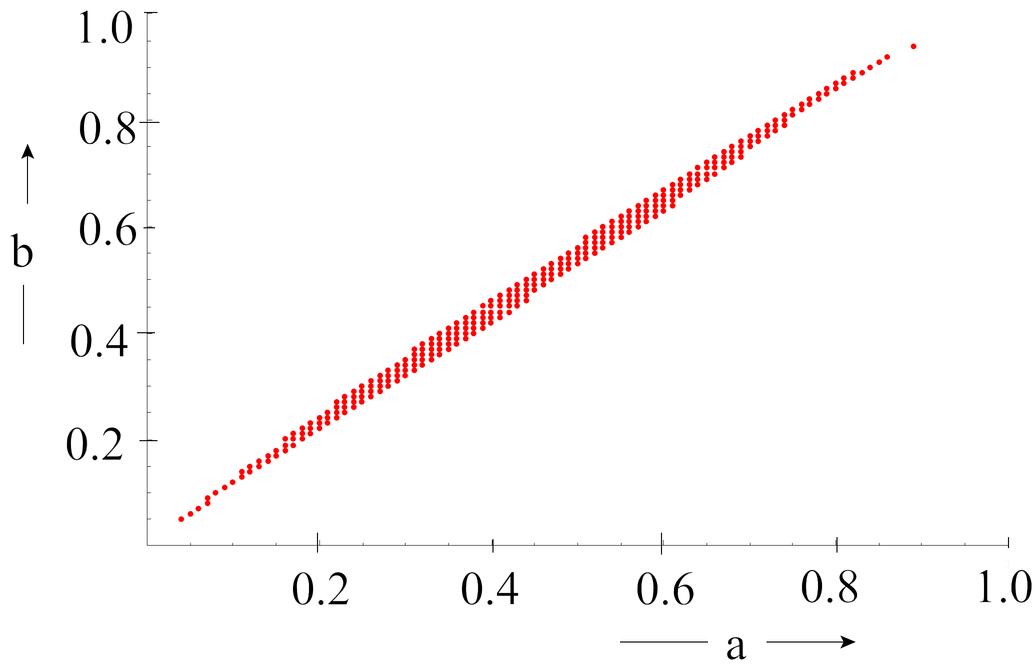

Figure 3: some values of price and wealth elasticities for which $z^S > z^F$

I choose parameter values $p = 1$, $c = .5$, and $M = 1$ and simulate using the computer. In Figure 3, the shaded area represents values of *(a,b)* for which $z^S > z^F$. As expected, the shaded area has $b > 0$ (*b* is a normal good) and $b > a$ (*x* is a gross complement). The area below the shaded area represents values of (a,b) for which $z^S < z^F$ and the area above the shaded area represents values of (a,b) which are not permitted because the implied utility function violates the Slutsky Restriction.

4. LABOR TAX

I now consider the case when the public service must be financed by a labor tax. In considering this case, I use the previous model but interpret ℓ to be leisure, x to be the numeraire commodity and M to be the household's leisure endowment; I follow traditional notation and rewrite the leisure endowment as H, $M \equiv H$. The household's before-tax wage w is his productivity or $w = 1/p$. The economy's resource constraint is obtained from Equation (1):

$$x + w\ell + kwz = wH.$$

Writing labor supply as $L \equiv H - \ell$, the economy's resource constraint is rewritten as:

$$x = wL - kwz.$$

4.1 First-Best Allocation

The first-best problem is:

$$\max_{z} \max_{L} U(H - L, wL - kwz)$$

The analysis is similar to that of Section 2. Write the optimal allocation of labor conditional on z as $L^F(z)$:

$$\frac{U_\ell(H - L^F(z), wL^F(z) - kwz)}{U_x(H - L^F(z), wL^F(z) - kwz)} = w.$$

The first-best public service level z^F is the value of z which solves

$$\frac{G_z(z)}{U_x(H - L^F(z)\,,\, wL^F(z) \,-\, kwz)} \;=\; kw\,.$$

The left-hand side is the marginal benefit (*MB*) and the right-hand side is the marginal resource cost (*MC*) where these quantities are now denominated as units of *x* per unit of the public service.

4.2 Second-best Allocation

If the public service is financed by a labor tax τ, the after-tax wage paid to the household is $\omega = w(1\text{-}\tau)$; with *w* being exogenous, ω may be used in lieu of the tax rate τ. Using the methodology described in Section 2, the household chooses its labor conditional on the after-tax wage rate, $L^S(\omega)$, as:

$$\frac{U_\ell(H - L^S(\omega)\,,\, \omega L^S(\omega))}{U_x(H - L^S(\omega)\,,\, \omega L^S(\omega))} \;=\; \omega\,.$$

ω is itself a function of *z*, $\omega(z)$, through the government budget requirement:

$$(w \,-\, \omega)L^S(\omega) \;=\; kwz\,.$$

The second-best public choice level is

$$\frac{G_z(z)}{U_x(H - L^S(z)\,,\, \omega L^S(z))} \;=\; kw\,\frac{1}{1 - \dfrac{w - \omega}{\omega}\,\varepsilon_L}$$

where $\varepsilon_L = (\omega/L)\, dL/d\omega$. The left-hand side is the marginal benefit of the public service (*MB'*) when the public service is being financed by a labor tax and the right-hand side (*MC'*) is the resource cost of the public service multiplied by the marginal cost of funds. As noted by Atkinson and Stern (1974), and by Ballard and Fullerton (1992), the wage distortion may cause the marginal cost of funds to rise above or fall below unity depending on whether an increase in the tax rate causes the tax base to decrease or increase. This is essentially the same argument as put forward by Stiglitz and Dasgupta (1971), that the marginal cost of funds is less than unity if the increase in the tax rate, which is needed to fund the higher public service, changes quantities so that more tax revenue is collected from the pre-existing tax structure. For the subsequent discussion, I restrict attention to the case of $dL/d\omega > 0$, so that the proportional labor tax increases the marginal cost of funds above unity.

ASSUMPTION 2: the wage elasticity of labor is positive, or $\partial L/\partial w > 0$.

4.3 Discussion of Labor Tax Case

Because of the model's simplicity, the labor tax is equivalent to a commodity tax. In particular, when the government uses the labor tax, the household problem is:

$$\max_{L, x} U(H - L, x) \quad s.t. \quad x = w(1 - \tau)L.$$

Remembering that $\ell = H - L$ this becomes:

$$\max_{\ell, x} U(\ell, x) \quad s.t. \quad x + w(1 - \tau)\ell = w(1 - \tau)H \tag{11}$$

Writing $w = 1/p$, and setting $t \equiv \tau/(1 - \tau)$, the household problem is:

$$\max_{\ell, x} U(\ell, x) \quad s.t. \quad \ell + p(1 + t)x = H.$$

Hence the household makes the same choices of ℓ and x when faced with a wage tax τ or a commodity tax t provided $t \equiv \tau/(1 - \tau)$; government revenue is also the same provided $t \equiv \tau/(1 - \tau)$.[12] Equation (11) shows the well-known fact that a labor tax can be considered as a lump-sum tax plus a leisure subsidy.

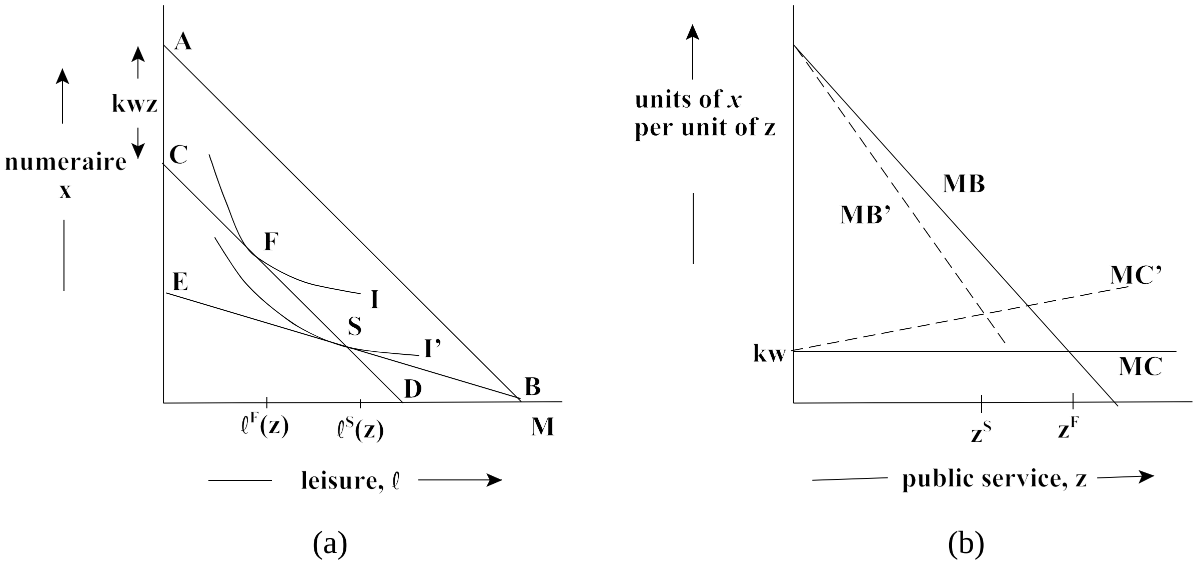

Figure 4: second-best efficiency with a labor tax

Figure 4(a) shows that, at any *z*, the replacement of the lump-sum tax by the labor tax shifts the household allocation from *F* to *S*. Because the labor tax is equivalent to the commodity tax, Figure 4(a) is identical to Figure 2(a): the lowering in the after-tax wage causes the household to increase his leisure and decrease his consumption of the numeraire.[13] The direct effect of the decrease in the numeraire is to raise the marginal utility of the numeraire, lowering the marginal benefit and shifting the marginal benefit curve *MB'* down in Figure 4(b). Unless there are large indirect effects, *MB'* lies below *MB*. In this case, the shifts in the marginal benefit and marginal cost schedules reinforce, causing the second-best public service level to fall below the first-best level.

I now show that the indirect effects cannot dominate if leisure is a normal good. *MB'* lies below *MB* if, as x decreases from *F* to *S*, U_x increases or if:

$$\frac{d}{dx}U_x\left(\frac{wH - x - kwz}{w}, x\right) \equiv -\frac{1}{w}U_{\ell x} + U_{xx} < 0 .$$

This is the condition which ensures that leisure is a normal good.[14] Hence we have Lemma 2:

LEMMA 2: *if the labor elasticity is positive and leisure is normal, then* $z^S < z^F$

Lemma 2 was derived by Gaube (2000). Intuitively, it is an extension of Lemma 1. With a tax levied on a commodity, *MB'* lies below *MB* if the commodity is inferior. A labor tax may be interpreted as a lump-sum tax plus a leisure subsidy and a subsidy may be interpreted as a negative tax: with a leisure subsidy, *MB'* lies below *MB* if leisure is a normal good (or labor is an inferior good).

Finally, because the commodity tax is equivalent to a labor tax and *vice-versa*, Lemma 1 and 2 apply whether the tax is levied on a commodity or on labor. I.e., $z^s < z^F$ if either the commodity has a positive price elasticity and is inferior, or if the labor elasticity is positive and leisure is normal.

5. CONCLUSION

Pigou, in making his conjecture of the effect of a commodity tax on the public service level, failed to take account of the price distortion on the marginal benefit of the public service. Because of the effect on the marginal benefit, examples may be found in which the second-best public service level exceeds the first-best level. However a sufficient condition to ensure that this does not occur is either the price elasticity of the taxed commodity is positive and the taxed commodity is inferior, or the wage elasticity of labor is positive and leisure is normal.

APPENDIX A : DIMINISHING MARGINAL BENEFIT

To show that the marginal benefit curve is downward sloping, differentiate the marginal benefit with respect to *z*

$$\frac{d}{dz}\;\frac{G_z(z)}{U_\ell(M - px^F(z) - kz, x^F(z))} \;=\; \frac{1}{U_\ell^2}\left(U_\ell G_{zz} - G_z\left(U_{\ell\ell}\left(-p\frac{dx^F}{dz} - k\right) \;+\; U_{\ell x}\frac{dx^F}{dz}\right)\right) \;.$$

To obtain dx^F/dz, differentiate Equation (2) with respect to z, and rearrange

$$\frac{dx^F}{dz} \;=\; k\frac{U_{\ell x} - pU_{\ell\ell}}{p^2U_{\ell\ell} - 2pU_{\ell x} + U_{xx}} \;.$$

Hence

$$\frac{d}{dz}\;\frac{G_z(z)}{U_\ell(M - px^F(z) - kz, x^F(z))} \;=\; \frac{1}{U_\ell^2}\left(U_\ell G_{zz} + kG_z\frac{(U_{\ell\ell}U_{xx} - U_{\ell x}^2)}{p^2U_{\ell\ell} - 2pU_{\ell x} + U_{xx}}\right) \;.$$

But $G_{zz} < 0$; concavity of the utility function implies $p^2U_{\ell\ell} - 2pU_{\ell x} + U_{xx} < 0$ and

$U_{\ell\ell}U_{xx} - U_{\ell x}^2 > 0$. Hence each term on the right-hand side is negative.

APPENDIX B: RECOVERING THE INDIRECT UTILITY FUNCTION[15]

This procedure follows Hausman (1981). Denote the expenditure required to achieve utility u at consumer price Q as $e(Q, u)$ and denote the compensated demand as $h(Q,u)$. Duality implies that $h(Q,u) = x(Q, e(Q,u))$ or $h(Q,u) = Ce(Q,v)^b/Q^a$. Shepherd's Lemma implies

$$\frac{\partial e(Q,u)}{\partial Q} = \frac{C\, e(Q,u)^b}{Q^a} .$$

Assume $b \neq 1$ and integrate from some reference price P,

$$\int_{e(P,v)}^{e(q,v)} \frac{de}{e^b} = C \int_P^q \frac{dQ}{Q^a} ,$$

or $$\frac{e(q,u)^{1-b} - e(P,u)^{1-b}}{1-b} = C\, \frac{q^{1-a} - P^{1-a}}{1-a} .$$

Set $u = v(q, M)$, the indirect utility achieved at consumer prices q and income M. By duality, $M = e(q, v(q, M))$. Rearranging,

$$e(P, v(q,M))^{1-b} - C\,\frac{1-b}{1-a} P^{1-a} = M^{1-b} - C\,\frac{1-b}{1-a} q^{1-a} .$$

The left-hand side is a monotonic transformation of the indirect utility function. Because tastes are unchanged by a monotonic transformation of the utility function, preferences may also be represented by the indirect utility function

$$V(q,M) \equiv e(P, v(q,M))^{1-b} - C\,\frac{1-b}{1-a} P^{1-a} = M^{1-b} - C\,\frac{1-b}{1-a} q^{1-a} . \qquad \text{(B.1)}$$

APPENDIX C: $U_{\ell x} > 0$

Equations (9) and (10) give (q, M) as implicit functions of (ℓ, x): write the functional dependance as $q\ (\ell, x)$ and $M\ (\ell, x)$. Duality implies

$$U(\ell,x) \equiv V(q(\ell,x), M(\ell,x)) \ .$$

Hence

$$\frac{\partial U}{\partial \ell} = \frac{\partial V}{\partial q}\frac{\partial q}{\partial \ell} + \frac{\partial V}{\partial M}\frac{\partial M}{\partial \ell} .$$

$\partial V/\partial q$ and $\partial V/\partial M$ are obtained by differentiating Equation (B.1); $\partial q/\partial \ell$ and $\partial M/\partial \ell$ are obtained by differentiating Equations (9) and (10). Hence

$$\frac{\partial U}{\partial \ell} = \frac{1 - b}{M^{b}}.$$

Differentiating with respect to x

$$\frac{\partial^2 U}{\partial x\, \partial \ell} = - \frac{b(1-b)}{M^{b+1}}\frac{\partial M}{\partial x} = \frac{b(1-b)}{M^{1+b}}\frac{1-a}{a}q\frac{1}{1 - \dfrac{bCq^{1-a}}{aM^{1-b}}} > 0$$

where $\partial M/\partial x$ is obtained by differentiating Equations (9) and (10) with respect to x. The last inequality follows from the Slutsky Restriction that $\partial h(Q,u)/\partial Q < 0$. Viz.: By duality, $h(Q,u) = x(Q,e(Q,u))$. Hence $h(Q,u) = Ce^{b}(Q,u)/Q^{a}$ or

$$\frac{\partial h}{\partial Q} = \frac{bC}{Q^a}\frac{1}{e^{1-b}}\frac{\partial e}{\partial Q} - \frac{aCe^b}{Q^{a+1}} . \tag{C.1}$$

By duality, $V(Q, e(Q,u)) = u$ or, using Equation (B.1)

$$u = e(Q,u)^{1-b} - C\frac{1-b}{1-a}Q^{1-a}$$

Differentiating with respect to Q :

$$0 = (1-b)\frac{1}{e^b}\frac{\partial e}{\partial Q} - C\frac{1-b}{1-a}(1-a)\frac{1}{Q^a}$$

or

$$\frac{\partial e}{\partial Q} = \frac{Ce^b}{Q^a} .$$

Substituting in Equation (C.1) and rearranging:

$$\frac{\partial h}{\partial Q} = \frac{Ce^b}{Q^a}\left(bC\frac{e^{b-1}}{Q^a} - \frac{a}{Q} \right) .$$

But evaluation is being made at the allocation achieved with income *M*, or *e(Q, u) = M*. Hence $\partial h / \partial Q < 0$ implies

$$\frac{bCM^{b-1}}{Q^a} - \frac{a}{Q} < 0 .$$

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ENDNOTES

1. "Marginal benefit" is used synonymously with "marginal rate of substitution" throughout the paper. Similarly, "first-best" and "second-best" are used synonymously with "first-best efficient" and "second-best efficient."

2. Gaube's (2000) and my result both assume that the price elasticity is positive.

3. The result presupposes that the labor supply elasticity is positive.

4. The topic is reviewed in Batina and Ihori (2005).

5. The assumption of a public service is made to simplify the presentation. The results apply if the government expenditure is on a public good.

6. Formally: using Equation (2), (5) and (4)

$$U_x(M - px^F(z) - kz, x^F(z)) - pU_\ell(M - px^F(z) - kz, x^F(z)) = 0 < \frac{kz}{x^S}U_\ell(M - qx^S, x^S)$$

$$= (q - p)U_\ell(M - qx^S, x^S) = U_x(M - px^S(q(z)) - kz, x^S(q(z))) - pU_\ell(M - px^S(q(z)) - kz, x^S(q(z))) .$$

But $\partial/\partial x \left[U_x(M - px - kz, x) - p\,U_\ell(M - px - kz, x)\right] = p^2 U_{\ell\ell} - 2pU_{\ell x} + U_{xx} < 0$,

where the last inequality follows from the concavity of $U(.,.)$. Combining with the first inequality implies $x^S(q(z)) < x^F(z)$. Using the resource constraint (Equation (1)):

$$\ell^S(z) \equiv M - px^S(q(z)) - kz > M - px^F(z) - kz \equiv \ell^F(z) .$$

7. The household chooses x to maximize $U(M - px - kz, x)$. The first-order condition is: $-pU_\ell(M - px - kz, x) + U_x(M - px - kz, x) = 0$. Differentiating with respect to M:

$$\frac{\partial x}{\partial M} = -\frac{U_{\ell x} - pU_{\ell\ell}}{p^2 U_{\ell\ell} - 2pU_{\ell x} + U_{xx}}.$$

The concavity of U implies that the denominator is negative and hence $\partial x/\partial M > 0$ implies $U_{\ell x} - pU_{\ell\ell} > 0$.

8. This is a necessary but not sufficient condition. Write the price of ℓ as w $(w = 1)$. The household chooses x to maximize $U((M - qx)/w\,,x)$. The first-order condition is: $-q/w\, U_\ell((M-qx)/w,x) + U_x((M-qx)/w,x) = 0$. Differentiating with respect to q and rearranging:

$$\frac{\partial x}{\partial w} = -q\,\frac{U_\ell + \ell U_{\ell\ell} - \frac{w\ell}{q}U_{\ell x}}{q^2 U_{\ell\ell} - 2qwU_{\ell x} + U_{xx}}$$

The concavity of U implies that the denominator is negative. Therefore $\partial x/\partial w < 0$ is favored if $U_{\ell x} > 0$.

9. See Footnote 7.

10. In drawing a marginal cost of funds schedule which increases with increasing public expenditure, it is normal to presuppose a constant elasticity ε.

11. To see this, calculate U_ℓ, $U_{\ell\ell}$ and $U_{\ell x}$ using the same procedure as is described in Appendix C, and then insert into the equation shown in Footnote 8. Hence

$$\frac{\partial x}{\partial w} = \frac{(1-b)(a-b)}{aM^b}\,\frac{1}{1 - \frac{bCq^{1-a}}{aM^{1-b}}}$$

The Slutsky Restriction ensures that the denominator is positive. Hence x is a gross complement if $b > a$.

12. With a labor tax, the government's tax revenue is τwL units of x. With a commodity tax rate t, the government's tax revenue is tx units of x. But $t = \tau/(1-\tau)$ and $x = (1-\tau)\,w\,L$, so the commodity tax collects tax revenue $\tau/(1-\tau)\;w(1-\tau)L = \tau wL$ units of x.

13. Although Figure 4(a) is identical to Figure 2(a), Figure 4(b) is not identical to Figure 2(b). The marginal cost of unit of the public service in Figure 4(b) is kw and the marginal benefit is measured in units of x per unit of z.

14. The household chooses leisure to maximize his utility: $\max_{\ell} U(\ell, \omega H + I - \omega\ell)$ where I is endowed income ($I = 0$). The first-order condition is: $U_{\ell}(\ell, H + I - \omega\ell) - \omega U_{x}(\ell, H + I - \omega\ell) = 0$.

Differentiating with respect to I and rearranging:

$$\frac{\partial\ell}{\partial I} = -\frac{U_{\ell x} - wU_{xx}}{U_{\ell\ell} - 2wU_{\ell x} + w^2 U_{xx}}.$$

Concavity of the utility function implies the denominator is negative. Therefore $\partial\ell / \partial M > 0$ implies $U_{\ell x} - wU_{xx} > 0$

15. For further details, see Varian (1992, p. 125-129).