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## Modeling Network Externalities, Network Effects, and Product Compatibility with Logit Demand

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**Abstract**

With the advent of the digital age, information goods characterized by strong positive network externalities and effects are playing an increasingly prominent economic role. A logit model of oligopolistic competition is presented with a focus on providing an accessible rigorous analytic framework for positive network externalities and effects.

In the presence of positive network externalities and effects, market behavior is quite different from that of traditional logit models. Multiple stable equilibria arise. Oligopoly producers respond to higher price elasticities with lower prices and markups. Markets tend to be highly concentrated and the dominant producer can remain dominant even while producing an inferior product. Strategic behaviors arise that do not exist in the absence of network externalities or effects.

*JEL classification*: C65; D11; D43; L13

*Keywords*: Discrete Choice, Network Externalities, Network Effects

# Contents

# 1 Introduction

As our economy moves into the digital age, products exhibiting network effects and externalities are playing an increasingly important economic role.

Network externalities are externalities that a consumer sees as a result of others' use of a product and similar or compatible products. If eBay[1] had a

[1]eBay is a popular web site (`http://www.ebay.com`) which enables individuals to coordinate the sale of goods and services.

single user, that user would find little use for the site. And it becomes more useful as more people use the site.

Another example, which is familiar to anyone who uses a computer in their day-to-day lives, is software. While Microsoft Word is intrinsically useful as a word processor, it also provides its users with benefits in the form of network externalities. The more users of Microsoft Word there are, the greater the ability for each user to send documents as e-mail attachments or to otherwise communicate via documents generated in Microsoft Word.

Network effects are externalities that arise from use of a network of compatible products. Microsoft's dominance in the personal computer operating system market was widely attributed to an "applications barrier to entry," a network effect in which a disproportionate number of applications are produced for the dominant operating system, which in turn makes the operating system more desirable.

Network externalities and effects can be either positive or negative, though the focus of this paper is on positive externalities and effects[2] and throughout the term network externalities and effects shall refer to positive network externalities and effects unless explicitly stated otherwise.

In addition to the network externalities and effects observed by the consumer, producers make decisions regarding compatibility of their product with other products on the basis of network externalities and effects. If a producer can introduce an adapter, it may take advantage of a competitor's network of goods. For example, the producer of a word processor may choose to develop a compatibility layer that allows a competitor's documents to be imported and/or exported. Compatibility decisions may even give rise to sophisticated competitive games in which a producer incurs costs or uses proprietary intellectual property to block competitors' abilities to build compatibility into their products.[3]

Since the early work of Rohlfs in [16], network effects and externalities have

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[2]Economists commonly use the terms network externalities and network effects interchangeably, and though they are conceptually similar, they should properly be considered distinct concepts. It is not uncommon in reality for both to be observed together.

[3]AOL's IM wars against AIM clones and Microsoft's often cited strategy of "embrace and extend" come to mind as notable recent examples.

spawned a wealth of literature. In a seminal modeling effort, Katz and Shapiro [8] introduced a model of network externalities in which a continuum of consumers make a discrete choice between the products of Cournot competitors. They show that the level of industry output is greatest under complete compatibility, that any privately profitable industry-wide standard is desirable, but that compatibility incentives may be either socially excessive or inadequate. In a simple but insightful overlapping generations model, Oz Shy [18] shows that the rate of technological adoption in the presence of network externalities depends on the degree of substitutability or complementarity between network size and technological advance. E. G. Kristiansen [9] uses a 2-period model with a single product and research costs to shows that in the presence of network externalities, R&D incentives for incumbents push them to make too risky an investment choice while entrants make too certain an investment choice and that incompatible technologies are adopted with a socially excessive probability. Network externalities are even mentioned in the context of logit demand by Anderson et al. [19]; however, only the symmetric case is considered and only briefly at that.[4]

[4]Notably, they comment that in the asymmetric case, "The major difficulty with these equilibria is to characterize them and prove existence." [19, p. 257] In this paper, proof of existence is given and equilibria are characterized.

Although a number of stylized models have been advanced, no framework has been established suitable for ready applied examination of information goods that exhibit network externalities or effects. To that end, a discrete choice random utility model of positive network externalities is presented with an emphasis on providing an analytic framework for the economic analysis of these markets.

The model introduces two innovations in a traditional logit setting: A utility function in which a consumer explicitly values the consumption of a product and compatible products by others and a production function with a compatibility decision and an associated cost of compatibility.[5]

In the presence of network externalities, multiple equilibria can arise. However, they can be easily identified and characterized numerically. Although second-order regularity cannot be assured, numerical techniques are found to be generally robust to perturbation of the calibration set.[6]

[5] Although the focus of the modeling effort is on network externalities, a simple reformulation of the network term will admit network effects and the results are roughly equivalent in either case.

[6] A logit formulation was chosen for its popularity and simplicity; however, the numerical techniques developed for this analysis are by no means specific to the logit model.

As a result of network externalities, producers compete in both price and network size and firms use compatibility decisions to strategically react to large, entrenched competitors. Markets are shown to be more responsive to the actions of both dominant and fringe producers. A dominant producer's tendency to extract monopoly profits is mitigated by the need to support a dominant network.

In section 2, a model of network externalities is presented. Numerical results from several scenarios are presented in section 3 and in section 4 the results are discussed and several avenues for future research are suggested.

## 2 The Model

A logit model of positive network externalities will be introduced in which consumers who consume a product explicitly value use of the same or a compatible product and producers compete in both price and compatibility.

The consumer optimization problem will be introduced in section 2.1, followed by the profit maximization problem of the producers in section 2.2. Then section 2.3 will provide first-order conditions for equilibrium and describe demand elasticities. Finally, to bring the theory into the realm of application, section 2.4 will outline calibration steps for analysis of real-world data.

## 2.1 Consumers

$N$ consumers make a discrete choice over a set of $I$ products in which each consumer first chooses a single product from among the available choices, then purchases a continuous quantity of the product.[7] Utility is modeled by a random utility function in which a consumer derives utility from an aggregation of consumption of a good, network externalities associated with others' consumption of that good or other compatible goods, and an idiosyncratic valuation that is independently and identically distributed according to a type 1 extreme value distribution with a zero location parameter

[7] A single consumer type is considered here; however, the model is easily extensible to multiple consumer types.

$$u_{i,n} = y + q_i - \gamma p_i + v(z_{i,n}) + \sigma \epsilon_{i,n} \tag{1}$$

where $i \in \{1, \ldots, I\}$ denotes product $i$, $n$ denotes consumer $n$ where $n \in \{1, \ldots, N\}$, $u_{i,n}$ is the utility of product $i$ for consumer $n$, $y$ is consumer income, $q_i$ is the perceived quality of product $i$, $p_i$ is the price of good $i$ with $\gamma$ its elasticity parameter,[8] $\sigma$ is a scaling parameter corresponding to the degree of heterogeneity across products, and $\epsilon_{i,n}$ is the consumer's idiosyncratic valuation of product $i$.

Consumer $n$'s perception of the value of the network of product $i$, $v$, is taken to be a continuous and strictly increasing function of consumer $n$'s perceived network size, $z_{i,n}$, of product $i$, that is, others' consumption of product $i$ and compatible products. It is also given that $v(0) = 0$. Compatibility between products $i$ and $j$ is given by $\rho_{i,j}$, where $\rho_{i,j} \in [0, 1]$ and $\rho_{i,j} \frac{\partial z_{i,m}}{\partial x_{i,n}} = \frac{\partial z_{i,m}}{\partial x_{j,n}}$ $\forall$ $i \neq j$, $m \neq n$. In other words, the parameter $\rho_{i,j}$ describes the impact on the size of the network of additional expected consumption of a compatible

[8]Notably, in [17], Saha and Simon apply a utility function that is polynomial in price to the analysis of mergers and find that the linear specification tends to over-estimate the post-merger price effect.

product, $j$, relative to additional expected consumption of product $i$. $\rho_{i,j} = 0$ represents complete incompatibility whereas $\rho_{i,j} = 1$ implies that products $i$ and $j$ are fully compatible.[9]

As we shall see in section 2.2, $\rho_{i,j}$ is determined by compatibility parameters $\phi_{i,j}$ and $\phi_{j,i}$, which reflect producers' compatibility decisions.

It is not uncommon to include an outside good, representing a numeraire, in the traditional logit model to represent the choice "none of the above." Good $I$ can serve as the outside good by assuming a price of zero, unitary intrinsic utility, no associated network externalities, and full incompatibility with all other products.

Implicit in the consumers' preferences are strongly additive[10] positive network externalities in which a consumer sees no network externalities unless at least some of a product is consumed by others. Also implicit in the preferences is the traditional logit formulation as the special case in which none

[9]See appendix B for examples of a variety of functional forms expressing consumer perceptions of network size.

[10]I.e., the cross-partial $\partial^2 u_{i,n}/\partial q_{i,n} \partial x_{i,n} = 0$.

of the products exhibit network externalities.[11]

The base quality can be considered the utility the consumer receives as a result of intrinsic attributes of the product. For example, the user of a word processor gains usefulness from the product by its ability to compose documents. The additional network effects are derived exclusively from the user's ability to interact with other users. Continuing with the word processor example, this may include the ability to send and receive documents[12] to and from other users of the same or compatible word processors and the ability to draw on the knowledge base of other word processor users to accomplish complex tasks. Note that some products may have no base quality. If there were a single fax machine in the world, it wouldn't be doing anybody much good.

The formulation, in concert with the producer decision outlined in section 2.2, will be referred to throughout as the Network MNL (Network Multinomial Logit) Model. Although the treatment of the model for the purposes of this paper is in the context of network externalities, the model would readily

[11] That is, $v_{i,n} \equiv 0 \; \forall \; i, n$.

[12] As e-mail attachments, for example.

apply to network effects with little modification.[13]

Based on the utility specification in equation 1, associated with each consumer $k$ and product $i$ is a probability $\mathcal{P}_n(i)$ where

$$\mathcal{P}_n(i) = P(u_{i,n} = \max_{j=1,\dots,I} u_{j,n}) \tag{2}$$

A further symmetry assumption, $\mathcal{P}_m(i) = \mathcal{P}_n(i) \ \forall \ m, n \in \{1, \dots, N\}$, is imposed on $\mathcal{P}_n(i)$ to provide both analytic and computational tractability which allows us to abbreviate $v_{i,n}$ as $v_i$ and $\mathcal{P}_n(i)$ as $\mathcal{P}(i)$.[14]

A simple integration will show that a closed-form solution for $\mathcal{P}(i)$ is given by[15]

[13] Consider a discrete choice over a set of operating systems and the "applications barrier to entry" (see [6, p. 28].) which received much attention in the Microsoft litigation. Consumers respond to the number of applications ported to any particular operating system, which in turn is a function of the number of users of the operating system and compatible operating systems. Equation 1 could then be rewritten $u_{i,n} = y + q_i - \gamma p_i + v(w(z_i)) + \sigma\epsilon_{i,n}$, with $w$ a function representing the number of applications created for operating system $i$ given network size $z_i$.

[14] This assumption is consistent with logit demand models that do not incorporate network effects and allows for a considerably reduced solution space.

[15] See appendix A.

$$\mathcal{P}(i) = \Psi_i(x; p, q, \rho) = \frac{e^{q_i - \gamma p_i + v_i(x)}}{\sum_{j=1}^{I} e^{q_j - \gamma p_j + v_j(x)}} \tag{3}$$

Equilibrium is given to be a Nash equilibrium; that is, in equilibrium, consumption decisions are made simultaneously taking prices, product compatibility, and other consumers' choices as given. In equilibrium, $x_i = N\mathcal{P}(i)$.

As with the traditional logit demand system, it is easy to show that an equilibrium exists.[16] Unlike the traditional logit demand system, due to the increasing returns inherent in positive network externalities, multiple equilibria can exist; indeed, they are to be expected as a fundamental characteristic of the system when the value of network externalities is sufficiently large and convex in perceived network size.

However, contrary to what one might expect, even in the presence of convex positive network externalities, multiple equilibria are not guaranteed. With weak network externalities and sufficient differentiation between products in terms of core attributes and/or pricing, a single stable equilibria will

[16] See appendix C.1.

be found. More precisely, consumers can be supposed to follow a discrete tatônnement process in which, in each time period, they make their consumption decisions taking prices, compatibility, and network sizes based on the previous time period's choices as given. When for all possible allocations $x$ at prices $p$, qualities $q$, and compatibility $\rho$,

$$\frac{du_i}{dx_i} < \frac{1}{2\Psi_i(x)(1-\Psi_i(x))} \; \forall \; i \tag{4}$$

the solution set reduces to a single, stable equilibrium.[17] In general, for any producer which does not sport a price or feature advantage, network externalities must be strong enough for a large network size to overcome the intrinsic disadvantage in that producer's product attributes or pricing to enjoy a dominant equilibrium.

Given multiple equilibrium, it is typically easy to establish the stability of the equilibrium. On the basis of the tatônnement process, it can be shown[18]

[17]See appendix C.2.
[18]For a more detailed overview of stability, see appendix C.3.

that an equilibrium $x^*$ is stable if

$$\left| \frac{du_i}{dx_i} \right|_{x^*} < \frac{1}{2\Psi_i(x^*)(1 - \Psi_i(x^*))} \; \forall \; i \tag{5}$$

Although the stability condition is not also sufficient, it provides a simple means by which the stability of any given equilibrium can be evaluated. From equations 4 and 5, if an equilibrium is unique, it is also stable.

## 2.2 Producers

Production of good $i$ involves a fixed cost, a cost associated with the level of product quality, a constant marginal cost, and a compatibility cost associated with making a product compatible with other competing products. Compatibility is not assumed to be an equivalence relation; that is, if product $i$ is compatible with product $j$, it does not imply that product $j$ is equally compatible with product $i$. In this sense, a compatibility decision can involve construction of either a one-way or two-way adapter or something in between.

Producers are oligopolists; however, their compatibility decisions enable them to draw on the size of the consumer base of other producers' products.

More formally,

$$\max_{p_i,\phi_i} \pi_i(p_i, \phi_i) = (p_i - b_i)y_i(p, \phi) - a(q_i) - \sum_{j \neq i} c_{i,j}\phi_{i,j} \tag{6}$$

where $\pi_i$ is the profit of producer $i$, $y_i$ is the production of good $i$, $b_i$ is the marginal cost of producing good $i$, $a$ is a strictly convex, increasing function representing the cost of producing quality $q_i$, $c_{i,j}$ is the cost of making product $i$ compatible with product $j$, $i \neq j$, and $\phi_{i,j}$ represents the level of spending on compatibility.

The level of spending on compatibility impacts compatibility through the continuous function $\rho$ in which $\rho_{i,j}$ denotes $\rho(\phi_{i,j}, \phi_{j,i})$.[19] It is assumed that $\rho$ is strictly increasing in $\phi_{i,j}$ and nondecreasing in $\phi_{j,i}$, concave in its arguments, and that $\rho(0,0) = 0$ and $\lim_{\phi_{i,j}\to\infty} \frac{\partial \rho_{i,j}}{\partial \phi_{i,j}} = \lim_{\phi_{j,i}\to\infty} \frac{\partial \rho_{i,j}}{\partial \phi_{j,i}} = 0$. Thus, with no spending on compatibility, products are fully incompatible, and producers experience diminishing marginal compatibility.

While a number of authors have explored the use of both one-way[20] and two-way adapters,[21] the specification of $\rho$ in this exposition assumes that each producer can to some extent control the degree of compatibility of their own product with respect to other products; however, each producer's compatibility decision may impact the relative compatibility of other products. In other words, the functional form of $\rho$ allows for both one-way and two-way adapters and two-way adapters do not necessarily impart equivalent compatibility both ways.

In fact two-way adapters are quite possible and need not arise from mutual

[19]See section 2.1 for a description of $\rho$ as it factors into consumer utility.

[20]A one way adapter only provides compatibility in one direction; for example, many popular word processors can save documents in PDF, Adobe's ubiquitous document format, but not open them.

[21]See, for example, [8].

benefit to compatibility by either producer (as would arise when the industry is not concentrated into a single dominant firm and a competitive fringe). Firms may also use side payments or strategic agreements to enhance compatibility, a strategy that is common in practice,[22] though that possibility is not explicitly considered in the following exposition.

When two-way adapters are not considered, analysis of a producer's decision-making process is somewhat simpler as the introduction of compatibility is of benefit solely to the producer that incurs the costs and to the detriment of all of that producer's competitors. When two-way adapters are allowed, the constraints on the derivatives of $\rho$ imply that a producer $i$'s decision to increase to the compatibility of its product with product $j$ may result in the increase in the effective network size or marginal contribution to network size of compatibility of producer $i$'s competitors.

[22]Microsoft, for example, has a long-standing agreement to provide AOL with a desktop icon in exchange for using Internet Explorer as the built-in browser for the AOL client [13].

## 2.3 Equilibrium

Equilibrium results from a simultaneous move Bertrand-Nash game. Producers and consumers form expectations regarding consumers' choices with complete information about the consumers' response functions.[23] Producers simultaneously choose price, product quality, and compatibility to maximize profit with complete information about the consumers' response functions. Consumers simultaneously maximize utility by choosing consumption taking prices, product quality, and compatibility as given. In equilibrium, both producers' and consumers' expectations of network size are realized; that is, expectations are rational.

The question of how consumers and producers form their expectations and why there may be a focus on one equilibrium over any other may be based on the problem under consideration. When small exogenous shocks are considered, market players may expect that the resultant equilibrium following an exogenous shock will occur an equilibrium connected by a continuous

[23] That is, producers and consumers form expectations regarding the size of product networks in response to price and compatibility choices.

"path" of stable equilibria to the current equilibria.[24] Alternatively, perhaps consumers expect change will be minimal or the equilibrium that leaves the most dominant players in the most dominant positions may be selected.[25] Another reasonable assertion would be that the equilibrium expectation is formed in a Stackleberg manner by the dominant producer. Whatever the coordination process, it is reasonable to assume that the set of admissible equilibria are stable as defined by a linearization about the equilibrium point of the tatônnement process outlined in appendix C.

First-order conditions for profit maximization in prices are given by

$$x_i + (p_i - b_i)\frac{dx_i}{dp_i} \leq 0 \;\perp p_i \;\geq 0 \tag{7}$$

where, from equation 3, firms face an own-price demand derivative of

[24]If one exists, that is.
[25]Again, if one exists.

$$\frac{dx_i}{dp_i} = -\gamma N \mathcal{P}(i)(1 - \mathcal{P}(i))[1 + \varepsilon_{p_i}] \tag{8}$$

$\varepsilon_{p_i}$ reflects the first-order impact of price changes on network size[26] and is given by

$$\varepsilon_{p_i} = e_i \left( \sum_{n=1}^{\infty} \mathcal{J}^n \right) \mathcal{D}_p e_i^T \tag{9}$$

where

$$\mathcal{J} = \left( \frac{N-1}{N} \right) \left[ \sum_{i=1}^{I} \frac{\partial \Psi_m}{\partial v_i} \frac{\partial v_i}{\partial x_n} \right] \tag{10}$$

and

[26] When consumers face a choice set without network externalities, the own-price demand elasticity of product $i$ is given by $\eta_{p_i} = \gamma p_i (1 - \mathcal{P}(i))$.

$$\mathcal{D}_p = \left[ \frac{\partial \Psi_m}{\partial p_n} \right] \tag{11}$$

From equation 8, the own-price elasticity of demand is given by[27]

$$\eta_{p_i} = \gamma p_i (1 - \mathcal{P}(i))[1 + \varepsilon_{p_i}] \tag{12}$$

In general, positive network externalities exacerbate consumers' price responses, often quite dramatically. With preferences convex in network size, producers anticipate that consumers are more responsive to changing prices or product attributes than they would otherwise be when considering products that do not exhibit network externalities.

In a regime with no product compatibility, the responsiveness is greatest as is the cost in terms of network benefits of switching from a dominant to a fringe product. With full compatibility, as the number of consumers becomes

[27] See appendix B for details regarding the derivation of demand elasticities.

large, the price elasticity approaches that found in a market without network externalities as any loss in network size is made up by a corresponding gain in the network size of compatible products, resulting in no net impact to the effective network size.[28] More generally, with full compatibility, as the number of consumers becomes large, the model converges to the traditional MNL formulation.

The firms face a cross-price elasticity of demand of

$$\eta_{p_{i,j}} = \gamma p_j \mathcal{P}(i)[1 + \varepsilon_{p_{i,j}}] \tag{13}$$

with

$$\varepsilon_{p_{i,j}} = e_i \left( \sum_{n=1}^{\infty} \mathcal{J}^n \right) \mathcal{D}_p e_j^T \tag{14}$$

[28]That is, the terms in $1 - \mathcal{P}(i) + \sum_{j \neq i} \rho_{i,j} \mathcal{P}(j)$ cancel.

where the term $\varepsilon_{p_{i,j}}$ reflects the first-order impact to demand for good $i$ of price changes to good $j$ on network size.[29]

As with the expression for own-price demand elasticity, the network term in equation 13 has the greatest impact in the absence of any product compatibility. Without product compatibility, consumers are wary of switching from a product with a strong network to one without. With full product compatibility, as in the case of own-price elasticity of demand, as the number of consumers becomes large, the cross-price elasticity approaches that of a market without network externalities.

The well-known IIA property of logit demand implies that [30] that the cross-price elasticity of demand is equal for any product $i$ with respect to a product $j$. However, with the introduction of network externalities, it is easy to see from equations 13 and 14 that in the absence of full compatibility, IIA does not necessarily hold in the presence of network externalities. As a result, the familiar behavior associated with the traditional logit model in the presence

[29] In the absence of network externalities, the cross-price elasticity of demand is given by $\eta_{p_{i,j}} = \gamma p_j \mathcal{P}(i)$.

[30] See [7, pp. 86–87] and [19, pp. 23–24 and 43–44].

of IIA, in which a marginal change in price causes consumers to substitute away from a product to other products in proportion to their existing market shares, no longer holds. In fact, as we shall see in section 3, a fringe producer who lowers its price may actually find consumers substituting away from the dominant producer toward its product and other fringe producers' products.

Similarly, first-order conditions for profit maximization with respect to product quality are given by

$$(p_i - b_i)\frac{dx_i}{dq_i} - \frac{da_i}{dq_i} \leq 0 \ \perp q_i \ \geq 0 \tag{15}$$

where, from equation 3, firms face an own-quality demand derivative of

$$\frac{dx_i}{dq_i} = N\mathcal{P}(i)(1 - \mathcal{P}(i))[1 + \varepsilon_{q_i}] \tag{16}$$

$\varepsilon_{q_i}$ reflects the first-order impact of quality changes on network size[31] and is given by

$$\varepsilon_{q_i} = e_i \left( \sum_{n=1}^{\infty} \mathcal{J}^n \right) \mathcal{D}_q e_i^T \tag{17}$$

where

$$\mathcal{D}_q = \left[ \frac{\partial \Psi_m}{\partial q_n} \right] \tag{18}$$

The cross-quality elasticity of demand is

$$\eta_{q_{i,j}} = q_j \mathcal{P}(i)[1 + \varepsilon_{q_{i,j}}] \tag{19}$$

[31] When consumers face a choice set without network externalities, the own-quality demand elasticity of product $i$ is given by $\eta_{q_i} = q_i(1 - \mathcal{P}(i))$.

with

$$\varepsilon_{q_{i,j}} = e_i \left( \sum_{n=1}^{\infty} \mathcal{J}^n \right) \mathcal{D}_q e_j^T \tag{20}$$

where the term $\varepsilon_{q_{i,j}}$ reflects the first-order impact to demand for good $i$ of price changes to good $j$ on network size.[32]

In addition to price and product quality, consumers respond to changes in network size. The first-order conditions for profit-maximization in compatibility for product $i$ with respect to product $j$, $j \neq i$, is given by

$$(p_i - b_i) \frac{dx_i}{d\phi_{i,j}} - c_{i,j} \leq 0 \;\perp\; \phi_{i,j} \geq 0 \tag{21}$$

The derivative of demand for good $i$ with respect to the compatibility of goods $i$ and $j$ is given by

[32] In the absence of network externalities, the cross-price elasticity of demand is given by $\eta_{q_{i,j}} = \gamma q_j \mathcal{P}(i)$.

$$\frac{dx_i}{d\phi_{i,j}} = N\mathcal{P}(i)\mathcal{P}(j)\left[\frac{\partial v_i}{\partial \rho_{i,j}}\frac{\partial \rho_{i,j}}{\partial \phi_{i,j}} + \mathcal{P}(i)\left(\frac{\partial v_j}{\partial \rho_{j,i}}\frac{\partial \rho_{j,i}}{\partial \phi_{i,j}} - \frac{\partial v_i}{\partial \rho_{i,j}}\frac{\partial \rho_{i,j}}{\partial \phi_{i,j}}\right) + \varepsilon_{\phi_{i,j}}\right] \tag{22}$$

From equation 22, the network elasticity of demand is given by[33]

$$\eta_{\phi_{i,j}} = \phi_{i,j}\mathcal{P}(j)\left[\frac{\partial v_i}{\partial \rho_{i,j}}\frac{\partial \rho_{i,j}}{\partial \phi_{i,j}} + \mathcal{P}(i)\left(\frac{\partial v_j}{\partial \rho_{j,i}}\frac{\partial \rho_{j,i}}{\partial \phi_{i,j}} - \frac{\partial v_i}{\partial \rho_{i,j}}\frac{\partial \rho_{i,j}}{\partial \phi_{i,j}}\right) + \varepsilon_{\phi_{i,j}}\right] \tag{23}$$

with

$$\varepsilon_{\phi_{i,j}} = e_i\left(\sum_{n=1}^{\infty}\mathcal{J}^n\right)\mathcal{D}_{\phi_i}e_j^T \tag{24}$$

reflecting the first-order impact of compatibility changes on network size

[33]See appendix B for a full derivation.

where

$$\mathcal{D}_{\phi_i} = \left[ \frac{\partial \Psi_m}{\partial v_i} \frac{\partial v_i}{\partial \rho_{i,n}} \frac{\partial \rho_{i,n}}{\partial \phi_{i,n}} + \frac{\partial \Psi_m}{\partial v_n} \frac{\partial v_n}{\partial \rho_{n,i}} \frac{\partial \rho_{n,i}}{\partial \phi_{i,n}} \right] \tag{25}$$

## 2.4 Calibration

The MNL is known as a "rough and ready" model [7] for the ease with which existing market data can be calibrated against the demand specification and counterfactuals introduced to analyze relevant policy decisions.

The Network MNL is no different in this regard, but involves additional steps to calibrate the scale of the network externalities or effects and incorporate costs of compatibility. While preferences are estimable by well-established econometric techniques and prices and market shares are typically readily observable, compatibility levels may not be. The means by which compatibility levels would be determined would likely be product-specific. For the purposes of this exposition, compatibility levels are assumed to be observable.

Given that the number of consumers, preferences, prices, product qualities, market shares, and compatibilities are observable,[34] and given a suitable functional form for the network term $v$, by equation 3, a system of simultaneous equations can be solved to yield the scale of the network externalities and from the scale parameterize the function $v$. Likewise, given a suitable functional form for compatibility $\rho$ and a set of compatibility levels, a system of equations can be solved to yield the compatibility activity of each producer $\phi$.[35]

Given that the number of consumers, preferences, prices, product qualities, market shares, and compatibilities are observable and given a suitable functional form for $v$, by expression 6, the implied marginal costs borne by a profit-maximizing producer are given by

$$b_i = p_i + x_i / \frac{dx_i}{dp_i}; \tag{26}$$

[34] Alternatively, elasticities may be used in place of market shares as a primitive of the model. Economists are more accustomed to working with elasticities and readily accessible econometric tools are available for their estimation.

[35] From a practical standpoint, if the matrix of compatibility levels is sparse, parameterization of the compatibility function $\rho$ may require some additional assumptions.

or, alternatively,

$$b_i = p_i(1 - \eta_i) \tag{27}$$

with $\frac{dx_i}{dp_i}$ given by equation 8 and $\eta_i$ given in equation 12. Typically, the dominant producer faces a higher elasticity of demand than the competitive fringe; however, due to the market concentration it enjoys, the calibration marginal costs reflects lower marginal costs (and higher margins and profits) for the dominant producer than for the fringe.

Quality costs can be recovered from the expression

$$(p_i - b_i)\frac{dx_i}{dq_i} - \frac{da}{dq_i} = 0 \tag{28}$$

with $\frac{dx_i}{dq_i}$ given by equation 16.

When the observed compatibility is positive, the implied marginal costs borne by producer $i$ relative to compatibility with product $j$ are given by

$$c_{i,j} = (p_i - b_i)\,\frac{dx_i}{d\phi_{i,j}} \tag{29}$$

with $\frac{dx_i}{d\phi_{i,j}}$ given by equation 22. It is not atypical that some products may be wholly incompatible, in which case a cost of compatibility must be extrapolated from reasonable assumptions and observed calibration costs with respect to similar sets of compatible products.

# 3 Results

It is easy to show that an equilibrium exists;[36] however, as is to be expected due to the increasing returns inherent in positive network externalities, multiple stable equilibria can arise.

[36] See appendix C.1.

When there are sufficiently strong, convex network externalities, the relative importance of the differentiating features of a product are subsumed by the need to standardize on one of the available choices. The user of a word processor may not care so much that an embedded spreadsheet is dynamically updated as they do that others can read their documents. Furthermore, contrary to traditional models of consumer demand, the dominant good may not even be the preferred good. In fact, in the presence of sufficiently strong network externalities, consumers can rationally choose to standardize on any of the available goods[37] and an equilibrium may exist where the features of the fringe producers' goods may be strongly preferred to the dominant good.[38]

With symmetric preferences convex in network size, the most preferred equilibria arise concentrating demand on any of the $I$ goods. A single least preferred equilibrium arises when demand is concentrated equally on all $I$ goods. Between these stable equilibria, there may exist saddle points in which several producers form a dominant set from which a small perturbation will

[37]Though due to the distribution of the random utility component $\epsilon_k$ there will always be a nonzero probability of choosing any given good.

[38]Much to the chagrin of the fringe producers, no doubt.

push the market toward a regime with a single dominant producer. Overall, consumers prefer equilibria that represent a more concentrated market.

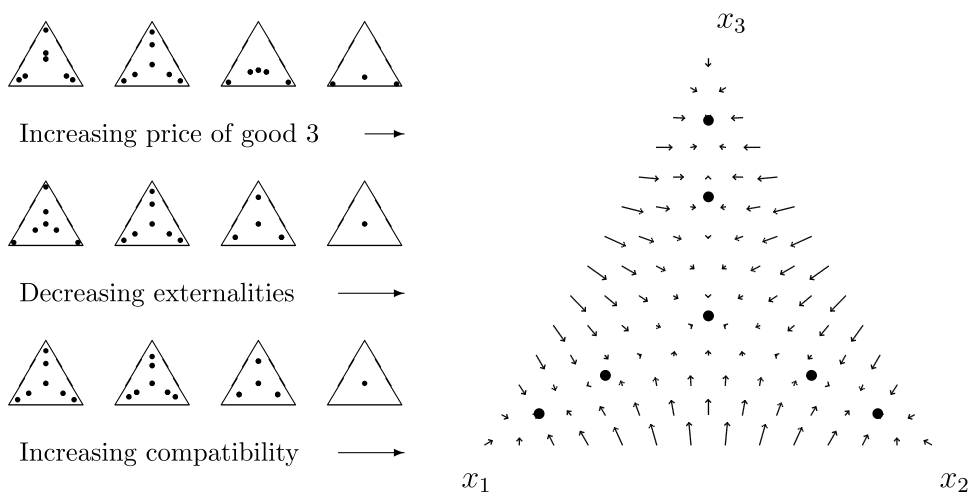


Figure 1: Demand equilibria and consumer tatônnement over the simplex

In figure 1, equilibria are plotted on a probability simplex representing a system with one consumer type and three producers. Each of the three vertices represents the consumers' choice probability with respect to the associated good. When network externalities are most pronounced, the stable equilibria involve a single firm capturing over 99% of the market, represented by three points almost touching each of the three vertices. As the strength of network externalities decrease and consumers progressively favor features over standardization, market dominance becomes less exaggerated and the extreme equilibria draw progressively farther away from the vertices and toward the center of the simplex. Likewise, as product compatibility increases, relative differences in value of network sizes converges for the products. In the extreme, with all products perfectly compatible, a unique equilibrium is found equivalent to the equilibrium that would arise in an equivalent market without network externalities. Without network externalities,[39] a single symmetric equilibria exists represented by a point in the center of the simplex in which each firm captures an equal portion of the market.

When asymmetries are introduced, the market dominance of the preferred good in the presence of network externalities becomes no less extreme. To the contrary, even with relatively mild network externalities, the logit demand formulation can admit fairly extreme dominant equilibria with the dominant firm commanding a substantial portion of the market. This is neither unreasonable nor out of line with anecdotal observed characteristics of mature markets with strong network externalities.

The dominant producer of a product exhibiting strong network externali-

[39]Or, alternatively, in the limit as the strength of network externalities decreases or compatibility increases.

ties is a very satisfied (and profitable) producer. Much more so than in a comparable industry that does not exhibit network externalities. The fringe producers, on the other hand, face a much harsher business climate than they would in an industry without network externalities.

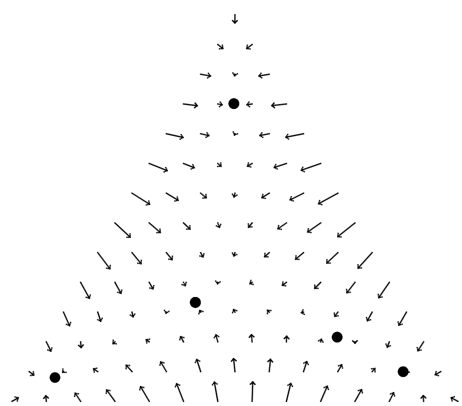

Figure 2: Consumer tatônnement with asymmetric preferences

In addition to multiple equilibria, the demand specification incorporating network externalities gives rise to discontinuities in stable demand equilibria. Consumers value the features of a product, the price of the product, and the network size associated with the product and compatible products. With strong network externalities, a dominant producer can charge a price significantly higher or offer a product with features significantly less desirable than competing producers and retain a dominant market position. However, as can be seen in figure 1, as differences in prices or feature sets become increasingly disparate, equilibria dominant in the least-preferred goods are no longer supported as the benefits of a dominant network are no longer able to overcome the disparities in prices or features.

While positive network externalities can exacerbate consumer price responses, convexities in the network externality function can fundamentally alter the character of the responses as well. As table 1 shows, in the absence of network externalities, cross-price elasticities in the traditional MNL reflect the IIA property in which a drop in price by a producer results in a gain in market share from competitors in proportion to their relative market shares.

Also shown in table 1, with producer 1 the dominant producer and producers 2 and 3 forming a competitive fringe, with convex positive network externalities, a drop in price by a fringe producer can see a gain in market share by other members of the competitive fringe at the expense of the dominant producer. When a fringe producer steals market share from a dominant producer, thereby eroding the value of a very attractive network, other fringe producers' networks become relatively more valuable as a result. This dynamic may give rise to counterintuitive strategic behavior in which fringe producers collude to drop prices below the individual profit-maximizing levels in the presence of a strongly dominant competitor or a dominant competitor seeks control of a fringe producer to raise the fringe producer's price to the detriment of other fringe producers.

| $\eta_{i,j}$ | $j = 1$ | $j = 2$ | $j = 3$ | $\eta_{i,j}$ | $j = 1$ | $j = 2$ | $j = 3$ |
|---|---|---|---|---|---|---|---|
| $i = 1$ | $-1.98$ | 0.99 | 0.99 | $i = 1$ | $-1.17$ | 0.58 | 0.58 |
| $i = 2$ | 31.84 | $-25.99$ | $-5.86$ | $i = 2$ | 18.83 | $-19.42$ | 0.58 |
| $i = 3$ | 31.84 | $-5.86$ | $-25.99$ | $i = 3$ | 18.83 | 0.58 | $-19.42$ |

Table 1: Price elasticities with and without network externalities

Equivalently, one fringe producer may even be incented to subsidize a competitor's product quality. While this may at first glance seem somewhat far-fetched, the recent phenomenon of "open source" software, in which the licensing terms of the software ensure that any improvements made by one vendor are shared with others, may in part reflect such complementarities. IBM was reported to have spent $1 billion in 2002 on Linux, subsidizing development of the Linux operating systems and many related "open source" applications. According to the terms of the Linux licensing arrangement, commonly known as the GPL (Gnu Public License), any improvements to Linux or derivative works must be returned to the Linux development team and freely licensed to any third party.

# 4 Discussion

Clearly, if we are to believe that market behavior in the presence of network externalities are sufficiently similar to the Network MNL, a market in which consumers explicitly value network size behaves quite differently from one which does not. Furthermore, common inferences about preferences no longer apply in the presence of network externalities. The dominant good may be a good of inferior quality. And while each producer will compete for the dominant market position, only one will be the dominant producer and it will likely be strongly dominant.

Particularly intriguing is the manner in which strong, convex network externalities can change the strategic landscape. Fringe producers may find it optimal to collude to reduce prices or even subsidize competitors' quality enhancements.

Since its introduction, the traditional MNL has been extended in a variety of ways. The core Network MNL formulation would benefit from incorporation of many of these extensions.

Product nesting [19, section 2.7.1] provides modelers with the ability to accommodate a more diverse prior with regards to elasticities as well as formally model sequential decision-making processes. The Network MNL would benefit from similar extension, allowing complex behaviors, such as the choice of computer operating system in which consumers first choose the set of applications they would like to use then choose the operating system which supports those applications, to be effectively modeled. Bearing in mind the caveat of Chou and Shy [4], nesting would also allow modelers to separate network effects and externalities from complementarities with such products masquerading as network effects and externalities.

In addition to product nesting, incorporation of an outside good representing "none of the above" and multiple consumer types have seen wide application. Multiple consumer types would allow for a richer modeling framework in which, for example, the preferences of tech-savvy "first adopters" of technology products could be separated from more casual users or business use could be separated from recreational use of information goods at home.

The introduction of an intertemporal framework would allow a rich set of strategic behaviors to be considered. With a fixed cost of production, entry and exit decisions can be effectively modeled. And the implications of changes to cross-price and cross-quality derivatives detailed in section 3 would give rise to complex strategic interactions.

Finally, the model would benefit from the introduction of uncertainty, particularly as it may impact decision making surrounding regime changes or other discontinuities in equilibria sets.

The traditional MNL demand formulation has seen wide use in theoretical analysis of differentiated products industries, in simulation analysis of the impact of mergers and acquisitions, including in support of antitrust litigation, and in marketing. The Network MNL should be of interest in many of these settings, particularly when information goods or technology products that naturally exhibit network externalities and effects are being considered.

# A Calculation of Choice Probabilities

Suppose $N$ consumers derive utility over $I$ products of the form

$$u_{i,n} = y_n + q_i + v_{i,n} + \sigma \epsilon_{i,n} \tag{30}$$

where $y_n$ is consumer income, $q_i$ is the intrinsic quality of product $i$, $v_{i,n}$ is the utility consumer $n$ derives from others' consumption of product $i$ and compatible products, $\epsilon_{i,n}$ is a random variable representing consumer $n$'s idiosyncratic valuation of product $i$, and $\sigma$ is a scaling parameter determining the degree of heterogeneity across products.

For the utility of good $i$ to exceed that of good $j$ for consumer $n$, we need

$$q_i + v_{i,n} + \sigma \epsilon_{i,n} > q_j + v_{j,n} + \sigma \epsilon_{j,n} \tag{31}$$

In other words, we need

$$P(\sigma\epsilon_{i,n} = \bar{\epsilon})P(\bar{\epsilon} \geq \sigma \sum_{j \neq i} (q_j - q_i + v_{j,n} - v_{i,n} + \epsilon_{j,n})) \tag{32}$$

With $\epsilon$ distributed type 1 extreme,

$$P_n(i) = \int_{-\infty}^{\infty} \frac{1}{\sigma} e^{\frac{\bar{\epsilon}}{\sigma}} e^{-e^{-(\frac{\bar{\epsilon}}{\sigma})}} \prod_{j \neq i} e^{-e^{-\left(\frac{q_j - q_i + v_{j,n} - v_{i,n} + \bar{\epsilon}}{\sigma}\right)}} d\bar{\epsilon} \tag{33}$$

Using the change of variable $\phi = e^{-\left(\frac{\bar{\epsilon}}{\sigma}\right)}$ and letting $g(\phi) = e^{-\phi}$, $dg = -\frac{1}{\sigma} e^{-\phi} d\bar{\epsilon}$, $\phi|_{\bar{\epsilon}=\infty} = 0$, $\phi|_{\bar{\epsilon}=-\infty} = \infty$,

$$
\begin{aligned}
P_n(i) &= \int_{-\infty}^{\infty} \frac{1}{\sigma} e^{-\left(\frac{\bar{\epsilon}}{\sigma}\right)} e^{-e^{-\left(\frac{\bar{\epsilon}}{\sigma}\right)}} \prod_{j\neq i} e^{-e^{-\left(\frac{\bar{\epsilon}}{\sigma}\right)} e^{-\left(\frac{q_j - q_i + v_{j,n} - v_{i,n}}{\sigma}\right)}} d\bar{\epsilon} \\
&= -\int_0^{\infty} g(\phi) \prod_{j\neq i} e^{-\phi e^{-\left(\frac{q_j - q_i + v_{j,n} - v_{i,n}}{\sigma}\right)}} dg \\
&= -\int_0^{\infty} g(\phi) e^{-\phi e^{-\left(\frac{q_i + v_{i,n}}{\sum_{j\neq i} q_j + v_{j,n}}\right)}} dg \\
&= -\int_0^{\infty} e^{-\phi\left(1 + \frac{\sum_{j\neq i} q_{j,n} + v_{j,n}}{q_{i,n} + v_{i,n}}\right)} dg \\
&= -\int_0^{\infty} e^{-\phi\left(\frac{\sum_j q_{j,n} + v_{j,n}}{q_{i,n} + v_{i,n}}\right)} dg \\
&= -\frac{e^{q_{i,n} + v_{i,n}}}{\sum_j e^{q_{j,n} + v_{j,n}}} \left[ e^{-\phi\left(\frac{\sum_j e^{q_{j,n} + v_{j,n}}}{q_{i,n} + v_{i,n}}\right)} \right]_0^{\infty} \\
&= -\frac{e^{q_{i,n} + v_{i,n}}}{\sum_j e^{q_{j,n} + v_{j,n}}} [0 - 1] \\
&= \frac{e^{q_{i,n} + v_{i,n}}}{\sum_j e^{q_{j,n} + v_{j,n}}}
\end{aligned}
$$

The equivalent derivation in the presence the symmetry assumptions of section 2.1 can be performed by dropping the subscript $n$.

# B Calculation of Demand Elasticities

Suppose utility for consumer $n \in \{1, \ldots, N\}$ for product $i \in \{1, \ldots, I\}$ based on intrinsic quality $q$, prices $p$, and compatibilities $\rho$ is given by

$$u_{i,k} = q_i - \gamma p_i + v(z_{i,n}) + \sigma \epsilon_n \tag{34}$$

where $\epsilon_n$ is a random variable distributed type 1 extreme and $z_{i,n}$ is the network size of product $i$ from the perspective of consumer $n$.

For example, if $z_{i,n}$ is taken to be a function of absolute network size,[40] it would be given by

$$z_{i,n} = \sum_{m \neq n} (x_{i,m} + \sum_{j \neq i} \rho_{i,j} x_{j,m}) \tag{35}$$

[40] That is, it is the sum of the expected consumption of product $i$ and all compatible products by all other consumers based on the degree of compatibility between product $i$ and other products

On the other hand, if consumers perceptions are in terms of relative network size, or effective market share, $z_{i,n}$ would be given by

$$z_{i,n} = \frac{\sum_{m \neq n}(x_{i,m} + \sum_{j \neq i} \rho_{i,j} x_{j,m})}{\sum_{m \neq n} x_{i,m}} \tag{36}$$

Assuming symmetric choice probabilities, a closed-form expression for the probability that any given consumer will choose product $i$ is given by

$$\mathcal{P}_i = \Psi_i(x) = \frac{e^{q_i - \gamma p_i + v(z_i)}}{\sum_{j=1}^{I} e^{q_j - \gamma p_j + v(z_j)}} \tag{37}$$

and demand for product $i$ is given by $x_i = N\mathcal{P}_i$.

From equation 37, the positive network externalities intrinsic to demand produce positive feedback effects in which an external shock (e.g., a change in price) affects demand both through its direct effect on demand and its indirect effect on demand through the network term.

The indirect effect on demand can be characterized by the Jacobian

$$\mathcal{J} = \left[ \sum_{k=1}^{I} \frac{\partial \Psi_i}{\partial v_k} \frac{\partial v_k}{\partial x_j} \right] \tag{38}$$

where $v_k$ denotes the network term associated with product $k$.

From 37, the first term of the Jacobian can be decomposed as

$$\frac{\partial \Psi_i}{\partial v_k} = \begin{cases} \quad \mathcal{P}_i \left(1 - \mathcal{P}_i\right) & \text{when } i = k \\ - \quad \mathcal{P}_i \mathcal{P}_k & \text{otherwise} \end{cases} \tag{39}$$

When absolute network size is assumed,[41] the second term in the Jacobian is given by

[41] That is, that network size is given by equation 35.

$$\frac{\partial v_k}{\partial x_j} = \begin{cases} \frac{N-1}{N}\frac{\partial v_k}{\partial z_j} & \text{when } j = k \\ \rho_{k,j} \quad \frac{N-1}{N}\frac{\partial v_k}{\partial z_j} & \text{otherwise} \end{cases} \tag{40}$$

where the term $(N-1)/N$ arises because consumers only derive value from the use of the same or compatible products by others.

On the other hand, when relative network size is assumed,[42] under the symmetry assumptions the term $\frac{N-1}{N}$ drops out. If the effective market share of product $j$ is given by

$$\mathcal{S}_j = \frac{x_j + \sum_{k\neq j} \rho_{j,k} x_k}{\sum_{k=1}^{I} x_k} \tag{41}$$

then $\frac{\partial v_k}{\partial x_j}$ is given by

[42] That is, that network size is given by equation 36.

$$\frac{\partial v_k}{\partial x_j} = \begin{cases} \frac{1}{\sum_{i=1}^{I} x_i} & \frac{\partial v_k}{\partial z_j} \quad \text{when } j = k \\ \frac{\rho_{k,j} - \mathcal{S}_j}{\sum_{i=1}^{I} x_i} & \frac{\partial v_k}{\partial z_j} \quad \text{otherwise} \end{cases} \tag{42}$$

The direct effect on demand $x_i$ from a change in price $p_j$ is given by the matrix

$$\mathcal{D}_p = \left[ \frac{\partial x_m}{\partial p_n} \right] \tag{43}$$

From equation 37, the terms of $\mathcal{D}_p$ can be decomposed as

$$\frac{\partial x_i}{\partial p_j} = \begin{cases} - & \gamma N \mathcal{P}_i \left(1 - \mathcal{P}_i\right) \quad \text{when } i = j \\ & \gamma N \mathcal{P}_i \mathcal{P}_j \quad \text{otherwise} \end{cases} \tag{44}$$

From the effects 38 and 44, the total derivative of demand with respect to price is given by[43]

[43]The discerning reader will recognize $(I - \mathcal{J})$ as a Markov matrix.

$$\frac{dx_i}{dp_j} = e_i \left(\mathcal{J}^0 + \mathcal{J}^1 + \ldots\right) \mathcal{D}_p e_j^T \tag{45}$$

$$= e_i \left(I - \mathcal{J}\right)^{-1} \mathcal{D}_p e_j^T \tag{46}$$

where $e_i$ is the row vector $[0 \; \ldots \; 0 \; 1 \; 0 \; \ldots \; 0\,]$ in which the unit value is in the $i$th column. For the matrix $(I - \mathcal{J})$ to be nonsingular, it must be the case that the largest eigenvalue of the Jacobian $\mathcal{J}$ is not equal to one; certainly, this is the case at any stable equilibrium.[44]

The direct effect on demand $x_i$ from a change in quality $q_j$ given by the matrix

$$\mathcal{D}_q = \left[\frac{\partial x_m}{\partial q_n}\right] \tag{47}$$

[44] The requirement for a nonsingular matrix $(I - \mathcal{J})$ can pose problems for a solver when the search path varies sufficiently from the benchmark calibration point (as would occur, for example, in simulating a sufficiently large exogenous shock). In practice, an alternative (but less intuitive) formulation can be used to mitigate problems surrounding singularities. Consult the author for more information.

From equation 37, the terms of $\mathcal{D}_q$ can be decomposed as

$$\frac{\partial x_i}{\partial q_j} = \begin{cases} -\quad N\mathcal{P}_i\left(1-\mathcal{P}_i\right) & \text{when } i=j \\ \quad N\mathcal{P}_i\mathcal{P}_j & \text{otherwise} \end{cases} \tag{48}$$

From the effects 38 and 48, the total derivative of demand with respect to quality is given by

$$\frac{dx_i}{dq_j} = e_i\left(\mathcal{J}^0+\mathcal{J}^1+\ldots\right)\mathcal{D}_q e_j^T \tag{49}$$

$$= e_i\left(I-\mathcal{J}\right)^{-1}\mathcal{D}_q e_j^T \tag{50}$$

Similarly, the derivative of demand with respect to compatibility involves both the direct impact of the compatibility on network sizes and the indirect effect as the change to network sizes propagate through the network term. The direct effect can be characterized by the matrix

$$\mathcal{D}_{\phi_k} = \left[ \frac{\partial x_i}{\partial v_k} \frac{\partial v_k}{\partial z_k} \frac{\partial z_k}{\partial \rho_{k,j}} \frac{\partial \rho_{k,j}}{\partial \phi_{k,j}} + \frac{\partial x_i}{\partial v_j} \frac{\partial v_j}{\partial z_j} \frac{\partial z_j}{\partial \rho_{j,k}} \frac{\partial \rho_{j,k}}{\partial \phi_{k,j}} \right] \tag{51}$$

where the second term characterizes the complementary nature of compatibility when two-way adapters can be constructed.[45] Note that since products are considered to be fully compatible with themselves,[46] $\frac{\partial \rho_{i,n}}{\partial \phi_{i,n}} = \frac{\partial \rho_{n,i}}{\partial \phi_{i,n}} = 0$ when $m = n$ and $\mathcal{D}_{\phi_i}$ has zeros on the diagonal. The expression for $\frac{\partial x_m}{\partial v_i}$ is given by 39 and assuming that network size is given by 35, $\frac{\partial z_i}{\partial \rho_{i,n}} = x_n$.

From expressions 38 and 51, the derivative of demand with respect to compatibility is given by

$$\frac{dx_i}{d\phi_{i,j}} = e_i \left( \mathcal{J}^0 + \mathcal{J}^1 + \ldots \right) \mathcal{D}_{\phi_i} e_j^T \tag{52}$$

$$= e_i \left( I - \mathcal{J} \right)^{-1} \mathcal{D}_{\phi_i} e_j^T \tag{53}$$

[45]See section 2.2 for a discussion of one-way versus two-way adapters.
[46]That is, $\rho_{i,i} \equiv 1 \; \forall \; i \in \{1, \ldots, I\}$.

for $i \neq j$.

It is worth noting that in the limit as the number of consumers becomes large, with full compatibility, the elasticity approaches that found in a market without network externalities. Intuitively, any loss in network size is made up by a corresponding gain in the network size of compatible products, resulting in no net impact to the effective network size.[47] Whether relative or absolute network size is considered, with full compatibility, as the number of consumers becomes large the model converges to the traditional MNL formulation.

# C Existence, Uniqueness, and Stability

Miyao and Shapiro [14] establish existence, uniqueness, and stability for the general case of models of discrete choice. Their results are extended here to incorporate the Network MNL framework.

[47] More precisely, in equation 38, $\partial v_k / \partial x_j = 0$ for all $j, k$.

Given a set of prices $p_i$ and product qualities $q_i$, we can consider consumer $n$'s utility to be a function of consumer $n$'s perceived network size of product $i$

$$u_{i,n} = q_i - \gamma p_i + v(z_{i,n}) + \sigma\epsilon_n \tag{54}$$

where the network size of product $i$ from the perspective of consumer $n$, $z_{i,n}$, is the consumer $n$'s perception of the network size of product $i$; for example, if $z_{i,n}$ represents the sum of expected consumption of product $i$ and all compatible products by all other consumers based on the degree of compatibility $\rho_{i,j}$ between products $i$ and $j$, $i \neq j$, $z_{i,n}$ would be given by

$$z_{i,n} = \sum_{m \neq n} (x_{i,m} + \sum_{j \neq i} \rho_{i,j} x_{j,m}) \tag{55}$$

where $x_{i,n}$ is the consumption of product $i$ by consumer $n$.

In considering questions of existence, uniqueness and stability, the nature of equilibrium can be thought of in terms of a dynamic adjustment process. Taking other consumers' choices from time $t$ as given and denoting the vector of consumption of product $i$ by $x^t$, each consumer's choice probability of selecting product $i$ at time $t+1$ is given by

$$\Psi_{i,n}^{t+1}(x^t) = P(u_{i,n}^{t+1}(x^t) = \max_{j=1,\ldots,I} u_{j,n}^{t+1}(x^t)) \tag{56}$$

and consumption of product $i$ at time $t+1$ is given by $x_{i,n}^{t+1} = N\Psi_{i,n}(x^t)$. In other words, in each time period, consumers simultaneously make a decision regarding choice probabilities based on the choice probabilities of their peers from the previous period.

Equilibrium is defined to be an allocation $x^* = [x_{1,1}^*, \ldots, x_{I,N}^*]'$ in which the expected number of consumers who select choice $i$ is given by $x_i^*$ where

$$x^*_{i,n} = \sum_{n=1}^{N} \Psi_{i,n}(x^*) \tag{57}$$

Choice probabilities are assumed to be symmetric; that is, $\Psi_{i,m} = \Psi_{i,n} \; \forall \; n, m \in \{1, \ldots, N\}, m \neq n$ which implies $z_{i,m} = z_{i,n}$.

This allows us to write $z_i$ in place of $z_{i,n}$, $\Psi_i$ in place of $\Psi_{i,n}$, and equilibrium to be redefined as an allocation $x^* = [x^*_1, \ldots, x^*_I]'$ in which the number of consumers who select choice $i$ is given by $x^*_i$ and $x^*_i = N\Psi_i(x^*)$.

A simple integration will show that a closed-form solution for $\Psi$ is given by[48]

$$\Psi_i^{t+1}(x^t) = \frac{e^{q_i - \gamma p_i + v(z_i^t)}}{\sum_{j=1}^{I} e^{q_j - \gamma p_j + v(z_j^t)}} \tag{58}$$

From this it is easy to show that at least one equilibrium exists.

[48]See appendix A

## C.1 Existence

**Proposition C.1 (Existence)** *An equilibrium exists.*

**Proof C.1** *From equation 58,* $0 < \Psi_i(x) < 1 \; \forall \; i$ *and* $\sum_i \Psi_i(x) = 1$*. And since* $v(z_i)$ *is continuous,* $\Psi_i(x)$ *is continuous for all* $i$*.*

*Then* $x_i = N\Psi_i(x)$ *is a function which maps the closed, convex ball* $\mathbb{B} = \{x \in \mathbb{R}^I : 0 \leq x_1, \ldots, x_I \leq N\}$ *onto itself and by Brouwer's fixed point theorem, a fixed point* $x$ *exists.* $\square$

While proposition C.1 guarantees the existence of at least one equilibrium, a natural follow-up to the question of existence is that of the nature of equilibria.

## C.2 Uniqueness

By virtue of the increasing returns nature of positive network externalities, multiple equilibria are to be expected and do, in fact, commonly arise. Conditions for uniqueness of equilibria can be established based on the strength of the network externalities.

On the basis of the tatônnement process described in appendix C.1, a sufficient condition for uniqueness can be established based on the strength of the network externalities.

**Proposition C.2 (Uniqueness)** *An equilibrium $x^*$ is unique when, for all possible $x$,*

$$\frac{du_i}{dx_i} < \frac{1}{2\Psi_i(1-\Psi_i)} \;\forall\, i \tag{59}$$

*where $\Psi_i$ is the probability of any consumer $k \in \{1, \ldots, N\}$ purchasing prod-*

*uct $i$ at allocation $x$.*

**Proof C.2** *Define the mapping $\Omega$ as*

$$\Omega_i(x^t) = x_i^t - x_i^{t+1} \tag{60}$$

*With $\Psi$ and $u$ differentiable, $\Omega$ is a function $\Omega : \mathbb{B}^I \rightarrow \mathbb{R}^I$ which maps the surface $\mathbb{B} = \{\omega \in \mathbb{R}^{I+} : \sum_{i=1}^{I} \omega_i = N\}$ onto $\mathbb{R}^I$.*

*The Jacobian of $\Omega$ is given by*

$$\mathcal{J} = N[\alpha_{i,j}] = N(I + A) \tag{61}$$

*where*

$$A = [a_{i,j}] = -\left[\frac{\partial \Psi_i}{\partial u_j}\frac{\partial u_j}{\partial x_j}\right] \tag{62}$$

*and, from equation 58,*

$$\frac{\partial \Psi_i}{\partial u_j} = \begin{cases} \quad \Psi_i(1-\Psi_i) & \textit{when } i = j \\ -\quad \Psi_i\Psi_j & \textit{otherwise} \end{cases} \tag{63}$$

*When network externalities are positive,* $\frac{\partial \Psi_i}{\partial u_i}\frac{\partial u_i}{\partial x_i} > 0$ *and* $\frac{\partial \Psi_i}{\partial u_i}\frac{\partial u_i}{\partial x_j} < 0$*. From assumption 59, we know that* $\alpha_{i,i} = 1 - \frac{\partial \Psi_i}{\partial u_i}\frac{\partial u_i}{\partial x_i} > \frac{\partial \Psi_i}{\partial u_i}\frac{\partial u_i}{\partial x_i}$*. And since* $\sum_i \Psi_i = 1$*, we know that* $\frac{\partial \Psi_i}{\partial u_i}\frac{\partial u_i}{\partial x_i} + \sum_{j\neq i}\frac{\partial \Psi_i}{\partial u_j}\frac{\partial u_j}{\partial x_j} = 0$*. Hence,* $\alpha_{i,i} > -\sum_{j\neq i}\frac{\partial \Psi_i}{\partial u_j}\frac{\partial u_j}{\partial x_j} = \sum_{j\neq i}\alpha_{i,j}$*.*

*Thus, because*[49] $\mathcal{J}$ *is a dominant diagonal matrix with positive diagonal elements,* $\Omega$ *is univalent in* $\mathbb{B}^I$ *and the equilibrium* $\Omega(x^*) = 0$ *is unique.* □

[49] *See [5, p. 84].*

When more than one equilibrium arises,[50] there tend to be a set of stable equilibria each with a dominant product facing a competitive fringe, an unstable equilibria in which no single firm plays the role of a dominant producer, and several saddle points dividing the stable and unstable manifolds. Sufficient conditions exist to identify an equilibrium as locally stable based on a simple condition placed on the magnitude of the network externalities and the degree of market concentration.

## C.3 Stability

Again, considering a dynamical system based on the tatônnement process described in section C.2, we can say that

**Proposition C.3 (Stability)** *Any equilibrium $x^*$ is stable when*

$$\left|\frac{du_i}{dx_i}\right|_{x^*} < \frac{1}{2\Psi_i^*(1-\Psi_i^*)}\ \forall\ i \tag{64}$$

[50] As shown in section 2.3.

*where $\Psi_i^*$ is the probability of any consumer $k \in \{1, \ldots, K\}$ purchasing product $i$ at an equilibrium allocation $x^*$.*

**Proof C.3** *The dynamic adjustment process can be linearized around $x^*$ with a matrix $\Omega$ such that $x^{t+1} \approx x^* + \Omega(x^t - x^*)$ where $\Omega = [\omega_{i,j}]$ and*

$$\omega_{i,j} = \left. \frac{\partial \Psi_i}{\partial u_i} \frac{\partial u_i}{\partial v} \frac{\partial v}{\partial x_j} \right|_{x^*} \tag{65}$$

*From [10][51], we know that magnitude of any eigenvalue $\lambda$ of $\Omega$ can be bounded by the following relationship: $|\lambda| \leq \max_i \sum_j |\omega_{i,j}|$. Since we know that $\sum_i \Psi_i = 1$, taking a derivative tells us that $\frac{d\Psi_i}{du_i} \frac{du_i}{dv} \frac{dv}{dx_i} = -\sum_{j \neq i} \frac{d\Psi_j}{du_j} \frac{du_j}{dv} \frac{dv}{dx_j}$. Thus we know that the equilibrium $x^*$ is stable if, at $x^*$,*

[51] *See theorem 3 p. 49.*

$$
\begin{aligned}
|\lambda| &\leq \max_i \sum_j |\omega_{i,j}| \\
&= \max_i \left| \frac{d\Psi_i}{du_i} \frac{du_i}{dv} \frac{dv}{dx_i} + \sum_{j \neq i} \frac{d\Psi_i}{du_i} \frac{du_i}{dv} \frac{dv}{dx_j} \right| \\
&\leq 2 \max_i \left| \frac{d\Psi_i}{du_i} \frac{du_i}{dv} \frac{dv}{dx_j} \right| \\
&< 1
\end{aligned} \tag{66}
$$

*which is true when* $\frac{du_i}{dv}\frac{dv}{dx_j} < \frac{1}{2\frac{d\Psi_i}{du_i}}\ \forall\ j$. *Since from equation 58,*

$$
\frac{\partial \Psi_i}{\partial u_j} = \begin{cases} \quad \Psi_i(1-\Psi_i) & \textit{when } i = j \\ -\quad \Psi_i \Psi_j & \textit{otherwise} \end{cases} \tag{67}
$$

*from equation 58, this condition can be restated as equation 64.* □

While there is no easy way to evaluate the stability of every equilibrium, this relationship provides a convenient testing procedure to determine whether any given equilibria can be shown to be locally stable without having to evaluate the eigenvalues of the Jacobian.

A full characterization of invariant sets is elusive in all but the simplest cases of dynamical systems; however, numerical methods exist to identify and characterize equilibria and their accompanying manifolds.

With a single consumer type and 3 producers, equilibria can be defined as fixed points on a 3-dimensional simplex of choice probabilities. The system dynamics of consumer behavior can be considered to be determined by the tatônnement process described above.

# D Local Uniqueness

Given that the increasing returns nature of the Network MNL gives rise to multiple equilibria, when considering an equilibrium set of prices, qualities, and product compatibilities, the modeler would undoubtedly find it comforting to know that equilibria are "locally isolated" or "locally unique" in the sense that there are no other arbitrarily close equilibria.

Formally,

**Definition D.1 (Locally Unique)** *Let $E(\omega)$ represent the set of equilibria demand vectors at a vector of demand parameters $\omega$. An equilibrium $\Sigma$ is locally unique if there exists neighborhoods $U$ of $\omega$ and $V$ of $\Sigma$ such that for all $\omega' \in U$, $|E(\omega') \cap V| = 1$ and the mapping $\sigma : U \to V$ defined by $\{\sigma(\omega')\} = E(\omega') \cap V$ is continuous.*

This appendix will introduce the concept of a regular equilibrium and show that equilibria of the Network MNL are regular equilibria, which will in turn establish local uniqueness.

**Definition D.2 (Regular Equilibrium)** *An equilibrium vector of prices $p$, product qualities $q$, and product compatibilities $\rho$ is a regular equilibrium if the matrices of price effects, quality effects, and compatibility effects given* *in equations 46, 50, and 53 are nonsingular.*

**Proposition D.1 (Regularity)** *Any equilibrium of the Network MNL at which price, quality, and compatibility derivatives are defined is also a regular equilibrium.*

**Proof D.1** *It is clear from equations 46, 50, and 53 that the Jacobians reflecting the indirect effect on demand as a result of changes to network size are invertible. Since the product of invertible matrices is itself invertible, it remains to show that the matrices of price, quality, and compatibility derivatives* $\mathcal{D}_p$*,* $\mathcal{D}_q$*, and* $\mathcal{D}_\phi$ *are invertible at any equilibrium.*

*The price and product quality derivatives* $\mathcal{D}_p$ *and* $\mathcal{D}_q$ *are simply matrices of derivatives as would represent a model without network externalities calibrated to equivalent prices and product qualities considering the network term to be an additional, exogenously specified product quality.*

*If either of these matrices were singular, price or quality changes could be made such that consumer demand could not be adjusted to bring the system* *back into equilibrium.*[52] *However, from [19],*[53] *we know that for any set of prices and product qualities a unique equilibrium exists. Furthermore, by [19] theorem 2.2, we know that demand is a continuous function of prices and product qualities. Thus, we know that Jacobians with respect to price and quality of the traditional multinomial logit model must be nonsingular.*

*Likewise, if the network term is considered in terms of an additional product quality term of the traditional multinomial logit demand system, the Jacobian of direct effect on demand of a change in product compatibility levels can be thought of as a linear combination of changes in the product quality term representing the network effects or externalities. By similar reasoning, the Jacobian of the direct demand effects of a change in product compatibility must be nonsingular as well.* □

Finally, by the definition of regular equilibrium, we can say

**Proposition D.2 (Local Uniqueness)** *Every regular equilibrium is locally*

[52] *A technical description of this intuition is beyond the scope of this appendix. The interested reader is referred to [15] for more detail.*

[53] *See section 7.10.1.*

*unique.*

**Proof D.2** *This follows as a direct consequence of the inverse function theorem.*[54]

which also tells us that any equilibrium of the Network MNL at which price, quality, and compatibility derivatives are defined is also locally unique.

## D.1 Dominance

As is common with dynamical systems with positive feedback, equilibria in the Network MNL are prone to discontinuities in the set of stable equilibria. For example, while positive network externalities may be strong enough to support a stable equilibria dominant in each producer's product at a given set of prices, if a producer raises its price sufficiently, the network externalities attributable to a dominant network are no longer sufficient to overcome the price disadvantage.

[54] *See [1] appendix M.E.*

The profits enjoyed by a dominant producer in a market with strong positive network effects or externalities can be considerably greater than those seen by the fringe producers.[55]

Given a stable equilibria $x^*$, a dominant producer (say, producer $i$) may be interested in whether a change in price (say, by $\delta$) would result in the loss of a stable, dominant equilibria in their good.

With the consumer response function $\Psi_i$ given by[56]

$$\Psi_i(x; q, p, \gamma, \phi) = \frac{e^{y+q_i-\gamma p_i+v_i(x)}}{\sum_{j=1}^{I} e^{y+q_j-\gamma p_j+v_j(x)}} \tag{68}$$

an equilibrium is defined as any $x^*$ such that $x^* - N\Psi(x^*; q, p, \gamma, \phi) = 0$. Define the difference between a consumption bundle $x$ and the consumer response as the function $\mathcal{F}$ such that $\mathcal{F}_i(x; q, p, \gamma, \phi) = x_i - N\Psi_i(x; q, p, \gamma, \phi)$.

[55] An order of magnitude difference was not uncommon in scenarios analyzed in section 3. For a real-world example, one need only compare the historical profits of Microsoft and Apple.

[56] See appendix A.

Then the producer is interested in whether there is a solution of

$$\mathcal{F}_i(x + \delta; q, p + \epsilon, \gamma, \phi) = 0 \tag{69}$$

for $\epsilon$ a vector with a positive number in row $i$ and zeros elsewhere.

This is the case when

$$\left[\frac{\partial \Psi}{\partial x}\right] \delta + \left[\frac{\partial \Psi}{\partial p}\right] \epsilon = 0 \tag{70}$$

which is true when

$$\delta = - \left[\frac{\partial \Psi}{\partial p}\right] \epsilon \left[\frac{\partial \Psi}{\partial x}\right]^{-1} \tag{71}$$

A solution does not exist when $\left[\frac{\partial \Psi}{\partial x}\right]$ is singular.

At the point of singularity, a bifurcation[57] will typically be observed.

# E Model Formulation

Equilibria were identified and characterized by a 3-stage numerical simulation in the presence of both symmetric and asymmetric preferences. In the first stage, consumer behavior was described as a discrete dynamical system on the basis of the consumer tatônnement process and box coverings of chain recurrent sets[58] were found using a multilevel subdivision technique via GAIO.[59] In the second, equilibrium conditions were described as an NLP and the box coverings were used to populate the set of initial conditions. The NLP was solved in GAMS[60] to identify the equilibria of the system. In the

[57] [11] pp. 59–69 contains a brief overview of bifurcations.

[58] The chain recurrent sets can be considered the invariant sets of the system, where the definition of chain recurrence is useful for numerical simulations. See [3] for more information.

[59] GAIO (Global Analysis of Invariant Objects), created by Michael Dellnitz and Oliver Junge, consists of a C library and Matlab or Python interfaces which can identify and characterize attributes of dynamical systems, such as equilibria, stable manifolds and other invariant sets. For more information on GAIO, see [2]

[60] GAMS (General Algebraic Modeling System) is a modeling system created by Alexander Meeraus which allows an optimization problem to be described algebraically and solved with any of a variety of solvers. For more information on GAMS, see [12].

third, the full model, with the producer optimization problem specified as an MCP and calibrated against benchmark prices and compatibilities.

## E.1 NLP

The NLP model was used to identify all of the equilibria associated with a particular parametrization of demand.

In practice, rather than performing a grid search of starting values to identify all possible equilibria, the set of initial values was determined through an external system call to Matlab which in turn solved for the set of equilibria through a set of calls to GAIO library functions. Matlab then exported the center points of box coverings of all chain recurrent sets in GAMS readable form as the set of initial values.

We assume $I = 3$, $N = 100$ and price and compatibility are held fixed at $p_i = 1 \; \forall \; i$ and $\phi_{i,ii} = 0 \; \forall \; i, ii$.[61] The set $ii$ and $iii$ alias the set $i \in \{1, 2, 3\}$.

[61] The model was originally formulated with more than one consumer type and is easily

From equation 3, we solve

$$\min_x \left\{ \sum_i \left( x_i - \frac{e^{q_i - \gamma p_i + (z_i/Z)^2}}{\sum_{ii} e^{q_{ii} - \gamma p_{ii} + (z_{ii}/Z)^2}} \right)^2 \right\} \tag{72}$$

where

$$z_i = (N-1)x_i + \sum_{ii \neq i} [(N-1)\phi_{i,ii} x_{ii}] \tag{73}$$

## E.2 MCP

We assume $I = 3$ and $N = 100$. [62] The set $ii$ and $iii$ alias the set $i \in \{1, 2, 3\}$.

Since we have a closed form for demand which does not support a corner solu-

generalizable. This would be useful, for example, when examining the market for operating systems in which one might want to consider both traditional desktop users and information technology users.

[62]The model was originally formulated with more than one consumer type and is easily generalizable. This would be useful, for example, when examining the market for operating systems in which one might want to consider both traditional desktop users and information technology users who may have differing intrinsic valuations but whose networks have a strong overlap.

tion, we do not need to specify utility maximization complementary slackness conditions and market clearing is defined as an equality relation taken from equation 3 with $x_i$ a positive variable.

The profit maximization conditions are given in sections E.2.1 and E.2.2. Since a closed-form solution for demand exists and consumers are constrained to purchase one and only one product, consumer first-order conditions and market clearing are implied by the demand equation given in section E.2.3. Finally, supplemental equations, necessary for the derivation of first-order conditions presented in sections E.2.1 through E.2.3, are presented in section E.2.4.

### E.2.1 Profit Maximization in Prices

$$x_i + (p_i - b_i)\frac{dx_i}{dp_i} \leq 0 \ \perp p_i \ \geq 0 \tag{74}$$

where

$$\frac{dx_i}{dp_{ii}} = \sum_{iii} a_{i,iii} \frac{\partial x_{iii}}{\partial p_{ii}} \tag{75}$$

#### E.2.2 Profit Maximization in Compatibility

$$N(p_i - b_i)\frac{dx_i}{d\phi_{i,ii}} - \frac{dc}{d\phi_{i,ii}} \leq 0 \ \perp \phi_{i,ii} \ \geq 0 \tag{76}$$

where

$$\frac{dx_i}{d\phi_{i,ii}} = \begin{cases} 0 & \text{when } i = ii \\ \sum_{iii} a_{i,iii} \frac{\partial x_{iii}}{\partial \phi_{i,ii}} & \text{otherwise} \end{cases} \tag{77}$$

#### E.2.3 Demand

$$x_i = \frac{e^{q_i - \gamma p_i + v_i}}{\sum_{ii} e^{q_{ii} - \gamma p_{ii} + v_{ii}}} \tag{78}$$

### E.2.4 Supplemental Equations

$$v_i = (z_i/Z)^2 \tag{79}$$

$$z_i = (N-1)(x_i + \sum_{ii \neq i} \rho_{i,ii} x_i) \tag{80}$$

$$\frac{\partial x_i}{\partial v_{ii}} = \begin{cases} \quad x_i(1-x_i) & \text{when } i = ii \\ - \quad x_i x_{ii} & \text{otherwise} \end{cases} \tag{81}$$

$$\frac{\partial x_i}{\partial p_{ii}} = \begin{cases} - \quad \gamma x_i(1-x_i) & \text{when } i = ii \\ \quad \gamma x_i x_{ii} & \text{otherwise} \end{cases} \tag{82}$$

$$\frac{dv_i}{dx_i} = (N-1)\frac{\partial v_i}{\partial z_i}\rho_{i,ii} \tag{83}$$

$$\rho_{i,ii} = \begin{cases} 1 & \text{when } i = ii \\ \phi_{i,ii}/(1+\phi_{i,ii}) & \text{otherwise} \end{cases} \tag{84}$$

$$\frac{\partial v_i}{\partial z_i} = 2z_i/Z^2 \tag{85}$$

$$\frac{\partial z_i}{\partial \rho_{i,ii}} = (N-1)x_i \tag{86}$$

$$\frac{\partial \rho_{i,ii}}{\partial \phi_{i,ii}} = \begin{cases} 0 & \text{when } i = ii \\ 1/(1+\phi_{i,ii})^2 & \text{otherwise} \end{cases} \tag{87}$$

$$\frac{\partial x_i}{\partial \phi_{ii,iii}} = \begin{cases} 0 & \text{when } ii = iii \\ \frac{\partial x_i}{\partial v_{ii}} \frac{\partial v_{ii}}{\partial z_{ii}} \frac{\partial z_{ii}}{\partial \rho_{ii,iii}} \frac{\partial \rho_{ii,iii}}{\partial \phi_{ii,iii}} & \text{otherwise} \end{cases} \tag{88}$$

Note that equation 88 reflects a one-way adapter. A two-way adapter would be reflected by an additional term resulting in entry $i, ii, iii$ being given by an expression such as $\frac{\partial x_i}{\partial v_{ii}} \frac{\partial v_{ii}}{\partial z_{ii}} \frac{\partial z_{ii}}{\partial \rho_{ii,iii}} \frac{\partial \rho_{ii,iii}}{\partial \phi_{ii,iii}} + \frac{\partial x_i}{\partial v_{iii}} \frac{\partial v_{iii}}{\partial z_{iii}} \frac{\partial z_{iii}}{\partial \rho_{iii,ii}} \frac{\partial \rho_{iii,ii}}{\partial \phi_{ii,iii}}$ where $ii \neq iii$.

$$\frac{dc}{d\phi_{i,ii}} = c_{i,ii} \tag{89}$$

$$J = \frac{N-1}{N} \left[ \sum_{iii} \frac{\partial x_i}{\partial v_{iii}} \frac{dv_{iii}}{dx_{ii}} \right] = [j_{i,ii}] \tag{90}$$

$$A = (I - J)^{-1} = [a_{i,ii}] \tag{91}$$

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