DISCUSSION PAPERS IN ECONOMICS

Working Paper No. 00-08

Capitalization and Community Income Distributions

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2000

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CAPITALIZATION AND COMMUNITY INCOME DISTRIBUTIONS

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ABSTRACT

Tiebout's (1956) model of fiscal competition suggests income sorting between communities while the Alonso (1964), Mills (1967) and Muth (1969) model of the monocentric city suggests income sorting over space. We add fiscal decentralization to the spatial model by considering a circular inner city surrounded by a suburban community. The fiscal difference between the communities and the commuting advantage of locations closer to the city center are capitalized into house prices. The model has equilibria in which there is income sorting between communities and equilibria in which there is income mixing between communities. The structure allows for the possibility of undeveloped land in the inner city.

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INTRODUCTION

An important issue in local public economics is whether a household's mobility within a metropolitan area leads to communities in which residents have similar incomes. In Tiebout's (1956) model of fiscal competition, communities are formed on a featureless plain and community boundaries may be freely adjusted. Each community provides a public service which is financed by a head-tax. A household's income does not depend on the community in which it resides. A household shops over communities, choosing the community which provides his preferred public service level. If the public service is a normal good, households with different incomes demand different public service levels. In consequence, households with different incomes chose different communities, or all households within each community have the same income (McGuire (1974), Berglas (1976a) and Wooders (1978)). The prediction of households sorting themselves by income between communities is robust if the model is changed to have a finite number of communities with fixed boundaries: incomes in each community lie in an interval and the income intervals associated with different communities do not overlap (Elickson (1971), Yinger (1982), Epple, Filimon and Romer (1984)).[1]

An alternative model of income distribution within a metropolitan area is Alonso (1964), Mills (1967) and Muth's (1969) spatial model of the "monocentric city". The metropolitan area is considered to be a circular area with the central business district located at its center. Each household commutes from its residence to the central business district. The advantage of a location closer to the metropolitan center is capitalized into the land price at that location and land prices fall as locations move further from the metropolitan center. The distribution of income within the metropolitan area depends on the

income elasticities of land demand and of commuting cost (Wheaton (1977)). If land demand is unresponsive to income changes and commuting costs increase with income, rich households outbid poor households for locations closer to the city's center. Conversely, if land demand is sufficiently income elastic, the saving achieved by the purchase of land further from the city's center is greater for the rich households and compensates them for the associated increase in commuting cost. In this case rich households choose to live in the low-priced locations further from the city's center. In both cases the prediction is of a monotonic relationship between household income and distance from the metropolitan center.[2] Therefore, if the metropolitan area is considered to be a system of annular communities, the model's prediction for income distribution is similar to that of the Tiebout model - incomes in each community lie in an interval and the income intervals associated with different communities do not overlap.

The model of income sorting between communities underlies much public policy. For example, programs which redistribute resources from rich school districts to poor school districts are often justified as income redistribution programs, it being considered self-evident that only rich households live in the rich school districts and only poor households live in the poor school districts. However, the prediction of strict income sorting does not fit well with the data. A significant percentage of families in both the inner city and in the suburbs have income below the poverty level (14.1% and 6% respectively in 1989).[3] Pack and Pack (1977 and 1978) find larger income variation within the towns of the metropolitan areas of Pennsylvania than is consistent with the homogeneous communities predicted by the Tiebout model. Persky (1992) examines Chicago and finds considerable evidence of income heterogeneity in both the city and the suburbs. Epple and Platt (1998) discuss the income variation

within the Boston metropolitan area: the incomes of the wealthiest households in a community of low average income typically exceed the incomes of the poorest households in a community of high average income.

In view of the large amount of federal and state aid directed at inner cities and of observed income heterogeneity within communities, a model which can predict income mixing between communities seems desirable. Various modifications have been made to Tiebout's model to ensure income heterogeneity within a community. Berglas (1976b), Stiglitz (1983), McGuire (1991) and Brueckner (1994) consider households to earn their incomes at firms which are located in the community. The firms have a production technology which requires the use of low- and high-skilled workers. In such a situation, which is perhaps best exemplified by the nineteenth century company town, low-wage and high-wage households coexist in the same town. Berglas and Pines (1981) suggest that communities provide several different public services and that the optimal community size for each public service is different. With the optimal community size for one public service being less than for another public service, it is desirable to add to the community households who are relatively large users of the second public service. Epple and Platt (1998) create income mixing by allowing households to differ both in their incomes and in their preferences for the public service. A community providing an intermediate level of the public service is chosen both by rich households who place a low weight on the public service and by poor households who place a high weight on the public service.

Although the "pure" models of fiscal competition and of the monocentric city both give similar predictions of income sorting between communities, we show that a model which combines elements of both models can predict income mixing between communities. We consider a metropolitan area to be

comprised of a circular inner city surrounded by the suburbs. At the center of the inner city is the central business district to which all households commute. We consider two income classes. A household's cost of commuting is proportional to the distance traveled and to its income, and land demand is wealth inelastic. These assumptions ensure that within each community rich households live closer to the metropolitan center. There is always an equilibrium with income sorting between communities: at least one community contains only one income level. This equilibrium may be associated with undeveloped land in the inner city. We also find some equilibria with income mixing between communities: both communities contain both income levels. In these equilibria, in the inner city poor households form the majority and vote a low public service level; in the suburbs rich households form the majority and vote a high public service level. Rich households live in the inner city because they value the smaller commuting times associated with being close to the central business district more than poor households, and they live in the suburbs because they value the higher public service of the suburbs more than poor households.

Capitalization underlies the equilibrium with income mixing between communities. As in the monocentric city model, within a community land prices rise as the location moves closer to the metropolitan center reflecting the advantage of the smaller commute. As in the models of fiscal decentralization, at the boundary between the two communities there is a discrete change in land prices reflecting the relative attractiveness of each community's public service to the marginal households. At each location within a community, the rate of increase in house prices reflects the commuting cost of the households who live there. In the outer areas of the inner city poor households live and, as the location moves towards the city's center, house prices rise at a rate which is less than the increase in the benefit

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to rich households of the smaller commute. As a result, a net surplus - commuting benefits less the increase in house prices - is created for rich households as the location in the inner city moves away from the suburban boundary and towards the metropolitan center. At a certain distance from the boundary this surplus equals the compensation required by rich households for the lower public service, and rich households are indifferent between such locations and their suburban locations. Capitalization similarly allows poor households to be indifferent about living in either community.

The income distribution in the equilibrium with income mixing is: in the inner city rich households live near the center and are surrounded by a ring of poor households. In the suburbs, adjacent to the boundary between the inner city and the suburbs, is a ring of rich households and these households are in turn surrounded by a ring of poor households. As the location moves out from the metropolitan center, household income falls, then rises at the boundary between inner city and suburbs, and then falls again. This distribution is descriptively similar to the empirical relationship found for "old" cities by Glaeser at al. (2000).

The paper is organized as follows. In Section 2 lays out the model and proves the existence of an equilibrium with income sorting. The presence of undeveloped land is highlighted. Section 3 presents the equilibrium with income mixing. Section 4 presents conditions sufficient to ensure the existence of an equilibrium with income mixing. Some welfare results are discussed in Section 5: we show that poor households may achieve higher utility in the equilibrium with income mixing than in the equilibrium with income sorting. Section 6 concludes.

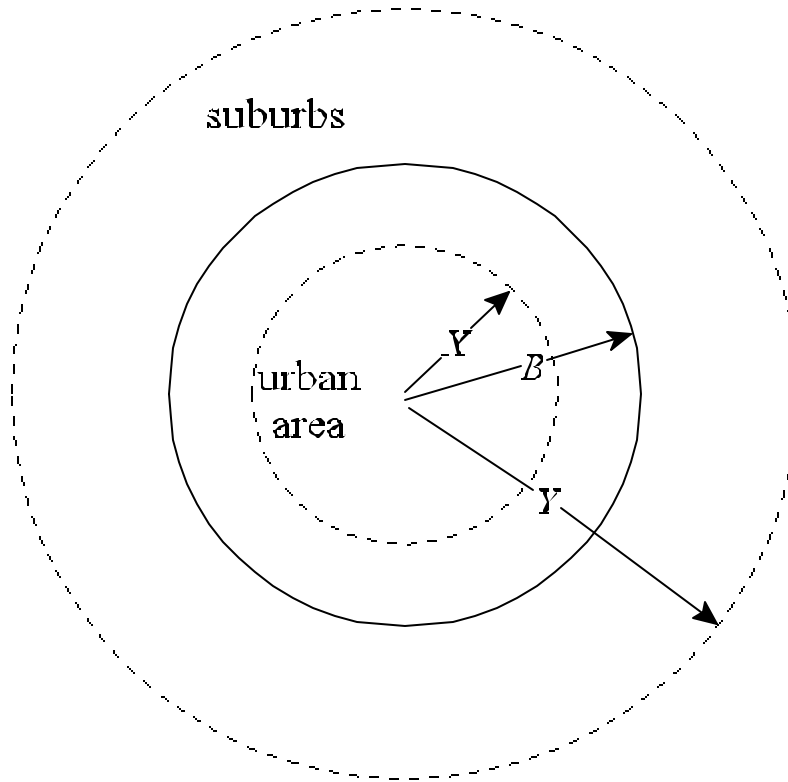## 2. INCOME SORTING BETWEEN COMMUNITIES

Figure 1: the metropolitan area

A metropolitan area is composed of a business district and two communities. For convenience of presentation, the business district is located at the metropolitan center and is assumed to be a point with no area.[4] It is surrounded by a circular community, termed the "urban area", whose jurisdictional boundary has radius B. The urban area is itself surrounded by another community, termed the "suburbs". The outer jurisdictional boundary of the suburbs is sufficiently distant that all households live

either in the urban area or in the suburbs. The limit of development in the suburbs is distance Y from the metropolitan center.

A household h obtains utility from consuming a private good c^h and from a public service g as $U(c^h) + V(g)$. The utility function has standard properties: $U(c^h: c^h \leq 0) = -\infty$; $U(c^h: c^h > 0) > -\infty$; $U'(c^h: c^h = 0) = +\infty$; $0 < U'(c^h: c^h > 0) < \infty$; $U''(c^h: c^h > 0) < 0$; $V(g: g \leq 0) = -\infty$; $V(g: g > 0) > -\infty$; $V'(g: g = 0) = +\infty$; $0 < V'(g: g > 0) < \infty$; $V''(g: g > 0) < 0$. The private good is the numeraire good and the household has a unit time endowment which he can use either for working or for commuting. If he uses all the time endowment for work, he earns income M^h. All firms are located in the business district so that, if the household lives distance s from the metropolitan center, the time spent commuting is proportional to s and his income is reduced by $tM^h s$ ($tY < 1$); the price of his house is $r(s)$. The community provides a public service g which is financed by a uniform residency tax. The resource cost of one unit of the public service per household is one unit of numeraire, so that the residency tax is g.[5] The central government may provide a lump-sum transfer T. The consumption of the private good by the household if he locates distance s from the city center is therefore

$$c^h = M^h - tM^h s - r(s) - g + T.$$

We consider all houses to have a fixed size. This assumption combined with the assumption of two income classes assures the existence of a sorting equilibrium. Under these assumptions, voter preferences are single peaked in a jurisdiction[6], and the fixed size assumption eliminates the possibility of an infinite regress in the adjustment process between the housing market and the voting equilibria.[7]

The use of a model in which a sorting equilibrium exists in general provides a simple basis for examining the sufficient conditions for the existence of an equilibrium with income mixing.

A household's housing choice within the community is restricted to its location. The household takes the housing price schedule $r(s)$ and the public service g in a community as given. His preferred location s within the community solves

$$\max_s \; U(M^k - tM^k s - r(s) - g + T) + V(g).$$

The benefit of locating closer to the metropolitan center is greater for the rich household or, within a community, rich households outbid the poor households for the locations closer to the metropolitan center. In consequence rich households live on the inside and poor households live on the outside of each community. This is formalized in Lemma A.[8]

LEMMA A: Within a community, household income is a non-increasing function of distance from the metropolitan center.

PROOF: See Appendix B.

Within a community, the decreased commuting time associated with locations closer to the business district is capitalized into house prices, or rent decreases as distance from the business district increases. In addition, the rent decrease must be continuous as otherwise a household which is located adjacent to the discontinuity on the side of high rent could increase his utility by moving across the discontinuity to the side of low rent: his rent would decrease by a discrete amount but his commuting

cost would change only marginally. This is formalized in Lemma B.[9]

LEMMA B: *If s_A and s_B both lie in a given community and $s_A < s_B$, then $r(s_A) > r(s_B)$ and $r(s)$ is a continuous function from s_A to s_B.*

PROOF: See Appendix B.

There are N households. A fraction θ are poor households with income M_1 and a fraction $(1-\theta)$ are rich households with income $M_2 : M_1 < M_2$. Each household inhabits a space a (fixed). We assume that the urban area cannot accommodate all households: $\pi B^2 < Na$. In the remainder of this section we consider income sorting between communities so that at least one community is homogeneous. There are 5 possible cases depending on the size of the two income groups and of the communities. For ease of presentation, we focus on the case in which the rich form the majority in the urban area but do not completely fill the urban area: $\pi B^2/2 \leq (1 - \theta)Na < \pi B^2$; the other cases are presented in Appendix A. The situation is non-standard because there is the possibility of undeveloped urban land. The outer limit of rich urban households is distance x_u from the metropolitan center ($x_u < B$) and the outer limit of urban development is distance X from the metropolitan center ($x_u \leq X \leq B$). Put differently and measuring distances from the metropolitan center, rich households live in the urban area up to distance x_u ; poor households live in the urban area between distances x_u and X, and in the suburbs at distances between B and Y. There is undeveloped land if $X<B$. Equilibrium in the land market implies

$$\pi X^2 + \pi(Y^2 - B^2) = Na ; \tag{1}$$

$$\pi x_u^2 = (1 - \theta) N a .$$

(2)

If households of income M_i locate in community j, they achieve the same utility at all points s at which they locate, or

$$U(M - tMs - r(s) - g + T) + V(g) = constant .$$

Hence the rent plus commuting cost paid by a household of income M_i locating at s in community j is

$$c_{ij} \equiv r(s) + tM_i \, s \, ,$$

where c_{ij} is a constant. Alternatively, $r(s) = c_{ij} - tM_i \, s$. Descriptively, as s decreases, the commuting cost for the household decreases and this advantage is exactly capitalized into house prices, so that all households of given income achieve the same utility.
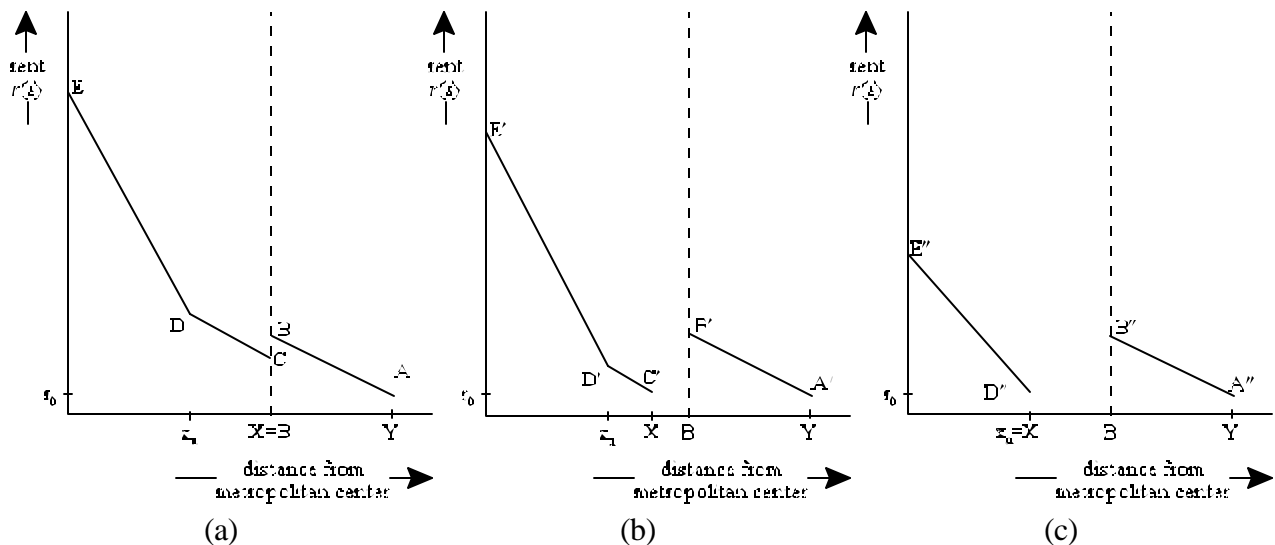

Figure 2: the rent schedule.

The reservation price of land is r_0 ($r_0 \geq 0$). Figure 2 shows possible rent schedules The curves may also be interpreted as the bid-rent curves of the households. Consider Figure 2(a) in which there is no undeveloped land. For poor households living at the suburban fringe, the rent is r_0. As the location moves inwards, the commuting advantage to the poor household is capitalized so that the rent rises at rate tM_1 or the rent schedule is AB. The public service (and associated taxes) changes discretely as the location moves across the urban boundary at distance B from the metropolitan center. Poor households vote their desired public service in the suburbs but in the urban area it is set by rich households. Hence, as the location moves across the urban boundary, rent falls by the amount represented by the line segment BC to reflect the cost to the poor household of the higher urban public service. Poor households live in the outer urban area and the rent gradient along CD is tM_1. However, at distance x_u from the metropolitan center, households become rich and the rent gradient along DE increases to tM_2 to reflect the advantage to them of a marginally smaller commute.

The reservation price r_0 is a rent floor. In Figures 2(b) and 2(c), the cost to the poor household of the high urban public service is sufficiently large that poor households are unwilling to pay rent r_0 to live at the boundary of the urban area. As the location moves inward, the benefit of the smaller commute increases and, in Figure 2(b), the location becomes attractive to poor households at distance X from the metropolitan center. In Figure 2(c), even at distance x_u a poor household is unwilling to pay rent r_0 and there are no poor urban households.

Poor households live at the suburban fringe where the price of land is the reservation price r_0. Therefore, for poor households living in the suburbs[10]

$$c_{1r} = r_0 + tM_1Y. \tag{3}$$

If there is no undeveloped urban land (case *a*), the rent at the urban boundary must be at least r_0 or

$$x_u < X = B : \quad c_{1u} \geq r_0 + tM_1B. \tag{4a}$$

If there is undeveloped urban land, the rent at the limit of urban development is r_0. If poor households live at the limit of urban development (case *b*)

$$x_u < X < B : \quad c_{1u} = r_0 + tM_1X; \tag{4b}$$

If there are no poor households in the urban area, it is the rich households who live at the limit of urban development (case *c*) and

$$x_u = X < B : \quad c_{2u} = r_0 + tM_2x_u. \tag{4c}$$

If there are poor households in the urban area, (cases *a* and *b*) rent continuity implies[11]

$$x_u < X \leq B : \quad c_{2u} - tM_2x_u = c_{1u} - tM_1x_u. \tag{5a}$$

If there are no poor households in the urban area (case *c*) and a poor household were to move to the urban area, the urban rent schedule implies that he would maximize his utility by locating at the edge of development, paying rent r_0, or

$$x_u = X < B : \quad c_{1u} = r_0 + tM_1x_u. \tag{5b}$$

If poor households live in each community, they achieve the same utility in either community, or

$$x_u < X \le B : \quad U(M_1 - c_{1s} - g_s + T) + V(g_s) = U(M_1 - c_{1u} - g_u + T) + V(g_u) . \tag{6a}$$

If there are no poor households in the urban area, a poor household must achieve at least as much utility in the suburbs as in the urban area, or

$$x_u = X < B : \quad U(M_1 - c_{1s} - g_s + T) + V(g_s) \ge U(M_1 - c_{1u} - g_u + T) + V(g_u) . \tag{6b}$$

Public service levels in each community are set by majority voting. We assume that each household votes to maximize his utility taking his rent as given.[12] With the rich forming the majority in the urban area and the poor forming the majority in the suburbs,

$$U'(M_2 - c_{2u} - g_u + T) = V'(g_u) ; \tag{7}$$

$$U'(M_1 - c_{1s} - g_s + T) = V'(g_s) . \tag{8}$$

To maintain a closed system, we assume that all rent paid is returned as the lump-sum transfer T.[13] As noted earlier, if a household of income M_i lives in community j at distance s from the metropolitan center, the rent is $r(s) = c_{ij} - t\, M_i s$. The rent collected from a circular element of radius s and width ds is $(r(s)/a)\, 2\pi s\, ds$, and therefore

$$T = \frac{1}{N}\left[\int_0^{x_u} \frac{c_{2u} - tM_2 s}{a} 2\pi s\, ds + \int_{x_u}^{X} \frac{c_{1u} - tM_1 s}{a} 2\pi s\, ds + \int_B^{Y} \frac{c_{1s} - tM_1 s}{a} 2\pi s\, ds \right]. \tag{9}$$

Note that $r_0 \ge 0$ implies $T > 0$.

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The equilibrium values of the nine variables $c_{2u}, c_{1u}, c_{1s}, g_u, g_s, x_u, X, Y$ and T solve Equations (1) - (9). In addition, equilibrium requires that rich urban households do not gain by moving to the suburbs (self-selection). We note that, if a rich household were to move to the suburbs, the suburban rent schedule implies that he would maximize his utility by moving to the inner boundary, paying rent $c_{1s} - tM_1B$. Therefore self-selection requires

$$U(M_2 - c_{2u} - g_u + T) + V(g_u) \geq U(M_2 - c_{1s} + tM_1B - tM_2B - g_s + T) + V(g_s).$$

We assume that a poor household has sufficient income to live in the suburbs, or $M_1 > r_0 + tM\hat{Y}$ where $\hat{Y}$ is the maximum possible value of Y: $\pi(\hat{Y}^2 - B^2) = \theta Na$.[14]

Proposition 1 states that there is a sorting equilibrium in the case considered above and in the other four cases determined by changing the sizes of the two income groups and of the communities.

PROPOSITION 1: *there is always an equilibrium in which one community is homogeneous in income.*
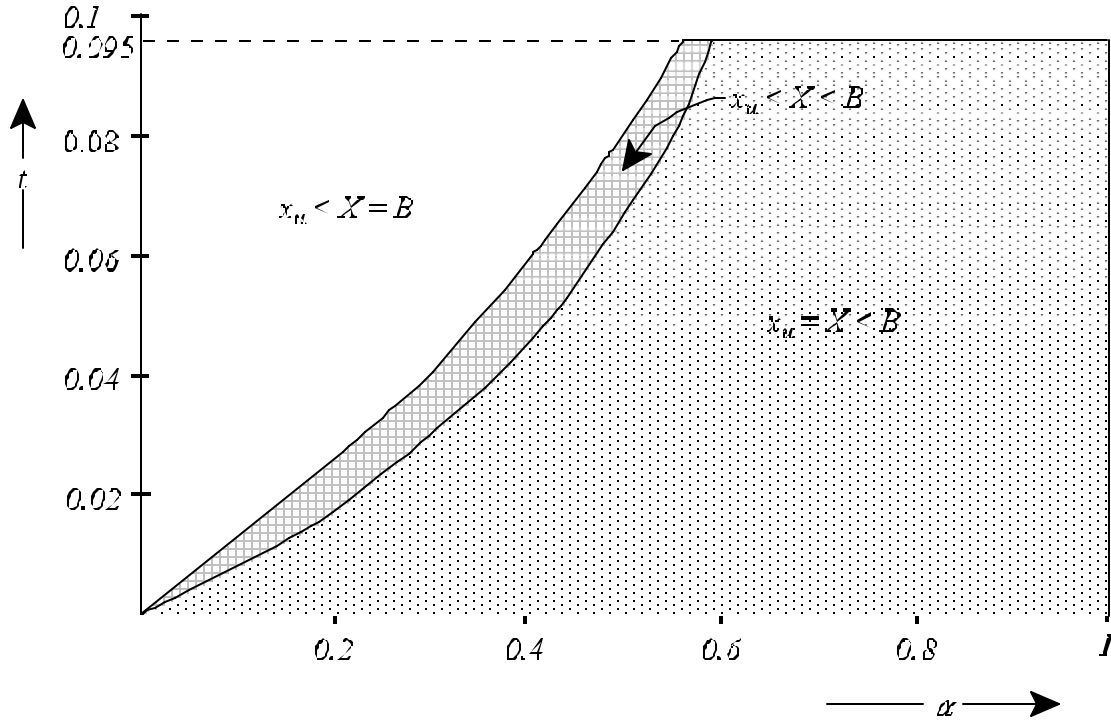
PROOF: See Appendix A.

Figure 3: types of equilibrium with income-sorting.

To illustrate the potential presence of undeveloped land, we consider an example in which

households have utility functions of Cobb-Douglas form $(1 - \alpha) \log_e c^h + \alpha \log_e g$. We consider

particular parameter values $M_1 = .3, M_2 = .6, \theta = .6, B = 7, N\alpha = \pi(10^2)$ and $r_0 = 0$, and vary α

and t.[15] Figure 3 shows the three regions $x_u < X = B$, $x_u < X < B$ and $x_u = X < B$, of which the last

two have undeveloped land and are shaded. As expected, the regions with undeveloped land have high

α which corresponds to a large difference in the willingness pay for the public service. As t increases,

the advantage of being close to the metropolitan center increases and the lowest value of α consistent

with undeveloped land increases accordingly.

3. INCOME MIXING BETWEEN COMMUNITIES

In this section we restrict attention to a possible second equilibrium with income mixing in which the urban area and the suburbs contain households of both income levels. Lemma A shows that the rich households live closer to the metropolitan center in each community. With both income classes living in both communities, the boundary between the rich and poor households occurs in the urban area at distance x_u from the metropolitan center and in the suburbs at distance y_s from the metropolitan center.

We use the structure developed in Section 2.[16] In particular, we continue to denote as c_{ij} the total rent plus commuting cost of a household of income M_i living in community j.[17] The reservation price of land is r_0 $(r_0 \geq 0)$ and poor households live at the suburban fringe, or

$$c_{1s} = r_0 + tM_1 Y. \tag{10}$$

Rent continuity in the urban area at x_u implies

$$c_{2u} - tM_2 x_u = c_{1u} - tM_1 x_u, \tag{11}$$

and rent continuity in the suburbs at y_s implies

$$c_{2s} - tM_2 y_s = c_{1s} - tM_1 y_s. \tag{12}$$

Equilibrium requires that poor households achieve the same utility in either community, or

$$U(M_1 - c_{1u} - g_u + T) + V(g_u) = U(M_1 - c_{1s} - g_s + T) + V(g_s). \tag{13}$$

and that rich households receive equal utility in either community, or

$$U(M_2 - c_{2u} - g_u + T) + V(g_u) = U(M_2 - c_{2s} - g_s + T) + V(g_s) \,. \tag{14}$$

As in Section 2, households vote taking the rent as given. If rich households were to form the majority in the urban area, rich households would prefer the urban area for both its commuting advantage and for its public service. Similarly, if poor households were to form the majority in both communities, rich households would achieve the same public service in either community and would prefer the urban area for its commuting advantage. In either case, rich households would migrate from the suburbs to the urban area until either no rich households live in the suburbs or until no poor households live in the urban area. Hence, if households with each income level live in both communities, rich households must form the majority in the suburbs and poor households must form the majority in the urban area. This is formalized in Lemma C.

LEMMA C: *If both communities contain households of each income level, poor households must form the majority in the urban area and rich households must form the majority in the suburbs.*

PROOF: See Appendix B

With poor households forming the urban majority and rich households forming the suburban majority, voting sets public service levels as

$$U'(M_1 - c_{1u} - g_u + T) = V'(g_u); \tag{15}$$

$$U'(M_2 - c_{2s} - g_s + T) = V'(g_s).$$ (16)

Poor households live in the suburbs because the disadvantages of the high public service level, with the associated high tax, and the high commuting cost are exactly offset by the low rent. If there were undeveloped land in the urban area, housing in the urban area would be available at the reservation price r_0 so that poor households could move from the suburban fringe to the urban area without a change in rent. This would benefit them because they would obtain a lower commuting cost and their favored public service level. Hence at equilibrium the urban area must have no undeveloped land. This is formalized in Lemma D.

LEMMA D: if rich and poor households live in both communities, there is no undeveloped urban land.

PROOF: See Appendix B.

With no undeveloped urban land, equating supply and demand of land implies

$$\pi Y^2 = Na;$$ (17)

$$\pi x_u^2 + \pi(y_s^2 - B^2) = (1 - \theta)Na.$$ (18)

To maintain a closed system, we continue to assume that all rent paid is returned as the lump-sum transfer T,

$$T = \frac{1}{N}\left[\int_b^{x_u} \frac{c_{2u} - tM_2 s}{a} 2\pi s x\, ds + \int_{x_u}^B \frac{c_{1u} - tM_1 s}{a} 2\pi s\, ds \right.$$

$$\left. \int_B^{y_s} \frac{c_{2s} - tM_2 s}{a} 2\pi s\, ds + \int_{y_s}^Y \frac{c_{1s} - tM_1 s}{a} 2\pi s\, ds \right]. \qquad (19)$$

Note that $r_0 \geq 0$ implies $T > 0$.

The ten endogenous variables c_{2u}, c_{1u}, c_{2s} c_{1s}, g_u, g_s, x_u, y_s, Y and T are determined by the
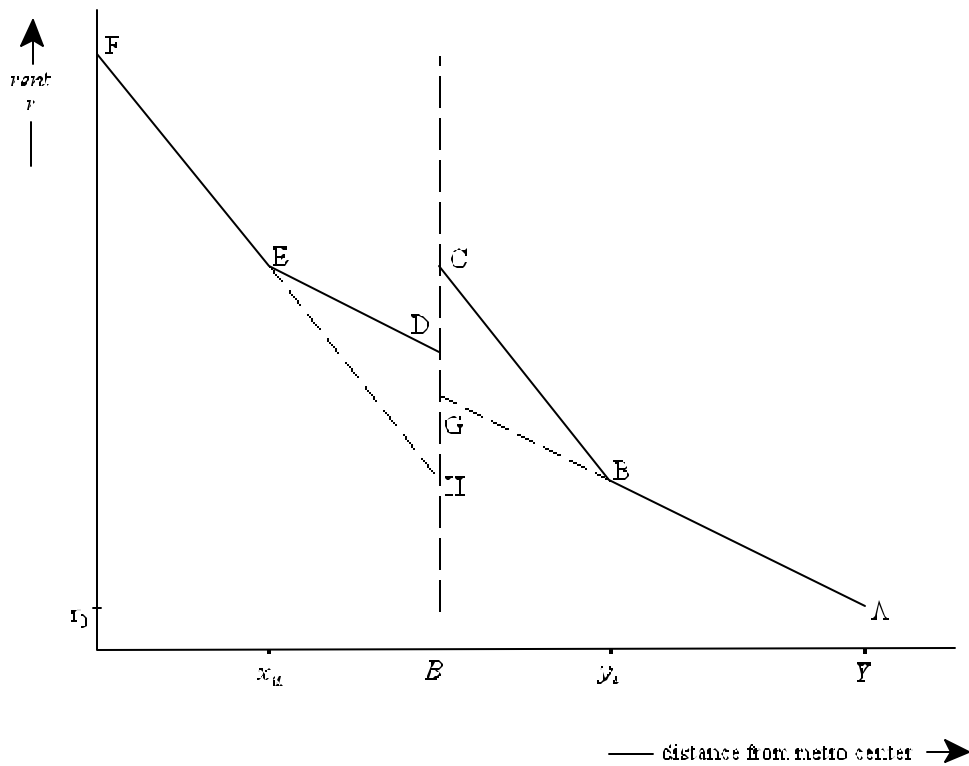
Equations (10)-(19).

Figure 4: the rent schedule

Figure 4 provides the intuition for the existence of an equilibrium with income mixing between

communities. At the limit of suburban development Y the rent schedule is anchored at the reservation

rent r_0. Poor households live in the outer suburban areas. As the location moves inwards from Y,

capitalization of the commuting advantage causes the rent to rise at rate tM_1. At distance y_s from the

metropolitan center, residents become rich and the rent gradient increases to tM_2. *ABG* is interpreted as the bid-rent curve of a poor household in the suburbs, and *BC* is interpreted as the bid-rent curve of a rich household in the suburbs.

As the location moves across the boundary between the suburbs and the urban area, the public service changes from the level set by rich households to the level set by poor households: poor households are willing to pay the premium *GD* to live in the urban area. *DE* is the bid-rent curve of a poor household in the urban area: the commuting advantage to the *poor* household of being closer to the metropolitan center is capitalized in rents, or rent rises along *DE* at rate tM_1

Rich households choose the suburban public service, so that rent at the boundary would have to fall by *CH* if a rich household were to be willing to live on the urban side of the boundary. *HEF* is the bid-rent curve of a rich household in the urban area. The vertical distance between *DE* and *HE* is the difference between actual rent and the rent a rich household is willing to pay, or is the net cost to a rich suburban household of moving to a location in the urban area. As the location moves closer to the metropolitan center, the willingness to pay of the rich household for the urban location rises faster (at rate tM_2) than the rent (rising at rate tM_1), reflecting the relatively low value poor households place on a marginally smaller commute. It is this difference, between the rate at which rich households value a shorter commute and the rate at which the shorter commute is being capitalized, which allows equilibrium to exist. At distance x_u from the metropolitan center, willingness to pay equals the rent, and rich households outbid poor households for all locations closer to the metropolitan center.

The solution to Equations (10)-(19) is an equilibrium solution if, at the solution values, poor households form the majority in the urban area

$$\pi x_u^2 \le \frac{1}{2}\pi B^2 , \tag{20}$$

and rich households form the majority in the suburbs

$$\pi(y_s^2 - B^2) \ge \frac{1}{2}\pi(Y^2 - B^2) . \tag{21}$$

The ordering of distances must satisfy

$$0 < x_u < B < y_s < Y . \tag{22}$$

All consumption values must be non-negative, or[18]

$$M_1 - c_{1s} - g_s + T \ge 0 . \tag{23}$$

We now state the main result of this section.[19]

PROPOSITION 2: *There exist equilibria in which the poor and the rich households are located in both communities.*

PROOF: The proof is by construction of an example (see below).

We construct examples with utility having Cobb-Douglas form,

$$U = (1 - \alpha)\log_e(M - r(s) - tMs - g + T) + \alpha\log_e g ,$$

and with parameter values: $M_1 = .3, M_2 = .6, \theta = .6, B = 7, Na = \pi (10^2)$ and $r_0 = 0$.
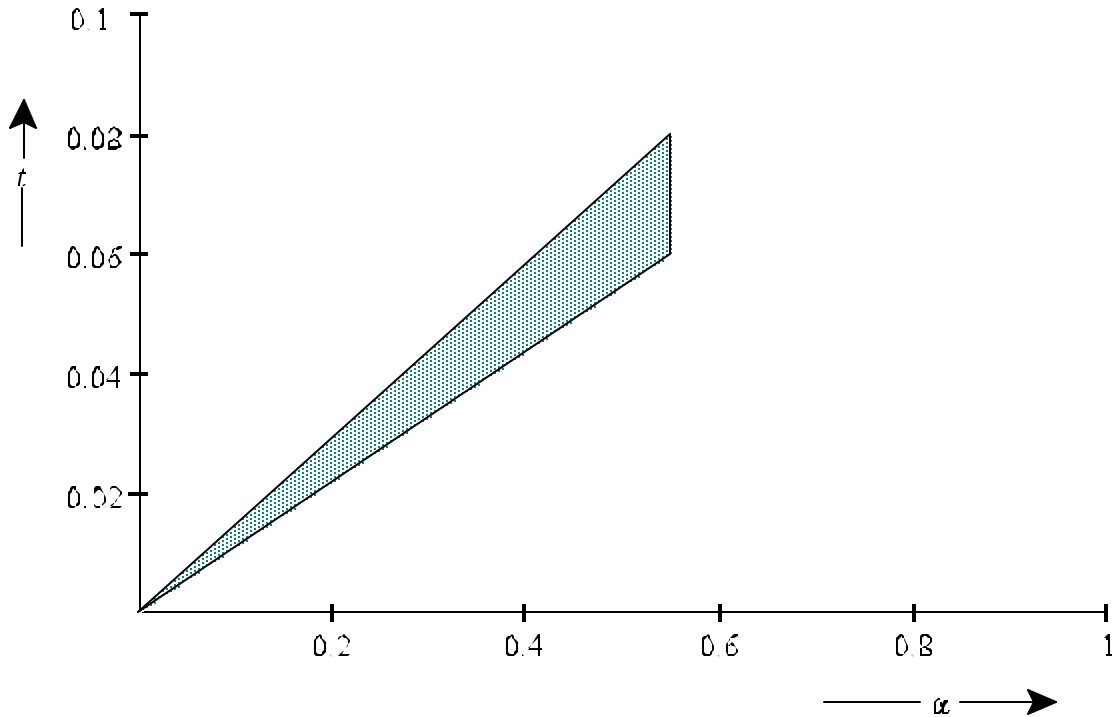
Figure 5: parameter values giving equilibrium with income-mixing.

Values for α and t for which there exists an equilibrium outcome in which households of both income levels live in both communities are shown in the shaded region of Figure 5. In addition to showing the existence of such outcomes, the diagram illustrates three additional points. Firstly, the equilibrium with income mixing does not exist at high levels of α: at high α the public service level voted in the suburbs is unaffordable to poor households (Inequality (23) is violated).

Secondly, the figure illustrates the trade-off between public service levels and transport costs. The region in which values of α and t support an equilibrium with income mixing lies "along" a diagonal. Increasing transport costs (increasing t) are associated with an increasing tendency for rich households to outbid poor households for locations in the urban area. Therefore, if the rich are to continue to live in the suburbs, increasing importance must be placed by the rich household on the higher public service

level of the suburbs (increasing α). Above the allowable region, commuting costs are sufficiently large that rich households migrate into the urban area and a required majority is reversed (Inequality (20) or (21) is violated). Below the allowable region, commuting costs are relatively unimportant so that rich households migrate into the suburban area to benefit from the higher public service and the urban area contains no rich households (Inequality (22) is violated) - a situation which descriptively corresponds to "urban flight".

Thirdly, Figure 5 shows that the region in which values of α and t support an equilibrium with income mixing between communities is "thick". Continuity implies that the equilibrium values change by only a small amount if there is a small change in the parameter values of the model. Therefore, provided the solution to Equations (10) - (19) lies strictly inside the allowable region defined by Inequalities (20)-(23), the equilibrium with both communities containing households of both income levels continues to exist if there is a small change in the parameter values of the model.

4. SUFFICIENT CONDITIONS

We now establish conditions which are sufficient to ensure the existence of an equilibrium with income mixing between communities. Consider the outcomes which would arise if the size of the rich suburban population were preset, and if rich households were immobile between communities but poor households could migrate between communities, i.e.,consider the solution to Equations (10)-(13) and (15)-(19) as a function of y_s. In particular, consider Allocation *A (B)* to be the allocation in which the rich suburban population has its lowest (highest) value consistent with income mixing between communities. If at Allocation *A (B)* rich suburban households achieve higher (lower) utility than rich

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urban households, then by continuity there is some rich suburban population at which rich households achieve equal utility in each community, or there is some y_s at which Equation (14) is satisfied. At this outcome Inequalities (20), (21) and (22) are satisfied by construction. Additional restrictions need to be imposed to ensure positive consumption (Inequality 23).

Formally, we consider first the conditions which are sufficient to ensure that a solution to Equations (10)-(19) and Inequalities (20)-(22) are satisfied (assuming Inequality (23) is satisfied). Because there is no undeveloped land in an equilibrium with income mixing, Y is determined by Equation (17) and is considered exogenous in this section. If poor households form the metropolitan majority, an equilibrium in which rich households form the suburban majority and live in the urban area requires that

$$\theta \geq .5 : \quad (1 - \theta)N\alpha > \frac{1}{2}\pi(Y^2 - B^2) . \tag{24}$$

Consider an allocation in which rich households form exactly half the suburbs (living in the area closest to the metropolitan center)

$$y_s = y_s^A : \quad \pi(y_s^{A\,2} - B^2) = \frac{1}{2}\pi(Y^2 - B^2)$$

and all remaining rich households are allocated to the inner urban area. With $y_s = y_s^A$, denote the values of $c_{2u}, c_{1u}, c_{2s}, c_{1s}, g_u, g_s, x_u, Y$ and T which solve Equations (10)-(13), (15)-(19) with a superscript A and suppose that at this allocation the rich households achieve higher utility in the suburbs

$$U(M_2 - c_{2s}^A - g_s^A + T^A) + V(g_s^A) > U(M_2 - c_{2u}^A - g_u^A + T^A) + V(g_u^A). \tag{25}$$

Now consider moving rich households from the urban area to the suburbs (increasing y_s) until either the

urban area contains no rich households or the suburbs contains no poor households,

$$y_s^B = \min y_s \text{ such that either } x_u = 0 \text{ or } y_s = Y.$$

With $y_s = y_s^B$, denote the values of $c_{2u}, c_{1u}, c_{2s}, c_{1s}, g_u, g_s, x_u$ and T which solve Equations (10)-(13), (15)-(19) with a superscript B, and suppose that at this allocation the rich households achieve higher utility in the urban area

$$U(M_2 - c_{2s}^B - g_s^B + T^B) + V(g_s^B) < U(M_2 - c_{2u}^B - g_u^B + T^B) + V(g_u^B). \qquad (26)$$

We show in the Appendix that the utility difference of a rich household varies continuously as y_s changes so that Inequalities (25) and (26) imply that there is some y_s at which a rich household achieves equal utility in either community, or at which Equation (14) is satisfied in addition to Equations (10)-(13), (15)-(19).

If rich households form the metropolitan majority, there only can be an equilibrium in which poor households form the majority in the urban area and live in the suburbs if

$$\theta \le .5 : \quad \theta Na > \frac{1}{2}\pi B^2. \qquad (27)$$

In this case the allocation denoted by the superscript A is considered as the allocation in which rich households form half the urban area, and the argument above is repeated to establish the sufficient conditions to ensure a solution to Equations (10)-(19) and Inequalities (20)-(22).

We now consider the requirement that consumption is positive. Using Equations (10) and (12),

$$c_{2s} = r_0 + tM_1Y + t(M_2 - M_1)y_s.$$

As y_s increases from y_s^A to y_s^B, a lower bound on c_{2s} is

$$c_{2s}^* = r_0 + tM_1Y + t(M_2 - M_1)y_s^A .$$

Consider the public service g_s^* desired by a rich suburban household paying rent plus commuting cost c_{2s}^* and receiving no lump-sum transfer

$$g_s^* = \underset{g}{argmax} \; U(M_2 - c_{2s}^* - g) + V(g) .$$

But $c_{2s} \geq c_{2s}^*$, $T > 0$ and normality imply $g_s - T < g_s^*$; hence

$$M_1 - c_{1s} - g_s + T > M_1 - r_0 - tM_1Y - g_s^* .$$

A sufficient condition to ensure that the solution to Equations (10)-(19) implies Inequality (23) is

$$M_1 > r_0 + tM_1Y + g_s^* . \tag{28}$$

We summarize by stating the sufficient conditions as Lemma E.

LEMMA E: *If Inequalities (24) or (27) hold, Inequalities (25), (26) and (28) are sufficient to ensure the existence of an equilibrium in which poor and rich households are located in both communities.*
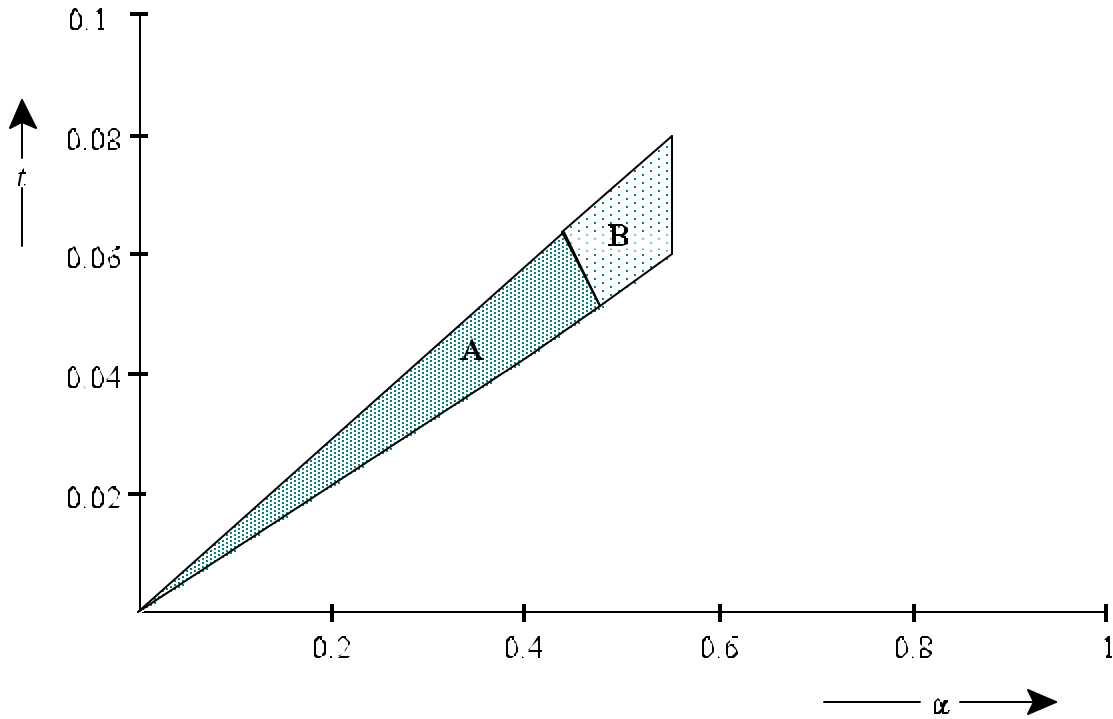

PROOF: See Appendix B.

Figure 6: overlap of the sufficient conditions and the equilibria with income-mixing.

To illustrate the power of the sufficient conditions, we use the example used earlier, in which utility has Cobb-Douglas form, $(1-\alpha) \, log_e \, c^h + \alpha \, log_e \, g$, and in which the parameter values are $M_1 = .3, M_2 = .6, \theta = .6, B = 7 , Na = \pi \, (10^2)$ and $r_o = 0$. Figure 6 overlays the values of α and t which satisfy the sufficient conditions on the values of α and t which support equilibria with income mixing between the communities. In Region A the sufficient conditions are satisfied. In Region B equilibria with income mixing exist but α is sufficiently large that the high implied value of g_t^* leads Inequality (23) to fail.

4. WELFARE

In the first-best problem, community boundaries are flexible. The first-best efficient outcome has income sorting with the urban boundary set so that only rich households live in the urban area, $\pi B^2 = (1-\theta)N\alpha$. In this way, transportation costs are minimized and there is perfect matching of households with their desired public service level.
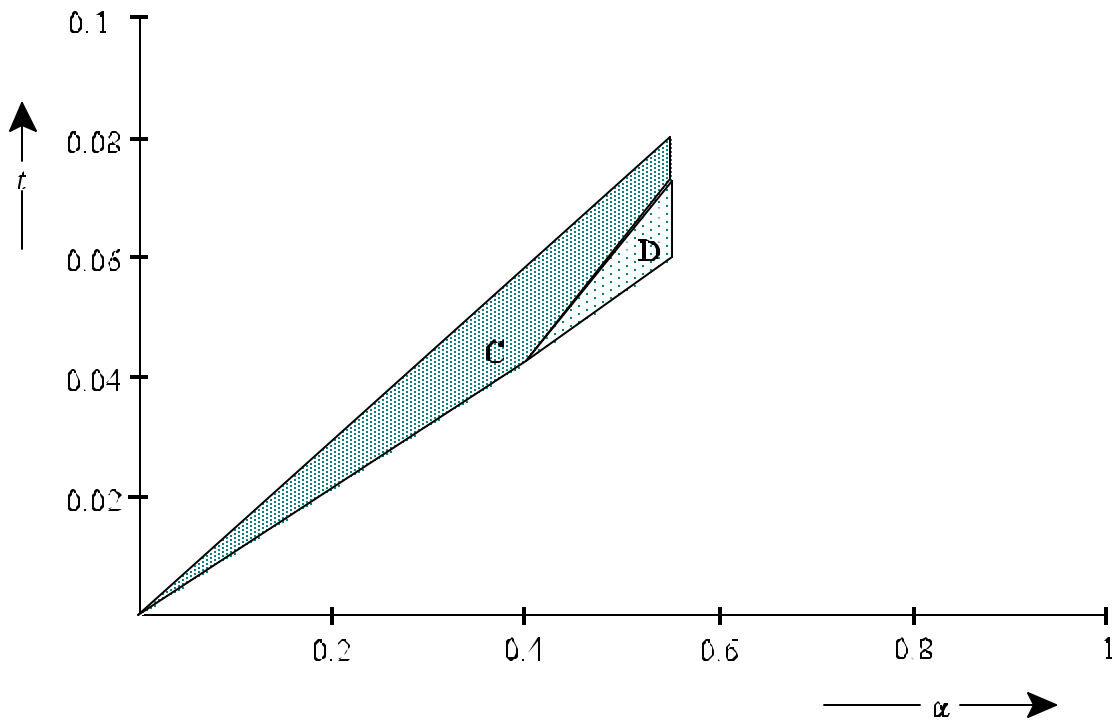

Figure 7: welfare comparison of equilibria with income sorting and income mixing.

In the second-best problem, the boundary of the urban area is fixed. In comparing the mixing equilibrium (if it exists) with the sorting equilibrium, there is a complex trade-off between commuting cost, rent paid and rents returned through the lump-sum transfer, and matching households with public

services. It is therefore not easy to make general statements.[20] Figure 7 shows two regions for the

example used earlier, in which utility has Cobb-Douglas form,

$(1-\alpha) \, log_e \, c^h + \alpha \, log_e \, g$, and in which the parameter values are $M_1 = .3, M_2 = .6, \theta = .6, B = 7$,

$Na = \pi \, (10^2)$ and $r_o = 0$. α and t are varied. In the region labeled C the higher total commuting costs

associated with mixing cause all utility levels to be lower in the mixing equilibrium than in the sorting

equilibrium. However the change in the transfer can mitigate the effect for the poor household - in the

region labeled D poor households achieve higher utility in the mixing outcome than in the sorting outcome

(but rich households achieve less utility).[21]

6. CONCLUSION

In this paper we have placed the model of fiscal competition inside a spatial model, using a

model with two communities and two income classes. We show that there is one equilibrium in which at

least one community is homogeneous in income, and we believe that this is the equilibrium on which most

of the literature has focused. This equilibrium may contain undeveloped land. We also show that there

can be another equilibrium in which both communities contain households of both income levels, and that

poor households may achieve higher utility in this equilibrium. This second equilibrium reconciles the

model of residential choice with the empirical finding that communities are heterogeneous in incomes.

The equilibrium with income mixing arises because the commuting advantage of locations nearer

the metropolitan center is capitalized into house prices at different rates. In the outer urban areas the rate

at which it is capitalized is less than the rate at which the benefit of a smaller commute is increasing for

rich households. In consequence the net benefits of locations close to the city's center area are able to

exactly offset the cost to rich households of the lower public service, so that households of both income levels reside in both communities.

Current metropolitan areas have grown out of much smaller cities. Might history have played a role in selecting the equilibrium with income mixing? The key to the development of an equilibrium with income mixing is the creation of a suburban area with a majority of rich households. Historically, when metropolitan populations were small compared to the size of the urban area, all households lived in the urban area and poor households formed the majority. This is the equilibrium in our model. We now consider the comparative statics of an increase in the population in the presence of an exogenous jurisdictional boundary. As the population grows,[22] or as the limit of urban development moves outwards, house prices near the metropolitan center rise and equilibria in our model have some rich households locating in the suburbs while there is still undeveloped urban land (provided rich households' willingness to pay for a higher public service is sufficiently strong). By moving, the rich avoid the high rents in the center of the urban area *and* obtain a public service level which better meets their needs. This second advantage is not present for poor households and hence, as required for an equilibrium with income mixing, it is the rich households who develop as the suburban majority. When the metropolitan population has grown so that there is no undeveloped urban land, further increases in population size result in poor and rich households locating in the suburbs.

PROOF: There are 5 potential cases. For the first four cases, poor households form the majority in the

suburbs and the assumption that poor households have enough income to live in the suburbs is taken to

imply that $M_1 > r_0 + t M_1 \hat{Y}$.

Case 1: poor households are the majority in both communities; $Na(1-\theta) < \frac{1}{2}\pi B^2$.

If poor households are the majority in both communities, they choose the public service level in each

community to maximize their utility, and equilibrium requires

$$\max_g \, [\, U(M_1 - c_{1u} - g + T) + V(g) \,] \;=\; \max_g \, [\, U(M_1 - c_{1s} - g + T) + V(g) \,] \,,$$

or $c_{1u} = c_{1s}$ and $g_u = g_s$. It is therefore exactly "as if" there is a single community for which it is

readily shown that an equilibrium exists.

Case 2: rich households form the majority in the urban area but do not completely fill the urban

area, and only poor households live in the suburbs: $\frac{1}{2}\pi B^2 \leq Na(1-\theta) < \pi B^2$.

Section 2 shows that equilibrium solves:[23]

land market: $\qquad\qquad \pi X^2 + \pi(Y^2 - B^2) = Na$; $\qquad\qquad\qquad$ (A.1)

$$\pi x_u^2 = (1-\theta)Na \; ; \qquad\qquad\qquad (A.2)$$

cost at suburban fringe: $c_{1s} = r_0 + t M_1 Y$; $\qquad\qquad\qquad$ (A.3)

urban rent must equal or exceed reservation value:

$$\text{either } x_u < X = B: \quad c_{1u} \geq r_0 + tM_1B \ ; \\ \text{or } x_u < X < B: \quad c_{1u} = r_0 + tM_1X \ ; \\ \text{or } x_u = X < B: \quad c_{1u} = r_0 + tM_1x_u \ ;$$

(A.4)

rent continuity at x_u:

$$\text{either } x_u < X = B: \quad c_{2u} = c_{1u} + t(M_2 - M_1)\,x_u \ ; \\ \text{or } x_u < X < B: \quad c_{2u} = c_{1u} + t(M_2 - M_1)\,x_u \ ; \\ \text{or } x_u = X < B: \quad c_{2u} = r_0 + tM_2x_u \ ;$$

(A.5)

equilibrium for poor:

$$\text{either } x_u < X = B: \quad U(M_1 - c_{1s} - g_s + T) + V(g_s) = U(M_1 - c_{1u} - g_u + T) + V(g_u) \ ; \\ \text{or } x_u < X < B: \quad U(M_1 - c_{1s} - g_s + T) + V(g_s) = U(M_1 - c_{1u} - g_u + T) + V(g_u) \ ; \\ \text{or } x_u = X < B: \quad U(M_1 - c_{1s} - g_s + T) + V(g_s) \geq U(M_1 - c_{1u} - g_u + T) + V(g_u) \ ;$$

(A.6)

urban voting:

$$U'(M_2 - c_{2u} - g_u + T) = V'(g_u) \ ;$$

(A.7)

suburban voting:

$$U'(M_1 - c_{1s} - g_s + T) = V'(g_s) \ ;$$

(A.8)

return of rents:

$$T = \frac{1}{N}\left[\int_0^{x_u} \frac{c_{2u} - tM_2s}{a} 2\pi s\,ds + \int_{x_u}^{x} \frac{c_{1u} - tM_1s}{a} 2\pi s\,ds + \int_B^{Y} \frac{c_{1s} - tM_1s}{a} 2\pi s\,ds \right].$$

(A.9)

In addition, we require that rich households prefer the urban location. We note that g_u maximizes

$U(M_2 - c_{2u} - g - T) + V(g)$ and that, if a rich household were to move to the suburbs, he would

move to the innermost location and pay rent $c_{1s} - tM_1X$. Therefore self-selection requires

$$\max_{g} [U(M_2 - c_{2u} - g + T) + V(g)] \geq U(M_2 - c_{1s} + tM_1X - tM_2X - g_s + T) + V(g_s). \quad (A.10)$$

The variables are: $c_{2u}, c_{1u}, c_{1s}, g_u, g_s, x_u, Y, T$ and X. In view of Equation (A.4), Equation (A.5) can be written as

$$c_{2u} = c_{1u} + t(M_2 - M_1)x_u. \quad (A.5')$$

We restrict attention to values of c_{1u} satisfying Inequality (A.4) and use Equations (A.1)-(A.3), (A.5'), (A.7)-(A.9) to write $c_{2u}, c_{1s}, g_u, g_s, x_u, Y$ and T as functions of X and c_{1u}. The Jacobian determinant associated with Equations (A.1)-(A.3), (A.5'), (A.7)-(A.9) is

$-(2t)^2 x_u Y(U''_{1s} U''_{2u} + U''_{2u} V''_s + U''_{1s} V''_u + V''_u V''_s)$, where the subscript on U'' and V'' is used to indicate the argument, e.g. $U''_{1s} = U'' (M_1 - c_{1s} - g_s + T)$ or $V''_u = V''(g_u)$. Provided

$M_1 - c_{1s} + T > 0$ and $M_2 - c_{2u} + T > 0$, the second derivatives exist and the Jacobian determinant is non-zero: by the Implicit Function Theorem, it is possible to write $c_{2u}, c_{1s}, g_u, g_s, x_u, Y$ and T as continuous functions of X and c_{1u}. $M_1 - c_{1s}$ is assured by the assumption that the poor have sufficient income to live in the suburbs. Equations (A.3) and (A.5') assure that all rents are positive (given Inequality (A.4)) and hence $T > 0$. Hence a sufficient condition to allow use of the Implicit Function Theorem is $M_2 - c_{2u} > 0$. Using Equation (A.5'),

$$M_2 - c_{2u} = M_1 - c_{1u} + (1 - tx_u)(M_2 - M_1) > M_1 - c_{1u}, \quad (A.11)$$

or (using $M_1 - c_{1s} > 0$) $M_2 - c_{2u} > 0$ provided $c_{1u} \leq c_{1s}$.

For $c_{1u} \leq c_{1s}$, define

$$Q(X; c_{1u}) = \max_g \, [\, U(M_1 - c_{1s}(X; c_{1u}) - g + T(X; c_{1u})) + V(g) \,]$$
$$- [\, U(M_1 - c_{1u} - g_u(X; c_{1u}) + T(X; c_{1u})) + V(g_u(X; c_{1u})) \,] .$$

Equations (A.4) and (A.6) demonstrate complementary slackness. Noting that g_s maximizes

$U(M_1 - c_{1u} - g + T) + V(g)$, a solution to Equations (A.1) - (A.9) implies

either $\quad x_u < X = B$: there exists c^A: $c^A \geq r_0 + tM_1B$ such that $Q(B; c^A) = 0$;

or $\quad x_u < X < B$: there exists X^A : $x_u < X^A < B$ such that $Q(X^A; r_0 + tM_1X^A) = 0$;

or $\quad x_u = X < B$: $Q(x_u; r_0 + tM_1x_u) \geq 0$.

First, consider $X = B$. In this case, Equations (A.1) and (A.3) imply that Y and c_{1s} are constants,

independent of c_{2u} ; consider the possibility $c_{1u} = c_{1s}$ for which the proposed relationships

$T(X; c_{1s})$ and $g_u(X; c_{1s})$ exist. Using Equation (A.3), $c_{1u} = c_{1s} = r_0 + tM_1Y > r_0 + tM_1B$, or Equation

(A.4) is satisfied. Using Equation (A.11), *if* $c_{1u} = c_{1s}$, Equations (A.7) and (A.8) imply $g_u > g_s$ and

therefore $Q(B; c_{1s}) > 0$.[24]

Second, maintain $X = B$ and lower c_{1u} from c_{1s} to $r_0 + tM_1B$. $c_{1u} \leq c_{1s}$ so that all variables

change continuously. $Q(B; c_{1u})$ changes continuously so that *either* Case A: there is come c_{1u} denoted

c^A, $r_0 + tM_1B \leq c_A < c_{1s}$ for which $Q(B; c^A) = 0$. This is an equilibrium (provided self-selection is

satisfied). *Or* Case B: $Q(B; r_0 + tM_1B) > 0$.

Third, consider Case B and lower X from B to x_u and change c_{1u} so that $c_{1u} = r_0 + tMX$

(Equation A.4). As above, $c_{1u} \leq c_{1s}$ and all variables change continuously so that *either* Case B1: there is

some X denoted X^A, $x_u \leq X < B$, for which $Q(X^A; r_0 + tM_1 X^A) = 0$. This is an equilibrium (provided

self-selection is satisfied). *Or* Case B2: $Q(x_u; r_0 + tM_1 x_u) > 0$. This is an equilibrium (provided self-

selection is satisfied).

To show that the solution satisfies self-selection, we show that a rich household would have an

increase in rent and commuting costs if he moved to the suburbs.[25] If he moved to the suburbs, he would

locate at the inner suburban boundary and pay rent c_{1s} - $tM_1 B$, or his total rent plus commuting cost would

be c_{1s} - $tM_1 B$ + $tM_2 B$. Using Equation (A.5') and the implicit result above that $c_{1u} < c_{1s}$,

$$c_{2u} = c_{1u} + t(M_2 - M_1)x_u < c_{1s} + t(M_2 - M_1)B;$$

Case 3: *rich households live in the urban area, poor households live in the suburbs, there is no*

undeveloped land and no income overlap: $\pi B^2 = Na(1-\theta)$.

This case is straightforward and is omitted for brevity.

Case 4:   rich households fill the urban area and live in both communities, and poor households

form the majority in the suburbs: $\pi B^2 < Na(1 - \theta) < \frac{1}{2}\pi(Y^2 + B^2)$.

Rich households choose the urban public service and have lower commuting costs if they locate in the

urban area. Ceteris paribus rich households therefore prefer the urban area and there is no undeveloped

urban land. The equations which define equilibrium are written as:

land market:
$$\pi Y^2 = Na;$$
(A.12)

$$\pi y_s^2 = (1 - \theta)Na;$$
(A.13)

cost at suburban fringe: $c_{1s} = r_0 + tM_1Y;$
(A.14)

urban rent must equal or exceed reservation level:
$$c_{2u} \geq r_0 + tM_2B;$$
(A.15)

rent continuity at y_s: $\quad c_{2s} = c_{1s} + t(M_2 - M_1)y_s;$
(A.16)

equal utility for rich: $\quad U(M_2 - c_{2s} - g_s + T) + V(g_s) = U(M_2 - c_{2u} - g_u + T) + V(g_u);$ (A.17)

urban voting: $\quad U'(M_2 - c_{2u} - g_u + T) = V'(g_u);$
(A.18)

suburban voting: $\quad U'(M_1 - c_{1s} - g_s + T) = V'(g_s);$
(A.19)

return of rents:
$$T = \frac{1}{N}\left[\int_0^B \frac{c_{2u} - tM_2s}{a} 2\pi s ds + \int_B^{y_s} \frac{c_{2s} - tM_2s}{a} 2\pi s ds + \int_{y_s}^Y \frac{c_{1s} - tM_1s}{a} 2\pi s ds \right].$$
(A.20)

In addition, we require that a poor suburban household prefers the suburbs. We note that g_s maximizes

$U(M_1 - c_{1s} - g + T) + V(g)$ and that, if a poor suburban household were to move to the urban area, he

would move to the outermost location and pay rent $c_{2u} - tM_2B$. Therefore self-selection requires

$$\max_g [U(M_1 - c_{1s} - g + T) + V(g)] \geq U(M_1 - c_{2u} + tM_2B - tM_1B - g_u + T) + V(g_u).$$

We restrict attention to values of c_{2u} satisfying Inequality (A.15) and use the Equations (A.12)-(A.14), (A.16), (A.18)-(A.20) to write $c_{2s}, c_{1s}, g_u, g_s, y_s, Y$ and T as functions of c_{2u}. The Jacobian determinant associated with Equations (A.12)-(A.14), (A.16), (A.18)-(A.20) is

$-(2\pi)^2 y_s Y(U''_{1s} U''_{2u} + U''_{2u} V''_s + U''_{1s} V''_u + V''_u V''_s)$, where the subscript on U'' and V'' is used to indicate the argument, e.g. $U''_{1s} = U'' (M_1 - c_{1s} - g_s + T)$ or $V''_u = V''(g_u)$. Provided

$M_1 - c_{1s} + T > 0$ and $M_2 - c_{2u} + T > 0$, the second derivatives exist and the Jacobian determinant is non-zero: by the Implicit Function Theorem, it is possible to write $c_{2s}, c_{1s}, g_u, g_s, y_s, Y$ and T as continuous functions of c_{2u}. $M_1 - c_{1s}$ is assured by the assumption that the poor have sufficient income to live in the suburbs. Equations (A.14) and (A.16) assure that all rents are positive (given Inequality (A.15)) and hence $T > 0$. Hence a sufficient condition to allow use of the Implicit Function Theorem is $M_2 - c_{2u} > 0$.

For $c_{2u} < M_2$, define

$$R(c_{2u}) = \underset{g}{\max}[U(M_2 - c_{2u} - g + T(c_{2u})) + V(g)]$$
$$- [U(M_2 - c_{2s}(c_{2u}) - g_s(c_{2u}) + T(c_{2u})) + V(g_s(c_{2u}))].$$

Noting that g_u maximizes $U(M_2 - c_{2u} - g + T) + V(g)$, a solution to Equations (A.12)-(A.20) implies that there exists c^B: $c^B \geq r_0 + tM_2X$ such that $R(c^B) = 0$.

Note that Equations (A.12)- (A.14) and (A.16) imply that y_s, Y, c_{2s} and c_{1s} are constant, independent of c_{2u}. First, consider the possibility $c_{2u} = c_{2s}$. Using Equation (A.16) and

$$M_2 - c_{2u} = M_2 - c_{1s} - t(M_2 - M_1)y_s = M_1 - c_{1s} + (1 - ty_s)(M_2 - M_1) > M_1 - c_{1s},$$

- 37 -

the assumption $M_1 - c_{1s} > 0$ implies that the proposed relationships $T(c_{2u})$ and $g_s(c_{2u})$ exist. Using Equations (A.14) and (A.16), if $c_{2u} = c_{2s}$,

$$c_{2u} = c_{2s} = c_{1s} + t(M_2 - M_1)y_s$$

$$= r_0 + tM_1Y + t(M_2 - M_1)y_s = r_0 + tM_2y_s + tM_1(Y - y_s) > r_0 + tM_2B.$$

or Inequality (A.15) is satisfied. $M_2 - c_{2u} > M_1 - c_{1s}$ implies $g_u > g_s$, and hence $R(c_{2s}) > 0$.[26]

Second, increase c_{2u} above c_{2s} (but maintaining $c_{2u} < M_2$). By assumption, $M_1 - c_{1s} > 0$, and hence, as c_{2u} increases, $M_1 - c_{1s} - g_s + T > 0$.[27] In addition,

$$M_2 - c_{2s} = M_2 - c_{1s} - t(M_2 - M_1)y_s = M_1 - c_{1s} + (1 - ty_s)(M_2 - M_1) > M_1 - c_{1s},$$

or $M_2 - c_{2s} - g_s + T > 0$. Therefore, as c_{2u} increases, $U(M_2 - c_{2s} - g_s + T) + V(g_s) > -\infty$.

However, noting that Y, y_s, c_{1s} and c_{2s} are constant and integrating Equation (A.20) gives

$$T = A + \frac{B^2}{Y^2}c_{2u}$$

where A is a constant. Therefore

$$M_2 - c_{2u} + T = M_2 - (1 - \frac{B^2}{Y^2})c_{2u} + A ,$$

or, as c_{2u} increases, $M_2 - c_{2u} + T$ decreases until, at $c_{2u} = (M_2+A)/(1 - (B^2/Y^2))$, it equals zero.

Therefore

$$\max_{g} \left[U(M_2 - c_{2u} + A + \frac{B^2}{Y^2}c_{2u}) + V(g) \right]\Big|_{c_{2u} = \frac{M_2 + A}{1 - \frac{B^2}{Y^2}}} = -\infty ,$$

or

$$R\left(\frac{M_2 + A}{1 - \dfrac{X^2}{Y^2}}\right) = -\infty.$$

By continuity, there exists some c_{2u} denoted c^B, $c_{2s} \leq c^B < (M_2 + A) / (1 - (B^2)/(Y^2))$, such that

$R(c^B) = 0$. This is an equilibrium (provided self-selection is satisfied)..

To show self-selection, we show that a poor household would have an increase in rent plus

commuting cost if he moved to the urban area. If he moved, he would locate at the outer urban boundary

and pay rent c_{2u} - $tM_2 B$; his total rent plus commuting cost would be c_{2u} - $tM_2 B + tM_1 B$. Using Equation

(A.16) and the implicit result above that $c_{2s} < c_{2u}$,

$$c_{1s} = c_{2s} - t(M_2 - M_1)y_s < c_{2u} - t(M_2 - M_1)B.$$

Case 5: rich households form the majority in both communities: $\dfrac{1}{2}\pi(Y^2 + B^2) \leq Na(1 - \theta)$.

If rich households are the majority in both communities, they choose the public service level in each

community to maximize their utility, and equilibrium requires

$$\max_g \left[U(M_2 - c_{2u} - g + T) + V(g) \right] = \max_g \left[U(M_2 - c_{2s} - g + T) + V(g) \right],$$

or $c_{2u} = c_{2s}$ and $g_u = g_s$. Provided a poor household has sufficient income to live in the suburbs

$(M_1 > r_0 + tM_1 Y + g_s)$, it is therefore exactly "as if" there is a single community for which it is

straightforward to show that an equilibrium exists.

APPENDIX B: PROOF OF LEMMAS

PROOF OF LEMMA A: consider two locations in a community, s_A and s_B with $s_A < s_B$. Suppose that

households of income M_A are located at s_A and households of income M_B are located at s_B. At equilibrium,

households of income M_A cannot get more utility by moving to location s_B or

$$U(M_A - tM_A s_A - r(s_A) - g + T) + V(g) \geq U(M_A - tM_A s_B - r(s_B) - g + T) + V(g),$$

or $\qquad tM_A s_A + r(s_A) \leq tM_A s_B + r(s_B)$.

Similarly, households of income M_B cannot get more utility by moving to location x_A or

$$U(M_B - tM_B s_B - r(s_B) - g + T) + V(g) \geq U(M_B - tM_B s_A - r(s_A) - g + T) + V(g),$$

or $\qquad tM_B s_B + r(s_B) \leq tM_B s_A + r(s_A)$.

Combining

$$tM_B(s_B - s_A) \leq r(s_A) - r(s_B) \leq tM_A(s_B - s_A).$$

But by assumption $s_A < s_B$ and $O \leq M_A, M_B$; it follows that $M_B \leq M_A$.

PROOF OF LEMMA B:

(a) Decreasing rents. Consider two locations in a community, s_A and s_B with $s_A < s_B$. If $r(s_A) < r(s_B)$, than for all M

$$U(M - tMs_A - r(s_A) - g + T) + V(g) > U(M - tMs_B - r(s_B) - g + T) + V(g)$$

or all households prefer location s_A to s_B, an outcome which contradicts equilibrium.

(b) *Continuity*: Suppose there is a discontinuity at s_C such that $\lim_{s \to s_C^-} r(s) = \lim_{s \to s_C^+} r(s) + a$

with $a > 0$. A household locating at the inner side of the discontinuity achieves

utility $\lim_{s \to s_C^-} U(M - tM(s) - r(s) - g + T) + V(g)$. If the household were to move to the outer side of the discontinuity, it would achieve utility

$$\lim_{s \to s_C^+} U(M - tM(s) - r(s) - g + T) + V(g)$$

$$= \lim_{s \to s_C^-} U(M - tM(s) - r(s) + a - g + T) + V(g),$$

or the household could increase its utility by moving. This contradicts equilibrium.

PROOF OF LEMMA C: The proof is by contradiction. The maintained assumption is that poor and rich

households live in both communities.

(a) *Suppose that rich households form the majority in the urban area and that poor households*

form the majority in the suburbs. A poor household obtains the same utility in the urban area and in the

suburbs, or

$$U(M_1 - c_{1u} - g_u + T) + V(g_u) = \max_{g} [\, U(M_1 - c_{1s} - g + T) + V(g)\,], \quad \text{or} \quad c_{1s} \geq c_{1u}. \quad \text{(B.1)}$$

A rich household must obtain the same utility in either community, or

$$\max_{g} [\, U(M_2 - c_{2u} - g + T) + V(g)\,] = U(M_2 - c_{2s} - g_s + T) + V(g_s), \quad \text{or} \quad c_{2u} \geq c_{2s}. \quad \text{(B.2)}$$

Rent continuity at y_s implies

$$c_{2s} - tM_2 y_s = c_{1s} - tM_1 y_s, \qquad \text{or} \qquad c_{2s} = c_{1s} + t(M_2 - M_1)y_s. \qquad \text{(B.3)}$$

Similarly, rent continuity at x_u implies

$$c_{2u} - tM_2 x_u = c_{1u} - tM_1 x_u, \qquad \text{or} \qquad c_{1u} = c_{2u} - t(M_2 - M_1)x_u. \qquad \text{(B.4)}$$

Hence, combining Inequality (B.2) and Equations (B.3) and (B.4)

$$c_{1u} = c_{2u} - t(M_2 - M_1)x_u \geq c_{2s} - t(M_2 - M_1)x_u = c_{1s} + t(M_2 - M_1)(y_s - x_u) > c_{1s},$$

which contradicts Inequality (B.1).

(b) *suppose that poor households form the majority in both communities.* A poor household obtains

the same utility in the urban area and in the suburbs or

$$\max_{g} [\, U(M_1 - c_{1u} - g + T) + V(g) \,] = \max_{g} [\, U(M_1 - c_{1s} - g + T) + V(g) \,].$$

Hence $c_{1u} = c_{1s}$ and $g_u = g_s$. Using the rent continuity equations at y_s and x_u:

$$c_{2u} = c_{1u} + t(M_2 - M_1)x_u = c_{1s} + t(M_2 - M_1)x_u = c_{2s} - t(M_2 - M_1)(y_s - x_u) < c_{2s} \quad \text{(B.5)}$$

At equilibrium a rich household must obtain the same utility in either community, or

$$U(M_2 - c_{2u} - g_u + T) + V(g_u) = U(M_2 - c_{2s} - g_s + T) + V(g_s).$$

With $g_u = g_s$, this implies $c_{2s} = c_{2u}$. This contradicts Inequality (B.5)

(c) *suppose that rich households form the majority in both communities.* A rich household obtains

the same utility in the urban area or in the suburbs, or

$$\max_{g} [\, U(M_2 - c_{2u} - g + T) + V(g) \,] = \max_{g} [\, U(M_2 - c_{2s} - g + T) + V(g) \,].$$

Hence $c_{2u} = c_{2s}$ and $g_u = g_s$. Using the rent continuity equations at y_s and x_u:

$$c_{1u} = c_{2u} - t(M_2 - M_1)x_u = c_{2s} - t(M_2 - M_1)x_u = c_{1s} + t(M_2 - M_1)(y_s - x_u) > c_{1s} \quad \text{(B.6)}$$

A poor household must obtain the same utility in either community, or

$$U(M_1 - c_{1u} - g_u + T) + V(g_u) = U(M_1 - c_{1s} - g_s + T) + V(g_s).$$

With $g_u = g_s$, this implies $c_{1u} = c_{1s}$. This contradicts Inequality (B.6).

PROOF OF LEMMA D: The proof is by contradiction. Suppose there is undeveloped urban land. If

poor households form the majority in the urban area, a poor household at the urban fringe achieves utility

$\max_{g} [U(M_1 - r_0 - tM_1X - g + T) + V(g)]$. A poor household at the suburban fringe achieves utility

$U(M_1 - r_0 - tM_1Y - g_s + T) + V(g_s)$. But $X < Y$ implies

$$\max_{g} [U(M_1 - r_0 - tM_1X - g + T) + V(g)] > U(M_1 - r_0 - tM_1Y - g_s + T) + V(g_s)$$

which is inconsistent with an equilibrium in which the poor live in both communities.

PROOF OF LEMMA E: If y_s is pre-determined, Equations (10)-(13), (15)-(19) constitute nine

equations in the nine unknowns $c_{2u}, c_{1u}, c_{2s}, c_{1s}, g_u, g_s, x_u, Y$ and T. The Jacobian associated with

the equation set is

$$4\pi^2 Y x_u \left[\frac{B^2}{Y^2}(U''_{1u} + V''_u)(U''_{2s}V'_s + V''_s U'_{1s}) + (1 - \frac{B^2}{Y^2})(U''_{2s} + V''_s)(U''_{1u}V'_u + V''_u U'_{1u}) \right]$$

where the subscript on U', V', U'' and V'' are used to indicate the argument (e.g.,

$U''_{1s} = U''(M_1 - c_{1s} - g_s + T)$). Footnote 18 shows that $M_1 - c_{1s} - g_s + T > 0$ (implied by

Inequality (28)) is sufficient to ensure that all consumptions are positive and hence that the derivatives

exist. Provided $x_u > 0$, the Jacobian determinant is non-zero:[28] by the Implicit Function theorem, it is

possible to write $c_{2u}, c_{1u}, c_{2s}, c_{1s}, g_u, g_s, x_u, Y$ and T as continuous functions of y_s, $y_s \in [y_s^A, y_s^B]$.

For $y_s \in [y_s^A, y_s^B]$, define

$$S(y_s) = U(M_2 - c_{2s}(y_s) - g(y_s) + T(y_s)) + V(g(y_s))]$$

$$- [U(M_2 - c_{1u}(y_s) - g_u(y_s) + T(y_s)) + V(g_u(y_s))].$$

With $c_{2u}, c_{1u}, c_{2s}, c_{1s}, g_u, g_s, x_u, Y$ and T being continuous functions of y_s (and M_1 being sufficient to ensure positive consumption), $S(y_s)$ is a continuous function of y_s. By assumption, $S(y_s^A) > 0$ and $S(y_s^B) < 0$. Hence, as y_s changes from y_s^A to y_s^B, there must be at least one y_s at which $S = 0$.

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ENDNOTES

1. Ross and Yinger (1999) review this literature.

2. If the income elasticities of land demand and of commuting costs are equal, the relationship between household income and distance from the metropolitan center is indeterminate. This is the case considered theoretically by Montesano (1972) and considered statistically relevant by Wheaton (1977).

3. Table 31990 Census of Population, Social and Economic Characteristics. Washington, D.C.: U.S. Department of Commerce.

4. Many cities have business districts dispersed throughout the metropolitan area in addition to the central business district. Our model of a circular metropolitan area and a central business district is therefore stylized. It is constructed to show how capitalization at different locations occurs at different rates and how this allows the income distribution in different communities to overlap. The logic can be extended to more complex spatial patterns.

5. For convenience of presentation, the public service is assumed to show constant returns to community population. Because each community contains a fixed number of households, no results change if the service is a local public good.

6. Without the fixed size assumption, demand for the public good would vary within an income class because housing price and income net of commuting cost vary over space.

7. Epple, Filimon, and Romer (1984) describe how such an infinite regress can arise with myopic voting.

8. Lemma A is a consequence of assuming that the income elasticity of land demand is zero and that per mile commuting costs increase with income. As noted in the Introduction, the pattern of sorting within a community is determined by comparing the income elasticity of land demand with the income elasticity of per mile transportation costs (Wheaton (1977)).

9. For a more general development, see Fujita (1989, Chap. 4).

10. Because c_{ij} is a constant for all households of income M_i living in community j, the analysis focuses on c_{ij} and not on the rent schedule $r(s)$. However, the full rent schedule is:
$r(s: s=Y) = r_0$. $r(s: B < s \leq Y) = c_{1s} - tM_1 s$. If $x_u < X = B$, $r(s: x_u \leq s \leq B) = c_{1u} - t M_1 s$.
If $x_u < X < B$, $r(s: s = X) = r_0$ and $r(s: x_u \leq s \leq X) = c_{1u} - t M_1 s$. If $x_u = X < B$, $r(s: s = x_u) = r_0$.
$r(s: 0 \leq s \leq x_u) = c_{2u} - t M_2 s$.

11. I.e., $\displaystyle \lim_{s \to x_u^-} r(s) = \lim_{s \to x_u^+} r(s)$.

12. This assumption is not important *per se*. What is important is that the two communities choose different public service levels. In addition, the proof of Proposition 1 requires that public service levels change continuously with c_{2u} and c_{1s}.

13. It is simple to change the model to allow rents to be paid to absentee landlords.

14. This restriction is unnecessarily strong if $T > 0$.

15. In this example the restriction $M_1 > r_0 + tM_1\hat{Y}$ implies $t < .095$

16. In particular we continue to assume that all houses have the same fixed size a. However, if housing is made endogenous, our result that equilibria can exist in which there is income mixing between communities is still true (de Bartolome and Ross (2000)).

17. The rent schedule is: $r(s: s = Y) = r_0$; $r(s: y_s < s \le Y) = c_{1s} - t M_1 s$;
$r(s: B < s \le y_s) = c_{2s} - t M_2 s$; $r(s: x_u < s \le B) = c_{1u} - t M_1 s$; $r(s: 0 < s \le x_u) = c_{2s} - t M_2 s$.

18. This is sufficient to ensure all households have positive consumption. Viz. using
$M_1 - c_{1s} - g_s + T > 0$ and the assumed form of the utility function, Equation (13) implies
$M_1 - c_{1u} - g_u + T > 0$. Equation (12) and $tY < 1$ implies
$M_2 - c_{2s} - g_s + T = M_1 - c_{1s} + (M_2 - M_1)(1 - ty_s) - g_s + T > M_1 - c_{1s} - g_s + T > 0$.
Equation (11) implies
$M_2 - c_{2u} - g_u + T = M_1 - c_{1u} + (M_2 - M_1)(1 - tx_u) - g_u + T > M_1 - c_{1u} - g_u + T > 0$.

19. In Figure 4 D lies above G which lies above A, or rent at the urban boundary necessarily equals or exceeds r_0. Viz, because poor households are the urban majority, Equation (13) is rewritten as:

$$\underset{g}{Max} \quad U(M_1 - c_{1u} - g + T) + V(g) = U(M_1 - c_{1s} - g_s + T) + V(g_s),$$

or $c_{1u} \ge c_{1s}$, or $r(B) + tM_1B \ge r_0 + tM_1Y$, or $r(B) > r_0$.

20. Without redistributed rents, the welfare of analysis of the two groups is fairly straightforward because the welfare of poor households is anchored by the rent and commute associated with living at the outer edge of the suburbs. However, such an approach cannot be used to consider the efficiency of the two possible equilibria since it ignores the welfare effects on absentee landlords. de Bartolome and Ross (1999) examine such welfare effects using a consumer surplus utility function and specific parameter values.

21. It is possible to envisage a situation in which community sizes are such that the mixing equilibrium gives better matching of households with public service levels. For example, in the sorting equilibrium the rich may only just form the majority in the urban area so that many poor households have "too much" public service. If the suburbs are only just larger than the number of rich households, it is possible to envisage a mixing equilibrium in which rich households almost fill the suburbs and poor households almost fill the urban area. In this case, the better matching could dominate. We have not

been able to find this equilibrium with the Cobb-Douglas utilities. Note that this outcome is impossible if rents are not redistributed because one group is always either being moved to a smaller jurisdiction or experiences a decrease in the number of jurisdictions in which it is a majority. However, both groups may benefit from a better allocation if rents are redistributed.

22. Historically, the increase in metropolitan populations was accompanied by an improvement in transport technology lowering commuting costs. LeRoy and Sonstelie (1983) suggest how the history of advances in urban transportation might have led the observed outcome to be equilibrium with income mixing between communities.

23. Note that Equations (A.4) and (A.5) are obtained by rearranging the order of Equations (4a)-(4c) and (5a), (5b).

24. $U(M_2 - c_{2u} - g + T) + V(g)$ is strictly concave in g so that the value of g which maximizes $U(M_2 - c_{2u} - g + T) + V(g)$ is unique.

25. This implies his utility is higher in the urban area as the urban public service level is set to maximize his utility.

26. $U(M_2 - c_{2u} - g + T) + V(g)$ is strictly concave in g so that the value of g which maximizes $U(M_2 - c_{2u} - g + T) + V(g)$ is unique.

27. g_s is being chosen to maximize $U(M_1 - c_{1s} - g + T) + V(g)$ where $U(c^h: c^h \leq 0) = -\infty$.

28. The requirement $x_u > 0$ reflects the discontinuity at $x_u = 0$.