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Technology Life-Cycles and Endogenous Growth

Murat F. Iyigun
Department of Economics, University of Colorado at Boulder
Boulder, Colorado

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Center for Economic Analysis
Department of Economics

University of Colorado at Boulder
Boulder, Colorado 80309

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Murat F. Iyigun
University of Colorado at Boulder
murat.iyigun@colorado.edu

Abstract

I develop a growth model in which R&D resources can be directed either to the discovery of new technologies (inventions) or to the improvement of the quality of machines without altering their underlying technology (innovations). Learning-by-doing is an important determinant of the relative share of resources allocated to inventive versus innovative activity. The dynamics generate endogenous economic growth driven by cycles of technological change where the pattern and timing of technological improvements are consistent with historical evidence. That is, (a) inventions and innovations play complementary roles in expanding the technology frontier; (b) when inventions occur they tend to arrive in clusters; and (c) a life cycle of technologies during the early stages of which a discovery is followed by a period of rapid economic growth and innovation, and the late stages of which dwindling innovations and slower growth set the stage for new discoveries.

Keywords: inventions, innovations, learning-by-doing, human capital.

JEL Classification Numbers: I20, J24, O11, O31, O40.

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Please send all correspondence to the University of Colorado at Boulder, Department of Economics, Campus Box 256, Boulder, CO 80309-0256. Phone: (303) 492-6653. Fax: (303) 492-8622.

1. Introduction

Besides the fact that rapid and sustained technological change is a relatively recent phenomenon, the history of technological progress reveals three distinct empirical regularities. First, there exists a strong complementarity between inventions and innovations in expanding the technology frontier. Without inventions, the innovative process will eventually be subject to decreasing returns, and absent productivity-enhancing innovations, new technologies may never be adopted. Second, major breakthroughs in technology arrive infrequently and in clusters. It is well documented, for example, that the height of the ancient Greek civilization between 400 B.C. and 100 A.D., the Ming dynasty rule in China during the 14th century, and more recently, the Industrial Revolution of the 18th century were periods during which many new and path-breaking discoveries were made. Finally, every new technology appears to go through a three-period life-cycle; the *early stages,* during which the newly discovered technology is adopted and the potential benefits of learning-by-doing are largest, the *improvement phase* in which rapid innovations make the existing technology more efficient and accessible, and the *maturity phase* during which dwindling innovative activity and slowing productivity gains set the stage for the discovery of new technologies.[1]

Despite these observations, the literature on economic growth and technological progress has, for the most part, continued to remain disjoint. In one of the very few exceptions, Young (1993) points out that, on the one hand, there exits models of invention in which technological progress is the outcome deliberate and costly R&D effort.[2] On the other hand, there are a number of other influential papers in which technological change occurs mostly as a by-product of learning-by-doing in the production of goods.[3] Moreover, when the emphasis is on deliberate and costly R&D effort as the prime engine of technological progress, an effort to distinguish between what Mokyr (1990) defines as

[1] To quote from Mokyr (1990, pp. 297-98), "The main reason why I have dwelled so long on the distinction between micro- [innovations] and macroinventions [inventions] is that both were very much part of the story. Here, then, is the most fundamental complementarity of the economic history of technological change. Without big new ideas, the drift of cumulative small inventions will start to run into diminishing returns...But more is involved. The historical survey has indicated that macroinventions rarely occurred alone, but rather tended to appear in clusters."

[2] For example, Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992).

[3] See, for example, Arrow (1962) and Lucas (1993).

macro- and microinventions (or inventions versus innovations) hardly exists.

In this paper, I develop a hybrid economic growth model in which resources can be directed to both the discovery of new technologies (inventions) and the improvement of existing ones (innovations). While both types of technological progress lead to quality improvements, the former involves the generation of new ideas and techniques which result in discrete jumps up the quality ladder–a la Grossman-Helpman and Aghion-Howitt, whereas the latter reflects more the type of Schumpeterian progress which manifests itself in small refinements to existing inventions.[4] By allowing serendipitous learning to co-exist with deliberate and costly R&D effort aimed at generating new inventions as well as innovations, I show how learning-by-doing guides the allocation of resources between inventive and innovative R&D activities and helps to generate cycles of technological change and endogenous growth.

More specifically, in the model I present below, the process of learning-by-doing alters the R&D incentives for the discovery of new technologies differently than those for improving existing ones. This is due to the fact that potential productivity gains from learning-by-doing is high but unrealized for newly discovered technologies. Thus, new inventions spur R&D effort aimed at the discovery of even more sophisticated technologies as, in periods immediately following the introduction and adoption of new technologies, their productivity remains relatively low. This raises the likelihood that a new invention will be made soon after the discovery of a new technology. However, if entrepreneurial efforts fail to generate a new invention following a recent discovery, productivity gains rise rapidly as a result of learning-by-doing with the existing technology. Consequently, the incentives to innovate and improve existing technologies–by introducing newer generation more efficient machines without altering the underlying technology–rise relative to the incentives to generate new ideas and technologies. This implies that, when the episode of new discoveries come to an eventual end, the likelihood of entering a phase of intensified innovative activity rises. Nonetheless, the productivity gains and the expected entrepreneurial profits associated with the introduction of newer generation machines are bounded from above as they are constrained by the underlying sophistication of the tech-

[4] Schumpeter (1934).

nology in use. As a result, while monopoly profits from the discovery of a new invention can remain positive in the long run, those from innovation monotonically approach zero with the introduction of each new generation machine. And when a technological breakthrough is made long after the introduction of an existing technology, it is likely to be in the form of a new discovery. Thus, the dynamics of the model below are such that they generate endogenous and stochastic growth cycles, which are driven at times by discovery and invention episodes, and at others by spurts of innovative activity. While the stochastic nature of the model allows long periods of technological stagnation during which neither new technologies are invented nor existing older ones are improved upon, the distinction between inventive versus innovative R&D activities and the impact of learning-by-doing on R&D incentives help to generate empirically consistent patterns of technological change.

In addition to the papers mentioned above, this paper is related to some others which have combined learning-by-doing with product innovation and endogenous growth. In a leading example, Stokey (1988) incorporates the introduction of newer goods into a learning-by-doing model. She shows how the set of goods produced varies according to the knowledge stock of the economy, which in turn is augmented by cumulative production experience. Thus, in her model, as cumulative work experience rises newer goods are introduced, older ones are dropped, and economic growth is endogenously sustained. Absent in the model, however, is deliberate and costly R&D effort aimed at either the discovery of new technologies or the improvement of existing ones.

Finally, because one can interpret the R&D effort directed at the discovery of totally new technologies as "ideas" or "basic research" and those directed at product or input innovation as "applied research," the model I present below is also related to papers that highlight the distinction between the creation of ideas and applied R&D. Helpman and Tratjenberg (1994), for example, develop a cyclical growth model in which the development of each new major discovery prompts investment in complementary inputs. The cyclicality of the long-run dynamics arises because productivity gains start to accumulate only after enough compatible inputs become available. While Helpman and Tratjenberg focus on the strategic complementarities between innovations and inventions, like I do here, they do not incorporate learning-by-doing into their model and therefore do not address what impact learning-by-doing might have on the relative resources allocated to inventive versus innovative activities. In a similar vein, Aghion and Howitt (1996) distinguish research that produces fundamental knowledge from development which generates secondary knowledge. They demonstrate that economic growth is positively correlated with research even in extreme cases where fundamental knowledge is created indirectly by secondary knowledge. Like Helpman and Tratjenberg, however, their model does not incorporate learning-by-doing.

The remainder of the paper is organized as follows: Section 2 describes the model, Section 3 discusses its dynamic behavior, Section 4 provides a couple of numerical examples in order to highlight the main results, and Section 5 concludes.

2. The Economy

2.1. Production

Consider an economy in which real economic activity extends over an infinite discrete time. Production is carried out by a continuum of firms indexed by j, $j \in [0,1]$. Each firm produces a single homogenous good, y_t^j, using a machinery aggregate, M_t^j, and a labor aggregate, L_t^j, with the following constant returns to scale (CRS) production technology:

$$y_t^j = A_t \, (M_t^j)^{\alpha} \, (L_t^j)^{1-\alpha}, \qquad 0 < \alpha < 1 \tag{1}$$

where A_t denotes an endogenously determined productivity parameter for the underlying level of technology of vintage v, $0 \leq v \leq t$.[5] By definition, aggregate output at time t, Y_t, equals $\int_0^1 y_t^j \, dj$.

The labor market is competitive. Thus, the wage rate paid to labor, w_t, equals its marginal product:

[5] I elaborate on the determination of these variables in more detail in section 2.3.

$$w_t = (1-\alpha)\ A_t\ \left(\frac{M_t}{L_t}\right)^{\alpha}. \tag{2}$$

2.2. Individuals

Individuals live for two periods in overlapping generations. They are endowed with one unit of time in every period. In both periods, individuals supply their labor inelastically; in the first period of life, they work and save, and in the second period, they work, dissave and consume. Individuals' preferences are represented by a utility function that is linear in consumption in the second period.[6] There is no population growth.

2.3. The Technology and Potential versus Actual Productivity

Firms must purchase new machines in every period t because machines depreciate fully in one period. Let z_t and q_t respectively denote the quality and quantity of machines utilized in production at time t, and let ϕ_t represent the underlying level of technological sophistication in the same period.[7] I assume that the machinery aggregate used in production at time t, M_t, is given by the following:

$$M_t \ = \ \frac{\phi_t\ z_t\ q_t}{\alpha^{\frac{1}{\alpha}}}. \tag{3}$$

Equation (3) implies that the machinery aggregate, M_t, increases with the number and quality of machines used in production as well as with their underlying level of technology.

Based on this specification, technological progress can come about in two ways: inventions and innovations. The former is the discovery of new technologies (or leaps

[6] This assumption pins down the interest rate at the discount rate. Neither relaxing this assumption nor allowing consumption in the first period would materially affect the main results.

[7] One can think of the term $\phi_t z_t$ as the overall quality of a given machine. With this formulation, I am essentially adopting the notion that changes in each machine's overall quality can be decomposed into two components, one of which is conditional on the level of the existing technology and a residual which is directly determined by the technology itself.

up the quality ladder a la Grossman-Helpman and Aghion-Howitt), while the latter involve the design of the next generation of machines in an attempt to improve machine quality without altering the underlying sophistication of the level of technology. To be more specific, I assume that a new invention moves the underlying level of technological sophistication per machine one step up the quality ladder. In particular, when there is an invention, the technology level ϕ_t increases such that $\phi_t = \bar{g}\phi_{t-1}$, where $\bar{g} > 1$.[8] In contrast, for any given level of underlying technology, ϕ_t, there exists a maximum attainable machine quality which I will denote by $\bar{z}$. The introduction of each new generation machine brings the actual machine quality closer to this maximum $\bar{z}$. More specifically, I assume that

$$z_t \;=\; \bar{z}\exp\left(-\frac{1}{x_t}\right), \tag{4}$$

where x_t, $x_t = 1, 2, 3...\infty$, is an index which identifies the machine generation used in production. The specification in (4) implies that

$$\lim_{x_t \to 1} z_t = \bar{z}\exp(-1) < \bar{z} \qquad \text{and} \qquad \lim_{x_t \to \infty} z_t = \bar{z}. \tag{5}$$

Even when there are no new inventions or innovations that raise the quality of machines available to the consumption-good producers, actual productivity can still improve through learning-by-doing that allows more efficient use of the existing vintage of machines. Thus, the older the technology, the more efficient its use:

$$A_t \;=\; \bar{A}\exp\left(-\frac{1}{1+t-v}\right), \tag{6}$$

[8]For simplicity, I have assumed that, whenever a discovery of a new technology is made, the leap up the ladder is by the same proportion. Under a more realistic setup in which the size of the jump or the number of jumps up the technology ladder in any one period is allowed to vary, the qualitative nature of the results presented below would be unchanged.

where $t - v$ is the length of time machines with underlying technology level of vintage v have remained in use. Note that, as with the effect of the introduction of a new generation machine on actual quality, the specification in (6) implies that

$$\lim_{(t-v)\to 0} A_t \;=\; \bar{A}\exp(-1) \;<\; \bar{A} \quad \text{and} \quad \lim_{(t-v)\to\infty} A_t \;=\; \bar{A} \tag{7}$$

Thus, to the extent that a given technology of vintage v stays in use and does not become obsolete, learning-by-doing allows the productivity derived from the use of a technology, A_t, to converge to its potential, $\bar{A}$. It is also important to note at this point that, since a first generation machine is introduced with each new invention that moves the underlying technological sophistication level up one notch, $t - v = 0 \Rightarrow x_t = 1$. To put it somewhat differently, a new invention can be interpreted as the introduction of the very first generation machine which utilizes a new underlying technology.

Figure 1 shows the effect of learning-by-doing on the productivity parameter, A_t. Figure 2 shows how the introduction of the next generation machine improves machine quality.

[Figures 1 and 2 about here.]

Before moving on to firms' decision to adopt new technologies versus newer generation machines of a given vintage, it is illuminating to examine the evolution of the economy described so far under three exogenously determined, non-stochastic, regimes. First, consider a world in which there is no technological change (either in the form of new inventions which raise ϕ_t or in the innovation of newer, more efficient, machines for a given technology, z_t) but there exists only learning-by-doing. In such a world, economic growth will be driven purely by learning-by-doing and the dynamic evolution of the economy will be fully described by the only state variable of the economy, $t - v$, the length of time the exogenously given initial level of the technology, ϕ_0, remains in production. Given that $v = 0$ and the initial technology stays in use indefinitely, $\forall t \geq 0$,

$t - v = t$. Thus, the underlying productivity of the machines in use, A_t, continues to improve according to $A_{t+1} = A_t \exp[1/(t+1)(t+2)] = \bar{A} \exp[-1/(t+2)]$. and economic growth remains positive. Over time, however, growth monotonically decreases and asymptotically converges to zero as the exogenously given level of technology constrains and exhausts the potential for learning-by-doing in the long run.

Second, consider the version of the model in which learning-by-doing and innovations occur in every period. In this case, there will be two sources of economic growth; one driven by learning-by-doing and the resulting gains in productivity, and the other via the introduction of more productive and efficient machines. The dynamic evolution of this economy will be guided by the evolution of the two state variables, $t - v$, and x_t. As in the previous case, $\forall t \geq 0$, $\phi_t = \phi_0$, $v = 0$, and the initial technology stays in use indefinitely. Thus, $\forall t \geq 0$, $t - v = t$, and the underlying productivity of the machines in use, A_t, continues to improve according to $A_{t+1} = A_t \exp[1/(t+1)(t+2)] = \bar{A} \exp[-1/(t+2)]$. Moreover, because a newer generation machine is introduced in every period t, $\forall t \geq 0$, $x_t = t + 1$, and $z_{t+1} = z_t \exp[1/(t+1)(t+2)] = \bar{z} \exp[-1/(t+2)]$. Not surprisingly, this economy grows faster than the one described above as both learning-by-doing and the introduction of more efficient machines raise productivity in every period t. Nonetheless, as in the learning-by-doing-only economy, growth monotonically decreases and asymptotically converges to zero in this case as well. That is, as indicated by (5) and (7), the exogenously given technology level binds from above not only the potential for learning-by-doing but also the productivity gains from the introduction of newer generation machines.

Finally, consider a world in which learning-by-doing and inventions–instead of innovations–take place in every period. The dynamic behavior of this economy will be very different than those described above as the invention of newer technologies in every period t allows the economy to grow at a constant rate. In this case, $\forall t \geq 0$, $v = t$, $x_t = 1$, and $t - v = 0$. In words, the discovery of a newer technology in every period does not allow the economy to benefit from learning-by-doing, and the dynamic evolution of the economy is fully described by the state variable ϕ_t only. And since the technology improves in every period t such that, $\forall t \geq 0$, $\phi_t = \bar{g}\phi_{t-1}$, the constant growth rate in output (which will be a function of $\bar{g}$) will be sustained and the economy will follow a Balanced Growth Path (BGP) in the long run.

2.4. Adoption of New Technologies and Next Generation Machines

The decision of a firm j, $j \in [0,\,1]$, is

$$\max_{\phi_t^j,\, z_t^j,\, q_t^j,\, L_t^j} \; A_t (M_t^j)^{\alpha} (L_t^j)^{1-\alpha} \; - p_t q_t^j - \; w_t L_t^j, \tag{8}$$

where A_t is given by (6) and p_t denotes the price per machine, which the firm takes as given. The solution to this problem yields, $\forall\ j \in [0,\,1]$,

$$q_t^j = \left[\bar{A} \exp\left(-\frac{1}{1+t-v} \right) \frac{(\phi_t^j z_t^j)^{\alpha} (L_t^j)^{1-\alpha}}{p_t} \right]^{\frac{1}{1-\alpha}}. \tag{9}$$

As (9) implies, the demand for machines is strictly increasing in their vintage, $t-v$, and the number of times machines with that underlying technology has been improved, x_t.

Lemma 1: $\frac{\partial q_t^j}{\partial (t-v)} > 0$ and $\frac{\partial q_t^j}{\partial x_t} > 0$.

Proof:

$$\frac{\partial q_t^j}{\partial x_t} \; = \; \frac{1}{1-\alpha} \frac{q_t^j}{(1+t-v)^2} \; > \; 0, \tag{10}$$

and,

$$\frac{\partial q_t^j}{\partial (t-v)} \; = \; \frac{\alpha}{1-\alpha} \, \frac{q_t^j}{x_t^2} \; > 0. \tag{11}$$

□

If at any given time t, machines embed a new invention or an innovation, a single firm will own the patent for them. For older vintages of technology, I assume that any prior patents have expired, the blueprints are readily available, and that any firm can produce machines that embed old technology at the constant marginal cost, c.[9]

Given that older vintages of technology are always available at a lower price, there is no guarantee that firms will prefer to buy machines which incorporate the newest inventions or innovations at a monopoly price. Firms will only be willing to pay a premium for new technology if the resulting increase in efficiency is large enough. Even when the net efficiency gains warrant final-goods producers to switch to the newest technology machines, however, the monopolist may not be able to charge c/α, which is the unconstrained optimal monopoly markup given the isoelastic demand for machines defined by (9). Hence, the monopolist's markup would equal the smaller of c/α and that which would make final goods producers indifferent between buying newer machines sold at monopoly markups (either newly invented machines or next generation machines with new innovations) and older machines for which the blueprints are readily available. Let p_t^x denote the monopoly price in period t of a newer generation machine of vintage v, p_t^ϕ represent that of a new technology machine invented in period t, and let τ, $t \geq \tau \geq 0$, be the previous period in which an invention was last made.

Lemma 2:

(i)

$$p_t^x = \min\left[\frac{c}{\alpha},\ c\exp\left(\frac{\alpha}{x_t(1+x_t)}\right)\right] \geq c. \tag{12}$$

[9] I have chosen to maintain a constant marginal cost for machine production to keep the analysis focused on the relevant dynamics. If the cost of machine production was allowed to vary over time, increased sophistication of the technology would argue for an increasing cost but higher production efficiency could have a potentially offsetting effect.

(ii)

$$p_t^{\phi} = \min\left[\frac{c}{\alpha},\ c\bar{g}^{\alpha}\exp\left(-\ \frac{\alpha(x_t-1)}{x_t}\ -\ \frac{t-\tau}{1+t-\tau}\right)\right] \geq c. \qquad (13)$$

Proof: See Appendix, Section 6.1.

Note that, the monopoly price of both newly invented machines and newer generation of vintage technology machines are non-increasing in the number of times a given technology machine has been improved via innovations. The reason for this is that, existing older machines, which are available at marginal cost c, become more productive with the introduction of each new generation machine. Consequently, regardless of whether newest machines embed a totally new invention or they belong to a newer generation of existing machines, the monopolist's price declines as existing, alternative, machines become more productive. Note also that the monopoly price of machines which embed a newly discovered technology is non-increasing in the length of time the previously-superior older technology has remained in production (i.e. $t - \tau$). This is due to the fact that learning-by-doing improves the productivity of technologies which have stayed in use longer (that is, regardless of how many different generation of machines with a common underlying level of technology have been introduced over a technology's tenure).

Lemma 3: *(i)* $\forall x_t \geq \check{x} \equiv [(1-4(\alpha/\ln\alpha))^{\frac{1}{2}}-1]/2$,

$$\frac{\partial p_t^x}{\partial(t-\tau)} = 0 \qquad \textit{and} \qquad \frac{\partial p_t^x}{\partial x_t} < 0;$$

(ii) $\forall x_t \geq \hat{x}_t \equiv \{\alpha/[(t-\tau)/(1+t-\tau) - \ln\alpha - \alpha(\ln\bar{g}-1)]\}$,

$$\frac{\partial p_t^{\phi}}{\partial(t-\tau)} < 0, \qquad \frac{\partial p_t^{\phi}}{\partial x_t} < 0 \qquad \textit{and} \qquad \frac{\partial p_t^{\phi}}{\partial \bar{g}} > 0.$$

Proof: (i) $\forall x_t \geq \check{x}$, $p_t^x = c\exp\left(\frac{\alpha}{x_t(1+x_t)}\right)$ and

$$\frac{\partial p_t^x}{\partial(t-\tau)} = 0;$$

$$\frac{\partial p_t^x}{\partial x_t} = -\frac{1+3x_t}{x_t^2(1+x_t)^2}\,\alpha p_{t,v}^x < 0. \tag{14}$$

(ii) $\forall x_t \geq \hat{x}_t$, $p_t^\phi = c\bar{g}^\alpha \exp\left(-\frac{\alpha(x_t-1)}{x_t} - \frac{t-\tau}{1+t-\tau}\right)$ and

$$\frac{\partial p_t^\phi}{\partial(t-\tau)} = -\frac{p_t^\phi}{(1+t-\tau)^2} < 0,$$

$$\frac{\partial p_t^\phi}{\partial x_t} = -\frac{\alpha p_t^\phi}{x_t^2} < 0, \tag{15}$$

$$\frac{\partial p_t^\phi}{\partial \bar{g}} = \frac{\alpha p_t^\phi}{\bar{g}} > 0.$$

□

Of course, Lemmas 2 and 3 imply that monopolists' profits are smaller when they cannot charge the unconstrained optimal monopoly markup. More specifically, because the optimal markup equals c/α, monopolists' profits–either from the invention of a new technology or the introduction of newer machines–depend strictly positively on monopoly prices when the latter are below c/α. Letting q_t^x and q_t^ϕ respectively denote the demand for newly innovated and invented machines, and π_t^x and π_t^ϕ respectively represent the monopoly profits from those machines,the following hold:

Lemma 4: *(i)* $\forall x_t \geq \check{x}$,

$$\frac{\partial \pi_t^x}{\partial p_t^x} > 0;$$

(ii) $\forall x_t \geq \hat{x}_t$,,

$$\frac{\partial \pi_t^\phi}{\partial p_t^\phi} > 0.$$

Proof: (i) $\pi_t^x = (p_t^x - c)q_t^x$ and $\forall x_t \geq \check{x}$,

$$\frac{\partial \pi_t^x}{\partial p_t^x} = -\frac{\alpha}{1-\alpha}\frac{q_t^x}{p_t^x}\left(p_{t,v}^x - \frac{c}{\alpha}\right) > 0, \tag{16}$$

(ii) $\pi_t^\phi = (p_t^\phi - c)q_t^\phi$ and $\forall x_t \geq \hat{x}_{t,v}$,

$$\frac{\partial \pi_t^\phi}{\partial p_t^\phi} = -\frac{\alpha}{1-\alpha}\frac{q_t^\phi}{p_t^\phi}\left(p_t^\phi - \frac{c}{\alpha}\right) > 0. \tag{17}$$

□

2.5. Equilibrium R&D Effort in Inventions versus Innovations

Inventions and innovations are the result of R&D carried out by research firms which use the final consumption good as their only input. In all time periods, there are a finite number of exogenously given R&D firms, N, who behave competitively.[10] Let I_t^ϕ denote the economy-wide probability that a new invention will actually occur in any given period t, and I_t^x denote the economy-wide probability that a next generation machine will be introduced in t. I assume that these probabilities, I_t^ϕ and I_t^x, depend positively on aggregate resources spent on R&D in inventions and innovations, respectively:

$$I_t^* = \bar{I}\left(\frac{\omega_t^*}{1 + \omega_t^*}\right)^\gamma; \qquad * = \phi,\ x; \quad 0 < \bar{I} \leq 1; \qquad 0 < \gamma < 1, \tag{18}$$

[10] The qualitative and quantitative nature of my main conclusions are not dependent on the assumption that all firms engage in R&D for the purposes of both invention and innovation. In other words, allowing for specialization in R&D would not alter the basic results.

where ω_t^* is the aggregate resources spent on R&D in period t for the purpose of innovating or inventing, and where $\bar{I}$, $0 < \bar{I} \leq 1$, denotes the maximum attainable economy-wide probability of invention or innovation in any given period t.

If aggregate inventive or innovative activity is successful in advancing the economy-wide level of technology in use, the probability that any given R&D firm lands the monopoly rights to sell new technology machines (or those which belong to the next generation of an existing technology) depends on the relative share of resources the firm spends on R&D, $\omega_t^{*,n}/\omega_t^*$, $(n = 1, 2, 3, ..., N)$. Put differently, conditional on the fact that an invention or an innovation has occurred in any given period t, the odds of a particular R&D firm being the inventor or the innovator of that new technology depends positively on the ratio of its R&D expenditures to that in aggregate for that R&D activity.

Monopolists' patents expire after one period. Thus, if the technology does not become obsolete after one period, consumption-goods firms can replace existing machines at their marginal cost, c. (Because the machines depreciate fully in one period, producers must purchase new machines in every period.) The decision of an R&D firm, n, $n = 1, 2, 3, ..., N$, is

$$\max_{\omega_t^{\phi,n},\, \omega_t^{x,n}} \left\{ I_t^{\phi}(1 - I_t^x)\pi_t^{\phi}\frac{\omega_t^{\phi,n}}{\omega_t^{\phi}} \ + \ I_t^x(1 - I_t^{\phi})\pi_t^x\frac{\omega_t^{x,n}}{\omega_t^z} - \ B(\omega_t^{\phi,n} + \omega_t^{x,n}) \right\}, \qquad (19)$$

where $I_t^{\phi}(1 - I_t^x)\pi_t^{\phi}(\omega_t^{\phi,n}/\omega_t^{\phi})$ and $I_t^x(1 - I_t^{\phi})\pi_t^x(\omega_t^{x,n}/\omega_t^x)$ respectively denote the expected monopoly profits from inventions and innovations, B, $B > 0$, is the marginal cost of the R&D effort in terms of the consumption good[11], and where $\pi_t^* = (p_t^* - c)q_t^* = \int_0^1 q_t^{*,j}\, dj$. Implicit in (19) is the notion that when a new invention and an innovation are made simultaneously, the monopolistic competition in setting prices generates a Cournot-Nash equilibrium outcome in which $p_t^{\phi} = p_t^x = c$. As a consequence, the expected monopoly

[11] The main results of this paper are not dependent on the constant marginal cost assumption either. As in a number of other related papers, one could assume that the marginal cost of R&D effort is a function of the sophistication of existing technology based on the notion that as the underlying technology improves, it gets more or less costly (in terms of the final consumption good) to improve it. In fact, in the numerical example I present below, I consider the case in which the marginal cost of R&D is increasing in the level of technology.

payoff from an invention when there is also an innovation, $I_t^{\phi} I_t^{v} \pi_t^{\phi}$, and the expected monopoly payoff from an innovation when there is also an invention, $I_t^{x} I_t^{\phi} \pi_t^{x}$, are both zero.

Proposition 1: *A solution to the problem specified above exists and, $\forall n = 1, 2, 3, ..., N$, one is given by*

$$\tilde{\omega}_t^{\phi,n} = \frac{I_t^{\phi}(1 - I_t^{x})\pi_t^{\phi}}{BN} \qquad \text{and} \qquad \tilde{\omega}_t^{x,n} = \frac{I_t^{x}(1 - I_t^{\phi})\pi_t^{x}}{BN}, \tag{20}$$

where (20) implicitly defines $\tilde{\omega}_t^{\phi,n}$ and $\tilde{\omega}_t^{x,n}$ as $I_t^{\phi} = \bar{I}[N\tilde{\omega}_t^{\phi,n}/(1 + N\tilde{\omega}_t^{\phi,n})]$ and $I_t^{x} = \bar{I}[N\tilde{\omega}_t^{x,n}/(1 + N\tilde{\omega}_t^{x,n})]$.

Proof: See Appendix, Section 6.2.

$\square$

Not surprisingly, aggregate equilibrium R&D effort in inventive or innovative activity, $\tilde{\omega}_t^{*}$, $\tilde{\omega}_t^{*} = N\tilde{\omega}_t^{*,n}$, $* = \phi$, x, is increasing in monopoly profits from that invention or innovation.[12] [13]

[12]By assumption, there is free-entry into research and development by relatively small firms. Those firms ignore their impact on both the economy-wide probability of success in generating new inventions and the total number of R&D firms (which in turn affect the conditional odds of landing monopoly rights). If there had been one large firm engaged in R&D, it would have taken into account the effect of changes in its R&D resources, ω_t^{*}, on the probability of invention, I_t, but the qualitative nature of the results would have been unaffected. Similarly if there had been barriers to entry into the R&D sector which would have restricted the number of firms engaged in research and development, I would have had to consider a game-theoretic solution but again the qualitative nature of the main results would have remained intact.

[13]As also implied by (20), the intensification of research and development activity might be related to more firms deciding to invest in R&D. This result would be consistent with Sokoloff and Kahn (1990) who discuss the historical pattern of entrepreneurial activity which eventually led to inventions. Late 18th and early 19th century patent data indicate that it was the broadening of the entrepreneurial pool, rather than the concentration of inventions in the hands of a limited group of researchers and professional inventors, that led to rapid technological change in the United States in the 19th century.

Proposition 2: $\forall n = 1, 2, 3, ..., N,$

$$\left.\frac{\partial \omega_t^{*,n}}{\partial \pi_t^*}\right|_{\omega_t^{*,n} = \tilde{\omega}_t^{*,n}} > 0 \qquad * \ = \ \phi, \ x.$$

Proof: See Appendix, Section 6.3.

3. The Dynamics

In this model, the length of time a technology of vintage v has stayed in use, $t - v$, and the number of times higher quality machines of that technology have been introduced, x_t, determine the relative amounts of R&D expenditure on inventions and innovations. The reason for both are relatively straightforward: The longer an existing technology stays in use or the more machines with a given level of underlying technology are improved, the higher is the opportunity cost of giving up the use of existing machines. Moreover, given that the learning curve is steeper for newly discovered technologies compared with that for new generation machines, the impact of $t - v$ and x_t on the amount of R&D resources spent on inventions and innovations are different.

In order to characterize the dynamics of the model more formally, I will now focus on a restricted set of parameter values that satisfy the following:

Assumption 1: $\ln \bar{g} > \frac{1+\alpha}{\alpha}$ and $-\frac{\ln \alpha}{\alpha} \leq \frac{1}{6}$. (A.1)

(A.1) ensures two things: First, leaps up the quality ladder via the invention of new technologies, $\bar{g}$, is sufficiently large that the constrained monopoly markup given by (13), $\bar{g}^{\alpha} \exp[-\ \alpha(x_t - 1)/x_t \ - \ (t-\tau)/(1+t-\tau)]$, is strictly greater than 1 even when the length of time a previously-superior technology of vintage τ has stayed in use, $t - \tau$, and the number of times machines with a given technology has been improved, x_t, approach infinity. The implication of this is that, $\forall t \geq 0$, the monopoly profits from the invention of new technologies, π_t^{ϕ}, remains strictly positive. Second, $-\ln \alpha/\alpha \leq 1/6$, ensures that

$\check{x} \geq 2$. More explicitly, the second term in (A.1) guarantees that the productivity of newly innovated early generation machines which rely on relatively young technologies are large enough that monopolists who innovate and introduce new early generation machines can actually charge the unconstrained monopoly markup, c/α.[14]

Regardless of parameter restrictions, the dynamic evolution of the economy will be determined by those of the three state variables defined above: The underlying level of technology, ϕ_t, the length of time a technology of vintage v stays in use, $t - v$, and the number of times machines for a given level of technology has been improved, x_t. Letting again τ, $\tau \geq 0$, denote the time period in which an invention is made and T, $T \geq \tau \geq 0$, represent the time period in which a new generation machine is introduced, we can state the following:

$$v = \tau \qquad \Rightarrow \qquad t - v = t - \tau, \tag{21}$$

$$\phi_t = \begin{cases} \bar{g}\phi_{t-1} & \text{if } t = \tau, \\ \phi_{t-1} & \text{if } t \neq \tau, \end{cases} \tag{22}$$

and,

$$x_t = \begin{cases} 1 & \text{if } t = \tau, \\ x_{t-1} + 1 & \text{if } t = T, \\ x_{t-1} & \text{if } t \neq \tau,\ T. \end{cases} \tag{23}$$

At the beginning of any given period $t + 1$, an economy will be in one of three possible regimes depending on the value of its state variables x_t and $t - v$ in the previous period.

[14] For the derivation of these parameter restrictions, see Appendix, Section 6.4.

(I) $x_t + 1 < \check{x}$ and $t - v = 0$: Period $t+1$ follows one in which a new invention was made. Thus, $t = v = \tau$ and $x_t = 1$. Under this regime, which covers only the period following a new invention, the demand for and the profits from the introduction of the next generation of machines with this new technology are at their lowest. The reason is twofold: Because the technology is very new, (a) potential productivity gains from learning-by-doing have not yet materialized, and (b) available machines belong to the earliest generation, which for the given new technology level $\phi_{\tau,}$ are of the lowest quality. Moreover, under Lemma 1 $\partial q^x_{t+1}/\partial(t-v) > 0$, $\partial q^x_{t+1}/\partial x_{t+1} > 0$, $\partial q^{\phi}_{t+1}/\partial(t-v) = 0$, and $\partial q^{\phi}_{t+1}/\partial x_{t+1} = 0$. This implies that under (I), the share of resources devoted to inventive R&D activity relative to that allocated to innovative activity is relatively high.

(II) $x_t + 1 \leq \check{x}$ and $t - v > 0$: Regime (II) follows (I). Any given period $t+1$ under this regime is such that the underlying technology level, ϕ_τ, has been in use for some time, $t - v = t - \tau > 0$, but the number of times machines utilizing that technology has been improved remains low enough that $x_t \leq \check{x}$. Given the latter, the productivity gains that can be generated from the introduction of newer generation machines remain high enough that a monopolist owning a patent for such machines can still charge the unconstrained monopoly markup c/α. Thus, when combined with Lemma 1, this suggests that the monopoly profits from the introduction of the next generation machine, $\pi^x_{t+1} = [(1-\alpha)/\alpha]cq^x_{t+1}$, is strictly increasing in x_{t+1} and $t - v$ (i.e. $\partial \pi^x_{t+1}/\partial x_{t+1} > 0$ and $\partial \pi^x_{t+1}/\partial(t-v) > 0$). In addition, since due to Lemmas 2 and 3 the profitability of an invention in period $t+1$ is either flat (when $x_t \leq \hat{x}_t$) or strictly decreasing in x_{t+1} and $t - v$ (when $x_t > \hat{x}_t$), the share of R&D resources devoted to inventive efforts relative to that devoted to innovative activities is declining under regime (II).

(III) $x_t + 1 > \check{x}$ and $t - v > 0$: Regime (III) follows (II). Any given period $t+1$ is such that the underlying technology level, ϕ_τ, has been in use for some time, $t - v = t - \tau > 0$, and the number of times machines utilizing that technology has been improved upon sufficiently many times that $x_t > \check{x}$. There are two implications of these. First, given that the existing technology has remained in use a relatively long time, most of the efficiency gains from learning-by-doing with that technology have been realized. Second, since R&D firms have updated the quality of machines numerous times, innovations have also led to greater efficiency in the use of the underlying level of technology, ϕ_τ. Thus, in this maturity phase, the productivity of machines in use is higher than that of machines in the two other regimes, and the marginal productivity gains that would result from the adoption of a newer-generation machine or a newly-invented technology is relatively low. As a result, a monopolist who has a patented innovation can no longer charge the unconstrained monopoly markup c/α. When combined with Lemmas 2 and 3, this suggests that the monopoly profits from innovation are strictly decreasing in x_{t+1} (i.e. $\partial\pi^x_{t+1}/\partial x_{t+1} < 0$). Most importantly, however, is the fact that under (A.1), $\forall t \geq \tau$, $\lim_{x_t\to\infty} \pi^x_{t+1} = 0$ and $\lim_{x_t\to\infty} \pi^\phi_{t+1} > 0$. Consequently, the longer the economy stays in this regime, the higher is the relative share of R&D resources devoted to inventive versus innovative activity and the higher is the likelihood that, when a technological breakthrough does materialize, it is likely to be in the form of a new discovery.

Figures 3 and 4 plot monopoly profits from inventions and innovations respectively as a function of the length of time a technology of vintage v has stayed in use, $t-v$, and the number of times machines with a given level of underlying technology has been improved, x_t. And Proposition 3 formalizes the long-run dynamic properties of the model.

[Figures 3 and 4 about here.]

Proposition 3: *$\forall t \geq 0$, the state variables $(\phi_t,\ x_t,\ t-v)$ determine the stochastic dynamic evolution of the economy. For any given ϕ_t, if x_t and v are s.t. (i) $x_t + 1 < \check{x}$ and $t - v = 0$, the probability of an invention, I^ϕ_t, relative to that of an innovation, I^x_t, is at its maximum; (ii) $x_t + 1 \leq \check{x}$ and*

$t-v>0$, the probability of an invention, I_t^{ϕ}, relative to that of an innovation, I_t^x, is strictly decreasing in the generation of machines, x_t, and the vintage of the technology in use, $t-v$; (iii) $x_t+1>\check{x}$ and $t-v>0$, the probability of an invention, I_t^{ϕ}, relative to that of an innovation, I_t^x, is strictly decreasing in the vintage of the technology in use, $t-v$, but non-decreasing in the generation of machines, x_t.

Proof: See Appendix, Section 6.5.

4. Numerical Examples

In this section I provide a quantitative analysis of the model to illustrate some of its main implications. Before I do so, however, some qualifications and minor modifications need to be made.

First, in order to focus on how the relationship between learning-by-doing and differentiated R&D affect the pattern of technological progress and economic growth, I abstract from population dynamics. Nonetheless, as some recent papers like Lucas (1998), Jones (1999) and Galor and Weil (forthcoming) convincingly demonstrate, there exists a strong interaction between population size and the discovery of new ideas and technologies. As a result, the quantitative analysis below cannot fully capture the magnitude and timing of improvements in the standards of living over the very long run. That noted, the exercise should still be helpful in demonstrating how the interactions between learning-by-doing and directed R&D investment can help to create empirically consistent *patterns* of technological change and standards of living improvements over the long run.

Second, as is implicit in the workings of the model, I also assume that institutions that promote technological change, such as physical and intellectual property rights, are securely in place. In fact, as Jones demonstrates, one needs to incorporate exogenous variations in the security of property rights to match better the long-term evolution of the world consumption and output growth. Again, I do not take this step to keep the focus of the exercise solely on the interplay between learning-by-doing, directed and differentiated R&D and endogenous growth cycles.

Finally, in order to keep the simulation tractable and somewhat more consistent with the magnitude of historical world real GDP growth over the last several thousand years, I let the asymptotic probability of inventions, $\bar{I}^{\phi}$, be less than that of innovations, $\bar{I}^{x}$. While this modification can be justified based purely on the definitions of inventions and innovations–with the former being the emergence of rather cumbersome and difficult path-breaking new ideas or methods, and the latter being the relatively easier productivity-enhancing refinements to existing techniques–it is also supported by the historical evidence which shows that innovations were far more ubiquitous than inventions.[15] I will also let the marginal cost of inventive and innovative R&D effort, B, be an increasing function of the level of technology. In other words, I will adopt the notion that it is more costly to expand the technology frontier when the existing technology level is already high. This final modification will serve two purposes. It will help to rule out explosive paths in which resources devoted to R&D in either inventions or innovations converge to infinity. It will also demonstrate the robustness of the main results to modifications in the R&D cost function, as I alluded to earlier in footnote 11.

I simulate the above-described economy for 6000 years, or given that average life expectancy over that period was roughly about 40 years historically, for 300 model periods. For reference, the available historical data on the evolution of the world per capita income suggest that there was little or no net increase in the standards of living over most of the simulation period, and that starting only around the late 16th century has there been steady and substantial improvements in world per capita income. Maddison (1982, 1995) estimates, for example, that average per capita incomes were stagnant in Europe between the years 500 and 1500 and that it has risen close to tenfold since 1500.

There are 10 variables that need to be parameterized and I simulate the economy for two different sets of parameter specifications. Table 1 presents the parameter choices.

[15] See Mokyr pp. 290-302.

[Table 1 about here.]

In both simulations, most parameter values such as the asymptotic technology and machine quality levels, $\bar{A}$ and $\bar{z}$, marginal cost of machine production, c, are arbitrary, and are set at their chosen values for convenience. The initial level of the technology, ϕ_0, is deliberately set very close to zero. In the first simulation, the asymptotic probabilities of invention and innovation, $\bar{I}^{\phi}$ and $\bar{I}^{x}$, are set at 10 percent and 90 percent respectively. That is, when resources devoted to R&D for invention and innovation are infinite, the economy-wide odds of a commercially successful invention are 10 percent and that of innovation are 90 percent. The values of the quality jumps, $\bar{g}$, and the parameter α are chosen such that (A.1) is satisfied. Specifically, $\ln \bar{g} = 2.3 > (1+\alpha)/\alpha = 2.14$ and $-\ln \alpha/\alpha = 0.145 < 1/6$. As a result of the latter, a monopolist who holds a patent for a newly innovated machine of generation 3 or less can charge the unconstrained monopoly markup c/α. And given that $\ln \bar{g} = 2.3$ in the first simulation, the invention of a new technology leads to an approximately tenfold increase in the parameter ϕ, holding constant machine quality, z_t, and vintage, $t - v$. It is important to note, however, that this does not imply a tenfold increase in machine productivity immediately after the introduction of a new technology. For example, due to the effects of learning-by-doing and machine vintage on actual productivity, the productivity of a new technology will be only about 30 percent higher than that of existing older vintage 20 machines (i.e. for which $x_t = t - v = 20$).

The simulations highlight the model's main implications. First, both the invention of new technologies and innovations to existing techniques contribute to improvements in the standards of living. As shown in the first panel of Figure 5, however, these technological advances lead to meaningful and significant improvements much later in the process starting around the late 18th century. Second, there exist periods during which inventions are clustered together as well as long stretches during which technological change comes mostly in the form of small innovations. As shown in the middle panel of Figure 5, for example, there are three distinct waves of inventive activity; one between roughly 1250 B.C. and 1050 B.C. during which six new discoveries are made, another one which lasts from 500 B.C. to 300 A.D. when there are 7 new discoveries, and finally a shorter stretch in the late 17th century when there are two new inventions. There are also two very long periods of time over which technologies essentially remain the same and progress, if there is any, is solely due to innovations. Both of these episodes last over a thousand years, and during each episode, 29 new generation machines are introduced. These long periods of "inventive stagnation" come to an end because the marginal productivity gains of learning-by-doing and newer machines, which remain high in the early stages of a new invention, decline rapidly when technologies become more mature. Thus, as shown in the final panel of Figure 5, the odds of invention relative to that of innovation rises during the maturity phase of a technology as the gains from the introduction of newer generation machines are exhausted and more and more R&D resources get directed to inventive activities.

In the second simulation, the asymptotic probabilities of invention and innovation, $\bar{I}^{\phi}$ and $\bar{I}^{x}$, are chosen to be lower at 5 percent and 75 percent respectively. In addition, α is now set at 0.67 and $\bar{g}$ at 20. One implication of this is that (A.1) is no longer fully satisfied as $\ln \bar{g} = 3 > (1+\alpha)/\alpha = 2.48$ but $-\ln \alpha/\alpha = 0.60 \not\leq 1/6$. Thus, a monopolist who holds a patent for a newly innovated machine can only charge the constrained monopoly markup $c\exp[\alpha/x_t(1+x_t)]$. In general, the results of this simulation match those discussed above. The primary difference from the earlier simulation is that, given the asymptotic probabilities of invention and innovation are both lower, there are fewer inventions as well as improvements in machine quality for a given technology.

[Figures 5 and 6 about here.]

5. Conclusion

In the model above, I combine learning-by-doing with R&D activity that can be directed either to the discovery of new technologies or to the improvement of existing ones without fundamentally altering the underlying technology. This hybrid endogenous growth model, which extends earlier work to combine learning-by-doing with differentiated R&D, demonstrates the role of learning-by-doing in generating cycles of technological change and endogenous growth.

I find that, because the learning curve is steeper for newly discovered technologies, the existence of learning-by-doing alters R&D incentives for the discovery of new technologies differently than those for improving existing ones. Consequently, the model's dynamics lead to stochastic and endogenous growth cycles which are driven at times by discovery and invention episodes and at others by spurts of innovative activity. While the stochastic nature of the model allows long periods of technological stagnation during which neither new technologies are invented nor existing older ones are improved upon, the distinction between inventive versus innovative R&D activities and the impact of learning-by-doing on R&D incentives help to generate empirically consistent patterns of technological change.

6. Appendix

- *6.1. Proof of Lemma 2:*

  (i) Given (9), the demand for newer generation machines, $q_t^x = \int_0^1 q_t^{x,j} dj$, is isoelastic and their unconstrained monopoly markup equals c/α. When the younger generation machines are in use, the quality adjusted price of these earlier generation machines, $c/\{\bar{A}\exp(-1/(1+t-\tau)[\phi\bar{z}\exp(-1/x_t)]^\alpha\}$, exceeds that of the newest generation machine, $c/\alpha\{\bar{A}\exp(-1/(1+t-\tau)[\phi\bar{z}\exp(-1/(x_t+1))]^\alpha\}$, and as a result, the monopolist can charge c/α. By setting the quality adjusted prices equal, I derive $\check{x} \equiv [(1-4(\alpha/\ln\alpha))^{\frac{1}{2}} - 1]/2$, the machine generation above which the monopolist can no longer charge the unconstrained monopoly markup c/α. Setting $p_t^x/\{\bar{A}\exp(-1/(1+t-\tau)[\phi\bar{z}\exp(-1/(x_t+1))]^\alpha\}$ equal to $c/\{\bar{A}\exp(-1/(1+t-\tau)[\phi\bar{z}\exp(-1/x_t)]^\alpha\}$ I derive the price at which consumer goods producers would be indifferent between the newest and the previous generation machines when $x_t \geq \check{x}$. Thus, charging this price, which equals $c\exp[\alpha/x_t(1+x_t)]$, the monopolist can still generate positive sales when $x_t \geq \check{x}$.

  (ii) As with (i), the unconstrained monopoly markup for newly invented machines equals c/α. When the older technology machines are in use, the quality adjusted price of these older vintages, $c/\{\bar{A}\exp(-1/(1+t-\tau)[\phi\bar{z}\exp(-1/x_t)]^\alpha\}$, exceeds that of the newest technology machine, $c/\alpha\{\bar{A}\exp(-1)[\bar{g}\phi\bar{z}\exp(-1)]^\alpha\}$, and as a result, the monopolist can charge c/α. By setting the quality adjusted prices equal, I derive $\hat{x}_t = \{\alpha/[(t-\tau)/(1+t-\tau) - \ln\alpha - \alpha(\ln\bar{g} - 1)]\}$, the machine generation above which the monopolist can no longer charge the unconstrained monopoly markup c/α. Setting $p_t^\phi/\{\bar{A}\exp(-1)[\bar{g}\phi\bar{z}\exp(-1)]^\alpha\}$ equal to $c/\{\bar{A}\exp(-1/(1+t-\tau)[\phi\bar{z}\exp(-1/x_t)]^\alpha\}$ I derive the price at which consumer goods producers would be indifferent between the newest and the previous technology machines when $x_t \geq \hat{x}_t$. Thus, charging this price, which equals $c\bar{g}^\alpha\exp[-\alpha/x_t(1+x_t) - (t-\tau)/(1+t-\tau)]$, the monopolist can still generate positive sales when $x_t \geq \hat{x}_t$. □

- *6.2. Proof of Proposition 1:*

  For each firm, n, $n = 1, 2, 3, ...N$, which takes as given the aggregate amount of resources devoted to R&D in inventive and innovative activities, ω_t^*, $* = \phi$, x, the following hold:

$$\omega_t^\phi \begin{cases} = \ 0 & \text{if } I_t^\phi(1-I_t^x)\pi_t^\phi \ < \ B\omega_t^\phi \\ \in \ (0, \ \infty) & \text{if } I_t^\phi(1-I_t^x)\pi_t^\phi \ = \ B\omega_t^\phi \\ = \ 1 & \text{if } I_t^\phi(1-I_t^x)\pi_t^\phi \ > \ B\omega_t^\phi \end{cases}, \tag{6.1}$$

and

$$\omega_t^x \begin{cases} = \ 0 & \text{if } I_t^x(1-I_t^\phi)\pi_{t,v}^x \ < \ B\omega_t^x \\ \in \ (0, \ \infty) & \text{if } I_t^x(1-I_t^\phi)\pi_{t,v}^x \ = \ B\omega_t^x \\ = \ 1 & \text{if } I_t^x(1-I_t^\phi)\pi_{t,v}^x \ > \ B\omega_t^x \end{cases}. \tag{6.2}$$

Given that all R&D firms are identical, $\omega_t^* = 0$ and $\omega_t^* = 1$, $* = \phi$, x, cannot hold in equilibrium, and $\omega_t^* \in (0, \infty)$, $* = \phi$, x, has to hold. And $\omega_t^* = N\omega_t^{*,n}$, $\omega_t^* \in (0, \infty)$, $* = \phi$, x, is a non-trivial equilibrium outcome. □

- *6.3. Proof of Proposition 2:*

Using (20) and invoking the implicit function theorem,

$$\frac{\partial\omega_t^{\phi,n}}{\partial\pi_t^\phi}\Bigg|_{\omega_t^{\phi,n}=\tilde{\omega}_t^{\phi,n}} = \frac{1}{N}\frac{I_t^\phi(1-I_t^x)}{B-\pi_t^\phi\bar{I}\left[\frac{(1-I_t^x)}{(1+\omega_t^{\phi,n})^2} + \frac{I_t^\phi}{(1+\omega_t^{x,n})^2}\frac{\partial\omega_t^{x,n}}{\partial\omega_t^{\phi,n}}\right]}, \tag{6.3}$$

and,

$$\frac{\partial\omega_t^{x,n}}{\partial\pi_t^x}\Bigg|_{\omega_t^{x,n}=\tilde{\omega}_t^{x,n}} = \frac{1}{N}\frac{I_t^x(1-I_t^\phi)}{B-\pi_t^x\bar{I}\left[\frac{(1-I_t^\phi)}{(1+\omega_t^{x,n})^2} + \frac{I_t^x}{(1+\omega_t^{\phi,n})^2}\frac{\partial\omega_t^{\phi,n}}{\partial\omega_t^{x,n}}\right]}. \tag{6.4}$$

In equilibrium R&D resources devoted to inventions and innovations are such that $I_t^\phi(1-I_t^x)\pi_t^\phi - B\omega_t^\phi = I_t^x(1-I_t^\phi)\pi_t^x - B\omega_t^x = 0$. This, together with the properties of I_t^*, $* = \phi,\, x$, suggests that $B - \pi_t^\phi \bar{I}\left[\frac{(1-I_t^x)}{(1+\omega_t^{\phi,n})^2} + \frac{I_t^\phi}{(1+\omega_t^{x,n})^2}\frac{\partial \omega_t^{x,n}}{\partial \omega_t^{\phi,n}}\right]$ and $B - \pi_t^x \bar{I}\left[\frac{(1-I_t^\phi)}{(1+\omega_t^{x,n})^2} + \frac{I_t^x}{(1+\omega_t^{\phi,n})^2}\frac{\partial \omega_t^{\phi,n}}{\partial \omega_t^{x,n}}\right]$ are strictly positive. Thus, (6.3) and (6.4) are both strictly positive. □

- *6.4. Deriving Assumption 1:*

Let $\underline{x}_t$ represent the generation of machines which set the constrained monopoly price of a machine with a newly invented technology $c\bar{g}^\alpha \exp[-\,\alpha(x_t - 1)/x_t - (t-\tau)/(1+t-\tau)]$ equal to its marginal cost, c : Using (13),

$$\underline{x}_t \;=\; \frac{\alpha}{\frac{t-\tau}{1+t-\tau} \;-\; \alpha(\ln \bar{g} - 1)}\;. \tag{6.5}$$

Note that $\lim_{(t-\tau)\to\infty} \frac{t-\tau}{1+t-\tau} = 1$. Thus, if $\ln \bar{g} > \frac{1+\alpha}{\alpha}$, $\forall t \geq 0$, $\underline{x}_t < 0$ and $x_t > \underline{x}_t$. As a result, the monopoly markup will never converge to zero. To derive $-\frac{\ln \alpha}{\alpha} \leq \frac{1}{6}$, I simply set $\breve{x} = [(1 - 4(\alpha/\ln \alpha))^{\frac{1}{2}} - 1]/2 \geq 2$. □

- *6.5. Proof of Proposition 3:*

(i) Referring again to (20), the following is satisfied in equilibrium:

$$\frac{\pi_t^\phi}{\pi_t^x} \;=\; \frac{\omega_t^\phi}{\omega_t^x}\,\frac{I_t^x(1-I_t^\phi)}{I_t^\phi(1-I_t^x)} \;=\; \frac{(\omega_t^\phi)^{1-\gamma}(1+\omega_t^\phi)^\gamma - \omega_t^\phi}{(\omega_t^x)^{1-\gamma}(1+\omega_t^x)^\gamma - \omega_t^x} \tag{6.6}$$

Combining (6.6) with Lemmas 1-4, π_t^ϕ/π_t^x is at its maximum attainable value $\forall$ $x_t \leq \breve{x}$ and $t - v = 0$. Consequently, ω_t^ϕ/ω_t^x and I_t^ϕ/I_t^x are also at their highest.

(ii) Given (6.6) and Lemmas 1-4, $\forall\; x_t \leq \breve{x}$ and $t - v > 0$,

$$\frac{\partial(\pi_t^\phi/\pi_t^x)}{\partial(t-v)} < 0 \qquad \text{and} \qquad \frac{\partial(\pi_t^\phi/\pi_t^x)}{\partial x_t} < 0, \tag{6.7}$$

which implies that, $\forall\; x_t + 1 \leq \breve{x}$ and $t - v > 0$, $\frac{\partial(I_t^\phi/I_t^x)}{\partial(t-v)} < 0$ and $\frac{\partial(I_t^\phi/I_t^x)}{\partial x_t} < 0$.

(iii) Finally, (6.6) and Lemmas 1-4, $\forall\ x_t + 1 \leq \check{x}$ and $t - v > 0$,

$$\frac{\partial(\pi_t^{\phi}/\pi_t^{x})}{\partial(t-v)} < 0 \qquad \text{and} \qquad \frac{\partial(\pi_t^{\phi}/\pi_t^{x})}{\partial x_t} < 0, \tag{6.8}$$

which implies that, $\forall\ x_t + 1 > \check{x}$ and $t - v > 0$, $\frac{\partial(I_t^{\phi}/I_t^{x})}{\partial(t-v)} < 0$ and $\frac{\partial(I_t^{\phi}/I_t^{x})}{\partial x_t} \geq 0$.

□

7. References

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Figure 1

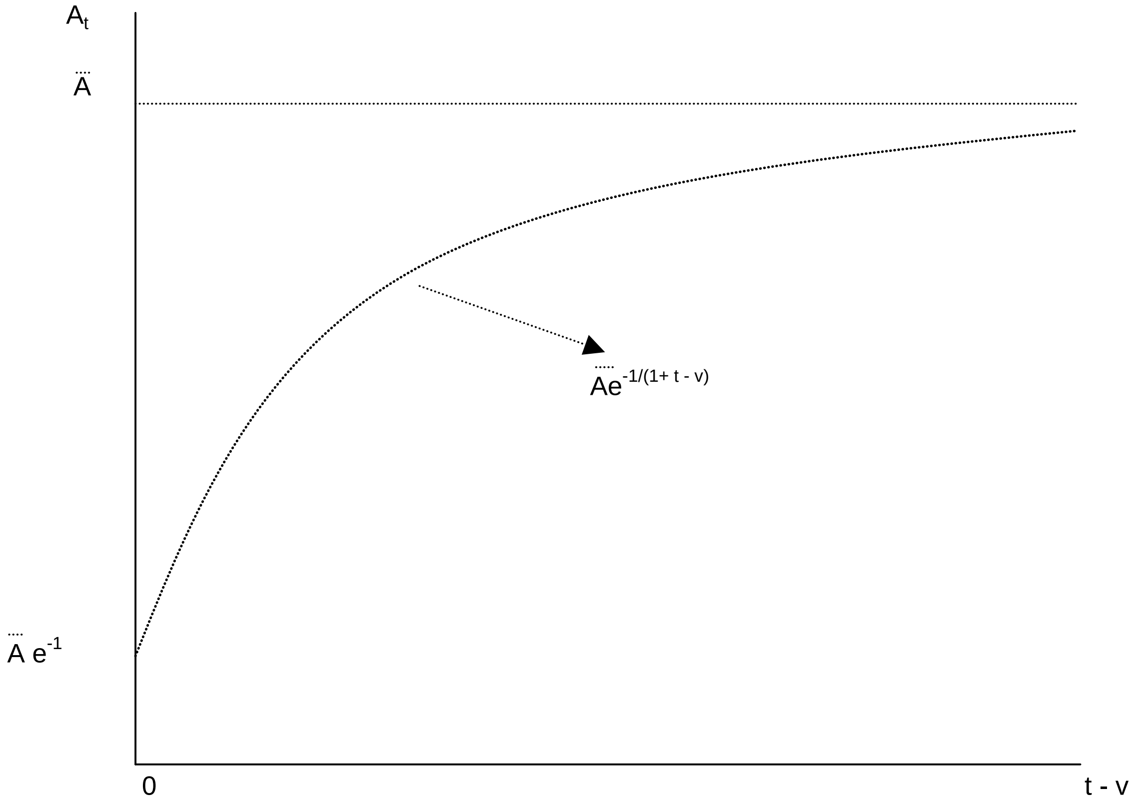


Figure 2

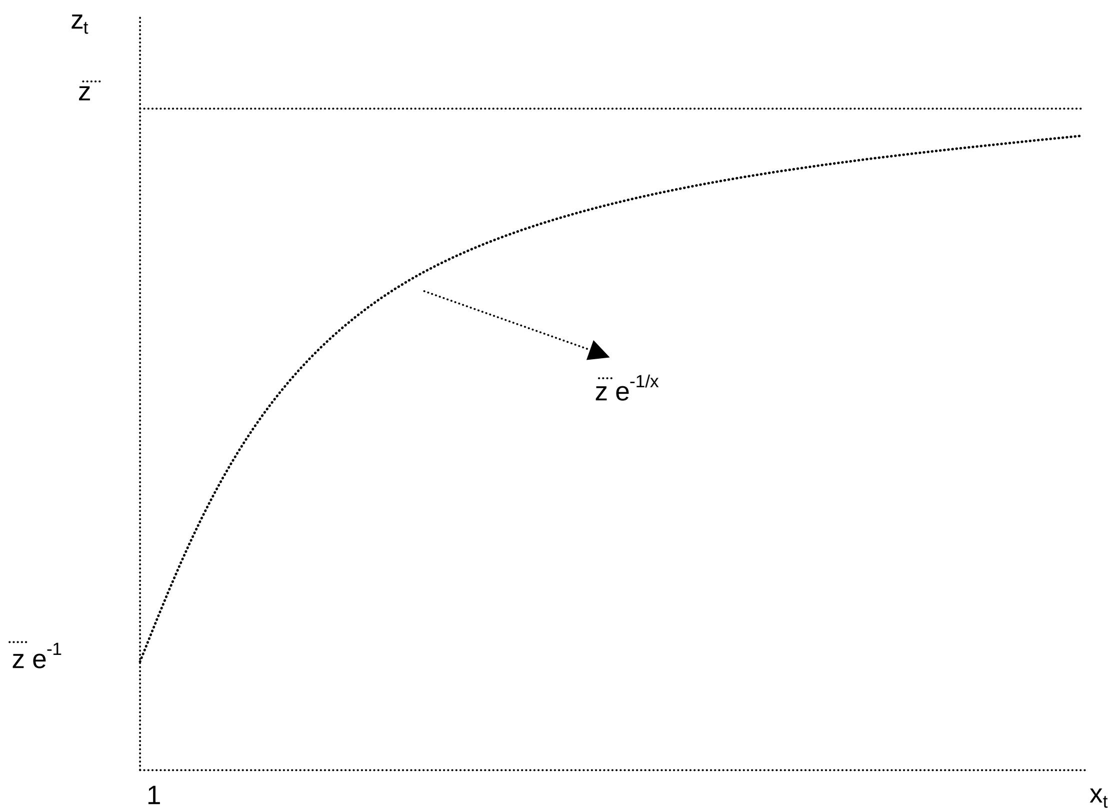
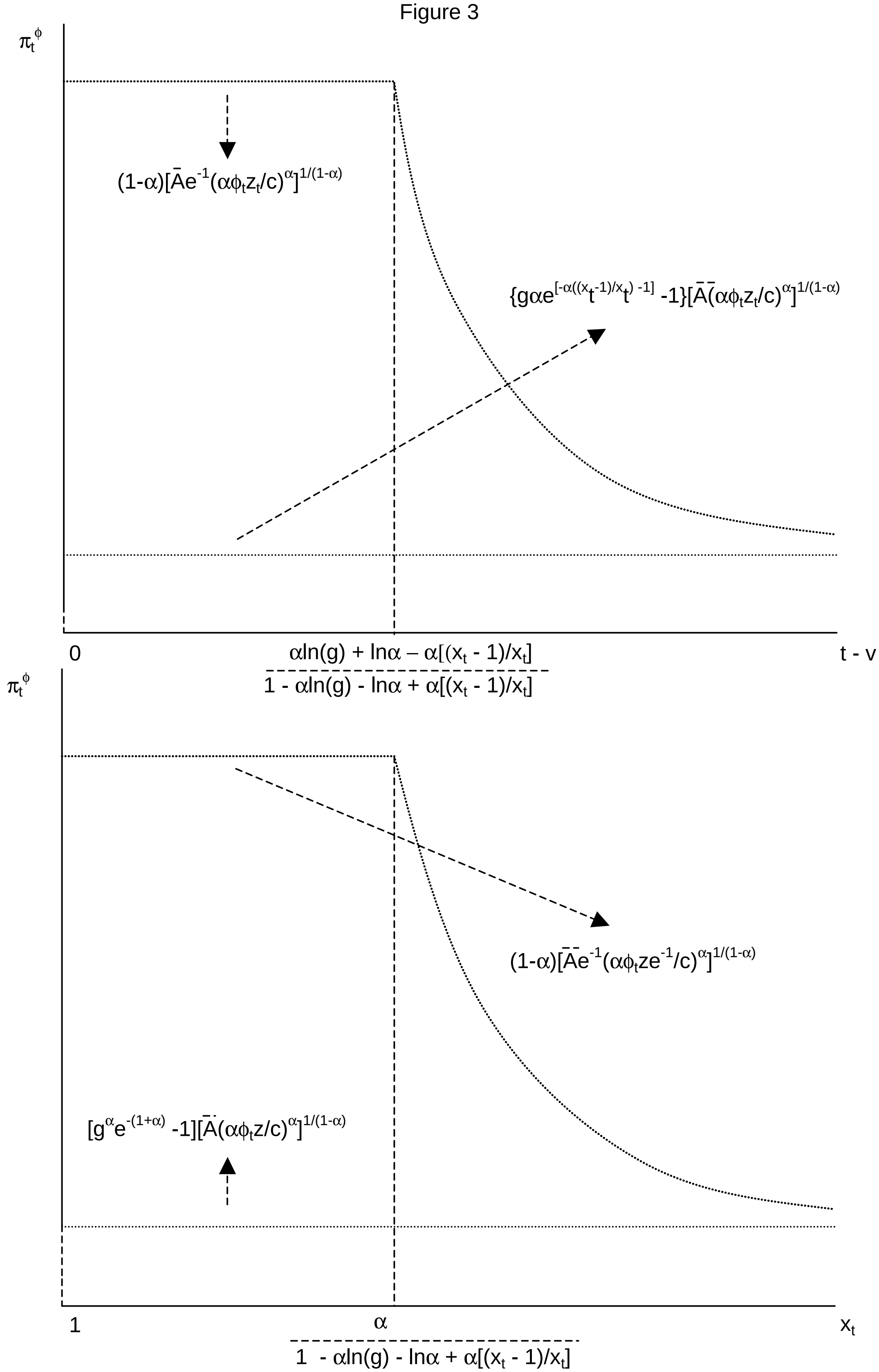
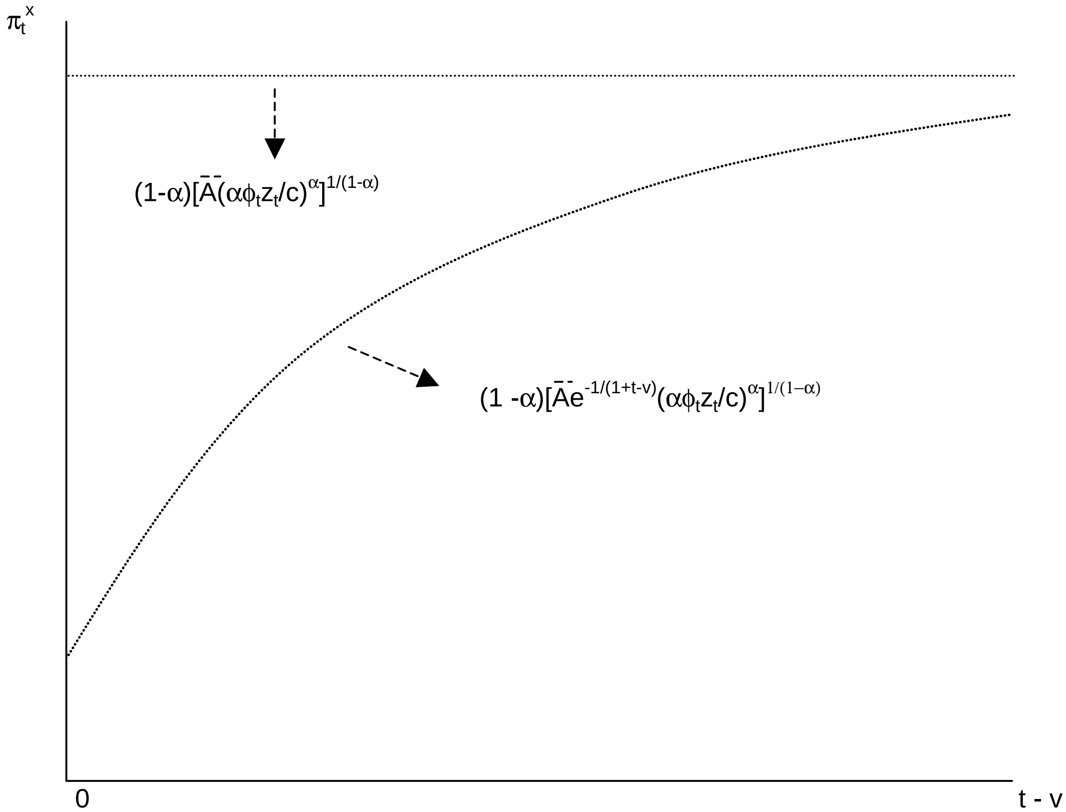


Figure 3



π_t^{ϕ}
$(1-\alpha)[\bar{A}e^{-1}(\alpha\phi_t z_t/c)^{\alpha}]^{1/(1-\alpha)}$
$\{g\alpha e^{[-\alpha((x_t-1)/x_t)-1]} - 1\}[\bar{A}(\alpha\phi_t z_t/c)^{\alpha}]^{1/(1-\alpha)}$
0
$\frac{\alpha\ln(g) + \ln\alpha - \alpha[(x_t - 1)/x_t]}{1 - \alpha\ln(g) - \ln\alpha + \alpha[(x_t - 1)/x_t]}$
t - v
π_t^{ϕ}
$(1-\alpha)[\bar{A}e^{-1}(\alpha\phi_t z e^{-1}/c)^{\alpha}]^{1/(1-\alpha)}$
$[g^{\alpha}e^{-(1+\alpha)} - 1][\bar{A}(\alpha\phi_t z/c)^{\alpha}]^{1/(1-\alpha)}$
1
$\frac{\alpha}{1 - \alpha\ln(g) - \ln\alpha + \alpha[(x_t - 1)/x_t]}$
x_t

Figure 4


π_t^x
$(1-\alpha)[\bar{A}(\alpha\phi_t z_t/c)^{\alpha}]^{1/(1-\alpha)}$
$(1-\alpha)[\bar{A}e^{-1/(1+t-v)}(\alpha\phi_t z_t/c)^{\alpha}]^{1/(1-\alpha)}$
0
t - v

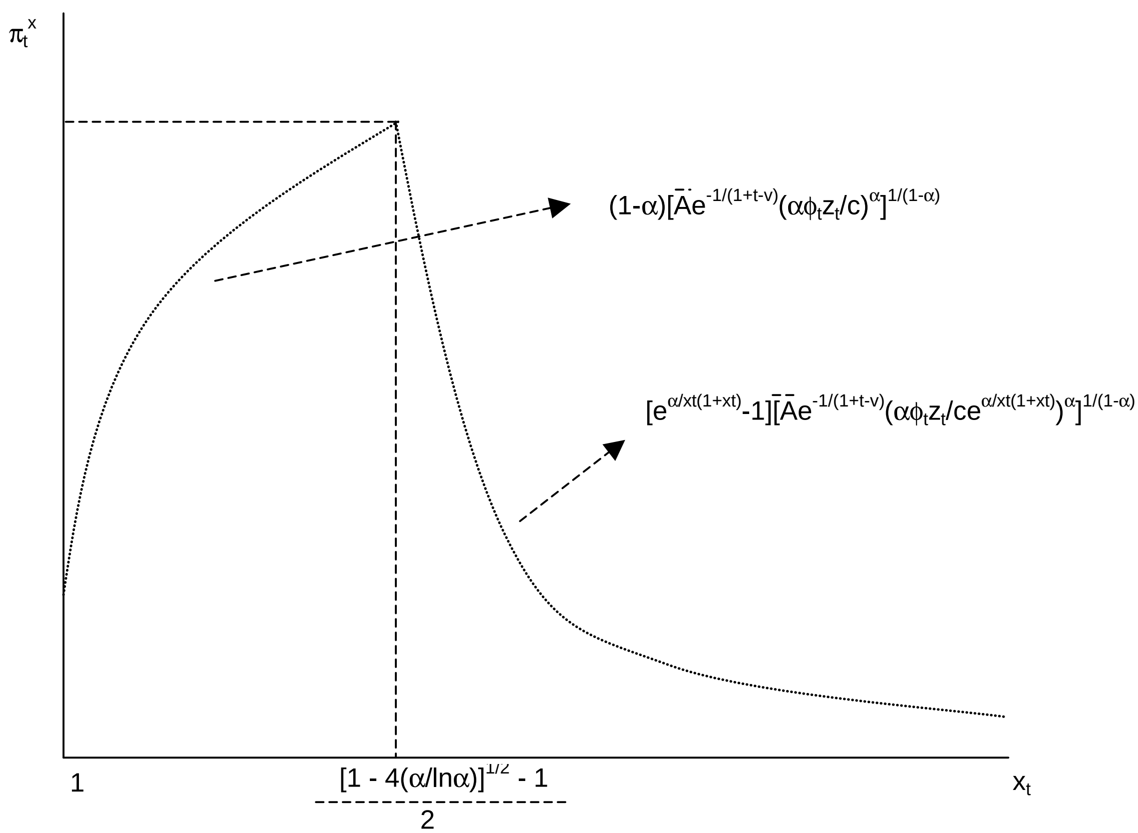
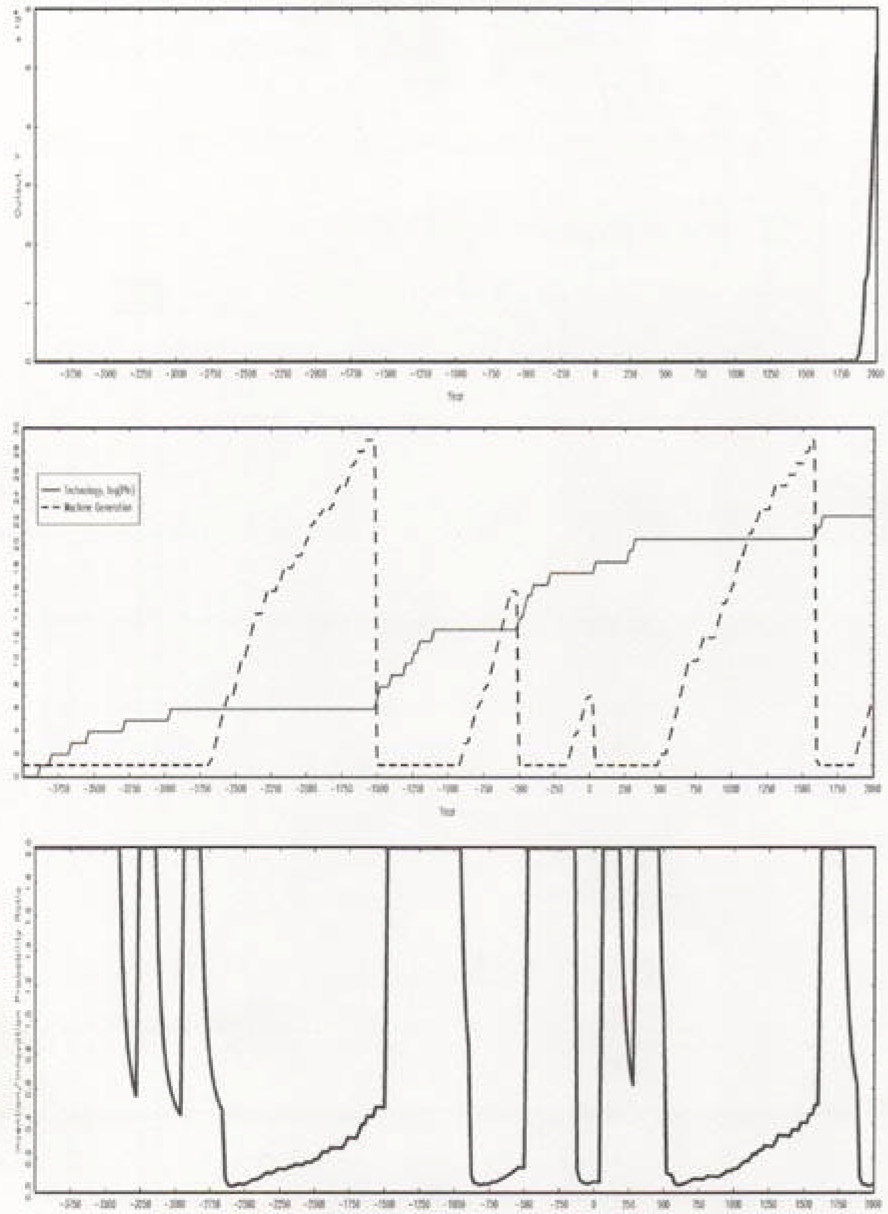

π_t^x
$(1-\alpha)[\bar{A}e^{-1/(1+t-v)}(\alpha\phi_t z_t/c)^{\alpha}]^{1/(1-\alpha)}$
$[e^{\alpha/xt(1+xt)}-1][\bar{A}e^{-1/(1+t-v)}(\alpha\phi_t z_t/ce^{\alpha/xt(1+xt)})^{\alpha}]^{1/(1-\alpha)}$
1
$\frac{[1-4(\alpha/\ln\alpha)]^{1/2}-1}{2}$
x_t

Figure 5:



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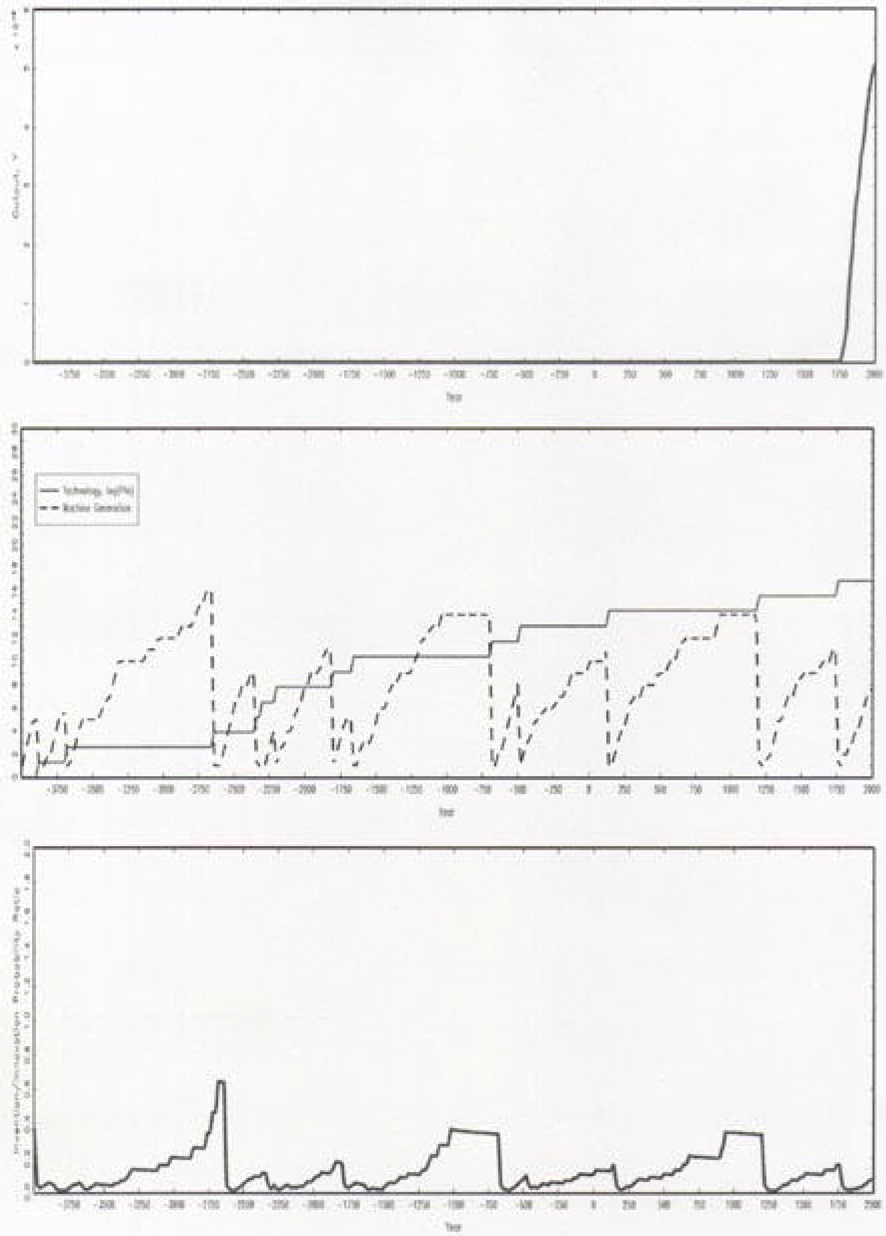

Table 1: Parameter Choices

Parameter	Value (I)	Value (II)
$\bar{A}$	1	1
$\bar{z}$	1	1
$\bar{I}^{\phi}$	0.10	0.05
$\bar{I}^{x}$	0.90	0.75
c	10	10
B	10	10
ϕ_0	10^{-20}	10^{-20}
$\bar{g}$	10	20
α	0.875	0.675
γ	0.10	0.10