Allocation Mechanisms with Mixture-Averse Preferences*

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Abstract

Consider an economy with equal amounts of N types of goods, to be allocated to agents with strict quasi-convex preferences over lotteries. We show that ex-ante, all Pareto efficient allocations give almost all agents lotteries over at most two outcomes. Therefore, even if all preferences are the same, some identical agents necessarily receive different lotteries. Our results provide a simple criterion to show that many popular allocation mechanisms are ex-ante inefficient. Assuming the reduction axiom, social welfare deteriorates by first randomizing over these binary lotteries. Efficient full ex-ante equality is achieved if agents satisfy the compound independence axiom.

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1 Introduction

Ten thousand children need to be allocated into ten schools, each accommodating one thousand of them. The schools are not the same, and parents may rank them in different ways. However, if all children are considered equal, then a social lottery, where each student has an equal chance to attend each of the ten schools, seems to be the best solution.[1] This procedure is egalitarian — everyone gets the same lottery — and feasible. But is it efficient? Specifically, is there no other procedure such that ex-ante, before people know their allocated school, they will get a better lottery?

If individual preferences over the schools are not the same, then this procedure may be inefficient, for example, if each school is ranked best by exactly 1000 parents. It is true that if all individuals are expected utility maximizers and have the same preferences over lotteries (and in particular, over the schools), then this procedure leads to an efficient allocation. This is also the case if all have the same quasi-concave preferences, i.e. preferences for randomization over lotteries. But if preferences are quasi-convex, and a mixture of two indifferent lotteries is inferior to the mixed lotteries, then we show that this procedure is never efficient, regardless of whether individual preferences are the same or not. Such preferences are implied by some well known alternatives to expected utility theory (for example, Tversky and Kahneman's [39] Cumulative Prospect Theory, where risk aversion implies quasi-convexity. See discussion below).

We analyze first an economy where N types of goods with k units each need to be allocated, one for each of Nk agents. All agents have strict preferences over the basic goods, and continuous, monotone (with respect to first-order stochastic dominance), and strictly quasi-convex preferences over

[1] For example, divide the students into ten groups $A_1, \ldots, A_{10}$ of size 1000 each. Choose with probability $\frac{1}{10}$ one of the ten permutations $\sigma_1, \ldots, \sigma_{10}$ of $(1, \ldots, 10)$, where $\sigma_j(i) = (i + j - 1) \pmod{10} + 1$, $j = 1, \ldots, 10$.

lotteries. Agents' preferences over the goods and over lotteries are not necessarily the same. Our first result (Theorem 1) shows that any feasible and ex-ante Pareto efficient allocation must give all but 'not too many' agents binary lotteries, that is, lotteries with not more than two outcomes. Moreover, the proportion of agents who hold non-binary lotteries vanishes as k increases.[2] In particular, even if all preferences are the same, some identical agents necessarily receive different lotteries. For the case of identical preferences, we establish existence of an ideal solution: a feasible and efficient solution, in which all the lotteries used are equally attractive (Theorem 2). We also derive an upper bound on the number of lotteries used.

We consider some extensions of our basic framework. We first analyze assignment problems where all agents have the same preferences, but the number of units to be allocated is not equal to the number of agents. We then study a continuum economy with the same mass of agents and goods, where each type has its own quasi-convex preferences over lotteries. In this part we are looking for no-envy allocations, that is, allocations of lotteries where no person prefers to receive a lottery obtained by someone else. We show that under a mild condition on preferences, a feasible and efficient allocation for the continuum economy with strictly quasi-convex preferences yields all agents binary lotteries and the set of no-envy allocations is not empty. Moreover, if all agents have the same preferences, then equality with such lotteries is obtained (Theorem 4).

The last part of the paper discusses possible merits of random allocations of the binary lotteries among individuals with identical preferences. The need for such an extra layer of randomization may be due to lack of confidence in policy makers' integrity or willingness of the allocating agencies to demonstrate they are unbiased. We show how individual preferences over two-stage lotteries imply different answers to this question. If they simplify

[2]As we show in Proposition 1 of Section 2.1, this result, with a small caveat, essentially also holds when individuals have different expected utility preferences.

such lotteries by multiplying the probabilities of the two stages, this extra randomization will reduce participants utilities. But if decision makers instead satisfy the compound independence axiom, according to which if they prefer q to q' they will prefer to replace q' with q in any compound lottery that includes the former as an outcome, then such randomizations will not change agents' welfare.

Our analysis depends on the assumption that individual preferences over lotteries are quasi-convex. Expected utility, where preferences are linear in the probabilities, is the boundary case. Strict quasi-convexity is for example the case with the popular family of rank-dependent utilities models (Quiggin [31]), which also includes Yaari's [46] dual theory, as well as Tversky and Kahneman's [39] Cumulative Prospect Theory, where risk aversion implies quasi-convexity. Other models which can exhibit quasi-convexity include quadratic utility (Chew, Epstein, and Segal [11]), and Kőszegi and Rabin's [24] models of reference-dependence. In addition, Machina [25] pointed out that quasi-convexity occurs if, as is common in many applications such as insurance purchasing, before the lottery is resolved agents can take actions that affect their final utility. If the optimal action depends on the probabilities, the induced maximum expected utility will be convex in the probabilities, meaning that even if the underlying preferences are expected utility, induced preferences over the optimal lotteries will be quasi-convex.

The experimental evidence on quasi-convexity versus quasi-concavity is mixed. Most of the experimental literature that documents violations of expected utility (e.g., Coombs and Huang [13]) found either preference for randomization or aversion to it. Camerer and Ho [9] find support for quasi-convexity over gains and quasi-concavity over losses. An example of behavior that distinguishes between the two attitudes to mixture is the probabilistic insurance problem of Kahneman and Tversky [23]. They showed that in contrast with experimental evidence, any risk averse expected utility maximizer must prefer probabilistic insurance to regular insurance. Sarver [33] pointed out that this result readily extends to the case of quasi-concave preferences. In contrast, quasi-convex preferences can accommodate aversion to probabilistic insurance together with risk aversion (for example, risk-averse rank-dependent utility; see Segal [34]). Sarver further illustrates that quasi-convex preferences are consistent with increasing marginal willingness to pay for insurance at some levels of coverage; another plausible property that in most models requires violation of risk aversion. In the context of group decision making, Dillenberger and Raymond [17] show that quasi-convexity of preferences in the individual level is equivalent to the consensus effect: individuals tend to conform to the choices of others in group decisions, compared to choices made in isolation.

The idea of using lotteries to allocate indivisible goods is not new (see, for example, Diamond [14], Hylland and Zeckhauser [22], and Rogerson [32]). Moreover, the *possible* existence of an optimal solution that induces each individual to face a binary lottery was already discussed in Hylland and Zeckhauser [22] under expected utility preferences. Our approach differs from these works. We show that in a large economy with quasi-convex preferences, any ex-ante efficient solution *must* use only binary lotteries. Also, as long as individuals simplify compound lotteries by multiplying the probabilities, randomizing among these binary lotteries (thus giving identical people the same ex-ante lottery) is always suboptimal.

In this paper we employ a strong notion of ex-ante efficiency, which takes into consideration individual preferences over lotteries. Two weaker notions of efficiency were previously studied, ordinal efficiency and ex-post efficiency, both only depend on ordinal rankings of the final goods. As we remark in Section 2.1, our results imply that many of the popular allocation mechanisms used in the literature are ex-ante inefficient. For example, random serial dictatorship, that assigns the order of individuals using uniform distribution, is inefficient as it typically implies that each individual will face a lottery with more than two elements in its support. Note that this inefficiency relies only on the ordinal property of the preferences over lotteries, namely that they are quasi-convex in probabilities.

The stronger notion of efficiency we consider, which is natural once individuals preferences over lotteries are taken into account, makes it harder to achieve strategy proofness, a property that ensures that it is always optimal for agents to truthfully report their preferences over lotteries. This raises the questions of who can use our results and how. We discuss this issue at length in Section 4. While not always possible, we argue that in many situations social planners can collect at least partial information about cardinal properties of preferences, and our results can guide them how to locally improve upon existing popular methods (a similar approach was suggested by Abdulkadiroğlu, Che, and Yasuda [2]). Moreover, empirical and experimental data regarding individual preferences can be collected and used in order to estimate ideal solutions. Such methods are used in various situations, for example, in medical decision making (Wakker [44]).

The paper is organized as follows. Section 2 lays out the basic problem in a finite environment and states our main results. Section 3 studies two possible extensions: the case where the number of agents and units is not the same, and the case of a continuum economy. Section 4 comments on the applicability of our approach. In Section 5 we discuss the benefit of a pre-randomization over the allocation lotteries. Section 6 concludes with a further discussion of binary lotteries and the applicability of our results. All proofs are in the Appendix.

2 Finite Economies

Consider an economy with Nk individuals and with k units of each of $N \geqslant 3$ basic goods $x_1, \ldots, x_N$. Denote by $q = (q_1, \ldots, q_N)$ the lottery $(x_1, q_1; \ldots; x_N, q_N)$ that yields x_i with probability q_i, $i = 1, \ldots, N$. With a little abuse of notation, we identify x_i with the lottery that yields it with probability 1. Each member n of society has preferences $\succeq_n$ over such lotteries, which are assumed to be continuous, strictly monotonic (with respect to first-order stochastic dominance), and strictly quasi-convex in probabilities. This last assumption captures a dislike of probabilistic mixtures of lotteries: $q \sim q'$ and $q \neq q' \implies q \succ \alpha q + (1-\alpha)q'$ for all $\alpha \in (0,1)$.

A solution is a list of N-dimensional probability vectors $\mathbf{q} = \left(q^1, \ldots, q^{Nk}\right)$, where q^n is the lottery faced by person n. We require for all $n = 1, \ldots, Nk$,

$$\sum_{i=1}^{N} q_i^n = 1 \tag{1}$$

That is, the probability that person n will get one of the items is 1. Also, for $i = 1, \ldots, N$,

$$\sum_{n=1}^{Nk} q_i^n = k \tag{2}$$

This condition means that with probability 1, each of the k items of each good will be allocated to someone. The last equation implies

$$\frac{1}{Nk} \sum_{n=1}^{Nk} q^n = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right) \tag{3}$$

That is, the average lottery faced by the participants is a uniform distribution over the N goods. Obviously, this distribution is feasible. The sum of its components must be 1, as the original lottery satisfies eq. (1). And if the average lottery is not uniform, then the original allocation is not feasible as it must violate eq. (2).

Any solution $\mathbf{q}$ specifies the probability distribution over final outcomes for each individual. The Birkhoff–von Neumann Theorem ([4],[43]) guarantees that for any $\mathbf{q}$ there is always a social lottery over all possible deterministic allocations of the final outcomes that induces the marginal probabilities of $\mathbf{q}$.[3]

[3] We assume throughout that each agent is indifferent between all units of the same

2.1 Ex-Ante Efficiency

We first characterize solutions $\mathbf{q}$ that are feasible, that is, satisfy equations (1) and (2), and are ex-ante Pareto efficient, in the sense that there is no solution $\tilde{\mathbf{q}}$ such that $\tilde{q}^n \succeq q^n$ for all n and $\tilde{q}^m \succ q^m$ for some m. As preferences are continuous over a compact domain, feasible efficient allocations exist. We show that in such allocations, and without any further assumptions on individual preferences, all but 'not too many' individuals obtain either a degenerate lottery or a lottery with positive probabilities on two goods only.

Definition 1 A lottery q^n is binary if $q_i^n > 0$ for *no more than* two outcomes.

Theorem 1 Let $\mathbf{q}$ be a feasible and Pareto efficient solution. Then for any three goods x_r, x_s, x_t, there is at most one person n such that q_r^n, q_s^n, $q_t^n > 0$.

This result implies that to detect violations of ex-ante efficiency, it is enough to observe an allocation in which two individuals receive lotteries that put positive probabilities on the same three goods. The exact probabilities are inconsequential.

To illustrate the main argument of the theorem, suppose that two agents m and n agree on their ranking of three goods $x_r \succ x_s \succ x_t$ and that they both receive lotteries with positive probabilities on these three goods, as in Figure 1. In this figure, each panel is the projection of the probability simplex on a normalized probability triangle. These triangles depict probability allocations over the three outcomes x_r, x_s, and x_t for individuals m and n that do not change the sums $\bar{q}^m = q_r^m + q_s^m + q_t^m$ and $\bar{q}^n = q_r^n + q_s^n + q_t^n$. Quasi-convex preferences have the property that along any line through a given point, preferences improve in at least one direction. Without loss of

good, so that we can confine our attention to the allocation of the goods themselves. This would not be the case if, for example, we were to allocate seats in different flights and travelers prefer sitting in a window or an aisle seat.

generality, one of the supporting slopes to the indifference curve of person n through (q_t^n, q_r^n) is weakly steeper than one of the supporting slopes to the indifference curve of person m through (q_t^m, q_r^m). Take a line with slope between these two values. To make both agents better off, transfer probabilities from one agent to another as depicted in the figure, a violation of the efficiency assumption. If, instead, individuals' ordinal rankings of the goods are *not* identical, then the two agents can trade in the probabilities of any two goods that they rank differently to improve ex-ante welfare.

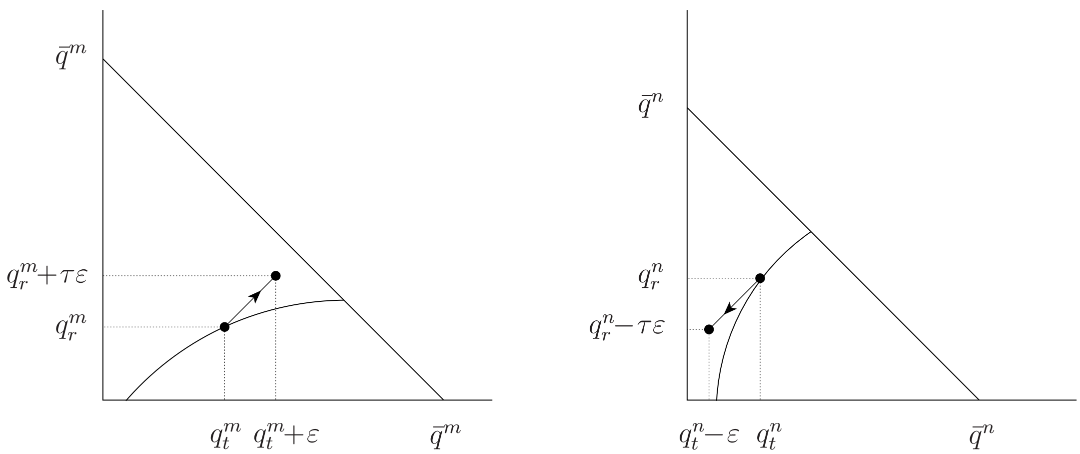

Figure 1: Changes in the allocations of individuals m and n

As we explain below, the arguments above also apply to agents with different expected utility preferences.

Theorem 1 implies a limit on the number of individuals who can receive a non-binary lottery.

Corollary 1 The number of individuals who hold non-binary lotteries in any feasible and efficient allocation is bounded above by $\binom{N}{3}$.

The number of subsets of $\{1, \ldots, N\}$ where no two elements have an intersection with more than two numbers is bounded above by $\binom{N}{3}$, which is the case where all subsets have three elements each.[4] Since the number of individuals who hold non-binary lotteries is bounded above by $\binom{N}{3}$ while the total population size is Nk, their fraction becomes arbitrarily small as k increases.

While for exposition purposes we confine our attention to the case of strict quasi-convex preferences, Theorem 1 generally also holds under expected utility, which is linear (and hence also weakly quasi-convex) in probabilities.[5] Under expected utility, if all agents have the same preferences over lotteries, then there are many efficient solutions, including interior ones. Our results are thus more prominent once preferences are cardinally different. More precisely,

Proposition 1 Consider two expected utility agents m and n with utility functions over final outcomes u_m and u_n, respectively. For any three goods x_r, x_s, and x_t, if $\mathbf{q}$ is a feasible allocation with both q_r^n, q_s^n, $q_t^n > 0$ and q_r^m, q_s^m, $q_t^m > 0$, and if

$$\frac{u_m(x_s) - u_m(x_t)}{u_m(x_r) - u_m(x_s)} \neq \frac{u_n(x_s) - u_n(x_t)}{u_n(x_r) - u_n(x_s)}$$

then $\mathbf{q}$ is inefficient ex-ante.

In words, if the slopes of the two agents' indifference curves in the corresponding probability triangles are not the same, then any allocation that gives both agents lotteries with positive probabilities on these three goods

[4]This bound may be tighter under further assumptions on individual preferences. See for example the case of same preferences in Section 2.2.

[5]Assuming that all individuals are expected utility maximizers, Hylland and Zeckhauser [22] use competitive equilibrium with equal incomes to show the existence of a solution in which almost all agents receive a binary lottery. Our result holds without relying on any market mechanism.

is inefficient. The proof is identical to the one given in the appendix for Theorem 1 and is omitted.

There are many popular mechanisms that can be used to allocate objects among a group of agents. One example that is broadly used and is easy to implement is random serial dictatorship. Randomly order the Nk individuals and let them choose in their turn the best good still available according to their personal ranking. It is well known that using this mechanism, the ultimate ex-post allocation of goods among agents is Pareto efficient (see for example Abdulkadiroğlu and Sönmez [3]). Theorem 1 implies, however, that ex-ante this mechanism is typically inefficient. To illustrate, suppose all individuals have the same ranking over the basic goods and that each individual has a probability $\frac{1}{Nk}$ to be the i^{th} in the order. Then, each individual will perceive this as a uniform lottery over all the goods (with probability $\frac{1}{N}$ each), which, according to Theorem 1 is inefficient. This argument is also valid if individuals don't have the same ordinal preferences over the goods, in which case the ex-ante lottery induced by random serial dictatorship for each agent is not necessarily uniform, yet typically has more than two goods in its support.[6] It thus follows that with quasi-convex preferences, random serial dictatorship is typically inefficient ex-ante.

This suggests a broader point. There are known results that imply the equivalence of different randomized mechanisms and random serial dictatorship (Abdulkadiroğlu and Sönmez [3]; see also Pathak and Sethuraman [29]), in the sense that they induce the same ex-ante probability distribution over the final goods. But then those seemingly identical mechanisms are also typically ex-ante inefficient. If social planners know the individuals' preferences over lotteries, and in particular that they are strictly quasi-convex, they can improve the agents' welfare ex-ante. For more on this, see Section 4 below.

[6] An extreme situation is where for each good i there are exactly k people who rank it first in their ordinal preferences. In this case the (degenerate) lottery is ex-ante efficient, but then there is no need for a mechanism in the first place.

Importantly, this argument only relies on simple, observable information: quasi-convexity of preferences and the size of the supports of the lotteries that are used.

Assuming expected utility, Bogomolnaia and Moulin [7] show how random serial dictatorship, which uses uniform distribution to rank agents, is not necessarily even ordinally efficient, as it may induce for each agent a distribution over the goods that is stochastically dominated, with respect to that agent's ordinal preferences, by another feasible distribution. Their suggested probabilistic serial mechanism (which is ordinally efficient) is typically not ex-ante efficient. It is also worth noting that their solution implies that agents with the same ordinal preferences must receive the same lottery over goods. In our case, even if all agents have the same cardinal preferences (and are strictly quasi-convex), *necessarily* not all of them receive the same lottery, as otherwise, the same binary lottery to all will not allocate all available goods.

2.2 Same Preferences

When all individuals have the same preferences, it is natural to require that a just mechanism will offer them the exact same outcome. But since, by Theorem 1, efficient allocations of lotteries with quasi-convex preferences are inconsistent with such a requirement, we instead impose equality in the sense that identical agents receive equally attractive outcomes. That is, if $\succeq_1 = \ldots = \succeq_{Nk} = \succeq$, then $q^1 \sim \ldots \sim q^{Nk}$. We assume in this subsection that wlog, all agents agree that $x_1 \succ \ldots \succ x_N$.

Definition 2 Let $\succeq_1 = \ldots = \succeq_{Nk} = \succeq$. A solution $\mathbf{q}$ is ideal if it is feasible, efficient, and satisfies equality.

The next result uses the floor function, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Theorem 2 Suppose that $\succeq_1 = \ldots = \succeq_{Nk} = \succeq$. Then:

1. Ideal solutions exist.

2. The number of different binary lotteries used in any ideal solution is bounded above by $M = \left\lfloor \frac{N^2}{4} \right\rfloor$.

3. An ideal solution yields all but at most $M = \left\lfloor \frac{N^2(N-2)}{8} \right\rfloor$ agents a binary lottery.

Below, we outline the main steps involved in the proof.

For part 1, let V be a continuous representation of $\succeq$. For a solution $\mathbf{q}$ satisfying equality, let $V(\mathbf{q}) := V(q^1) = \ldots = V(q^{Nk})$. Let $v = \sup\{V(\mathbf{q}) : \mathbf{q}$ is a feasible solution satisfying equality$\}$. We show first that since the domain of possible allocations is compact, there is a solution $\mathbf{q}^*$ for which v is obtained. We then show that if in a feasible allocation two agents do not receive the same utility level, say n receives higher utility than m, then there is a another feasible allocation in which m's utility goes up, n's utility goes down but is still higher than m's, while the allocation of no one else is affected. Suppose now that $\mathbf{q}^*$ is inefficient. Then there is an allocation $\tilde{\mathbf{q}}$ that is better than $\mathbf{q}^*$ for some and worse for none. Using the above result we can assume that w, the lowest utility in $\tilde{\mathbf{q}}$, is greater than the common utility in $\mathbf{q}^*$. Define b to be the inf of the size of the utility range for the set of feasible allocations that give everyone utility w or more. We show that b is obtained in a feasible allocation $\hat{\mathbf{q}}$, hence b must be zero, as otherwise, by the above result, the distance between two extreme agents can be reduced. In other words, $\hat{\mathbf{q}}$ is a feasible allocation satisfying equality, a contradiction to the definition of $\mathbf{q}^*$, hence $\mathbf{q}^*$ is efficient.

For parts 2 and 3, note that the number of binary lotteries used in any optimal solution is bounded above by the number of pairwise non-dominated binary lotteries that can simultaneously be used. If one of the binary lotteries used involves outcomes x_i and x_j with $i < j$, then since all lotteries on outcomes better than x_i dominate it and all lotteries on outcomes inferior to x_j are dominated by it, such lotteries cannot be part of the ideal solution.

Similarly, the bound on the number of agents who hold non-binary lotteries (which for $N > 4$ is lower than the $\binom{N}{3}$ bound from the general case of Theorem 1) is the number of non-dominated lotteries with three possible outcomes that can simultaneously be used. Note that many individuals may hold the same binary lottery, but only one individual can hold any non-binary lottery.

The proofs of parts 2 and 3 of Theorem 2 only use the requirement that the lottery received by one person cannot dominate the lottery received by another. The actual number of binary lotteries used in an ideal solution can be much smaller than the upper bound suggested by the theorem. Theorem 3 of Section 3.1 identifies conditions under which the set of binary lotteries in $\mathbf{q}$ is either $\{(q_1, q_i)\}_{i=2}^{N}$ or $\{(q_i, q_N)\}_{i=1}^{N-1}$. The number of binary lotteries used in these cases is $N-1$, significantly less than the bound obtained in part 2 of Theorem 2. For example, for $N = 10$, the conditions of Theorem 3 imply 9 binary lotteries, whereas the bound of Theorem 2 is 25. Note that the lower bound on the number of binary lotteries to be used is $\lceil \frac{N}{2} \rceil$, where $\lceil x \rceil$, the ceiling of the real number x, is the lowest integer greater than or equal to x. This will be the case when a feasible solution is obtained by a set of lotteries $(q_1, q_N), (q_2, q_{N-1}), \ldots$ that are all equally attractive in $\succeq$.

3 Extensions

We discuss two possible extensions to our basic framework. We first analyze assignment problems where all agents have the same preferences, but the number of units to be allocated is not equal to the number of agents. Second, we consider a continuum economy with the same mass of agents and goods.

3.1 Different Number of Units and Agents

Consider again the case where all agents have the same preferences over lotteries (as in section 2.2) and suppose as before that $x_1 \succ \ldots \succ x_N$. When there are more units than agents, efficiency implies that if $x_i \succ x_j$, then ex-post it cannot be the case that units of x_j are assigned while some units of x_i are not. Moreover, since quasi-convexity of preferences implies that if $x_i \succ x_j$, then $x_i \succ (x_i, \alpha; x_j, 1-\alpha)$ for all $\alpha \in (0,1)$, units of x_j will not be used in any ex-ante lottery if units of x_i are not exhausted.

More interesting is the case where there are more agents than units, which we analyze in Theorem 3 below. We say that the outcome x_N is terrible if even an allocation that yields everybody a lottery that is as good as the second-worst outcome x_{N-1} is not feasible, as it is using too much of the desired goods. The reason why a certain outcome is terrible may be that it is very bad compared to other results. Another reason may be that individuals are extremely risk averse, in which case the values of lotteries are heavily tilted in the direction of the least attractive outcome, even if it is not much worse than other outcomes.[7] In such cases, everyone has to receive the worst outcome with some positive probability, as otherwise one allocation will have to be better than x_{N-1}. Equality then requires everyone to receive a lottery which is better than x_{N-1}, a violation of feasibility.

Formally, let $L_i^T = (q_i^T, q_N^T) \sim \delta_{x_{N-1}}$, $i = 1, \ldots, N-1$ where $q_i^T + q_N^T = 1$. That is, L_i^T is the binary lottery over x_i and x_N that is indifferent to x_{N-1}. (For $i = N-1$ it is the degenerate lottery yielding x_{N-1} with probability 1). The outcome x_N is *terrible* if the hyperplane H^T through $L_1^T, \ldots, L_{N-1}^T$ is above the point $(\frac{1}{N}, \ldots, \frac{1}{N})$. Similarly, let $L_i^E = (q_1^E, q_i^E) \sim \delta_{x_2}$, $i = 2, \ldots, N$ where $q_1^E + q_i^E = 1$. The outcome x_1 is *excellent* if the hyperplane H^E through $L_2^E, \ldots, L_N^E$ is below $(\frac{1}{N}, \ldots, \frac{1}{N})$.

[7] We thank Todd Sarver for this insight.

Theorem 3 Suppose that all Nk agents have the same preferences and that $x_1 \succ \ldots \succ x_N$. If x_N is a terrible outcome, then all the binary lotteries used by an ideal allocation $\mathbf{q}$ are of the form (q_i, q_N), $i = 1, \ldots, N-1$, and for a sufficiently large k, they are all used. Parallel results hold for the case where x_1 is an excellent outcome, with the binary lotteries (q_1, q_i), $i = 2, \ldots, N$.

If there are more agents than units, define a new good x_{N+1} which is "receive nothing." At least in the case of school allocation, this may well be a terrible outcome. Theorem 3 implies that in that case, almost all children face a lottery where there are two possible outcomes: either they go to a specific school, or they stay at home. In other words, they face uncertainty regarding acceptance, but not regarding the school into which they will be accepted. Equality implies that the better the school, the less likely is a holder of a lottery for this school going to win.

3.2 Continuum Economies

Consider a continuum economy with a unit mass $\mathcal{A}$ of N equally sized (with respect to the Lebesgue measure μ) types of agents $\mathcal{A}_1, \ldots, \mathcal{A}_N$. There is a unit mass $\mathcal{B}$ of N goods $x_1, \ldots, x_N$ to be allocated among them, where the mass of each unit is $\frac{1}{N}$.[8] Each of the individuals of type i has strictly quasi-convex preferences $\succeq_i$ over lotteries over the N goods.

Our aim in this paper is to analyze possible mechanisms for the allocation of goods which are desired by all, as otherwise there is no need for a

[8]In fact, we can assume J types of goods, and that both the N types of individuals, as well as the J types of goods, are not of same size. However if the sizes are rational numbers, we can assume without loss of generality that $J = N$ and the sizes of the different goods are the same; and if they are irrational, we'll obtain our results using continuity, where the economy is the limit of economies with rational sizes. We therefore assume throughout $J = N$ and that the sizes of the types of agents and of the goods are all $\frac{1}{N}$. See Footnote 10 below for a further generalization.

compromise. Our analysis therefore fits best a situation where everyone has the same preferences over the N goods (even if not the same preferences over lotteries over these goods). Nevertheless, our mathematical results hold on a wider range of preferences, with the only restriction that all agents agree that a certain good, say x_1, is best. That is, for all $i = 1, \ldots, N$ and $j = 2, \ldots, N$, $x_1 \succ_i x_j$, but there are no other restrictions on the way individuals rank the outcomes $x_2, \ldots, x_N$.

With a little abuse of notation, a point q in the $(N-1)$-dimensional probability simplex Δ^{N-1} represents the lottery $(x_1, q_1; \ldots; x_{N-1}, q_{N-1}; x_N, 1 - \sum_{i=1}^{N-1} q_i)$ and we now denote by $q^a \in \Delta^{N-1}$ the lottery obtained by person a. An allocation is a measurable function $f : \mathcal{A} \to \Delta^{N-1}$. The allocation f is feasible if $\int_{\mathcal{A}} f_i(a) d\mu = \frac{1}{N}$, $i = 1, \ldots, N-1$ (this is the analogue condition to eq. (3) of Section 2). It is efficient if there is no allocation g such that $\forall i$ and $\forall a \in \mathcal{A}_i$, $g(a) \succeq_i f(a)$, and a positive mass of agents strictly prefer their outcome under g to their outcome from f. To simplify the presentation, we'll use the term "all" for "all but a zero measure of agents." We are interested in characterizing allocations that are efficient and satisfy the following No-Envy criterion.

No-Envy For all a and b, $q^a \succeq_a q^b$.

No-Envy postulates that in the allocation of lotteries, no individual would prefer to replace their lottery with that of any other agent.[9] Clearly, if $\succeq_1 = \ldots = \succeq_N = \succeq$, then No-Envy implies equality, in the sense that for all $a, b \in \mathcal{A}$, $f(a) \sim f(b)$.

No-Envy is appealing on normative grounds. Furthermore, in a standard (convex) Walrasian setting, it is compatible with the Efficiency requirement (see, for example, Varian [41]). But in a non-convex economy as ours, it is not guaranteed that the two coexist (see, for example, Vohra [42] and

[9] The definition is again in the ex-ante sense, before agents know the realization of the lotteries they receive.

Maniquet [26]). We show however that in the present context, the continuum economy guarantees the existence of no-envy allocations.

Theorem 4 A feasible and efficient allocation for the continuum economy with strictly quasi-convex preferences yields all agents a binary lottery. The set of no-envy such allocations is not empty, and if all agents have the same preferences, then equality with such lotteries is obtained.

We offer here an outline of the proof. The first step shows, similarly to the proof of Theorem 1, that efficient allocations must yield all agents a binary lottery. Next, we start from an allocation where everyone is facing the lottery that gives them an equal chance for each of the goods and employ a known technique of demand-sets convexification (see Mas-Colell, Whinston, and Green [28, Section 17.I] which is based on Starr [37]) to obtain a competitive market equilibrium prices and allocations. Given these prices, all agents will maximize their utility along the same budget set, so No-Envy is guaranteed. Competitive equilibria are feasible and efficient, hence the claim of the theorem.

There is however one issue that requires special attention in which our analysis of the market equilibrium differs from the literature. Formally, the lottery $(x_1, q_1; \ldots; x_N, 1 - \sum_{i=1}^{N-1} q_i)$ is represented as the vector $(q_1, \ldots, q_{N-1})$ in the $N-1$-dimensional simplex. This is different from the standard model, where the domain of preferences is not bounded from above. To see why this may create a problem, consider Example 1 in the Appendix with $N = 3$ where $x_1 \succ x_2 \succ x_3$. The preferences of this example are monotonic in the probabilities q_1 and q_2 in the sense that if $(q'_1, q'_2) \gneqq (q_1, q_2)$, then $(q'_1, q'_2) \succ (q_1, q_2)$. But they do not satisfy monotonicity with respect to first order stochastic dominance, in the sense that for $\varepsilon > 0$, $(q_1 + \varepsilon, q_2 - \varepsilon) \succ (q_1, q_2)$, and equilibrium does not exist. We show in the proof of Theorem 4 that this stronger version of monotonicity eliminates the existence problem.

Remark 1 Let T be the number of lotteries used in the proposed solution. Then for $h = 1, 2, ..., T$ there is a continuum of agents who receive the same binary lottery, say $(x^h, \rho^h; y^h, 1-\rho^h)$ for some outcomes x^h, y^h and $\rho^h \in [0, 1]$. The implementation of this, so that the fraction of the people in this group that receives x^h is ρ^h, can be guaranteed by using the appropriate law of large numbers for a continuum of independent random variables. Such approach appears, for example, in Sun [38], and we adopt here his measure theoretic framework.[10]

4 Is the Data Available?

In this paper we are interested in properties of the induced allocation of lotteries for any given set of preferences, as our aim is to emphasize the implications of taking individual preferences over lotteries into consideration in evaluating stochastic allocation mechanisms. To that extent, we ignore the question of strategy-proofness, that is, how to guarantee that agents truly reveal their preferences. As Zhou [47] shows, under expected utility, which is a subset of all quasi-convex preferences, there exists no mechanism that satisfies symmetry, ex-ante Pareto optimality, and strategy-proofness. As global strategy proofness cannot be achieved, we ask instead whether there is any reliable data available to policy makers, and if only vague information is available, can it still be useful?

The first question to answer is how do decision makers evaluate lotteries? There is a lot of empirical research trying to answer this question, mostly with respect to lotteries with monetary payoff (see, for example, the surveys

[10] We assumed that there are N blocks of agents so that the analysis of the continuum will parallel the finite case. If there is a continuum of types where the measure of each type is zero, then as in Mas-Colell, Whinston, and Green [28, p. 629] the actual allocation doesn't require the analysis of this remark, as almost all agents will have a unique lottery in their demand set.

of Camerer [8] and Starmer [36]). One of the most popular group of models is based on the idea that the evaluation of the probability of an outcome depends on its rank in the support of the lottery. This family includes Quiggin's [31] rank-dependent utilities, Yaari's [46] dual theory, and Tversky and Kahneman's [39] cumulative prospect theory. For $x_1 \succ x_2 \succ \ldots \succ x_N$, the rank dependent functional form is given by

$$V(q) = u(x_1)\pi(q_1) + \sum_{i=2}^{N} u(x_i)\left[\pi\left(\sum_{j=1}^{i} q_j\right) - \pi\left(\sum_{j=1}^{i-1} p_j\right)\right]$$

where $\pi : [0,1] \to [0,1]$ is strictly increasing and onto. To use the rank dependent model, one needs to know the utilities from the outcomes and the transformation function the decision maker is using to evaluate the (cumulative) probabilities.

Although not always possible, there are situations where getting information about cardinal ranking of alternatives is possible. For example, rankings of schools are posted yearly in trusted journals. In recent years these rankings include descriptions of some quantitative parameters like total cost and students to faculty ratio, which may help candidates to infer their intensity of preference among them.

For the probability transformation function π, one can use known techniques to estimate parameters of specific functional forms or even perform a parameter-free elicitation within a class of preferences. For example, the analyst may assume either a general probability weighting function or the more specific function $\pi(p) = p^\alpha$ for some $\alpha > 0$ that will be calibrated together with the cardinal utility of the goods.[11] Likewise, it may be possible to get cardinal utilities using lab experiment. In fact, elicitations of the probability weighting function were implemented in medical decision analysis (Bleichrodt and Pinto [5], Bleichrodt, Pinto, and Wakker [6]) and were

[11]In this specification, quasi-convexity is implied if $\alpha > 1$, so that π is convex. Convex π captures "pessimism": for any x, increasing the probability of receiving a prize $y \preceq x$ decreases the probability weight of x. Furthermore, within the family of rank-dependent utilities models, convex π is necessary for risk aversion (Chew, Karni, and Safra [12]).

practically used to improve medical decisions (Wakker [44]). It is well known that measures of risk aversion tend to be context-specific and vary across domains (see, among others, Weber, Blais, and Betz [45] or Hanoch, Johnson, and Wilke [20]). Yet, and even though there is not yet a consensus about the "best" approach to use (Charness, Gneezy, and Imas [10]), methods analogous to those employed in financial, health/safety, recreational, ethical, and social decisions (as in Weber, Blais, and Betz [45], or the ones discussed in Finkelstein, Luttmer, and Notowidigdo [18] to estimate health state dependence of the utility functions) can be used in the domain of lotteries over apartments or schools.

Even without any information about the specific model used by members of society our results suggest ways for welfare improvements. Sometimes the social planner has information about other characteristics of the agents that can be used to assess their intensity of preferences over allocations. For example, it is plausible that a resident of a certain neighborhood would put higher premium on attending a school in close proximity compared to someone who considers only remote schools. Similarly, a religious person will naturally have stronger preferences for schools that have religious components in their operations or curriculum compared to someone who does not take this dimension into consideration.

Such information can be used in the following way. Starting from an allocation that results from a strictly strategy-proof mechanism with respect to the ordinal rankings (for example, random serial dictatorship), ex-ante allocations may yield two agents positive probabilities over the same three outcomes. The social planner can use insights about individual intensity of preferences for a local welfare improvement as described in Figure 1, without scarifying this form of strategy proofness. A related approach was suggested by Abdulkadiroğlu, Che, and Yasuda [2] to improve individuals' welfare over mechanisms that randomly break ties between agents with identical ordinal preferences over the goods. Confining their attention to expected utility preferences and large economies, the authors offered a mechanism that allows students to signal their cardinal preferences, and showed that it (ex-ante) Pareto dominates the popular deferred acceptance mechanism. Their new mechanism is typically ex-ante inefficient and is only ordinally strategy-proof.

5 Ex-ante Lotteries

If preferences are strictly quasi-convex, then giving two identical agents the same interior outcome must be inefficient, as moving in opposite directions along a supporting plane of the indifference curve will make both better off. Instead of equality in outcomes, allocation mechanisms will seek a weaker notion of equality, where identical agents will be indifferent between their respective outcomes. This is indeed the conclusion from Theorem 2, where everyone is indifferent between all allocated lotteries, even though they are not the same.

But indifference between outcomes does not imply indifference to the procedures used to allocate these outcomes. A person may be indifferent between two seemingly identical objects of art left by his grandparents. Yet realizing that at least one of them must be a faked copy of the original, he will not trust his cousin, a museum curator, to choose first. In the context of the school allocation problem, parents may suspect the social planner of having some private information regarding the schools which will imply better lotteries for some families favored by the authorities.

There is a simple way to avoid such potential mistrust: Everyone will face the same lottery P over the set of the binary lotteries. The learned cousin may know which of the two vases is Ming and which is a modern counterfeit, but she will not be able to use this information if the allocation is dictated by the outcome of a fair coin. Similarly, even if the social planner favors some families, inside information about the schools becomes useless if the lotteries of Theorem 2 are allocated by a lottery.[12] Given that all individuals will face the exact same lottery, this procedure guarantees full equality in the ex-ante stage.

The effectiveness of this procedure crucially depends on the agents' attitude towards multi-stage lotteries. Denote the relevant binary lotteries $q^{(1)}, \ldots, q^{(T)}$. If agents care only about the overall probability distribution over final outcomes, then they will perceive a compound lottery over lotteries as a simple lottery over final outcomes, where the probability of each x_i is $\sum_j P(q^{(j)})q^{(j)}(x_i)$. But then, if preferences over simple lotteries are strictly quasi-convex, all individuals will be worse off compared to their initially held lotteries.

Suppose however that individuals do not reduce compound lotteries using the laws of probability but satisfy instead the compound independence axiom (Segal [35], Dillenberger [15]). This axiom prescribes that if a person prefers receiving q to q' for sure, then they will prefer to replace q' with q in any compound lottery that has q' in its support. This implies that if initially the agent is indifferent between q and q', they will also be indifferent to any such replacement. Since, by construction, equality implies that all agents are indifferent between all lotteries in the suggested allocation, they will also be indifferent to any lottery over them. In other words, compound independence guarantees full ex-ante equality among agents without reducing their welfare. Whether or not compound independence holds — while the reduction of compound lotteries axiom does not — is an empirical question, which has been studied in various settings (see, among others, Halevy [19], Abdellaoui, Klibanoff, and Placido [1], Harrison, Martínez-Correa, and Swarthout [21], and Masatlioglu, Orhun, and Raymond [27].) To our knowledge, however,

[12] The emphasize here is on a real randomization rather than an imaginary randomization that each agent may entertain about his possibility to receive any of the objects. Only the former will remove agents' concerns for unfairness or of a biased use of planer's private information in the allocation decision.

it is yet to be examined in scrutiny for the specific context of allocation mechanisms.[13]

6 Concluding Remarks

The use of binary lotteries is pervasive in economics. Many experimental works are conducted with choices among such lotteries (or between them and sure outcomes), where the main rationale for using binary lotteries is that they are easily interpretable. Some recent theoretical papers use simplicity criteria to argue for the attractiveness of binary lotteries in terms of minimizing complexity costs (for example, Puri [30]), and of binary acts, that are always 'well-understood' and can be used as a tool for making difficult comparisons (Valenzuela-Stookey [40]).

In our setting, that (almost) everyone should receive a binary lottery follows mathematically from the assumption that all individual preferences are quasi-convex. As argued above, this gives us a simple necessary condition that can be used to assess whether an allocation of lotteries is ex-ante efficient. But as a social mechanism, binary lotteries have another independent attraction of their own. When facing a lottery over a set of outcomes on which they do not have full information, it is quite natural for people to look for such information before the lottery is played. If so, it is clearly better for them to face a lottery with fewer outcomes.

In Section 5 we suggested another layer of social randomization over the lotteries that will be used. If people reduce lotteries by multiplying the probabilities then they will probably need to evaluate all N outcomes. But if they use the compound independence axiom, then they view the first stage as a lottery over lotteries and will defer evaluating the outcomes till the next

[13]For a theoretical analysis of allocation mechanisms where the reduction of compound lotteries axiom is replaced with compound independence, see Dillenberger and Segal [16].

stage, when they'll face a lottery over two outcomes only.

Our aim in this paper is to suggest a new way to assess and think of existing mechanisms. Our approach would be most relevant to applied researchers if indeed agents have quasi-convex preferences in the context of lotteries over allocations and if our insights can be used even in the absence of full strategy proofness. While quasi-convex preferences (and obviously the limit case of expected utility) were heavily used in many theoretical applications, no comprehensive tests of quasi-convexity in our domain of interest have been conducted thus far.[14] Due to the simplicity of the assumption, such tests should be easy to perform. But if one may extrapolate from the appearance of quasi-convex preferences in other domains, then identifying ways and situations in which local improvements over current methods can be performed, as we discuss in Section 4, would be our main *applied* message.

Appendix

Proof of Theorem 1: Suppose that for $a = n, m$, $q_r^a, q_s^a, q_t^a > 0$. If the two individuals do not have the same ordinal preferences over the three goods, for example, if $x_r \succ_n x_s$ but $x_s \succ_m x_r$, then transfer ε probability of x_r from person m to n and ε probability of x_s from n to m to obtain a feasible allocation which is strictly preferred to the original one by n and m and indifferent to the original one by everyone else.

Suppose now that for $a = m, n$, $x_r \succ_a x_s \succ_a x_t$ and as before, that $q_r^a, q_s^a, q_t^a > 0$. For $a = m, n$, let $\bar{q}^a = q_r^a + q_s^a + q_t^a$. As explained in Section 2.1, the two triangles of Figure 1 depict probability allocations over the three outcomes for individuals m and n that do not change the sums of these probabilities. All the changes in this proof are sufficiently small so that

[14] Similarly, as we pointed out in Section 5, agents' attitudes towards multi-stage lotteries in our context have not been studied.

they can be done without violating eqs. (1) and (2). In both panels, the probability of x_t is measured on the horizontal axis and that of x_r on the vertical one. The only values of $\mathbf{q}$ that will be changed are those of q_i^a for $a = m, n$ and $i = r, s, t$. We will therefore deal with the induced preferences over the above triangles and ignore the rest of the probabilities. To simplify notation, we write (q_t^a, q_r^a) for $(q_t^a, \bar{q}^a - q_t^a - q_r^a, q_r^a)$, which by itself stands for $(q_t^a, \bar{q}^a - q_t^a - q_r^a, q_r^a, q_{-(t,s,r)}^a)$.

Without loss of generality, one of the supporting slopes to the indifference curve of person n through (q_t^n, q_r^n) is weakly steeper than one of the supporting slopes to the indifference curve of person m through (q_t^m, q_r^m) (such slopes exist by the quasi-convexity of the preferences). Let τ be a slope between these two values. Since preferences are strictly quasi-convex, we get that for a sufficiently small $\varepsilon > 0$, $(q_t^m + \varepsilon, q_r^m + \tau\varepsilon) \succ_m (q_t^m, q_r^m)$ and $(q_t^n - \varepsilon, q_r^n - \tau\varepsilon) \succ_n (q_t^n, q_r^n)$. Observe that eqs. (1) and (2) are still satisfied and everyone else is indifferent between the new and the old lotteries, a violation of efficiency. ■

Proof of Theorem 2: Let V be a continuous representation of the common preferences $\succeq$.

1. Ideal solutions exist: We prove this part of the theorem through a sequence of lemmas.

Lemma 1 There is a feasible solution $\mathbf{q}^* = (q^{1,*}, \ldots, q^{Nk,*})$ satisfying equality such that for any solution $\mathbf{q} = (q^1, \ldots, q^{Nk})$ satisfying equality, $q^{n,*} \succeq q^n$, $n = 1, \ldots, Nk$.

Proof: The set of feasible solutions satisfying equality is not empty, for example, $q^1 = \ldots = q^{Nk} = (\frac{1}{N}, \ldots, \frac{1}{N})$. For a solution $\mathbf{q}$ satisfying equality, let $V(\mathbf{q}) := V(q^1) = \ldots = V(q^{Nk})$. Let $v = \sup\{V(\mathbf{q}) : \mathbf{q}$ is a solution satisfying equality$\}$ and for $h = 1, \ldots$, let $\mathbf{q}^h = (q^{1,h}, \ldots, q^{Nk,h})$ be a sequence of solutions satisfying equality such that $V(\mathbf{q}^h) \to v$. For each n and h, $q^{n,h}$ is a vector in the compact probabilities simplex Δ^{N-1}, hence it follows by standard arguments that there is a subsequence of $\mathbf{q}^h$, without loss of generality the sequence itself, such that for all $n = 1, \ldots, Nk$, $q^{n,h} \to q^{n,*}$. The vector $\mathbf{q}^* = (q^{1,*}, \ldots, q^{Nk,*})$ satisfies eqs. (1) and (2), hence it is a solution. Since V is continuous it satisfies equality, and by the continuity of V, $V(q^{n,*}) = v$. It follows by the definition of v that for any solution $\mathbf{q} = (q^1, \ldots, q^{Nk})$ satisfying equality, $q^{n,*} \succeq q^n$, $n = 1, \ldots, Nk$. $\square$

Lemma 2 Let $\mathbf{q}$ be a feasible solution in which for some two individuals m and n, $q^n \succ q^m$. Then there is a feasible solution $\bar{\mathbf{q}}$ where $q^n \succ \bar{q}^n \succeq \bar{q}^m \succ q^m$, and for $\ell \neq n, m$, $\bar{q}^\ell = q^\ell$.

Proof: Since $q^n \succ q^m$, it follows by monotonicity with respect to first-order stochastic dominance (in short, by FOSD) that there are goods r and s such that $x_r \succ x_s$ and such that $\varepsilon = \min\{q^n_r, q^m_s\} > 0$, as otherwise $q^m \succeq q^n$. In both profiles below, q^ℓ does not change for all $\ell \neq n, m$. For $\varepsilon' \leqslant \varepsilon$, let $\bar{q}^n = (q^n_r - \varepsilon', q^n_s + \varepsilon', q^n_{-r,s})$ and $\bar{q}^m = (q^m_r + \varepsilon', q^m_s - \varepsilon', q^m_{-r,s})$. For a sufficiently small $\varepsilon' > 0$, $q^n \succ \bar{q}^n \succeq \bar{q}^m \succ q^m$. $\square$

Lemma 3 The solution $\mathbf{q}^*$ as in Lemma 1 is efficient.

Proof: Let $\mathbf{q}^*$ be as in Lemma 1, and suppose that there is $\tilde{\mathbf{q}} = (\tilde{q}^1, \ldots, \tilde{q}^{Nk})$ such that wlog $V(\tilde{q}^1) \geqslant \ldots \geqslant V(\tilde{q}^{Nk}) \geqslant V(q^{1,*}) = \ldots = V(q^{Nk,*})$, where at least one of these inequalities is strict. Applying Lemma 2 $Nk - 1$ times at most, we can create a feasible allocation $\bar{\mathbf{q}}$ such that for all n, $V(\bar{q}^n) > V(q^{1,*})$. Let $w = \min\{V(\bar{q}^n)\}$ and define

$$b = \inf\left\{\max_n\{V(q^n)\} - \min_n\{V(q^n)\} : \mathbf{q} \text{ is feasible and } \min_{\mathbf{n}}\{\mathbf{V(q^n)}\} \geqslant \mathbf{w}\right\}$$

As in the proof of Lemma 1, there is a feasible solution $\hat{\mathbf{q}}$ for which b is obtained. By Lemma 2, $b = 0$. This means that $\hat{\mathbf{q}}$ satisfies equality, a contradiction to the definition of $\mathbf{q}^*$. $\square$

By Lemma 1, the feasible solution $\mathbf{q}^*$ satisfies equality, and by Lemma 3 it is efficient, hence it is an ideal solution. $\Diamond$

2. The number of different binary lotteries used in any ideal solution is bounded above by $M = \left\lfloor \frac{N^2}{4} \right\rfloor$: Let B be the set of binary lotteries used by an ideal solution $\mathbf{q}$. We show that there is t^* such that for all non-degenerate $(q_r, q_s) \in B$, $r \leqslant t^* < s$.

Case 1: If one of the lotteries in B is degenerate, say δ_{t^*}, then by equality and FOSD, for any $(q_r, q_s) \in B$ it must be the case that $r < t^* < s$.

Case 2: There is no t for which there exists $(q_r, q_s) \in B$ such that $t < r < s$. In particular there is no $(q_r, q_s) \in B$ such that $1 < r < s$. Then all lotteries in B must have x_1 as one of their outcomes, and therefore for $t^* = 1$ we get that for all $(q_r, q_s) \in B$, $r \leqslant t^* < s$.

Case 3: There is t for which there exists $(q_r, q_s) \in B$ such that $t < r < s$. Suppose that there is no t^* as above. Then for every t either there is $(q_r, q_s) \in B$ such that $r < s \leqslant t$ or there is $(q_r, q_s) \in B$ such that $t < r < s$. We show that this requirement leads to a violation of equality. Let $\bar{t}$ be the highest value of t for which there is $(q_r, q_s) \in B$ such that $\bar{t} < r < s$. Let $(q_{r'}, q_{s'}) \in B$ such that $\bar{t} < r' < s'$. By the definition of $\bar{t}$, there is no $(q_r, q_s) \in B$ such that $\bar{t}+1 < r < s$, hence there is $(q_{r''}, q_{s''}) \in B$ such that $r'' < s'' \leqslant \bar{t}+1 \leqslant r' < s'$. By FOSD we obtain a violation of equality, as $(q_{r''}, q_{s''})$ dominates $(q_{r'}, q_{s'})$.

The maximal number of such pairs given t^* is $t^*(N - t^*)$. This term is maximized at $t^* = \frac{N}{2}$, where it is equal to $\left\lfloor \frac{N^2}{4} \right\rfloor$. $\Diamond$

3. An ideal solution yields all but at most $M = \left\lfloor \frac{N^2(N-2)}{8} \right\rfloor$ agents a binary lottery: As in Theorem 1, if $\mathbf{q}$ is an ideal solution, then for any three goods x_r, x_s, x_t there is at most one person n such that $q^n_r, q^n_s, q^n_t > 0$, otherwise $\mathbf{q}$ is inefficient. Given an ideal solution $\mathbf{q}$, let C be the set of the indexes of the

non-binary lotteries allocated by it. That is, $C = \{\{r_1 < r_2 < \ldots < r_d\} : (q_{r_1}, q_{r_2}, \ldots, q_{r_d})$ is one of the lotteries allocated by $\mathbf{q}\}$.

Similarly to part 2 above, we show that there is ℓ^* such that for all $\{r_1 < \ldots < r_d\} \in C$, $r_1 \leqslant \ell^* < r_d$. If there is no ℓ for which there exists $\{r_1 < \ldots < r_d\} \in C$ such that $\ell < r_1$, then all lotteries with indexes in C must have x_1 as one of their outcomes, in which case we set $\ell^* = 1$.

Suppose now that there is ℓ for which there exists $\{r_1 < \ldots < r_d\} \in C$ such that $\ell < r_1$, but there is no ℓ^* as above. Then for every ℓ either there is $\{r_1 < \ldots < r_d\} \in C$ such that $r_d \leqslant \ell$, or there is $\{r_1 < \ldots < r_d\} \in C$ such that $\ell < r_1$. Let $\bar{\ell}$ be the highest index for which there is $\{r_1 < \ldots < r_d\} \in C$ such that $\ell < r_1$. Let $\{r'_1 < \ldots < r'_d\} \in C$ be such a set, hence $\bar{\ell} + 1 \leqslant r'_1$. By the definition of $\bar{\ell}$, there is no $\{r_1 < \ldots < r_d\} \in C$ such that $\bar{\ell} + 1 < r_1$. There is therefore $\{s_1 < \ldots < s_c\} \in C$ such that $s_c \leqslant \bar{\ell}$. But as by FOSD all lotteries with support $(x_{s_1}, \ldots, x_{s_c})$ are strictly preferred to all lotteries with support $(x_{r'_1}, \ldots, x_{r'_d})$, equality of all lotteries in C cannot be satisfied, hence such ℓ^* exists.

Suppose that there is $\{r_1 < \ldots < r_d\} \in C$ where $d > 3$. Then either $r_1 \leqslant \ell^* < r_2$, or $r_1 < r_2 \leqslant \ell^* \leqslant r_{d-1} < r_d$ where at least one of the two weak inequalities is strict, or $r_{d-1} \leqslant \ell^* < r_d$. In the first case, replace $\{r_1, \ldots, r_d\}$ with $\{r_1, r_2, r_3\}$ and $\{r_1, r_2, r_4\}$. In the second case, replace it with $\{r_1, r_2, r_d\}$ and $\{r_1, r_{d-1}, r_d\}$. And in the third case, replace it with $\{r_1, r_2, r_d\}$ and $\{r_1, r_3, r_d\}$ to expand C while still maintaining the position of ℓ^*.[15] It thus follows that $|C|$ is bounded above by the number of triplets $\{r_1 < r_2 < r_3\}$ such that $r_1 \leqslant \ell^* < r_3$. The maximal number of such triplets is

$$\binom{\ell^*}{2} \times (N - \ell^*) + \ell^* \times \binom{N - \ell^*}{2} = \frac{\ell^*(N - \ell^*)(N - 2)}{2}$$

This expression is maximized at $\ell^* = \frac{N}{2}$, where it is equal to $\left\lfloor \frac{N^2(N-2)}{8} \right\rfloor$. ■

[15] Observe that these new lotteries will no longer necessarily satisfy feasibility and efficiency, but they help us establish the bound of the theorem.

Proof of Theorem 3: Denote by q^T and q^E the degenerate lotteries that yield x_N and x_1 with probability 1, respectively. For q, q', let $[q, q'] = \{\alpha q + (1-\alpha)q' : \alpha \in [0,1]\}$. A set A of lotteries is above set B if for all $q \in A$, $[q, q^T] \cap B \neq \varnothing$. It is below B if for all $q \in A$, $[q, q^E] \cap B \neq \varnothing$. Since H^T is the convex hull of points $\{(q_i, q_N)\}_{i=1}^{N-1}$, every lottery q is either above or below H^T. By quasi-convexity, if $q \sim \delta_{x_{N-1}}$ then q is above H^T, hence so is q such that $q \succeq \delta_{x_{N-1}}$. By FOSD, for $i > j \neq N$ with $q_i > 0$, $(q_i, q_j) \succ \delta_{x_{N-1}}$. If it is part of a solution $\mathbf{q}$ satisfying equality, then by the above argument all lotteries allocated by $\mathbf{q}$ are above H^T, hence $\mathbf{q}$ is not a feasible solution. It thus follows that all binary lotteries in an ideal solution must have x_N as one of its two outcomes.

By Theorem 2 part 3, an outcome x_i, $i \neq N$, can receive positive probability at no more than $\left\lfloor \frac{N^2(N-2)}{8} \right\rfloor$ lotteries, hence for a sufficiently large k, some of its occurrences must be in binary lotteries. By the first part of the theorem, the only possible such lottery is (q_i, q_N), hence the theorem. The proof of the case where x_1 is excellent is similar. ■

Proof of Theorem 4: We show first that an efficient solution yields everyone a binary lottery. Suppose that $\mathbf{q}$ is an efficient solution with $\gamma > 0$ mass of individuals receiving non-binary lotteries. Since N is finite, we may assume without loss of generality that they all receive with positive probabilities each of the three outcomes x_r, x_s, x_t where $r > s > t$. That is, $\mu\{a : f_i(a) > 0,\, i = r, s, t\} > 0$. As μ is σ-additive, it follows that for some $\varepsilon > 0$, $\mu(A) > 0$, where $A = \{a : f_i(a) > \varepsilon,\, i = r, s, t\}$.

For every $a \in A$, let D_a be the triangle $\{(q_t, q_r) \in \Re^2_+ : q_t + q_r \leqslant \bar{q}_a = f_r(a) + f_s(a) + f_t(a)\}$. Let τ_a be the slope of a supporting line to the indifference curve in D_a through $(f_t(a), f_r(a))$. Let τ^* be such that $\mu(a : \tau_a > \tau^*)$, $\mu(a : \tau_a < \tau^*) \leqslant \frac{1}{2}\mu(A)$. Divide A into two sets A_1 and A_2 such that $\mu(A_1) = \mu(A_2) = \frac{1}{2}\mu(A)$, for all $a \in A_1$, $\tau_a \geqslant \tau^*$ and for all $a \in A_2$, $\tau_a \leqslant \tau^*$. We now follow the procedure described in the proof of Theorem 1, where individuals m and n are replaced with A_1 and A_2. It follows that all agents receive a binary lottery.

Let $\Pi^{N-1} = \{(\pi_1, \ldots, \pi_{N-1}) \in \Re_+^{N-1} : \sum_{i=1}^{N-1} \pi_i = 1\}$ be a prices simplex. For every $\pi \in \Pi^{N-1}$, $\pi \cdot (\frac{1}{N}, \ldots, \frac{1}{N}) = \frac{1}{N}$. For $\pi \in \Pi^{N-1}$, let $D_i(\pi) = \{q \in \Delta^{N-1} : \pi \cdot q \leqslant \frac{1}{N} \text{ and } \pi \cdot q' \leqslant \frac{1}{N} \Longrightarrow q \succeq_i q'\}$, $i = 1, \ldots, N$, and let $D_i^*(\pi) = \text{Conv}(D_i(\pi))$. These are the convexified demand sets of the various types given prices π and endowments $(\frac{1}{N}, \ldots, \frac{1}{N})$. Observe that since the preferences $\succeq_i$ are strictly quasi-convex, the set $D_i(\pi)$ is a finite set of binary lotteries. In the continuum economy, these lotteries can be allocated to the type-i individuals in such proportions to obtain any point in $D_i^*(\pi)$.

Suppose that for some $q \in D_i(\pi)$, $\pi \cdot q < \frac{1}{N}$. If $q = (1, 0, \ldots, 0) := \delta_{x_1}$, then since for all i, x_1 is the best outcome, it follows that for all i, $D_i(\pi) = \delta_{x_1}$ and π cannot be a Walrasian equilibrium price-vector. If $q \neq \delta_{x_1}$, then there is $\alpha \in (0, 1]$ such that $\pi \cdot [\alpha\delta_{x_1} + (1-\alpha)q] = \frac{1}{N}$. By monotonicity with respect to FOSD, $\alpha\delta_{x_1} + (1-\alpha)q \succ_i q$, a contradiction to the definition of $D_i(\pi)$. It thus follows that $D_i(\pi) = \{q \in \Delta^{N-1} : \pi \cdot q = \frac{1}{N} \text{ and } \pi \cdot q' \leqslant \frac{1}{N} \Longrightarrow q \succeq_i q'\}$. Clearly the correspondences $D_i(\pi)$ (and therefore $D_i^*(\pi)$) are upper hemi-continuous, hence there exists an equilibrium vector π and allocations q_i^* in $D_i^*(\pi)$, $i = 1, \ldots, N$, such that $\sum_{i=1}^{N} q_i^* = (\frac{1}{N}, \ldots, \frac{1}{N})$.

These allocations are efficient, feasible, and since all agents face the same "price" vector π, they satisfy no-envy. The first part of the proof implies that all agents receive a binary lottery, hence the claim of the theorem. ■

Example 1 Consider a continuum economy as in Section 3.2 with $N = 3$. The preferences $\succeq_1$, $\succeq_2$, and $\succeq_3$ over $\Delta^2 = \{(q_1, q_2) \in \Re_+^2 : q_1 + q_2 \leqslant 1\}$ can be represented by $V_1 = V_2 = 3q_1 + q_2$ and $V_3 = 6.25q_1^2 + q_2^2$. The initial lottery held by each person is represented by the point $(\frac{1}{3}, \frac{1}{3}) \in \Delta^2$. Let the price of q_2 be 1, and denote the price of q_1 by π. The *convexified* demand correspondences of the various agents are given by

$$D_1(\pi) = D_2(\pi) \quad = \quad \begin{cases} (1,0) & \pi \leqslant \frac{1}{2} \\ (\frac{1+\pi}{3\pi}, 0) & \frac{1}{2} < \pi < 3 \\ \{(\frac{3+5\alpha}{18}, \frac{5(1-\alpha)}{6}) : \alpha \in [0,1]\} & \pi = 3 \\ (\frac{\pi-2}{3\pi-3}, \frac{2\pi-1}{3\pi-3}) & \pi > 3 \end{cases}$$

$$D_3(\pi) \quad = \quad \begin{cases} (1,0) & \pi \leqslant \frac{1}{2} \\ (\frac{1+\pi}{3\pi}, 0) & \frac{1}{2} < \pi < 5 \\ \{(\frac{2\alpha}{5}, 1-\alpha) : \alpha \in [0,1]\} & \pi = 5 \\ (0,1) & 5 < \pi < 6.8 \\ (\frac{8\alpha}{29}, \frac{29-8\alpha}{29}) : \alpha \in [0,1] & \pi = 6.8 \\ (\frac{\pi-2}{3\pi-3}, \frac{2\pi-1}{3\pi-3}) & \pi > 6.8 \end{cases}$$

Clearly, there is no π such that $\frac{1}{3}[D_1(\pi) + D_2(\pi) + D_3(\pi)] = (\frac{1}{3}, \frac{1}{3})$. $\square$

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