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The Primal Hamiltonian: A New Global Approach to Monetary Policy

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Abstract

I develop a framework for optimal interest rate policy by which all of the traditional assumptions of monetary policy (quadratic objective function, linear dynamics, and Gaussian shocks) can be broken simultaneously while still determining the interest rate as a rule. This allows for deeper and more general policy questions to be posed and answered. The problem of optimal interest rates is formulated as a two-stage problem. Constrained by the market equilibrium conditions, the central bank maximizes household utility, expressed in terms of agent decision rules, through the choice of interest rates. I make use of "the Primal Hamiltonian Method" developed in a companion paper to solve issues related to stochastic two-stage optimal control problems and simplify the state space. The equilibrium conditions for the central bank's optimal control problem are then used to computationally determine a globally optimal closed-feedback control function for the interest rate using deep learning methods. I find that nonlinearities matter in that the policy response should also be nonlinear, and so normal linearization methods could be insufficient. In addition, I find that variables such as price dispersion normally considered orthogonal to the optimal policy decision should be considered. With efficient price dispersion the policy response to inflation is close to linear, and as price dispersion inefficiencies increase, then the central bank should more aggressively target inflation through an increasingly nonlinear response function.

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1 Introduction

Monetary policy research has traditionally operated under a framework of a quadratic objective function, linear dynamics, and Gaussian shocks, and has tried to formulate optimal interest rates in terms of a rule, or function of the state variables or shocks.[1] Although research on optimal policy in the last two decades, particularly after the Great Recession, has expanded that framework by breaking one or two of these assumptions, these efforts have only resulted in highly case-specific approaches. Because these methodologies must be highly tailored to the question at hand, they become inadequate for other applications.[2] In addition, certain technical strategies for dealing with these problems have their own, often fatal issues. The most glaring examples are higher order pertubation methods being unable to accommodate inequality constraints (Swanson et al 2006) and discretation, as well as projection succumbing to sensitivity to initial conditions and the curse of dimensionality.[3]

This paper is the first to develop a new, overarching framework by which the assumptions of quadratic objective function, linear dynamics, and Gaussian shocks can be broken simultaneously while still obtaining optimal interest rate policy as a rule. The contributions then are fundamentally methodological. Because the methodology does not have to be highly modified for each case, with this new framework deeper and more general questions can be posed and answered.[4] In addition, with this new framework the full range of intermediate distortions or extensions can be explored instead of being restricted to only extreme cases. This general-

[1]The now-classic reference on this approach is Woodford (2003). See also Rotemberg and Woodford (1997); Clarida, Gali, and Gertler (1999); Giannoni and Woodford (2005); Levin, Onatski, and Williams (2005); and Schmitt-Grohe and Uribe (2005), among many others.

[2]An exhaustive literature review is impossible here, but see for example Kim and Ruge-Murcia (2019) for a model with nonlinearities and assymetric shocks, where higher order pertubation around the steady state is used; Curdia and Woodford (2015) for a modification the NK model to take credit frictions into account without fundamentally altering the LQ framework; Wu and Li (2014) for rational inattention implications in a log-linearized model; Swanson (2006) for monetary policy under parameter uncertainty in a linear model; Faulwasser, T. et al. (2020) for a nonlinear quadratic model used to study unconventional monetary policy; Nobay and Peel (2000) and Huh, Lee, and Lee (2009) for monetary policy in the context of a nonlinear Phillips curve; and Bilbiie and Ragot (2020) and Chale (2020) for an LQ framework modified for heterogenous agents and liquidity constraints, respectively.

[3]It should be noted that when full Ramsey policies are computed without these kinds of assumptions, they are done so without an explicit rule or are formulated in terms of a function paramaterized beforehand.

[4]This includes, for example, to what extent optimal policy should display inertia or adjust rapidly, how skewed or non-normal uncertainty should matter (i.e. how forward guidance should take place given endogenous kurtosis of inflation expectations or uncertainty with regards to the model), or how non-additive uncertainty should

ity, however, is not at the expense of the interest rate as a rule (the original rationale for the traditional approach), meaning that interest rate policy is still expressed as a function of state variables of the model.

My approach formulates the problem of optimal interest rates as a two-stage problem. Similar to the logic of mechanism design, the policymaker takes the equilibrium conditions of the economic agents as given, and the economic agents in turn treat the policy as exogenous. In order to make this possible, the objective function is written in terms of the underlying rules of the private market equilibrium. This allows analysis beyond a small neighborhood of the steady state. Higher order terms, if relevant, are not ignored, and I avoid convoluted construction inherent in traditional methods of deriving the objective function that makes comparisons even between very similar non-standard model environments and substandard policies difficult (Benigno & Woodford, 2008). This is carried out here within modelling environments set in continuous time, which helps us to characterize much of the equilibrium dynamics analytically and to avoid needing to compute expectations even within a global solution.[5]

In order to make this two-stage approach possible, this paper makes use of a new technique developed within a companion paper (Hennigan 2021) called "The Primal Hamiltonian Approach", which leverages solution methods of backward stochastic differential equations (BSDE). This paper is not only the first to make use of these techniques for optimal monetary policy, but also the first to apply them in a two-stage optimal control problem setting as a way of reducing the state space to allow tractability. To briefly summarize the "Primal Hamiltonian" approach, terms relating to uncertainty are treated in a similar way as taxes are treated in the optimal taxation literature: absorbed into new proxy variables that make the problem more tractable without loss of generality. More specifically, here the duality principle and solution techniques for BSDEs are manipulated to create a proxy costate variable to be used in the central bank's optimal control problem in lieu of second order terms of the value functions of

influence policy. The method also allows us to accommodate non-standard but realistic setups like preferences which respond to worst-case scenarios, rational inattention, or endogenous shock variances. See Mishkin (2010) for a more detailed list of monetary policy questions that are difficult to address by traditional techniques.

[5]There is nothing essential, however, about continuous time. This method could be done in discrete time, though at a much higher computational cost. See Achdou et al. (2017) for an explanation of the advantages of continuous time in this manner.

private economic agents of the model. This framework allows also for more complex stochastic processes to be included without fundamentally changing the methodology and maintaining a tractability not present in traditional approaches, as well as for optimal dynamics between policy regimes to be more fully explored. Beyond optimal monetary policy, this new technique has applications to any two-stage problem where private economic agents take policy as exogenous and the policy maker in turn takes those agents' equilibrium conditions as constraints in the formulation of policy. This would include, for example, New Dynamic Public Finance, and so the methodological contribution of this paper extends beyond the narrower (though still broad) question of optimal monetary policy.

The last innovation is that, on the computational side, deep learning methods are used to extract an explicit rule, avoiding simplification for the sake of computation. Deep learning techniques (Kang et al 2020, Raissi 2018) can be used to approximate an unknown function by a deep neural network. Although I could have used many other nonlinear solutions methods, machine learning techniques are convenient for a number of reasons. First, by the universal approximation theorem (Bach 2017), a neural network can approximate any unknown Borel measurable function and are less sensitive to good initial guesses than collocation methods. Second, a neural network method allows one to avoid the curse of dimensionality, allowing for my method to be extended to even much wider state environments in a way grid methods would not. Last, the neural network can be efficiently trained using back-propogation and gradient descent. Here to train the network I make use of the connection between PDEs and FBSDEs, and instances of Brownian motion (or randomly chosen initial states in the case of the deterministic problem) are used to create sample paths. There are other papers that used neural networks in the context of solution methods to dynamic economic models. [6] To my knowledge, this is the first paper to make use of these techniques for optimal monetary policy research.

The current paper is split into three sections which hope to give an intuition and set of examples of the range of this approach. All three analyze optimal interest rate policy in slightly different model environments but the fundamental methodology is the same. Section 2 of the

[6]Recent applications include Scheidegger and Bilionis (2017), Duarte (2018), Maliar et al. (2019), Ferandez-Villaverde et al. (2020), and Azinovic et al. (2020).

paper analyzes optimal interest rate policy in a deterministic New Keynesian model without capital. This section most explicitly lays out the basic framework and intuition of the technique. We will see that the choice of an optimal interest rate should be thought of instead through the lens of the optimal choice of firm marginal cost and its derivative. I find that when considering globally optimal interest rate policy, that price dispersion, normally considered orthogonal to the optimal policy decision, matters, as do nonlinearities more generally. As price dispersion becomes less efficient, the central bank should do more to target inflation, and the policy function becomes more and more nonlinear. Section 3 adds uncertainty in the technology state variable through a controlled diffusion process and "the Primal Hamiltonian Method" is demonstrated.

2 Deterministic Case: Introduction to the Method and Deep Learning Technique

This section is designed to give an understanding and intuition for the overall methodology. The underlying model environment is taken from Fernandez-Villaverde et al (2012). I will first illustrate the important elements and equilibrium conditions of the model environment. Then I will use those equilibrium conditions to both set up the constraint set and rewrite the reward function in a convenient fashion for the central bank's problem. I believe the analytical derivations help to better understand how optimal interest rate policy actually functions within the environment, a point I will seek to make clearer through a rewrite of the central bank's problem in terms of marginal cost. I will finally numerically approximate the model solution using deep learning techniques.

2.1 Underlying Model

The underlying model is a continuous time New Keynesian model with labor as the production input, Calvo pricing and monopolistic competition, and no uncertainty. I will summarize the important aspects of this model for clarity in the ensuing analysis. For full derivations, please reference Fernandez-Villaverde et al (2012).

2.1.1 Consumption

A representative consumer seeks to maximize lifetime utility, represented by a utility function separable in consumption (c) and hours worked (n).

$$\int_0^\infty e^{-\rho t}\{\ln(c_t) - \psi\frac{n_t^{1+\gamma}}{1+\gamma}\}$$

Where ρ is the subjective rate of time preference, ψ is the disultity of labor, and γ is the inverse of Frisch labor supply elasticity.

The household can trade on Arrow securities and on nominal government bonds b_t at a nominal interest rate r_t. The household earns a disposable income of $r_t b_t + p_t w_t n_t + p_t \Pi_t$, where p_t is the price of the consumption good, w_t is the real wage, and Π_t represents firm profits. Household financial wealth evolves as follows. Note that $\dot{b}$ refers to $\mathrm{d}b/\mathrm{d}t$

$$\dot{b} = r_t b_t - p_t c_t + p_t w_t n_t + p_t \Pi_t$$

Inflation is defined as:

$$\pi_t = \frac{\dot{p}}{p_t}$$

Let us define real financial wealth as $a_t \equiv \frac{b_t}{p_t}$. Real wealth then evolves as follows:

$$\begin{aligned}\dot{a} &= \frac{r_t b_t - p_t c_t + p_t w_t n_t + p_t \Pi_t}{p_t} - \frac{b_t}{p_t^2} \pi_t p_t \\ &= ((r_t - \pi_t) a_t - c_t + w_t n_t + \Pi_t\end{aligned}$$

2.1.2 Production

Final good production is competitive. A representative producer purchases intermediate goods and produces the final good with the production function:

$$y_t = (\int_0^1 y_{it}^{\frac{\varepsilon-1}{\varepsilon}})^{\frac{\varepsilon}{\varepsilon-1}}$$

where ε is the elasticity of substitution.

The input demand functions associated with the final good producer's problem are given as:

$$\begin{aligned}y_{it} &= (\frac{p_{it}}{p_t})^{-\varepsilon} y_t \qquad \forall i \\ p_t &= (\int_0^1 p_{it}^{1-\varepsilon} di)^{\frac{1}{1-\varepsilon}}\end{aligned}$$

Each intermediate firm i produces differentiated goods out of labor using:

$$y_{it} = An_{it}$$

where n_{it} is the amount of labor rented by firm i and A is a technology paramter. The intermediate good producer is a monopolistic firm and price setting is carried out via the Calvo formulation. At rate θ, intermediate firm i get the opportunity to reset their price. Any firm which does not receive such signal does not have the opportunity to change their price. The probability of receiving such a signal is independent of the timing of the last signal.

Prices are set to maximize expected discounted profits. Note that an expectation operator is used because although there is no uncertainty in the aggregate, because the timing of individual firm price changes is random there is uncertainty on the individual firm level. Note also that real marginal cost, $\mathrm{mc}_\tau = w_\tau / A$, is common across firms because firms share a common technological parameter.

The intermediate firm's problem is:

$$\max_{p_{it}} \mathrm{E}_t \int_t^\infty \frac{\lambda_\tau}{\lambda_t} e^{-\theta(\tau - t)} [\frac{p_{it}}{p_\tau} y_{i\tau} - \mathrm{mc}_\tau y_{i\tau}] d\tau$$

where λ_τ is the time t value of consumption in period τ to the household.

The first order conditions of the firm is as follows. The ratio of the optimal new price, common across all firms able to reset their prices, and the prices of the final good, is given by:

$$\frac{p_{it}}{p_t} = \frac{\varepsilon}{\varepsilon - 1} \frac{\Sigma_{Ct}}{\Sigma_{Rt}}$$

where:

$$\Sigma_{Rt} = \int_t^\infty \lambda_\tau e^{-\theta(\tau - t)} (\frac{p_t}{p_\tau})^{1-\varepsilon} y_\tau d\tau$$

represents expected present discounted value of total future revenue, and

$$\Sigma_{Ct} = \int_t^\infty \lambda_\tau e^{-\theta(\tau-t)} \mathrm{mc}_\tau (\frac{p_t}{p_\tau})^{-\varepsilon} y_\tau d\tau$$

represents expected present discounted value of total future costs. In both cases the λ term refers to the discount factor in terms of consumer valuation. This will refer to a stochastic discount factor in later sections. I maintain the same framework here for consistency.

This means that the optimal reset price equals the desired markup $\frac{\varepsilon}{\varepsilon-1}$ multiplied by the ratio of the future cost index Σ_{Ct} and future revenue index Σ_{Rt}. Because any firm has virtually no effect on aggregate terms, both of these indexes are exogenous to the firm.

The other variable of interest is that for price dispersion, v, which can be viewed as the inefficiency associated with not all firms having the same price at the same time. In practical terms it acts as a wedge between production in terms of inputs and in terms of output after aggregation.

$$y_t = \frac{An_t}{v_t}$$

where

$$v_t = \int_0^1 (\frac{p_{it}}{p_t})^{-\varepsilon} di$$

Note that $1 \leqslant v_t$, where $v_t = 1$ would imply efficiency. Price dispersion acts as the point of inefficiency flowing from staggered price setting. Price dispersion can also be thought of as a misalignment between decisions made on the basis of marginal cost and those made on the basis of marginal utility. This theme of a wedge between benefits and costs will be revisited when I examine the central bank's problem.

2.1.3 Equilibrium Results

The equilibrium results from the above model will be used later to formulate the constraints of the central bank's problem. These equilibrium results are expressed by the following differential equations and equality constraints.

$$
\begin{aligned}
\dot{\Sigma_R} &= (\theta - (\varepsilon - 1)\pi_t)\Sigma_{Rt} - 1 \\
\dot{\Sigma_C} &= (\theta - \varepsilon\pi_t)\Sigma_{Ct} - \mathrm{mc}_t \\
\dot{v} &= \theta(1 + \pi_t \frac{1-\varepsilon}{\theta})^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon\pi_t - \theta)v_t \\
\dot{\lambda} &= (\rho - r_t + \pi_t)\lambda_t \\
\mathrm{mc}_t &= \psi(A\lambda_t)^{-(1+\gamma)}v_t^{\gamma} \\
(1 + \pi_t \frac{1-\varepsilon}{\theta})^{\frac{1}{1-\varepsilon}} &= \frac{\varepsilon}{\varepsilon - 1}\frac{\Sigma_{Ct}}{\Sigma_{Rt}}
\end{aligned}
$$

The above conditions correspond to the development of the firm's revenue and cost expectations, aggregate price dispersion, the household Euler equation, and equations which determine equilibrium marginal cost and inflation. An important point here is that marginal cost acts as a sort of key. Indeed, for any given level of marginal cost and state variables the partial equilibrium is determined for consumption and labor (here essentially the consumption and labor decision is equivalent to a joint determination of marginal utility and marginal cost). This latter point will be exploited for the central bank's problem and will continue to act as a guiding principle throughout all our analysis later on.

There are other points of interest here. The sign of the relationship between the time derivative of the future cost and revenue indexes and the indexes themselves is dependent on the current level of inflation. We see that for very low, near zero or negative inflation levels, that higher current values of these indexes increases the time derivative, and that higher inflation levels flip that relationship. Note also that there are knife-edge cases of inflation where the time derivative loses all relationship with current levels.

2.2 Model

Now let us develop the workhorse of the current analysis, the central bank's problem. The central bank must choose interest rates over the infinite horizon in order to optimize household utility. In a certain sense the central bank is more constrained than the household and firms because they have to take into consideration those latter actor's decision making process when they optimize for them. In other words, they can only make a decision which the actors of the economy will follow. This is manifested in the central bank facing not only the state constraints that these actors faced, but also their equilibrium conditions as another constraint set. Note that the object function is exactly the consumer's utility function, though this no longer is expressed in terms of consumption and labor, but in terms rather of marginal utility and the product of price dispersion and marginal cost. These are derived from the equilibrium conditions of the underlying model. The central bank is thus trying to maximize

$$\int_0^\infty e^{-\rho t}\{\ln(c_t) - \psi \frac{n_t^{1+\gamma}}{1+\gamma}\}$$

From the equilibrium conditions we have:

$$\begin{aligned} \lambda_t &= 1/c_t \\ c_t &= 1/\lambda_t \\ \mathrm{mc}_t &= \psi n_t^{1+\gamma}/v_t \\ \psi n_t^{1+\gamma} &= v_t \mathrm{mc}_t \end{aligned}$$

This means that we arrive at the objective function:

$$\int_0^\infty e^{-\rho t}[\ln(1/\lambda_t) - \frac{v_t \mathrm{mc}_t}{1+\gamma}]dt$$

2.2.1 The Problem of the Central Bank

Even though the central bank only directly chooses the interest rate, I expand the choice set to include inflation and marginal cost. Because these are functions of underlying state variables, this is valid in terms of the optimal control problem (see Appendix 7.1 for proof of this claim).

$$\max_{r_t,\pi_t,\mathrm{mc}_t} \int_0^\infty e^{-\rho t}[\ln(1/\lambda_t) - \frac{v_t \mathrm{mc}_t}{1+\gamma}]dt$$

s.t.

$$\dot{\Sigma_R} = (\theta - (\varepsilon - 1)\pi_t)\Sigma_{Rt} - 1 \quad (1)$$

$$\dot{\Sigma_C} = (\theta - \varepsilon\pi_t)\Sigma_{Ct} - \mathrm{mc}_t \quad (2)$$

$$\dot{v} = \theta(1 + \pi_t \frac{1-\varepsilon}{\theta})^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon\pi_t - \theta)v_t \quad (3)$$

$$\dot{\lambda} = (\rho - r_t + \pi_t)\lambda_t \quad (4)$$

$$\mathrm{mc}_t = \psi(A\lambda_t)^{-(1+\gamma)} v_t^\gamma \quad (5)$$

$$(1 + \pi_t \frac{1-\varepsilon}{\theta})^{\frac{1}{1-\varepsilon}} = \frac{\varepsilon}{\varepsilon - 1}\frac{\Sigma_{Ct}}{\Sigma_{Rt}} \quad (6)$$

2.2.2 The Traditional Way

Before continuing, it may be useful to compare our approach with what is commonly done in the legacy of Woodford (2003). The central bank would be attempting to minimize a quadratic loss function. The following is a typical case:

$$\min_{r_t} \frac{1}{2} \int_0^\infty e^{-\rho t} [\alpha_\pi \pi_t^2 + \alpha_x x_t^2] dt$$

subject to:

$$d\pi_t = (\rho_\pi(\pi_t - \bar{\pi}) - \kappa_x x_t)dt + \rho_\pi dZ_t$$
$$dx_t = \frac{1}{\gamma}[r_t - \tilde{r} - (\pi_t - \bar{\pi})] + \rho_x dZ_t$$

Where x_t is the output gap and π_t is inflation (here it is assumed that the natural rate of inflation is zero).

A few things are obvious. First, this derivation is dependent on the underlying model in a way that must be derived and is not apparent a priori. It also relies on shocks being relatively small and assumes symmetry of effects as well as Gaussian shocks. The approach of this current paper maintains generality in these aspects. In addition, if one would like to change the underlying model, for example to introduce a distorted steady state, non-standard preferences, or rational inattention, then our approach makes this easier to accommodate in a tractable way.

2.2.3 The Hamiltonian

I will solve the central bank's problem using optimal control theory rather than dynamic programming. By duality, the results are equivalent, and I believe optimal control theory makes the intuition behind the result slightly clearer.

I thus construct the Hamiltonian as follows:

$$
\begin{aligned}
H =& \ln(1/\lambda_t) - \frac{v_t \mathrm{mc}_t}{1+\gamma} \\
&+ \Lambda_{\Sigma_R}[(\theta - (\varepsilon - 1)\pi_t)\Sigma_{Rt} - 1] \\
&+ \Lambda_{\Sigma_C}[(\theta - \varepsilon\pi_t)\Sigma_{Ct} - \mathrm{mc}_t] \\
&+ \Lambda_v[\theta(1 + \pi_t \frac{1-\varepsilon}{\theta})^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon\pi_t - \theta)v_t] \\
&+ \Lambda_\lambda[(\rho - r_t + \pi_t)\lambda_t] \\
&+ \mu_{mc}[\mathrm{mc}_t - \psi(A\lambda_t)^{-(1+\gamma)} v_t^\gamma] \\
&+ \mu_\pi[(1 + \pi_t \frac{1-\varepsilon}{\theta})^{\frac{1}{1-\varepsilon}} - \frac{\varepsilon}{\varepsilon - 1}\frac{\Sigma_{Ct}}{\Sigma_{Rt}}]
\end{aligned}
$$

Let us pause to consider the intuition of the above system, particularly of μ_{mc} and μ_π. μ_{mc} gives us the shadow price on allowing consumer marginal utility to decrease (increasing consumption) in terms of firm marginal cost. μ_π becomes clearer once we realize that $\frac{\varepsilon}{\varepsilon-1}\frac{\Sigma_{Ct}}{\Sigma_{Rt}} = \frac{p_{it}}{p_t}$, meaning that this is giving us the shadow price on inflation in terms of decreased overall efficiency (causing the ratio between optimal new prices and the current market price to further deviate from unity).

2.2.4 First Order Conditions of the Hamiltonian System

Control variables:

$$\frac{\partial H}{\partial r_t} = 0 = -\Lambda_\lambda \lambda_t \Longrightarrow \Lambda_\lambda = 0 \tag{7}$$

$$\frac{\partial H}{\partial \pi_t} = 0 = -\Lambda_{\Sigma_R}(\varepsilon - 1)\Sigma_{Rt} - \Lambda_{\Sigma_C}\varepsilon\Sigma_{Ct} - \Lambda_v[\varepsilon(1 + \pi_t(1-\varepsilon)/\theta)^{-\frac{1}{1-\varepsilon}} - \varepsilon v_t]$$
$$+ \Lambda_\lambda \lambda_t + \frac{\mu_\pi}{\theta}((1 + \pi_t \frac{1-\varepsilon}{\theta})^{\frac{\varepsilon}{1-\varepsilon}} \tag{8}$$

$$\frac{\partial H}{\partial \mathrm{mc}_t} = 0 = -\Lambda_{\Sigma_C} + \mu_{mc} - \frac{v_t}{1+\gamma} \Longrightarrow \mu_{mc} = \Lambda_{\Sigma_C} + \frac{v_t}{1+\gamma} \tag{9}$$

Equation (7) states that there is no value in expanding marginal utility. This makes intuitive sense. The interest rate acts on the rate of change of marginal utility, not on instantaneous marginal utility itself. This condition is also redundant, given that we have already locked in the relationship between marginal cost, price dispersion, and marginal utility in equation (5).

Equation (9) is of particular interest. It expresses that the shadow price linking firm marginal cost and consumer marginal utility is in terms of the shadow value on a marginal increase in firm expected marginal costs over the infinite horizon and instantaneous price dispersion. This relationship will be useful to us later on.

State variables:

$$\frac{\partial H}{\partial \Sigma_{Rt}} = -\dot{\Lambda}_{\Sigma_R} + \rho\Lambda_{\Sigma_R} = \Lambda_{\Sigma_R}(\theta - (\varepsilon - 1)\pi_t) + \mu_\pi \frac{\varepsilon}{\varepsilon - 1}\frac{\Sigma_{Ct}}{(\Sigma_{Rt})^2} \tag{10}$$

$$\frac{\partial H}{\partial \Sigma_{Ct}} = -\dot{\Lambda}_{\Sigma_C} + \rho\Lambda_{\Sigma_C} = \Lambda_{\Sigma_C}(\theta - \varepsilon\pi_t) - \frac{\mu_\pi}{\Sigma_{Rt}}\frac{\varepsilon}{\varepsilon - 1} \tag{11}$$

$$\frac{\partial H}{\partial v_t} = -\dot{\Lambda}_v + \rho\Lambda_v = -\frac{\mathrm{mc}_t}{1+\gamma} + \Lambda_v(\varepsilon\pi_t - \theta) - \mu_{mc}\gamma\psi(\lambda_t A)^{-(1+\gamma)}v_t^{\gamma-1} \tag{12}$$

$$\frac{\partial H}{\partial \lambda_t} = -\dot{\Lambda}_\lambda + \rho\Lambda_\lambda = -\frac{1}{\lambda_t} + \Lambda_\lambda(\rho - r_t + \pi_t) + \mu_{mc}(1+\gamma)\psi A^{-(1+\gamma)}\lambda_t^{-(2+\gamma)}v_t^{\gamma}] \tag{13}$$

Transversality Conditions:

$$\lim_{T\to\infty} e^{-pT}\Lambda_{\Sigma_R} \geqslant 0 \tag{14}$$

$$\lim_{T\to\infty} e^{-pT}\Lambda_{\Sigma_C} \geqslant 0 \tag{15}$$

$$\lim_{T\to\infty} e^{-pT}\Lambda_{vt} \geqslant 0 \tag{16}$$

$$\lim_{T\to\infty} e^{-pT}\Lambda_{\lambda t} \geqslant 0 \tag{17}$$

$$\lim_{T\to\infty} e^{-pT}\Lambda_{\Sigma_R}\Sigma_{Rt} = 0 \tag{18}$$

$$\lim_{T\to\infty} e^{-pT}\Lambda_{\Sigma_C}\Sigma_{Ct} = 0 \tag{19}$$

$$\lim_{T\to\infty} e^{-pT}\Lambda_{vt}v_t = 0 \tag{20}$$

$$\lim_{T\to\infty} e^{-pT}\Lambda_{\lambda t}\lambda_t = 0 \tag{21}$$

Note that by from (7), (12), and (13):

$$\dot{\Lambda_v} = (\rho - \varepsilon\pi_t + \theta)\Lambda_v + \frac{\mathrm{mc}_t}{1+\gamma} + \frac{\gamma}{(1+\gamma)v_t} \tag{22}$$

2.2.5 Determining the interest rate

From (7) and (13), we have:

$$\begin{aligned}
\frac{1}{\lambda_t} &= \mu_{mc}\psi(1+\gamma)v_t^{\gamma}A(\lambda_t A)^{-(2+\gamma)} \\
1 &= \mu_{mc}\psi(1+\gamma)v_t^{\gamma}(\lambda_t A)^{-(1+\gamma)} \\
(\lambda_t A)^{1+\gamma} &= \mu_{mc}\psi(1+\gamma)v_t^{\gamma} \qquad (23)
\end{aligned}$$

From (9) and (23), we have:

$$(\lambda_t A)^{1+\gamma} = (\Lambda_{\Sigma_C} + \frac{v_t}{1+\gamma})\psi(1+\gamma)v_t^{\gamma} \tag{24}$$

$$\lambda_t = \frac{1}{A}[(\Lambda_{\Sigma_C} + \frac{v_t}{1+\gamma})\psi(1+\gamma)v_t^{\gamma}]^{\frac{1}{1+\gamma}} \tag{25}$$

Taking the time derivative of λ_t, we arrive at:

$$\dot{\lambda} = \frac{\psi(1+\gamma)}{A}[(\Lambda_{\Sigma_C} + \frac{v_t}{1+\gamma})\psi(1+\gamma)v_t^{\gamma}]^{\frac{-\gamma}{1+\gamma}}[(\Lambda_{\Sigma_C} + \frac{v_t}{1+\gamma})\gamma v_t^{\gamma-1}\dot{v} + v_t^{\gamma}(\dot{\Lambda_{\Sigma_C}} + \frac{\dot{v}}{1+\gamma})]$$

Dividing by λ_t:

$$\begin{aligned} \frac{\dot{\lambda}}{\lambda_t} &= \frac{(\Lambda_{\Sigma_C} + \frac{v_t}{1+\gamma})\gamma v_t^{\gamma-1}\dot{v} + v_t^{\gamma}(\dot{\Lambda_{\Sigma_C}} + \frac{\dot{v}}{1+\gamma})}{(\Lambda_{\Sigma_C} + \frac{v_t}{1+\gamma})v_t^{\gamma}} \\ &= \gamma\frac{\dot{v}}{v_t} + \frac{(1+\gamma)\dot{\Lambda_{\Sigma_C}} + \dot{v}}{((1+\gamma)\Lambda_{\Sigma_C} + v_t)} \end{aligned} \tag{26}$$

Taking now the original equilibrium condition for marginal cost (5) and (24), we get:

$$\mathrm{mc}_t = \frac{1}{(1+\gamma)\Lambda_{\Sigma_C} + v_t} \tag{27}$$

Taking the time derivative,

$$\dot{\mathrm{mc}} = -\frac{(1+\gamma)\dot{\Lambda_{\Sigma_C}} + \dot{v}}{((1+\gamma)\Lambda_{\Sigma_C} + v_t)^2} \tag{28}$$

From (18), (19), and (20):

$$\frac{\dot{\lambda}}{\lambda_t} = \gamma\frac{\dot{v}}{v_t} - \frac{\dot{\mathrm{mc}}}{\mathrm{mc}_t} \tag{29}$$

From (4), the equilibrium condition for $\frac{\dot{\lambda}}{\lambda_t}$, and (29), we arrive at the rule for the optimal interest rate:

$$r_t = \rho + \pi_t - \gamma\frac{\dot{v}}{v_t} + \frac{\dot{\mathrm{mc}}}{\mathrm{mc}_t} \tag{30}$$

Thus the optimal interest rate is a function of time preferences of consumers, the inflation rate, the percent change in price dispersion, and the percent change in marginal cost.

For the sake of estimation it will be easier to rewrite the previous equation in the following way:

$$r_t = \rho + \pi_t - \gamma \frac{\dot{v}}{v_t} + \frac{(1+\gamma)\dot{\Lambda_{\Sigma_C}} + \dot{v}}{((1+\gamma)\Lambda_{\Sigma_C} + v_t)} \tag{31}$$

By duality, I may rewrite the previous equation in terms of the Bellman equation:

$$r_t = \rho + \pi_t - \gamma \frac{\dot{v}}{v_t} + \frac{(1+\gamma)\dot{V_{\Sigma_C}} + \dot{v}}{(1+\gamma)V_{\Sigma_C} + v_t} \tag{32}$$

Where V_{Σ_C} refers to the value function of the Bellman equation in terms of Σ_C, and $\dot{V_{\Sigma_C}}$ is the time derivative of that object. Note that these objects are functions of the underlying state variables, meaning that we have arrived at a rule for the interest rate in terms of the underlying state variables.

2.2.6 The System

We arrive thus at the following system of differential equations which may allow us to solve for the optimal interest rate through a determination of equilibrium marginal cost. The border conditions are given by the steady state values of the relevant variables. Note that our costate variable, the marginal utility of wealth, now vanishes in the system governing our complete results. Because marginal cost is allowed to freely move, the optimal interest rate is what I "back out" of the process controlling the costate term. It in a sense "controls" the marginal cost term. This realization is the essence of our analytical results and will carry over to the next section.

$$\dot{\Lambda_{\Sigma_R}} = (\rho - (\theta - (\varepsilon - 1)\pi_t))\Lambda_{\Sigma_R} - \mu_\pi \frac{\varepsilon}{\varepsilon - 1} \frac{\Sigma_{Ct}}{(\Sigma_{Rt})^2} \tag{33}$$

$$\dot{\Lambda_{\Sigma_C}} = (\rho - (\theta - \varepsilon\pi_t))\Lambda_{\Sigma_C} + \mu_\pi \frac{\varepsilon}{\varepsilon - 1} \frac{1}{\Sigma_{Rt}} \tag{34}$$

$$\dot{\Lambda_v} = (\rho - \varepsilon\pi_t + \theta)\Lambda_v + \frac{\mathrm{mc}_t}{1+\gamma} + \frac{\gamma}{1+\gamma}\frac{1}{v_t} \tag{35}$$

$$\dot{\Sigma_R} = (\theta - (\varepsilon - 1)\pi_t)\Sigma_{Rt} - 1 \tag{36}$$

$$\dot{\Sigma_C} = (\theta - \varepsilon\pi_t)\Sigma_{Ct} - \mathrm{mc}_t \tag{37}$$

$$\dot{v} = \theta(1 + \pi_t(1-\varepsilon)/\theta)^{-\varepsilon/(1-\varepsilon)} + (\varepsilon\pi_t - \theta)v_t \tag{38}$$

$s.t.$

$$(1 + \pi_t \frac{1-\varepsilon}{\theta})^{\frac{1}{1-\varepsilon}} = \frac{\varepsilon}{\varepsilon - 1}\frac{\Sigma_{Ct}}{\Sigma_{Rt}} \tag{39}$$

$$\mathrm{mc}_t = \frac{1}{(1+\gamma)\Lambda_{\Sigma_C} + v_t} \tag{40}$$

$$\frac{\mu_\pi}{\theta}(1 + \pi_t(1-\varepsilon)/\theta)^{\frac{\varepsilon}{1-\varepsilon}} = \Lambda_{\Sigma_R}(\varepsilon - 1)\Sigma_{Rt} + \Lambda_{\Sigma_C}\varepsilon\Sigma_{Ct} + \Lambda_v[\varepsilon[(1 + (1-\varepsilon)\pi_t/\theta)]^{\frac{-1}{1-\varepsilon}} - \varepsilon v_t] \tag{41}$$

I will use this system in our numerical computation. The steady state values are also computationally determined.

2.3 Comparison with the Ordinary Taylor Rule

Before moving on to computation, let us analytically compare the dynamics of our result with a rule of the Taylor Rule variety.

$$r_t = \phi_\pi \pi_t + r_{ss} \tag{42}$$

Ican now plug this value in to arrive at the dynamics of marginal cost, the key variable in our underlying system.

$$\begin{aligned}
\frac{\dot{\lambda}}{\lambda_t} &= \rho + \pi_t - \phi_\pi \pi_t - r_{ss} \\
&= \gamma \frac{\dot{v}}{v_t} - \frac{\dot{\mathrm{mc}}}{\mathrm{mc}_t} \\
&\Rightarrow \\
\frac{\dot{\mathrm{mc}}}{\mathrm{mc}_t} &= \gamma \frac{\dot{v}}{v_t} - ((1 - \phi_\pi)\pi_t)
\end{aligned}$$

Compare the above equation with our earlier results in the nonlinear case, which was:

$$\frac{\dot{\mathrm{mc}}}{\mathrm{mc}_t} = \frac{(1+\gamma)\dot{V_{\Sigma_C}} + \dot{v}}{(1+\gamma)V_{\Sigma_C} + v_t} \tag{43}$$

2.4 Numerical Solution: A Deep Learning Approach

In recent years deep learning has been applied to problems of PDEs (Raissi et al 2018). More recently it has been applied to explicit economics questions (Duarte 2018, Fernandez-Villaverde et al 2020). My approach follows most closely the method of Nakamura-Zimmerer et. al (2021a, 2021b), who use neural networks to approximate two-part boundary problems by first solving a system of ordinary differential equations, though in that paper the authors limit themselves to an LQ framework. I then use this data to train a neural network for approximating the relevant portions of the value function along the optimal path. The insight is that because I am only interested in the interest rate, I do not need to fully approximate the entire value function. In other words, I first numerically compute the optimal path, and then use the optimal path to train an additional model for the interrelation of state and control variables. This approach of solving the system of equations defining the monetary policy problem as a boundary problem and then using the results to train an additional model of the interest rate significantly aids with computational speed.

As mentioned in the introduction, there are advantages of this deep learning approach compared to more familiar methods. By the universal approximation theorem (Bach 2017), a neural network can approximate any unknown Borel measurable function, and neural networks are less sensitive to good initial guesses than collocation methods. A neural network method (largely) allows one to avoid the curse of dimensionality that define grid based methods, which forms the bulk of economic numerical methods.

2.4.1 Deep Learning: A Brief Overview

At the lowest level, a neural network is composed of "neurons", functions of the form:

$$n(x;\Theta) \equiv \phi(\theta_0 + \sum_i^N \theta_i x_i)$$

The function takes input x and is paramaterized by the weight vector Θ. The activation function $\phi(.)$ is a nonlinear function. Common functions include the hyperbolic tangent.

I create a "layer" by stacking N_1 neurons on top of each other:

$$N(x;\Theta) \equiv (n(x;\Theta_1),...,n(x;\Theta_{N1}))^T$$

A neural network now combines multiple such layers, by feeding the output of the previous layer as the input into the next layer. Finally, the output of the last layer is fed into an output layer.

$$NN(x;\Theta) \equiv \theta_0^{out} + \theta_1^{out} N(N(...N(x;\Theta_1)...;\Theta_{m-1});\Theta_m)$$

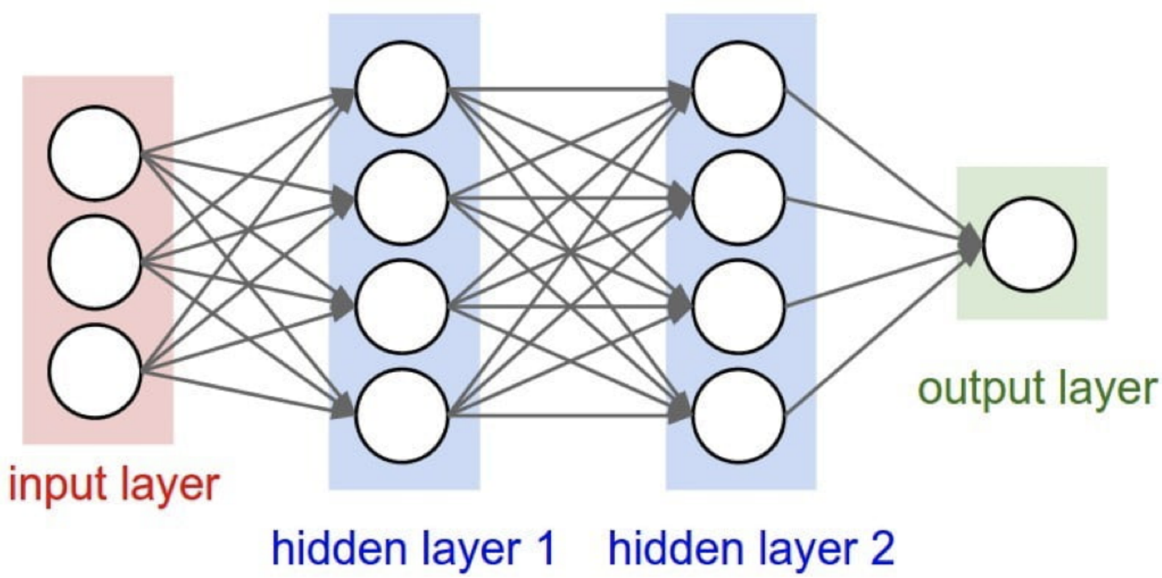


Figure 1: A neural network with two hidden layers.

2.4.2 Deep Learning: Application

Let us now look at how to actually implement the deep learning framework. To get a feel for the approach, consider a subset of our equilibrium conditions:

$$
\begin{aligned}
0 &= -\dot{\Lambda}_{\Sigma_R} + (\rho - (\theta - (\epsilon - 1)\pi_t))\Lambda_{\Sigma_R} - \mu_\pi \frac{\epsilon}{\epsilon - 1}\frac{\Sigma_{Ct}}{(\Sigma_{Rt})^2} \\
0 &= -\dot{\Lambda}_{\Sigma_C} + (\rho - (\theta - \epsilon\pi_t))\Lambda_{\Sigma_C} + \mu_\pi \frac{\epsilon}{\epsilon - 1}\frac{1}{\Sigma_{Rt}} \\
0 &= -\dot{\Lambda}_v + (\rho - \epsilon\pi_t + \theta)\Lambda_v + \frac{\mathrm{mc}_t}{1+\gamma} + \frac{\gamma}{1+\gamma}\frac{1}{v_t}
\end{aligned}
$$

I will express each variable as a neural network with time as the only input variable. I will define the error associated with each equilibrium condition in the following way:

$$
\begin{aligned}
err_{\Lambda_{\Sigma_R}} =& -\frac{\partial \Lambda_{\Sigma_R}(t,\Theta)}{\partial t} + (\rho - (\theta - (\epsilon - 1)\pi(t,\Theta)))\Lambda_{\Sigma_R}(t,\Theta) \\
& - \mu_\pi(t,\Theta)\frac{\epsilon}{\epsilon - 1}\frac{\Sigma_C(t,\Theta)}{(\Sigma_R(t,\Theta))^2} \\
err_{\Lambda_{\Sigma_C}} =& -\frac{\partial \Lambda_{\Sigma_C}(t,\Theta)}{\partial t} + (\rho - (\theta - \epsilon\pi(t,\Theta)))\Lambda_{\Sigma_C}(t,\Theta) \\
& + \mu_\pi(t,\Theta)\frac{\epsilon}{\epsilon - 1}\frac{1}{\Sigma_R(t,\Theta)} \\
err_{\Lambda_v} =& -\frac{\partial \Lambda_v(t,\Theta)}{\partial t} + (\rho - \epsilon\pi(t,\Theta) + \theta)\Lambda_v(t,\Theta) \\
& + \frac{\mathrm{mc}(t,\Theta)}{1+\gamma} + \frac{\gamma}{1+\gamma}\frac{1}{v(t,\Theta)}
\end{aligned}
$$

These error terms relate to deviations in the neural model from the dynamic path constraints and train the model to trace the optimal path as defined by optimality conditions.

I can do the same for path equality constraints defining certain variables and associated with Lagrange multipliers:

$$
\begin{aligned}
err_{mc} =& -\mathrm{mc}(t,\Theta) + \frac{1}{(1+\gamma)\Lambda_{\Sigma_C}(t,\Theta) + v(t,\Theta)} \\
err_{\mu_\pi} =& -\frac{\mu_\pi(t,\Theta)}{\theta}(1+\pi(t,\Theta)(1-\epsilon)/\theta)^{\frac{\epsilon}{1-\epsilon}} \\
&+ \Lambda_{\Sigma_R}(t,\Theta)(\epsilon-1)\Sigma_R(t,\Theta) + \Lambda_{\Sigma_C}(t,\Theta)\epsilon\Sigma_C(t,\Theta) \\
&- \Lambda_v(t,\Theta)[\epsilon[\theta(1+(1-\epsilon)\pi(t,\Theta)/\theta)]^{\frac{2\epsilon-1}{1-\epsilon}} + \epsilon v(t,\Theta)]
\end{aligned}
$$

As well as for the dynamics of each variable and the constraint defining inflation:

$$
\begin{aligned}
err_{\Sigma_R} =& -\frac{\partial \Sigma_R(t,\Theta)}{\partial t} + (\theta - (\epsilon-1)\pi(t,\Theta))\Sigma_R(t,\Theta) - 1 \\
err_{\Sigma_C} =& -\frac{\partial \Sigma_C(t,\Theta)}{\partial t} + (\theta - \epsilon\pi(t,\Theta))\Sigma_C(t,\Theta) - \mathrm{mc}(t,\Theta) \\
err_v =& -\frac{\partial v(t,\Theta)}{\partial t} + \theta(1+\pi(t,\Theta)(1-\epsilon)/\theta)^{\epsilon/(1-\epsilon)} \\
&+ (\epsilon\pi(t,\Theta) - \theta)v(t,\Theta) \\
err_\pi =& -(1+\pi(t,\Theta)\frac{1-\epsilon}{\theta})^{\frac{1}{1-\epsilon}} + \frac{\epsilon}{\epsilon-1}\frac{\Sigma_C(t,\Theta)}{\Sigma_R(t,\Theta)}
\end{aligned}
$$

Finally I define the error at the boundary conditions.

$$
\begin{aligned}
err_{\Sigma_R,0} =& \Sigma_R(0,\Theta) - \Sigma_{R0} \\
err_{\Sigma_C,0} =& \Sigma_C(0,\Theta) - \Sigma_{C0} \\
err_{v,0} =& v(t,\Theta) - v_0 \\
err_{\Lambda_{\Sigma_R},T} =& e^{-pT}\Lambda_{\Sigma_R}(T,\Theta) \\
err_{\Lambda_{\Sigma_R},T} =& e^{-pT}\Lambda_{\Sigma_C}(T,\Theta) \\
err_{\Lambda_{\Sigma_R},T} =& e^{-pT}\Lambda_v(T,\Theta)
\end{aligned}
$$

For estimation another technique was also used to increase efficiency in cases where error at the boundaries was unacceptably large. Instead of directly including error terms for the boundary conditions, I reformulate the neural network as a neural form to incorporate "hard boundaries", as shown in Lagari et al (2020).

A neural form is any construction that is built upon a neural network. For our purposes, consider for example the neural form associated with inflation:

$$(\frac{T-t}{T})\pi_0 + t(t-T)\pi(t,\Theta))) + (\frac{t}{T})\pi_{ss}$$

We can see that for the above at either boundary - the terminal steady state or the initial condition - the neural form is constructed by design to fit the boundary condition with complete accuracy. The central component, the actual neural network within the neural form, is what is trained to fit the path conditions.

The infinite-horizon variation is obtained with the limit $T \rightarrow \infty$. I will use the error above defined for a particular value of T, then extend that value as I solve if the terminal errors are above a certain tolerance.

The total loss is defined as:

$$\begin{aligned} loss(t;\Theta) =& err^2_{\Lambda_{\Sigma_R}} + err^2_{\Lambda_{\Sigma_C}} + err^2_{\Lambda_v} + err^2_{mc} + err^2_{\mu_\pi} \\ &+ err^2_{\Sigma_R} + err^2_{\Sigma_C} + err^2_{\pi} + err^2_{\Sigma_R,0} + err^2_{\Sigma_C,0} + err^2_{v,0} \\ &+ err^2_{\Lambda_{\Sigma_R},T} + err^2_{\Lambda_{\Sigma_R},T} + err^2_{\Lambda_{\Sigma_R},T} \end{aligned}$$

To solve the model, I choose the parameter set Θ to minimize the above global loss function over a set of time points.

$$\frac{1}{|D|}\sum_{i=1}^{|D|} loss(t_i;\theta)$$

The solution will be the open loop solution, in other words I will have the optimal path. Once I have the optimal path, then I can define another neural network to approximate the optimal interest rate as a function of the state variables:

$$r(x) = NN(x;\theta)$$

Note that I had previously defined the optimal interest rate as:

$$r_t = \rho + \pi_t - \gamma\frac{\dot{v}}{v_t} + \frac{\dot{\text{mc}}_t}{\text{mc}_t}$$

I thus already have the optimal interest rate defined on the optimal path. To train the neural network, I first obtain a set of optimal paths with randomly chosen initial points. I then take a set of points along these optimal paths and minimize the error to approximate the interest rate as a function of the state variables. In other words, I convert a set of open loop solutions to a closed loop one.

$$err_r(x,t) = \frac{1}{|D|}\sum_{i=1}^{|D|}[r(x;\theta) - r(t)]^2$$

2.4.3 Technical Details

All coding was done in python using the Tensorflow library. For the first round, I construct a fully connected neural network with 8 hidden layers of 120 neurons each. The library DeepXDE was used for the first round (Lu et al 2021). The sigmoid activation function and Adam stochastic gradient descent-type algorithm are adopted in the neural network. For the second round, I construct a fully connected neural network with 4 hidden layers of 64 neurons each. The tanh activation function and Adam stochastic gradient descent-type algorithm are adopted in the neural network. This work utilized the Summit supercomputer, which is supported by the National Science Foundation (awards ACI-1532235 and ACI-1532236), the University of Colorado Boulder, and Colorado State University. The Summit supercomputer is a joint effort of the University of Colorado Boulder and Colorado State University.

2.4.4 Parameterization and Steady State Values

Table 1: Parameterization

γ	1	Frisch labor supply elasticity
ρ	0.01	Subjective rate of time preference, $\rho = -4log0.9975$
γ	1	Frisch labor supply elasticity
ψ	1	Preference for leisure
θ	0.65	Calvo parameter for probability of firms receiving signal, $\theta = -4log0.85$
ϵ	25	Elasticity of substitution intermediate goods

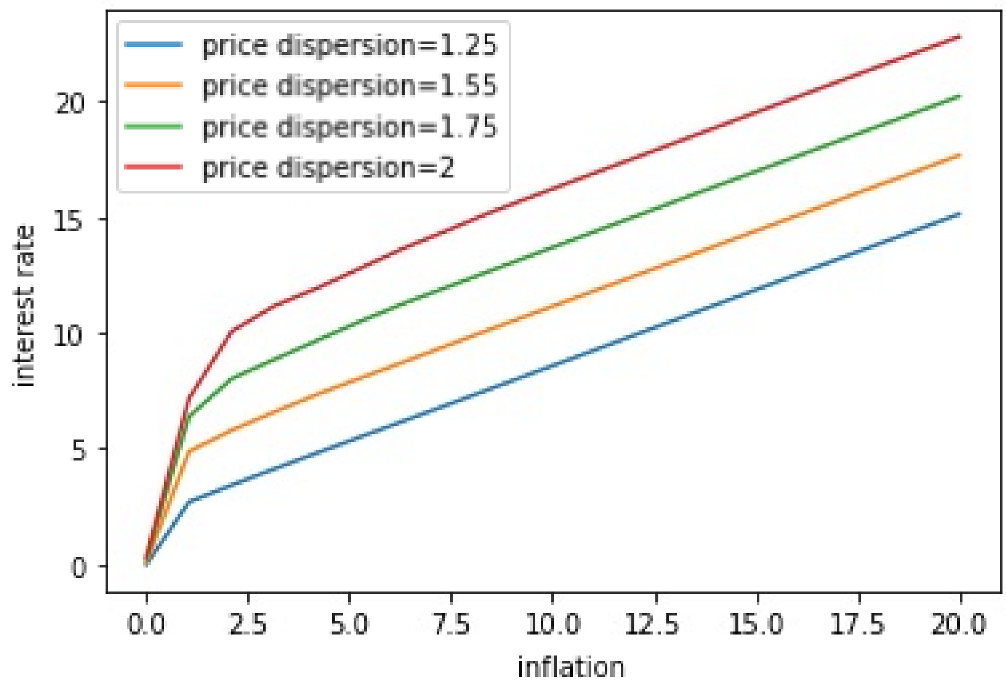
The following steady state values are computationally determined.

Table 2: Steady State Values

$\Lambda_{\Sigma_{RSS}}$	-0.02	Costate, Discounted Future Revenues
$\Lambda_{\Sigma_{CSS}}$	0.02	Costate, Discounted Future Costs
Λ_{vSS}	-1.48	Costate, Price Dispersion
Σ_{RSS}	1.54	Discounted Future Revenues
Σ_{CSS}	1.48	Discounted Future Costs
v_{SS}	1.00	Price Dispersion
$\mu_{\pi SS}$	0.02	Lagrange coefficient, inflation and auxiliary variables
mc_{SS}	0.96	Marginal Cost
π_{SS}	0.00%	Inflation
r_{SS}	1.00%	Interest Rate

2.4.5 Computational Results

We arrive then at preliminary results, the interest rate as a function of relevant state variables. The following diagram is an illustration of the interest rate rule. Note that here the interest rate is given as a function of inflation (π) at particular values of price dispersion (v). Remember that v=1 implies total efficiency, with v being bounded below by 1.



A few things are of note here. First, is the confirmation that nonlinearities matter. It also supports the idea that the price dispersion term does in fact influence optimal monetary policy. Traditionally, this term is considered orthogonal to the policy decision, or at the very least it is discarded as being second order. The relationship is intuitive: as inefficiencies associated with price dispersion become more pronounced, then the central bank should be more aggressive with respect to inflation.

3 Stochastic Case: The Primal Hamiltonian Method

Now let us introduce indeterminacy into the model through the technology parameter. The underlying model environment is again taken from Fernandez-Villaverde (2012). This section serves primarily to demonstrate our novel method for two-stage optimal control problems with underlying uncertainty. I restrict myself to uncertainty in one dimension to cultivate greater intuition.

As in the previous section, I will illustrate the important (new) elements and equilibrium conditions of the model environment. I will then illustrate the problem with naively proceeding in the same way as I did in the previous section and state our technique for overcoming this issue, which will involve rewriting the equilibrium conditions in a more usable way. I will then use those equilibrium conditions to rewrite the reward function and constraint set for the central bank's optimal control problem.

3.1 Underlying Model and Reformulation of the Equilibrium Conditions

Instead of being a constant term, technology is now defined as A_t and follows an Ornstein-Uhlenback process:

$$dlogA_t = -\rho_A logA_t dt + \sigma_A dB_{At} \tag{44}$$

where dB_{At} is a standard Brownian motion, also a Wiener process. By Ito's lemma:

$$dA_t = -(\rho_A logA_t - \frac{1}{2}\sigma_A^2)A_t dt + \sigma_A A_t dB_{At} \tag{45}$$

Everything else for the underlying model will be as described in the previous section. Note that nothing in our results is dependent upon the process being Ornstein-Uhlenback. Indeed, any Lipschitz diffusion pro-cess can be handled this way, meaning that there is wide generality. I choose to parameterize the process in this way to give greater intuition into the method.

As Fernandez-Villaverde et al (2012) show, this change in the underlying system results in the following equation for the evolution of the co-state variable associated with the marginal value of wealth:

$$d\lambda_t = (\rho - r_t + \pi_t)\lambda_t dt + \sigma_A \lambda_{At} dB_{At} \tag{46}$$

3.1.1 The Primal Hamiltonian

Take careful note of the term λ_{At}. For us to be able to make use of these co-state variables for the purposes of the central bank's problem, we need to know the time derivative of both variables. This, however, is impossible to do apriori, as the co-state variables are themselves the solution to the problem with unknown derivatives. To proceed, I make use of "The Primal Hamiltonian Approach", which leverages solution methods of backward stochastic differential equations (BSDE) and makes use of insights of stochastic optimal control using the Maximum Principle. Details of this method, as well as an elaboration to much more complex environments, are provided in a companion paper (Hennigan 2021).

Understanding that we may view λ_t specifically as a linear BSDE, the costate variable λ_t is given by the closed formula :

$$\Gamma_t \lambda_t = E(\lim_{t\to\infty} \Gamma_t \lim_{t\to\infty} \lambda_t + \int_t^{\infty} \Gamma_s \phi_s ds) \tag{47}$$

Γ_t is a process defined by the following forward LSDE:

$$d\Gamma_t = -(\rho - r_t + \pi_t)\Gamma_t dt, \quad \Gamma_0 = 1 \tag{48}$$

Note in the above that $\phi_s = 0 \ \ \forall \ s$. When the limits are defined by steady state values, we have thus that:

$$\Gamma_t \lambda_t = \Gamma_{ss} \lambda_{ss} \tag{49}$$

I may make use of the above as a sufficient statistic for the evolution of both adjoint variables from the original problem.

3.2 Model

I next expand upon the formulation of section (2.2.1), maintaining the same constraint set but adding in the technology process (45) as a constraint and substitute the process governing the evolution of the marginal utility of wealth (adjoint variable) (46) with that of our primal costate term (48). I also use the relationship between the primal costate and the original costate (49) to substitute throughout the entire system.

3.2.1 The Problem of the Central Bank

$$\max_{r_t,\pi_t,\mathrm{mc}_t} E_0 \int_0^\infty e^{-\rho t}[\ln(\Gamma_t) - \frac{v_t \mathrm{mc}_t}{1+\gamma}]dt$$

s.t.

$$\dot{\Sigma_R} = (\theta - (\varepsilon - 1)\pi_t)\Sigma_{Rt} - 1 \tag{50}$$

$$\dot{\Sigma_C} = (\theta - \varepsilon\pi_t)\Sigma_{Ct} - \mathrm{mc}_t \tag{51}$$

$$\dot{v} = \theta(1 + \pi_t \frac{1-\varepsilon}{\theta})^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon\pi_t - \theta)v_t \tag{52}$$

$$\dot{\Gamma} = (\rho - r_t + \pi_t)\Gamma_t \tag{53}$$

$$\mathrm{mc}_t = \psi(\frac{\Gamma_t}{A_t \Gamma_{ss}\lambda_{ss}})^{(1+\gamma)} v_t^{\gamma} \tag{54}$$

$$(1 + \pi_t \frac{1-\varepsilon}{\theta})^{\frac{1}{1-\varepsilon}} = \frac{\varepsilon}{\varepsilon - 1}\frac{\Sigma_{Ct}}{\Sigma_{Rt}} \tag{55}$$

$$dA_t = -(\rho_A log A_t - \frac{1}{2}\sigma_A^2)A_t dt + \sigma_A A_t dB_{At} \tag{56}$$

We see here the similarity of the central bank's problem here and in the non-stochastic case. We see the stochastic process governing the evolution of technology. We also see that the evolution of the marginal utility variable now contains a stochastic element (the Brownian motion term).

3.2.2 The Stochastic Hamiltonian

I may thus construct the Hamiltonian as follows:

$$
\begin{aligned}
H =& \ln(\Gamma_t) - \frac{v_t \text{mc}_t}{1+\gamma} \\
&+ \Lambda_{\Sigma_R}[(\theta - (\varepsilon - 1)\pi_t)\Sigma_{Rt} - 1] \\
&+ \Lambda_{\Sigma_C}[(\theta - \varepsilon\pi_t)\Sigma_{Ct} - \text{mc}_t] \\
&+ \Lambda_v[\theta(1 + \pi_t \frac{1-\varepsilon}{\theta})^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon\pi_t - \theta)v_t] \\
&+ \Lambda_\Gamma[(\rho - r_t + \pi_t)\Gamma_t] \\
&+ \Lambda_A(-\rho_A log A_t + \frac{1}{2}\sigma_A^2)A_t \\
&+ \Lambda_{A\sigma}\sigma_A A_t \\
&+ \mu_{mc}[\text{mc}_t - \psi(\frac{\Gamma_t}{A_t\Gamma_{ss}\lambda_{ss}})^{(1+\gamma)} v_t^\gamma] \\
&+ \mu_\pi[(1 + \pi_t\frac{1-\varepsilon}{\theta})^{\frac{1}{1-\varepsilon}} - \frac{\varepsilon}{\varepsilon - 1}\frac{\Sigma_{Ct}}{\Sigma_{Rt}}]
\end{aligned}
$$

3.2.3 First Order Conditions of the Hamiltonian System

Control variables:

$$\frac{\partial H}{\partial r_t} = 0 = -\Lambda_\Gamma \Gamma_t \Longrightarrow \Lambda_\Gamma = 0 \tag{57}$$

$$
\begin{aligned}
\frac{\partial H}{\partial \pi_t} = 0 = & -\Lambda_{\Sigma_R}(\varepsilon - 1)\Sigma_{Rt} - \Lambda_{\Sigma_C}\varepsilon\Sigma_{Ct} - \Lambda_v[\varepsilon(1 + \pi_t(1-\varepsilon)/\theta)^{-\frac{1}{1-\varepsilon}} + \varepsilon v_t] \\
& + \Lambda_\Gamma\Gamma_t + \frac{\mu_\pi}{\theta}(1 + \pi_t\frac{1-\varepsilon}{\theta})^{\frac{\varepsilon}{1-\varepsilon}}
\end{aligned} \tag{58}
$$

$$\frac{\partial H}{\partial \text{mc}_t} = 0 = -\Lambda_{\Sigma_C} + \mu_{mc} - \frac{v_t}{1+\gamma} \Rightarrow \mu_{mc} = \Lambda_{\Sigma_C} + \frac{v_t}{1+\gamma} \tag{59}$$

We see that the first order conditions are identical to those of the deterministic case once I make use of our primal costate term.

State variables:

$$\frac{\partial H}{\partial \Sigma_{Rt}} = -\dot{\Lambda}_{\Sigma_R} + \rho \Lambda_{\Sigma_R} = \Lambda_{\Sigma_R}(\theta - (\varepsilon - 1)\pi_t) + \mu_\pi \frac{\varepsilon}{\varepsilon - 1} \frac{\Sigma_{Ct}}{(\Sigma_{Rt})^2} \tag{60}$$

$$\frac{\partial H}{\partial \Sigma_{Ct}} = -\dot{\Lambda}_{\Sigma_C} + \rho \Lambda_{\Sigma_C} = \Lambda_{\Sigma_C}(\theta - \varepsilon\pi_t) - \mu_\pi \frac{\varepsilon}{(\varepsilon - 1)\Sigma_{Rt}} \tag{61}$$

$$\frac{\partial H}{\partial v_t} = -\dot{\Lambda}_v + \rho \Lambda_v = -\frac{\text{mc}_t}{1+\gamma} + \Lambda_v(\varepsilon\pi_t - \theta) - \mu_{mc}\gamma\psi\left(\frac{\Gamma_t}{A_t \Gamma_{ss} \lambda_{ss}}\right)^{(1+\gamma)} v_t^{\gamma - 1} \tag{62}$$

$$\frac{\partial H}{\partial \Gamma_t} = -\dot{\Lambda}_\Gamma + \rho \Lambda_\Gamma = \frac{1}{\Gamma_t} + \Lambda_\Gamma(\rho - r_t + \pi_t) + \mu_{mc} \frac{\psi(1+\gamma)}{A_t \Gamma_{ss} \lambda_{ss}} \left(\frac{\Gamma_t v_t}{A_t \Gamma_{ss} \lambda_{ss}}\right)^\gamma \tag{63}$$

$$-d\Lambda_A = [\Lambda_A(-\rho_A(\ln(A_t) + 1) + \frac{1}{2}\sigma_A^2 - \rho) + \Lambda_{A\sigma}\rho_A + \mu_{mc}\frac{\psi(1+\gamma)\Gamma_t}{A_t^2 \Gamma_{ss} \lambda_{ss}}]dt - \Lambda_{A\sigma} dB_{At} \tag{64}$$

One obvious difference from the analysis of the previous section is that I cannot express each constraint purely in terms of time derivatives because I have the stochastic term.

Transversality Conditions:

$$\lim_{T \to \infty} E_0[e^{-pT} \Lambda_{\Sigma_R}] \geqslant 0 \tag{65}$$

$$\lim_{T \to \infty} E_0[e^{-pT} \Lambda_{\Sigma_C}] \geqslant 0 \tag{66}$$

$$\lim_{T \to \infty} E_0[e^{-pT} \Lambda_{vt}] \geqslant 0 \tag{67}$$

$$\lim_{T \to \infty} E_0[e^{-pT} \Lambda_{\Gamma t}] \geqslant 0 \tag{68}$$

$$\lim_{T \to \infty} E_0[e^{-pT} \Lambda_{At}] \geqslant 0 \tag{69}$$

$$\lim_{T \to \infty} E_0[e^{-pT} \Lambda_{\Sigma_R} \Sigma_{Rt}] = 0 \tag{70}$$

$$\lim_{T \to \infty} E_0[e^{-pT} \Lambda_{\Sigma_C} \Sigma_{Ct}] = 0 \tag{71}$$

$$\lim_{T \to \infty} E_0[e^{-pT} \Lambda_{vt} v_t] = 0 \tag{72}$$

$$\lim_{T \to \infty} E_0[e^{-pT} \Lambda_{\Gamma t} \Gamma_t] = 0 \tag{73}$$

$$\lim_{T \to \infty} E_0[e^{-pT} \Lambda_{At} A_t] = 0 \tag{74}$$

3.2.4 Determining the interest rate

From the above, we have:

$$
\begin{aligned}
\frac{1}{\Gamma_t} &= -\mu_{mc}\frac{\psi(1+\gamma)}{A_t\Gamma_{ss}\lambda_{ss}}(\frac{\Gamma_t v_t}{A_t\Gamma_{ss}\lambda_{ss}})^{\gamma} \\
1 &= -\mu_{mc}\psi(1+\gamma)(\frac{\Gamma_t}{A_t\Gamma_{ss}\lambda_{ss}})^{1+\gamma}v_t \\
1 &= -(\Lambda_{\Sigma_C}+\frac{v}{1+\gamma})\psi(1+\gamma)(\frac{\Gamma_t}{A_t\Gamma_{ss}\lambda_{ss}})^{1+\gamma}v_t \\
(\frac{A_t}{\Gamma_t})^{1+\gamma} &= (\Lambda_{\Sigma_C}+\frac{v_t}{1+\gamma})\psi(1+\gamma)(\Gamma_{ss}\lambda_{ss})^{-(1+\gamma)}v_t^{\gamma}
\end{aligned}
\tag{75}
$$

By looking at the total derivative of (75), we arrive at:

$$
d(\frac{A_t}{\Gamma_t}) =
$$

$$
\psi(1+\gamma)(\Gamma_{ss}\lambda_{ss})^{-(1+\gamma)}[(\Lambda_{\Sigma_C}+\frac{v_t}{1+\gamma})\psi(1+\gamma)v_t^{\gamma}]^{\frac{-\gamma}{1+\gamma}}[(\Lambda_{\Sigma_C}+\frac{v_t}{1+\gamma})\gamma v_t^{\gamma-1}dv+v_t^{\gamma}(d\Lambda_{\Sigma_C}+\frac{dv}{1+\gamma})] \tag{76}
$$

By Ito's Product Rule and given Γ_t is of finite variance (by the Lipschitz condition holding on the underlying processing of A_t), I may rewrite (76) as:

$$
\frac{dA}{\Gamma_t}-\frac{A_t d\Gamma}{\Gamma_t^2} =
$$

$$
\psi(\Gamma_{ss}\lambda_{ss})^{-(1+\gamma)}(1+\gamma)[(\Lambda_{\Sigma_C}+\frac{v_t}{1+\gamma})\psi(1+\gamma)v_t^{\gamma}]^{\frac{-\gamma}{1+\gamma}}[(\Lambda_{\Sigma_C}+\frac{v_t}{1+\gamma})\gamma v_t^{\gamma-1}dv+v_t^{\gamma}(d\Lambda_{\Sigma_C}+\frac{dv}{1+\gamma})] \tag{77}
$$

Thus, with a little substitution and manipulation:

$$
r_t = \rho+\pi_t-\frac{\gamma}{1+\gamma}\frac{dv}{v_t}+\frac{1}{1+\gamma}\frac{d\mathrm{mc}}{\mathrm{mc}_t}+\frac{1}{1+\gamma}\frac{dA}{A_t} \tag{78}
$$

This, again, shows us that optimal choice of marginal cost is really what is driving the optimal choice of the interest rate. The evolution of marginal cost contains within it the evolution of the technology parameter, and so is itself stochastic.

As in the deterministic case, by duality, I may rewrite the previous equation in terms of the Bellman equation:

$$r_t = \rho + \pi_t - \frac{\gamma}{1+\gamma}\frac{dv}{v_t} + \frac{(1+\gamma)dV_{\Sigma_C} + dv}{(1+\gamma)V_{\Sigma_C} + v_t} + \frac{1}{1+\gamma}\frac{dA}{A_t} \tag{79}$$

3.2.5 The System

We arrive thus at the following system of differential equations which may allow us to solve for the optimal interest rate through a determination of equilibrium marginal cost. The border conditions are given by the steady state values of the relevant variables. Note that just as in the previous section, our primal costate variable now vanishes in the system which dictates our results. I "back out" the interest rate from the process controlling the primal costate term.

$$\dot{\Lambda_{\Sigma_R}} = (\rho - (\theta - (\varepsilon - 1)\pi_t))\Lambda_{\Sigma_R} - \mu_\pi \frac{\varepsilon}{\varepsilon - 1}\frac{\Sigma_{Ct}}{(\Sigma_{Rt})^2} \tag{80}$$

$$\dot{\Lambda_{\Sigma_C}} = (\rho - (\theta - \varepsilon\pi_t))\Lambda_{\Sigma_C} + \mu_\pi \frac{\varepsilon}{\varepsilon - 1}\frac{1}{\Sigma_{Rt}} \tag{81}$$

$$\dot{\Lambda_v} = (\rho - \varepsilon\pi_t + \theta)\Lambda_v + \frac{\mathrm{mc}_t}{1+\gamma} + \frac{\gamma}{1+\gamma}\frac{1}{v_t} \tag{82}$$

$$\dot{\Sigma_R} = (\theta - (\varepsilon - 1)\pi_t)\Sigma_{Rt} - 1 \tag{83}$$

$$\dot{\Sigma_C} = (\theta - \varepsilon\pi_t)\Sigma_{Ct} - \mathrm{mc}_t \tag{84}$$

$$\dot{v} = \theta(1 + \pi_t(1-\varepsilon)/\theta)^{-\varepsilon/(1-\varepsilon)} + (\varepsilon\pi_t - \theta)v_t \tag{85}$$

$$dA_t = -(\rho_A log A_t - \frac{1}{2}\sigma_A^2)A_t dt + \sigma_A A_t dB_{At} \qquad s.t.$$

$$(1 + \pi_t \frac{1-\varepsilon}{\theta})^{\frac{1}{1-\varepsilon}} = \frac{\varepsilon}{\varepsilon - 1}\frac{\Sigma_{Ct}}{\Sigma_{Rt}} \tag{86}$$

$$\mathrm{mc}_t = \frac{1}{(1+\gamma)\Lambda_{\Sigma_C} + v_t} \tag{87}$$

$$\frac{\mu_\pi}{\theta}(1 + \pi_t(1-\varepsilon)/\theta)^{\frac{\varepsilon}{1-\varepsilon}} = \Lambda_{\Sigma_R}(\varepsilon - 1)\Sigma_{Rt} + \Lambda_{\Sigma_C}\varepsilon\Sigma_{Ct}$$

$$- \Lambda_v[\varepsilon[\theta(1 + (1-\varepsilon)\pi_t/\theta)]^{\frac{2\varepsilon-1}{1-\varepsilon}} + \varepsilon v_t] \tag{88}$$

3.3 Computational Results

Just as before, to solve the model, we choose the parameter set Θ to minimize the below global loss function over a set of time points. Because the uncertainty in this case does not directly affect the paths of the other state variables, the first round is computed the same way as before, through a fully connected neural network with 8 hidden layers of 120 neurons each. The sigmoid activation function and Adam stochastic gradient descent-type algorithm are adopted in the neural network.

The stochastic component appears in the equation for the interest rate. The optimal interest rate rule "soaks up" the stochastic element of technological change. The optimal rule follows the standard result of the literature, in that it leans against the wind.

$$r_t = \rho + \pi_t - \gamma \frac{dv}{v_t} + \frac{d\text{mc}}{\text{mc}_t} + \frac{dA}{A_t}$$

We now run the second round of approximation. I construct a fully connected neural network with 4 hidden layers of 64 neurons each. The tanh activation function and Adam stochastic gradient descent-type algorithm are adopted in the neural network. We must optimize over not just the deterministic paths generated from different initial points, but also over the set of path realizations:

$$\frac{1}{|M||D|} \sum_{j=1}^{|M|} \sum_{i=1}^{|D|} loss(t_i, A_j^T; \theta)$$

with A_j^T referring to the jth sample path, defined according to the stochastic process:

$$dA_t = -(\rho_A log A_t - \frac{1}{2}\sigma_A^2) A_t dt + \sigma_A A_t dB_{At}$$

The stochastic paths only affect the path of the interest rate, not inflation or marginal cost. This means the function of interest rate to inflation and price dispersion is the same as in the deterministic case once we control for technological change.

4 Conclusion

I have developed in two distinct situations the basic formulation of my method for determining optimal interest rates, which is able to simultaneously break the traditional assumptions of a quadratic objective function, linear dynamics, and Gaussian shocks present in the optimal monetary policy literature. By reformulating the market equilibrium conditions as state constraints and replacing the control variables by their state feedback representation I was able to write the problem of the central bank of choosing the interest rate as maximizing household utility subject to those state constraints and formulate the problem of optimal interest rates as a two-stage, Ramsey optimal problem.

In order to make this possible, important innovations were developed. First, I used Deep Learning Methods to give a global approximation of the value function and thus derive an explicit interest rate as a function of the state variables of the model. Section 2.4 fleshes out the technical details of the methodology, but the overall intuition is straightforward. The optimal paths are first approximated with each variable being represented as the output of a neural network. The optimal paths are then used to train a neural network approximating the interest rate as a function of relevant state variables. This technique has applications far beyond the scope of this paper and can help with any framework with nonlinearities. As far as the deterministic monetary policy decision, I showed that these nonlinearities do in fact matter. First, price dispersion, which loses importance in linearized systems, influences how aggressive the central bank should be with respect to inflation, as well as the shape of the policy function. With efficient price dispersion, the policy function becomes a linear, increasing function of inflation. As price dispersion becomes less efficient, the central bank should do more to target inflation, and the policy function becomes more and more nonlinear.

In addition, in section 3 I demonstrated the use of a technique for solving this class of problems through the creation of a ”primal costate” variable which allowed us to overcome the difficulty inherent in a stochastic setting. In the ”Primal Hamiltonian method”, terms relating to uncertainty are treated in a similar way as taxes are treated in the optimal taxation literature: absorbed into new proxy variables that make the problem more tractable without loss of generality.

More specifically, here the duality principle and solution techniques for BSDEs are manipulated to create a proxy costate variable to be used in the central bank's optimal control problem in lieu of second order terms of the value functions of private economic agents of the model. This framework allows also for more complex stochastic processes to be included without fundamentally changing the methodology and maintaining a tractability not present in traditional approaches, as well as for optimal dynamics between policy regimes to be more fully explored.

Although this method has obvious possible applications, this paper focused on showcasing the technique through relatively simple model environments. The next step then would be to apply the methodology to a variety of more complex cases to show its flexibility. This includes, for example, to what extent optimal policy should display inertia or adjust rapidly, how skewed or non-normal uncertainty should matter, or how non-additive uncertainty should influence policy. The method also allows us to accommodate non-standard but realistic setups like preferences which respond to worst-case scenarios, rational inattention, or endogenous shock variances. Exploring these questions with the framework developed in this paper is another line of future research.

5 Appendix

5.1 Equivalency of Optimal Control Techniques

I will now prove that treating functions of state variables as controls with a Lagrangian is equivalent to working directly with them as state variables. This was used implicitly in the central bank's problem.

Consider the problem:

$$
\begin{aligned}
maxF(X,Y,Z)& \\
s.t.& \\
\dot{y} = G(X,Y,Z)& \\
Z = H(Y)&
\end{aligned}
$$

Consider the method of substitution:

$$
\begin{aligned}
maxF(X,Y,H(Y))& \\
s.t.& \\
\dot{y} = G(X,Y,H(Y))&
\end{aligned}
$$

Take the Hamiltonian:

$$H = F(X,Y,H(Y)) + \Lambda G(X,Y,H(Y))$$

FOC:

$$
\begin{aligned}
x :& F_x = 0 \\
y :& -\dot{\lambda} + \rho\Lambda = F_y + F_{H(y)}H_y + \Lambda[G_y + G_{H(y)}H_y]
\end{aligned}
$$

Let us take the Lagrangian method:

$$H = F(X,Y,Z) + \Lambda G(X,Y,Z) + \mu(H(Y) - Z)$$

FOC:

$$\begin{aligned}
x :& F_x + \Lambda G_x = 0 \\
z :& F_z + \Lambda G_z - \mu = 0 \\
y :& -\dot{\lambda} + \rho\Lambda = F_y + \Lambda G_y + \mu H_y \\
\mu :& G(Y) = Z
\end{aligned}$$

This implies:

$$\begin{aligned}
& F_z + \Lambda G_z = \mu \\
& -\dot{\lambda} + \rho\Lambda = F_y + \Lambda G_y + \mu H_y \\
& -\dot{\lambda} + \rho\Lambda = F_y + \Lambda G_y + [F_z + \Lambda G_z] H_y \\
& -\dot{\lambda} + \rho\Lambda = F_y + F_z H_y + \Lambda(G_y + G_z H_y) \\
& -\dot{\lambda} + \rho\Lambda = F_y + F_z H_y + \Lambda[G_y + G_z H_y] \\
& -\dot{\lambda} + \rho\Lambda = F_y + F_{H(y)} H_y + \Lambda[G_y + G_{H(y)} H_y]
\end{aligned}$$

We see therefore that for the purposes of optimal control analysis that we can consider any function of the state variables as a control variable without loss.

5.2 A Note on Marginal Cost

At first glance equation the equation governing the interest rate might seem to be nothing more than a sort of IS equation for the underlying model. Indeed, taking the time derivative of the equation linking firm marginal cost and consumer marginal utility would produce this equation. This, however, is a slightly misleading interpretation. Marginal cost is an equilibrium object, and the central bank could be seen as having the job of choosing marginal cost (or giving greater leeway, rather, to the possible positions marginal cost could take in equilibrium). This interpretation also is still not enough. To see this, let us reformulate the original problem of the central bank, taking the time derivative of equation before we begin. I may rewrite the problem in terms of choosing the *time derivative* of marginal cost instead of the interest rate. This is the true insight of the analytical exercise.

Let $\xi \equiv \dot{\text{mc}}$. Now let ξ be the relevant control variable and mc be a state. This let's us clearly see that whereas the equilibrium problem given a set interest rate can be reduced to choosing an optimal marginal cost, the bank's problem can be reduced to a choice of optimal marginal cost *and* the optimal time derivative of marginal cost.

$$\max_{\xi,\pi_t} \int_0^\infty e^{-\rho t}[\ln(1/\lambda_t) - \frac{v_t \text{mc}_t}{1+\gamma}]dt$$

s.t.

$$\begin{aligned}
\dot{\Sigma_R} &= (\theta - (\varepsilon - 1)\pi_t)\Sigma_{Rt} - 1 \\
\dot{\Sigma_C} &= (\theta - \varepsilon\pi_t)\Sigma_{Ct} - \text{mc}_t \\
\dot{v} &= \theta(1 + \pi_t \frac{1-\varepsilon}{\theta})^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon\pi_t - \theta)v_t \\
\dot{\lambda} &= [\gamma \frac{\dot{v}}{v_t} - \frac{\xi}{\text{mc}_t}]\lambda_t \\
\dot{\text{mc}} &= \xi \\
(1 + \pi_t \frac{1-\varepsilon}{\theta})^{\frac{1}{1-\varepsilon}} &= \frac{\varepsilon}{\varepsilon - 1}\frac{\Sigma_{Ct}}{\Sigma_{Rt}}
\end{aligned}$$

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