Hicksian Demand Functions, Expenditure Functions & Shephard's Lemma

Consider a world with 2 goods (x and y), where Wilbur has well-defined preferences over bundles of those two goods, and those preferences can be represented by the utility function

$$u = u(x, y).$$

Wilbur has income m and faces the parametric prices p_x and p_y.

So Wilbur chooses the bundle that

$$\max \quad u(x, y) \quad \text{s.t.} \quad m \geq p_x x + p_y y.$$

Consider Wilbur's Hicksian (conditional) demand functions for x and y.

$$x^h = x^h(u, p_x, p_y)$$

$$y^h = y^h(u, p_x, p_y)$$

where x^h is the amount of x Wilbur would purchase to achieve utility level u given the prices p_x and p_y.

The problem is

$$\min \quad p_x x + p_y y$$

wrt to x and y

$$\text{s.t.} \quad u = u(x, y)$$

The solution is

that x and y $(x^h \text{ and } y^h)$ that

min the cost of producing u utility

given preference $(u(x, y))$ and the prices p_x and p_y

$$x^h = x^h(u, p_x, p_y)$$

$$y^h = y^h(u, p_x, p_y)$$

So, Hicksian demand functions are the solution to a cost minimum problem.

What would one get if one plugged $x^h(u, p_x, p_y)$ and $y^h(u, p_x, p_y)$ into the expenditure level

$$p_x x + p_y y?$$

$$E = E(u, p_x, p_y)$$

$$= p_x x^h(u, p_x, p_y)$$

$$+ p_y y^h(u, p_x, p_y)$$

≡ the minimum expenditures required to produce u given p_x and p_y.

That is, $E = E(u, p_x, p_y)$ is the <u>cost</u> function to produce u given p_x and p_y.

We call it the expenditure function

$$E = E(u, p_x, p_y)$$

It identifies minimum expenditure to produce u as a function of p_x and p_y.

Properties of the expenditure function $E(u, p_x, p_y) = E(u, p)$ where $p = (p_x, p_y)$.

1. Nondecreasing in p. That is, if $p' \geq p$ then $E(u, p') \geq E(u, p)$.

2. Homogenous of degree one in p. That is, $E(u, \lambda p) = \lambda E(u, p)$ for $\lambda > 0$.

3. Concave in p. That is, $E(u, \lambda p + (1-\lambda)p) \geq \lambda E(u, p) + (1-\lambda) E(u, p)$ for $0 \leq \lambda \leq 1$.

4. Continuous in p. That is, $E(u, p)$ is continuous as a function of p for $p \geq 0$.

The expenditure function has the same properties as the cost function.

Since it has all the properties of a cost function (for producing *u* using the goods *x* and *y*) Shephard's Lemma applies and

$$\frac{\partial E\left(u,\, p_x,\, p_y\right)}{\partial p_x} = x^h\left(u,\, p_x,\, p_y\right)$$

and

$$\frac{\partial E\left(u,\, p_x,\, p_y\right)}{\partial p_y} = y^h\left(u,\, p_x,\, p_y\right)$$

This gives us a very simple and straightforward way of deriving the Hicksian demand function.

e.g. if

$$E\left(u,\, p_x,\, p_y\right) = 2\; u \;\; p_x^{.5}\; p_y^{.5}$$

derive the Hicksian demand functions.

By Shepard's Lemma

$$x^h = x^h\left(u,\, p_x,\, p_y\right) = \frac{\partial E\left(u,\, p_x, p_y\right)}{\partial\, p_x} = u\left(\frac{p_y}{p_x}\right)^{.5}.$$

And by analogy

$$x^h = y^h\left(u,\, p_x,\, p_y\right) = \frac{\partial E\left(u,\, p_x, p_y\right)}{\partial\, p_y} = u\left(\frac{p_x}{p_y}\right)^{.5}.$$

Can you prove Hicksian demand functions do not slope up if

$E\left(u,\, p_x,\, p_y\right)$ if non ↓ in *p*, and concave in *p* and twice differentiable?

Yes, by Shepard's Lemma

$$\frac{\partial x^h}{\partial\, p_x} = \frac{\partial^2 E\left(u,\, p_x,\, p_y\right)}{\partial p_x^2} \leq \; 0 \text{ (by concavity)}$$

That is, the "substitution effect" is not positive, but not necessarily strictly negative.

Duality between $u(x, y)$ and $E(u, p_x, p_y)$

$$u(x, y) \text{ and } p_x \text{ and } p_y \Rightarrow E(u, p_x, p_y)$$

$$\text{and } E(u, p_x, p_y) \Rightarrow u(x, y)$$

That is, we can, in theory, derive the direct utility function from the expenditure function (and vice versa)

How?

The same way we derived the production function $y = f(K, L)$ from $c = c(y, w, r)$

using Shepard's Lemma.

What would you get if you solved $E = E(u, p)$ for *u*?

$$u = E^{-1}(E, p)$$

Name this inverse function *v*, so

$$u = v(E, p).$$

$v(E, p)$ identifies maximum utility, *u*, as a function of prices, *p*, and the level of expenditures, *E*.

If one sets the level of expenditures equal to income, *m*

$$u = v(m, p).$$

$v(m, p)$ identifies maximum utility as a function of income and prices.

$v(m, p)$ is called the indirect utility function.

$u = v(m, p)$ can be shown to have the following properties:

1) $v(m, p)$ is nonincreasing in p. That is, if $p' \geq p$, then $v(m, p') \leq v(m, p)$.

2) $v(m, p)$ is homogenous of degree zero in (m, p). That is, $v(\lambda m, \lambda p) = v(m, p)$ for $\lambda > 0$.

3) $v(m, p)$ is quasiconvex in p. That is, $\{p : v(m, p) \leq k\}$ is a convex set for all k.

4) $v(m, p)$ is continuous $\forall\ p >> 0,\ m > 0$.

Looking ahead one can show that

$$x_i(m,\ p) \;=\; -\,\frac{\dfrac{\partial v(m,\ p)}{\partial p_i}}{\dfrac{\partial v(m,\ p)}{\partial m}}$$

where

$x_i(m,\ p)$ is the demand function for good *i*.

This result is know as Roy's Identity. We will soon prove Roy's Identity.

Note that the demand function $x_i(m,\ p)$ is sometimes referred to as the Marshallian demand function.

What is the difference between the Marshallian demand function

$x_i(m,\ p)$

and

the Hicksian demand function

$x_i^h(u,\ p)$?