On the propagation of light in media with periodic structure*

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In his theory of double refraction and natural optical activity, Sarrau[1] tried to deduce, from considerations based on the periodic structure of the medium, the vector equality

$$\mathbf{E} = \mathbf{f}_1\left(\mathbf{D}, \frac{\partial \mathbf{D}}{\partial x}, \frac{\partial \mathbf{D}}{\partial y}, \frac{\partial \mathbf{D}}{\partial z}\right)$$

Around the same time (1867-1868), Potier[2] was in possession of an integration method that gives more complete results. I intend to establish that Potier's analysis gives the vector equality

$$\mathbf{P} = \mathbf{f}_2\left(\mathbf{E}, \frac{\partial \mathbf{E}}{\partial x}, \frac{\partial \mathbf{E}}{\partial y}, \frac{\partial \mathbf{E}}{\partial z}\right)$$

Let us first suppose, as do Briot,[3] Sarrau and Potier, that we are dealing with an isotropic medium with a periodic structure. The equations to be integrated are

$$\nabla^2 \mathbf{E} - \nabla\nabla \cdot \mathbf{E} = K\frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{1}$$

$$\nabla \cdot (K\mathbf{E}) = 0 \tag{2}$$

*This is an English translation by E. F. Kuester of a paper which originally appeared in *Comptes Rendus de l'Académie des Sciences (Paris)*, vol. 178, pp. 319-321 (1924). For easier readability, mathematical notation has been made somewhat more consistent with that normally used today. Thus, vector quantities appear in bold, while grad $\rightarrow \nabla$, etc. Some clarifying comments have also been made, indicated as Translator's Notes. The translator is grateful to Dr. Alain Bossavit and Dr. Sébastien Rondineau for their comments on the translation.

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[1] [Translator's note: E. Sarrau, "Sur la propagation et la polarisation de la lumière dans les cristaux," *J. Math. Pures Appl.*, ser. 2, vol. 12, pp. 1-46 (1867) and vol. 13, pp. 59-110 (1868).]

[2] [Translator's note: A. Potier, "Recherches sur l'integration d'un système d'équations aux différentielles partielles à coefficients périodiques," *Comptes Rendus de l'Association Française pour l'Avancement des Sciences (Bordeaux)*, Sess. 1, pp. 255-272 (1872); also in A. Potier, *Mémoires sur l'Électricité et l'Optique*. Paris: Gauthier-Villars, 1912, pp. 239-256.]

[3] [Translator's Note: C. Briot, *Essais sur la Théorie Mathématique de la Lumière*. Paris: Mallet-Bachelier, 1864, especially livre III.]

where the dielectric constant K is a periodic function of (x, y, z).

Neglecting terms due to structural scattering[4], we easily see that the solution to the problem is

$$\mathbf{E}_m e^{iQ} + \nabla(\Phi e^{iQ})$$

where

$$Q = \frac{2\pi}{\tau}(n\alpha x + n\beta y + n\gamma z - t),$$

$\mathbf{E}_m$ denotes a constant vector and Φ a periodic function of (x, y, z). In what follows, $\mathbf{E}_m$ will be written simply as $\mathbf{E}$.

Putting

$$\Theta = \alpha E_x + \beta E_y + \gamma E_z$$

we obtain the system $(1')$, $(2')$:

$$n^2(\mathbf{E} - \boldsymbol{\nu}\Theta) = K\left[\mathbf{E} + e^{-iQ}\nabla\left(\Phi e^{iQ}\right)\right] \tag{$1'$}$$

$$\nabla \cdot \left[K\nabla\left(\Phi e^{iQ}\right)\right] + (\mathbf{E}\cdot\nabla)\left(Ke^{iQ}\right) = 0 \tag{$2'$}$$

Consistent with the order of approximation already used, $(2')$ will be replaced by $(2'')$:

$$\nabla\cdot(K\nabla\Phi) + (\mathbf{E}\cdot\nabla)K + \frac{2\pi ni}{\tau}\left[K\Theta + (\boldsymbol{\nu}\cdot\nabla)(K\Phi) + K(\boldsymbol{\nu}\cdot\nabla)\Phi\right] = 0 \tag{$2''$}$$

The symbols $(\mathbf{E}\cdot\nabla)$ and $(\boldsymbol{\nu}\cdot\nabla)$ in these equations represent the differential operators[5]

$$(\mathbf{E}\cdot\nabla) = E_x\frac{\partial}{\partial x} + E_y\frac{\partial}{\partial y} + E_z\frac{\partial}{\partial z}, \qquad (\boldsymbol{\nu}\cdot\nabla) = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \gamma\frac{\partial}{\partial z}$$

Limiting ourselves to the first two terms $\Phi_0 + \frac{2\pi ni}{\tau}\Phi_1$ of an expansion of Φ, we obtain from $(2'')$ equations (3) and (4) that must be satisfied by Φ_0 and Φ_1:

$$\nabla\cdot(K\nabla\Phi_0) + (\mathbf{E}\cdot\nabla)K = 0 \tag{3}$$

$$\nabla\cdot(K\nabla\Phi_1) + (\boldsymbol{\nu}\cdot\nabla)(K\Phi_0) + K(\boldsymbol{\nu}\cdot\nabla)\Phi_0 + K\Theta = 0 \tag{4}$$

From $(1')$, we end up with equation (5) for the average values E_x, E_y and E_z:

$$n^2(\mathbf{E} - \boldsymbol{\nu}\Theta) = \langle K\left(\mathbf{E} + \nabla\Phi_0\right)\rangle + \frac{2\pi ni}{\tau}\langle K\left(\boldsymbol{\nu}\Phi_0 + \nabla\Phi_1\right)\rangle \tag{5}$$

where the symbol $\langle g\rangle$ denotes the average value of the function g over a period cell.

[4] [Translator's Note: i. e., Bragg scattering.]

[5] [Translator's Note: That is, the vector $\boldsymbol{\nu} = \alpha\mathbf{u}_x + \beta\mathbf{u}_y + \gamma\mathbf{u}_z$, where $\mathbf{u}_{x,y,z}$ are the cartesian unit vectors. The author appears to intend that $\boldsymbol{\nu}$ is a unit vector. Clearly, $\Theta = \boldsymbol{\nu}\cdot\mathbf{E}$.]

Consequences

1. *The equations for the average values E_x, E_y and E_z are of Boussinesq's type.*[6]

   Indeed, the solutions of (3) and (4) are

   $$\Phi_0 = E_x\varphi_0' + E_y\varphi_0'' + E_z\varphi_0''', \qquad \Phi_1 = E_x\varphi_1' + E_y\varphi_1'' + E_z\varphi_1''' \tag{6}$$

   where φ_0', φ_1', ... satisfy the partial differential equations

   $$\nabla\cdot(K\nabla\varphi_0') + \frac{\partial K}{\partial x} = 0, \qquad \nabla\cdot(K\nabla\varphi_1') + (\boldsymbol{\nu}\cdot\nabla)(K\varphi_0') + K\alpha = 0, \tag{7}$$

   etc.

   Upon using (6), equations (5) can be written

   $$n^2(E_x - \alpha\Theta) = K_{11}E_x + K_{12}E_y + K_{13}E_z + \frac{2\pi n i}{\tau}\left(H_{11}E_x + H_{12}E_y + H_{13}E_z\right) \tag{5$'$}$$

   and so on, with

   $$K_{11} = \left\langle K\left(1 + \frac{\partial\varphi_0'}{\partial x}\right)\right\rangle, \quad K_{12} = \left\langle K\frac{\partial\varphi_0''}{\partial x}\right\rangle, \quad K_{21} = \left\langle K\frac{\partial\varphi_0'}{\partial y}\right\rangle, \quad \ldots$$

   $$H_{11} = \left\langle K\left(\alpha\varphi_0' + \frac{\partial\varphi_1'}{\partial x}\right)\right\rangle, \qquad H_{12} = \left\langle K\left(\alpha\varphi_0'' + \frac{\partial\varphi_1''}{\partial x}\right)\right\rangle$$

   $$H_{21} = \left\langle K\left(\beta\varphi_0' + \frac{\partial\varphi_1'}{\partial y}\right)\right\rangle, \qquad \ldots$$

   By (7) the K_{ij} are constants while the H_{ij} are linear functions of (α, β, γ).

2. *The equations for the average values conform to the notion of a potential.*

   In other words, the matrix formed from the K_{ij} is symmetric, and that formed from H_{ij} is skew-symmetric.

   This important property is a consequence of Potier's formulas (*Oeuvres*, pp. 250-256). For example, the relation $H_{12} + H_{21} = 0$ results from the two equalities (8):

   $$\begin{cases} \left\langle K\frac{\partial\varphi_1''}{\partial x}\right\rangle + \left\langle\varphi_0'\left[K\beta + (\boldsymbol{\nu}\cdot\nabla)(K\varphi_0'') + K(\boldsymbol{\nu}\cdot\nabla)\varphi_0''\right]\right\rangle = 0 \\ \left\langle K\frac{\partial\varphi_1'}{\partial y}\right\rangle + \left\langle\varphi_0''\left[K\alpha + (\boldsymbol{\nu}\cdot\nabla)(K\varphi_0') + K(\boldsymbol{\nu}\cdot\nabla)\varphi_0'\right]\right\rangle = 0 \end{cases} \tag{8}$$

   which is obtained upon integration by parts and using equations (5$'$). Taking the sum of the equations in (8) we get

   $$H_{12} + H_{21} = -2\left\langle(\boldsymbol{\nu}\cdot\nabla)(K\varphi_0'\varphi_0'')\right\rangle = 0$$

[6] [Translator's Note: M. Boussinesq, "Théorie nouvelle des ondes lumineuses," *J. Math. Pures Appl.*, ser. 2, vol. 13, pp. 313-339 (1868).]

Remark

Potier's method of integration can be applied to anisotropic media with periodic structure. For such media we have the equations

$$\nabla^2\mathbf{E} - \nabla\nabla\cdot\mathbf{E} = \frac{\partial^2\mathbf{D}}{\partial t^2}, \qquad \nabla\cdot\mathbf{D} = 0, \qquad D_x = k_{11}E_x + k_{12}E_y + k_{13}E_z, \qquad \ldots$$

with the k_{ij} $(= k_{ji})$ being periodic functions of (x, y, z). We again arrive at $(5')$ with $K_{ij} = K_{ji}$, $H_{ij} + H_{ji} = 0$; as far as dispersion is concerned, we conclude that optical activity of structural origin obeys the same laws as optical activity of atomic origin.