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ON A RELATION PERMITTING REPLACEMENT OF SUMMATION BY INTEGRATION, AND ITS APPLICATION TO SOME PROBLEMS OF DIFFRACTION THEORY*

by

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[Translation by E.F. Kuester**]

INTRODUCTION

In the solution of certain problems in the theory of diffraction of electromagnetic waves, particularly those encountered in antenna theory, shielding theory, or other areas of electrodynamics, one often has to deal with summations containing oscillating terms. The direct evaluation of such sums is not infrequently fraught with difficulties, and a relation permitting replacement of these sums with integrals could be quite useful here.

We shall first quote the result without proof, show how it can be applied to a particular example, and then give its derivation.

Let f(x) along with its first derivative be a continuous function given on the interval $1 \leq x \leq N$, where N is some natural number. Then the following relation holds:

$$\sum_{m=1}^{N} f(m)\, e^{-i\nu m} = \frac{\nu^3 \cos\frac{\nu}{2}}{8\sin^3\frac{\nu}{2}} \left\{ \int_1^N f(x)\, e^{-i\nu x} dx + \frac{1}{i\nu}\left(1 - \frac{4e^{-i\nu/2}\sin^2\frac{\nu}{2}}{\nu^2\cos\frac{\nu}{2}}\right) e^{-i\nu N} f(N) - \right.$$
$$\left. - \frac{1}{i\nu}\left(1 - \frac{4e^{i\nu/2}\sin^2\frac{\nu}{2}}{\nu^2\cos\frac{\nu}{2}}\right) e^{-i\nu} f(1) - \frac{1}{\nu^2}\left(1 - \frac{2}{\nu}\operatorname{tg}\frac{\nu}{2}\right)[f'(N)e^{-i\nu N} - f'(1)e^{-i\nu}] \right\} + R. \qquad (1)$$

Here $f'(m) = df/dx\big|_{x=m}$; R is a remainder term; and ν is an arbitrary, generally complex, number.

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*Translated from *Voprosy Mathematicheskoi Fiziki,* Leningrad: Nauka, 1976, pp. 79-93.

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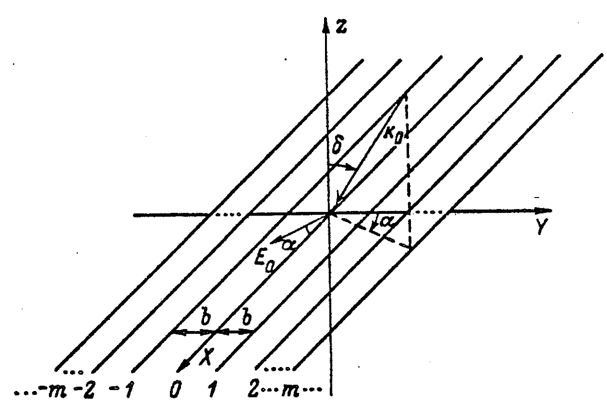
We note in passing that we can always choose $|\text{Re}\,\nu| \leq \pi$, inasmuch as subtracting an integer multiple of 2π from Re ν has no effect on the value of the left hand side of (1).

For real ν, the following estimate can be obtained for the remainder term R:

$$|R| \leqslant \frac{1}{16}\left|1 - 4\,\text{ctg}\,\frac{\nu}{2}\left[1 - \frac{\sin\nu}{\nu} + \sin\frac{\nu}{2}\left(1 - \frac{\sin^2\frac{\nu}{2}}{\left(\frac{\nu}{2}\right)^2}\right)\right]\right| |f''(1) - f''(N)|.$$

1. DIFFRACTION OF A PLANE WAVE FROM A WIRE ARRAY

As an example to illustrate the application of formula (1), we consider the problem of diffraction of a plane wave by an array consisting of parallel equidistant wires. To this end we consider an infinite plane array formed by parallel cylindrical wires of radius r_o; the wires are uniformly spaced a distance b from adjacent wires; we assume that $r_o \ll \lambda$, b; where λ is the wavelength. We choose a rectangular coordinate system XYZ such that the plane XOY coincides with the plane array; the X axis is parallel to the wires as shown in the figure.



Electromagnetic plane wave incident onto a wire array.

Let a plane wave be incident from the half-space z > 0, its direction of propagation given by the wave vector $\underset{\sim}{k}_o$, which forms the angles δ and α with the coordinate axes. The vector direction of the incident wave $\underset{\sim}{E}^{inc}$ is taken to be perpendicular to the Z axis. Under these assumptions

$$E_x^{inc} = E_0 \cos \alpha e^{-ik(x\sin\delta \cdot \sin\alpha + y\sin\delta \cdot \cos\alpha - z\cos\delta)} \tag{2}$$

where k = ω/c (ω is the angular frequency and c the speed of light).

We number the wires as shown in the figure. Thus the current in the mth wire can be represented as

$$I_m = I_0 e^{-ikx\sin\delta \cdot \sin\alpha} e^{-ikmb\sin\delta\cos\alpha}, \tag{3}$$

where I_o is the current in the zeroth wire at x = 0.

Before we can evaluate I_o we must obtain the field induced by all the wires on the surface of the zeroth wire. We have

$$\mathbf{E} = -\frac{i\omega}{c}\left(\mathbf{A} + \frac{1}{k^2}\,\mathrm{grad\,div}\,\mathbf{A}\right),^{1} \tag{4}$$

where

$$\mathbf{A} = \sum_{m=-\infty}^{\infty} \mathbf{A}_m = \sum_{m=-\infty}^{-1} \mathbf{A}_m + \sum_{m=1}^{\infty} \mathbf{A}_m + \mathbf{A}_0, \tag{5}$$

$\underset{\sim}{A}_m$ is the vector-potential of the field induced by the mth wire on the zeroth wire.

For m ≠ 0,

$$\mathbf{A}_m = \frac{1}{c}\int_{-\infty}^{\infty} \mathbf{I}_m \frac{e^{-ikr}}{r}\,dx' = \frac{\mathbf{I}_0}{c} e^{-ikmb\sin\delta \cdot \cos\alpha} \int_{-\infty}^{\infty} \frac{e^{-ik\sqrt{(x-x')^2+(mb)^2}}}{\sqrt{(x-x')^2+(mb)^2}} \times$$
$$\times e^{-ikx'\sin\delta\sin\alpha}\,dx',$$

where x' is a point on the wire axis.

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[1]The Gaussian system of units is used here; time dependence is taken to be $e^{i\omega t}$.

This integral reduces by means of the substitution $\sqrt{(x-x')^2+(mb)^2} =$ mb ch t to a known integral representation for the Hankel function

$$\int_{-\infty}^{\infty} e^{-i\rho \operatorname{ch} t} dt = -\pi i H_0^{(2)}(\rho).$$

As a result, we obtain

$$A_m = -\frac{i\pi}{c} I_0 e^{-ik\sin\delta(x\sin\alpha + mb\cos\alpha)} H_0^{(2)}\left[k\,|\,m\,|\,b\sqrt{1-\sin^2\delta\sin^2\alpha}\right]. \tag{6}$$

Also, in particular,

$$A_0 = -\frac{i\pi}{c} I_0 e^{-ikx\sin\delta\sin\alpha} H_0^{(2)}\left[kr_0\sqrt{1-\sin^2\delta\sin^2\alpha}\right].$$

The summations in (5) may now be written as

$$\sum_{m=-\infty}^{-1} A_m = -\frac{i\pi}{c} I_0 e^{-ikx\sin\delta\sin\alpha} \sum_{m=-\infty}^{-1} H_0^{(2)}\left[k\,|\,m\,|\,b\sqrt{1-\sin^2\delta\sin^2\sigma}\right] \times \times e^{ik|m|b\sqrt{1-\sin^2\delta\sin^2\alpha}} e^{-ikmb(\sin\delta\cos\alpha - \sqrt{1-\sin^2\delta\sin^2\alpha})}, \tag{7}$$

$$\sum_{m=1}^{\infty} A_m = -\frac{i\pi}{c} I_0 e^{-ikx\sin\delta\sin\alpha} \sum_{m=1}^{\infty} H_0^{(2)}\left[k\,|\,m\,|\,b\sqrt{1-\sin^2\delta\sin^2\alpha}\right] \times \times e^{ik|m|b\sqrt{1-\sin^2\delta\sin^2\alpha}} e^{-ikmb(\sin\delta\cos\sigma + \sqrt{1-\sin^2\delta\sin^2\alpha})}. \tag{8}$$

To evaluate the sums of (7) and (8), we make use of formula (1). We introduce the continuous functions

$$e^{-ik\gamma_1\eta} f_1(\eta) = H_0^{(2)}(pk\,|\,\eta\,|)\, e^{ipk|\eta|} e^{-ik\gamma_1\eta},$$
$$e^{ik\gamma_2\eta} f_2(\eta) = H_0^{(2)}(pk\,|\,\eta\,|)\, e^{ipk|\eta|} e^{ik\gamma_2\eta}.$$

Here $p = \sqrt{1 - \sin^2\delta\ \sin^2\alpha}$, $\gamma_1 = p + \sin\delta\ \cos\alpha$, $\gamma_2 = p - \sin\delta\ \cos\alpha$.

For $\eta = mb$, where m is an integer, these functions reduce to the expressions under the summations on the right sides of (7) and (8).

Consider now the case of "large" distances between wires, when $pkb \gg 1$; we use the asymptotic form of the Hankel function and put

$$H_0^{(2)}(pkb) = \sqrt{\frac{2}{\pi pkb}}\, e^{-ipkb} e^{i\pi/4}. \tag{9}$$

In this case $f_1(\eta)$ and $f_2(\eta)$ can be considered to be slowly varying. After applying formula (1) to (7) and (8) and some manipulations, we obtain the following expression:

$$A = -I_0 \varkappa e^{-ikx \sin\delta \sin\alpha}, \tag{10}$$

where

$$\varkappa = \frac{i\pi}{c} e^{i\pi/4} \left\{ \frac{(\gamma_2 kb)^3 \cos\frac{\gamma_2 kb}{2}}{8 \sin^3 \frac{\gamma_2 kb}{2}} \left[\frac{1}{kb\sqrt{\gamma_2 p}} \left[(1 - 2C(\gamma_2 kb)) - i(1 - 2S(\gamma_2 kb)) \right] + \right.\right.$$

$$\left. + i\sqrt{\frac{2}{\pi pkb}} \frac{e^{-ikb\gamma_2}}{\gamma_2 kb} \left(1 + \frac{i}{2\gamma_2 kb} - \frac{4e^{\frac{ikb\gamma_2}{2}} \sin^2 \frac{kb\gamma_2}{2} + i \sin\frac{\gamma_2 kb}{2}}{(\gamma_2 kb)^2 \cos\frac{\gamma_2 kb}{2}} \right) \right] +$$

$$+ \frac{(\gamma_1 kb)^3 \cos\frac{\gamma_1 kb}{2}}{8 \sin^3 \frac{\gamma_1 kb}{2}} \left[\frac{1}{kb\sqrt{\gamma_1 p}} \left[(1 - 2C(\gamma_1 kb)) - i(1 - 2S(\gamma_1 kb)) \right] + \right.$$

$$\left. + i\sqrt{\frac{2}{\pi pkb}} \frac{e^{-i\gamma_1 kb}}{\gamma_1 kb} \left(1 + \frac{i}{2\gamma_1 kb} - \frac{4e^{i\frac{\gamma_1 kb}{2}} \sin^2 \frac{\gamma_1 kb}{2} + i \sin\frac{\gamma_1 kb}{2}}{(\gamma_1 kb)^2 \cos\frac{\gamma_1 kb}{2}} \right) \right] +$$

$$\left. + e^{-i\pi/4} H_0^{(2)}(pkr_0) \right\},$$

in which $C(\gamma_{1,2}kb)$ and $S(\gamma_{1,2}kb)$ are Fresnel integrals.

The value of the tangential electric field component at the surface of the zeroth wire is now equal, according to (4), to

$$E_x = \frac{i\omega}{c} I_0 \varkappa (1 - \sin^2\delta \sin^2\alpha) e^{-ikx \sin\delta \sin\alpha}. \tag{11}$$

On the surface of a perfectly conducting wire, we must have

$$E_x^{\text{inc}} + E_x = 0,$$

and consequently, as a result of (11),

$$E_0 \cos\alpha + \frac{i\omega}{c} I_0 \varkappa (1 - \sin^2\delta \sin^2\alpha) = 0,$$

whence we find the desired expression for the current

$$I_0 = icE_0 \frac{\cos\alpha}{\omega\varkappa (1 - \sin^2\delta \sin^2\alpha)},$$

and consequently, the current in the mth wire, according to (3), is

$$I_m = icE_0 \frac{\cos\alpha}{\omega\varkappa(1-\sin^2\delta\sin^2\alpha)} e^{-ik\sin\delta(x\sin\alpha+mb\cos\alpha)}.$$

2. FIELD SCATTERED BY THE ARRAY

After determining the currents induced in the wires of the array by the incident wave, it is not difficult to determine the field produced by these currents in the surrounding space. Performing a few manipulations and and then using the Poisson summation formula[2], we obtain

$$E_x = -\frac{2\pi\omega I_0}{c^2 b} e^{-ikx\sin\delta\sin\alpha}(1-\sin^2\delta\sin^2\alpha)\times \\ \times\sum_{n=-\infty}^{\infty}\frac{e^{\mp i\sqrt{(kp)^2-\left(\frac{2\pi n-d}{b}\right)^2}\,z}}{\sqrt{(kp)^2-\left(\frac{2\pi n-d}{b}\right)^2}} e^{i\frac{2\pi n-d}{b}y}, \tag{12}$$

$$E_y = -\frac{2\pi\omega I_0}{c^2 kb}\sin\delta\sin\alpha e^{-ikx\sin\delta\sin\alpha}\times \\ \times\sum_{n=-\infty}^{\infty}\frac{2\pi n-d}{b}\frac{e^{\mp i\sqrt{(kp)^2-\left(\frac{2\pi n-d}{b}\right)^2}\,z}}{\sqrt{(kp)^2-\left(\frac{2\pi n-d}{b}\right)^2}} e^{i\frac{2\pi n-d}{b}y}, \tag{12a}$$

$$E_z = \pm\frac{2\pi\omega I_0}{c^2 kb}\sin\delta\sin\alpha e^{-ikx\sin\delta\sin\alpha}\sum_{n=-\infty}^{\infty} e^{\mp i\sqrt{(kp)^2-\left(\frac{2\pi n-d}{b}\right)^2}\,z} e^{i\frac{2\pi n-d}{b}y}, \tag{12b}$$

where $d = kb\sin\delta\cos\alpha$.

The minus sign in the exponents under the summation sign refers to the field for $z > 0$, the plus sign to $z < 0$; the plus sign in front of E_z refers to $z > 0$, the minus sign to $z < 0$.

Thus, the field scattered by the array is a superposition of plane waves (spatial harmonics) with various propagation constants along the Z-axis. It should be noted that not all the spatial harmonics defined in relations (12) are propagating. Those harmonics whose propagation constants along the Z-axis have a purely imaginary value, i.e., those harmonics

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[2]See, e.g., [1].

with orders for which the inequality

$$\left(\frac{b}{\lambda}\right)^2 (1 - \sin^2 \delta \sin^2 \alpha) < \left(n - \frac{b}{\lambda} \sin \delta \cos \alpha\right)^2,$$

is satisfied, decay with distance from the surface of the array.

3. DERIVATION OF FORMULA (1)

Let $\phi(x)$ and $f(x)$ be functions continuous along with their first derivatives, given on the interval $0 \leq x \leq N$, where N is some natural number. Let us also introduce the step-function $\phi_1(x)$, defined by the relations

$$\varphi_1(x) = a_k \text{ for } k \leqslant x < k + 1 \quad (k = 0, 1, \ldots, N),$$

with a_k some constants, in general complex.

The function $f(x)$ is to be slowly-varying in the sense that its second derivative is somehow nearly constant on each interval $k \leq x < k + 1$ (but can be different on different intervals).

Consider now an integral of the form

$$\int_0^N \Phi'(x) f(x)\, dx,$$

where $\Phi(x) = \phi(x) + \phi_1(x)$, $\Phi'(x) = d\Phi(x)/dx$. Integrating by parts twice, we get

$$\int_0^N f(x) \varphi'(x)\, dx = -\sum_{k=1}^{N-1} \Delta a_{k-1} f(k) + [\Phi(N) f(N) - \Phi(0) f(0)] - \\ - [U(N) f'(N) - U(0) f'(0)] + \int_0^N U(x) f''(x)\, dx. \tag{13}$$

Here we have used the notations

$$U(x) = \int_0^x \Phi(\xi)\, d\xi + C, \quad \Delta a_{k-1} = a_k - a_{k-1},$$

where C is an arbitrary constant.

If $f''(x)$ is constant (to a sufficient degree of accuracy) on each interval $k < x < k + 1$, then the last integral in (13) can be made to vanish by requiring that on each interval the inequality

$$\int_{k-1}^{k} U(x)\,dx = 0 \quad (k = 1, 2, \ldots, N-1). \tag{14}$$

holds. Then (13) takes the form

$$\int_{0}^{N} f(x)\varphi'(x)\,dx = -\sum_{k=1}^{N-1} \Delta a_{k-1} f(k) + \Phi(N) f(N) - \Phi(0) f(0) + U(0) f'(0) - U(N) f'(N). \tag{15}$$

The value of U(x) on $k - 1 \leq x \leq k$ can be cast in the form

$$U(x) = \int_{0}^{x} \varphi(\xi)\,d\xi + \sum_{s=0}^{k-2} a_s + (x - k + 1) a_{k-1} + C, \tag{15a}$$

and equation (14) becomes

$$\sum_{s=0}^{k-1} a_s - \frac{1}{2} a_{k-1} = -\int_{k-1}^{k} dx \int_{0}^{x} \varphi(\xi)\,d\xi - C. \tag{16}$$

Relation (16), valid for every k = 1,2,...,N, gives a system of equations for determining the a_k.

Consider now a special case (though very important for applications), where $\phi(x) = e^{\mu x}$, and μ is an arbitrary complex number. In this case (16) takes the form

$$\sum_{s=0}^{k-1} a_s - \frac{1}{2} a_{k-1} = -\left[\int_{k-1}^{k} \frac{e^{\mu x} - 1}{\mu}\,dx + C\right] = -\frac{1}{\mu^2}\left[2e^{\mu(k-1/2)}\,\mathrm{sh}\,\frac{\mu}{2} - \mu\right] - C. \tag{17}$$

Equation (17) can be satisfied by putting $a_s = a_o e^{\mu s}$, then

$$a_0\left[\sum_{s=0}^{k-1} e^{\mu s} - \frac{1}{2} e^{\mu(k-1)}\right] = a_0\left[\frac{1 - e^{\mu k}}{1 - e^{\mu}} - \frac{1}{2} e^{\mu(k-1)}\right] = \frac{a_0}{1 - e^{\mu}}\left[1 - e^{\mu(k-1/2)}\,\mathrm{ch}\,\frac{\mu}{2}\right]. \tag{18}$$

It is clear that (17) is satisfied by putting

$$C=\frac{1}{\mu}-\frac{a_0}{1-e^{\mu}}, \qquad a_0\frac{e^{-\mu/2}\operatorname{ch}\frac{\mu}{2}}{1-e^{\mu}}=-2e^{-\mu/2}\frac{\operatorname{sh}\frac{\mu}{2}}{\mu^2},$$

and consequently,

$$a_0=-\frac{4e^{\mu/2}}{\mu^2}\frac{\operatorname{sh}^2\frac{\mu}{2}}{\operatorname{ch}\mu/2}, \qquad C=-\frac{1}{\mu}\left[1-\frac{2}{\mu}\operatorname{th}\frac{\mu}{2}\right]. \tag{19}$$

Thus the coefficients a_k are defined by

$$a_k=-\frac{4e^{\mu/2}}{\mu^2}\frac{\operatorname{sh}^2\frac{\mu}{2}}{\operatorname{ch}\frac{\mu}{2}}e^{\mu k}. \tag{20}$$

and therefore

$$\Delta a_{k-1}=a_k-a_{k-1}=-\frac{8}{\mu^2}\cdot\frac{\operatorname{sh}^3\frac{\mu}{2}}{\operatorname{ch}\frac{\mu}{2}}e^{\mu k}. \tag{21}$$

Now we can evaluate U(k):

$$U(k)=\int_0^k\varphi_1(x)\,dx+\int_0^k e^{\mu x}dx+C=\sum_{s=0}^{k-1}a_k+\frac{e^{\mu k}-1}{\mu}+C=$$

$$=a_0\sum_{s=0}^{k-1}e^{\mu s}+\frac{e^{\mu k}-1}{\mu}+C=a_0\frac{1-e^{\mu k}}{1-e^{\mu}}-\frac{1-e^{\mu k}}{\mu}+C=$$

$$=(1+e^{\mu k})\left(\frac{a_0}{1-e^{\mu}}-\frac{1}{\mu}\right)+C=-C(1-e^{\mu k})+C=-\frac{1}{\mu}\left(1-\frac{2}{\mu}\operatorname{th}\frac{\mu}{2}\right)e^{\mu k}.$$

The coefficients in (15) are then determined by the relations

$$\Phi(0)=1-\frac{4}{\mu^2}e^{\mu/2}\frac{\operatorname{sh}^2\frac{\mu}{2}}{\operatorname{ch}\frac{\mu}{2}},$$

$$\Phi(N)=e^{\mu N}+a_{N-1}=e^{\mu N}\left(1-\frac{4e^{-\mu/2}}{\mu^2}\cdot\frac{\operatorname{sh}^2\frac{\mu}{2}}{\operatorname{ch}\frac{\mu}{2}}\right), \qquad U(0)=C, \quad U(N)=Ce^{\mu N}.$$

We can now rewrite (15) as

$$\int_0^N f(x)\,e^{\mu x}dx=\frac{8\operatorname{sh}^3\frac{\mu}{2}}{\mu^3\operatorname{ch}\frac{\mu}{2}}\sum_{k=1}^{N-1}f(k)\,e^{\mu k}+\frac{1}{\mu}\left(1-\frac{4e^{-\mu/2}}{\mu^2}\cdot\frac{\operatorname{sh}^2\frac{\mu}{2}}{\operatorname{ch}\frac{\mu}{2}}\right)e^{\mu N}f(N)-$$
$$-\frac{1}{\mu}\left(1-\frac{4e^{\mu/2}}{\mu^2}\cdot\frac{\operatorname{sh}^2\frac{\mu}{2}}{\operatorname{ch}\frac{\mu}{2}}\right)f(0)-\frac{1}{\mu^2}\left(1-\frac{2}{\mu}\operatorname{th}\frac{\mu}{2}\right)[e^{\mu N}f'(N)-f'(0)]. \tag{22}$$

Consider the case of purely imaginary μ, a case of great interest for problems in applied electrodynamics, for instance. We put $\mu = -i\nu$, where ν is real. Then formula (22) takes the form

$$\int_0^N f(x)\, e^{-i\nu x} dx = \frac{8 \sin^3 \frac{\nu}{2}}{\nu^3 \cos \frac{\nu}{2}} \sum_{k=1}^{N-1} f(k)\, e^{-i\nu k} - \frac{1}{i\nu}\left(1 - \frac{4e^{i\nu/2}}{\nu^2} \cdot \frac{\sin^2 \frac{\nu}{2}}{\cos \frac{\nu}{2}}\right) e^{-i\nu N} f(N) +$$
$$+ \frac{1}{i\nu}\left(1 - \frac{4e^{-i\nu/2}}{\nu^2} \cdot \frac{\sin^2 \frac{\nu}{2}}{\cos \frac{\nu}{2}}\right) f(0) + \frac{1}{\nu^2}\left(1 - \frac{2}{\nu} \operatorname{tg} \frac{\nu}{2}\right) [f'(N)\, e^{-i\nu N} - f'(0)], \tag{23}$$

or

$$\sum_{k=1}^{N} f(k)\, e^{-i\nu k} = \frac{\nu^3 \cos \frac{\nu}{2}}{8 \sin^3 \frac{\nu}{2}} \left\{ \int_1^N f(x)\, e^{-i\nu x} dx + \frac{1}{i\nu}\left(1 - \frac{4e^{-i\nu/2} \sin^2 \frac{\nu}{2}}{\nu^2 \cos \frac{\nu}{2}}\right) e^{-i\nu N} f(N) - \right.$$
$$\left. - \frac{1}{i\nu}\left(1 - \frac{4e^{i\nu/2} \sin^2 \frac{\nu}{2}}{\nu^2 \cos \frac{\nu}{2}}\right) e^{-i\nu} f(1) - \frac{1}{\nu^2}\left(1 - \frac{2}{\nu} \operatorname{tg} \frac{\nu}{2}\right) [f'(N)\, e^{-i\nu N} - f'(1)\, e^{-i\nu}] \right\}.$$

Let us make some additional remarks:

1.) For $\nu \to 0$, carrying out the appropriate calculations in (23), we obtain

$$\int_0^N f(x)\, dx = \sum_{k=1}^{N-1} f(k) + \frac{1}{2}[f(N) + f(0)] - \frac{1}{12}[f'(N) - f'(0)],$$

i.e., the Euler-Maclaurin summation formula, if all derivative terms of higher order than two at the ends of the interval are neglected. Applications of this formula are well-known.

2.) Formula (23) is suitable for summing series with oscillating terms, i.e., for computing sums of the form

$$S = \sum_{k=1}^{N-1} f(k)\, e^{-i\nu k}.$$

It is important, however, to note the following point. If the value of ν is close to $2\pi q$ for some integer $q \neq 0$, then $8 \sin^3(\nu/2)/\nu^3 \cos(\nu/2)$

is close to zero, and at first glance, this would seem to render (23) useless. But in fact we can always take $|\nu| \leq \pi$ because an integer multiple of 2π can always be subtracted from ν without changing the value of the sum S.

3.) Let us also remark upon the case when we must sum an alternating series of terms whose absolute value varies slowly. Putting $\nu = \pi$, we obtain

$$-\sum_{k=1}^{N-1} f(k)\, e^{-i\pi k} = \frac{1}{2}[f(0) + (-1)^N f(N)] + \frac{1}{4}[f'(0) - (-1)^N f'(N)].$$

As an example, we evaluate

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \ldots = 1 - \frac{1}{2} + \frac{1}{3} + \sum_{s=1}^{\infty} (-1)^s \frac{1}{s+3} =$$

$$= 0.500 + 0.333 - \frac{1}{2 \cdot 3} + \frac{1}{4 \cdot 3^2} = 0.694.$$

In fact, $\ln 2 \approx 0.693$.

4.) We now give an estimate of the error involved in replacing the sum by the integral according to (23). To this end we return to equation (13). It is clear that the error is determined by the last term on the right side of (13), i.e., the quantity

$$T = \int_0^N f''(x)\, U(x)\, dx, \tag{24}$$

which must be estimated.

We restrict ourselves to the case when $f'''(x)$ does not change sign on the interval of interest, i.e., when $f''(x)$ is a monotonic function. The results obtained can be extended in an obvious manner to the case when the interval can be broken up into a finite number of sub-intervals on each of which $f''(x)$ is monotonic.

Integrating (24) by parts, we have

$$T = f''(N) \int_0^N U(x)\, dx - \int_0^N f'''(x) \left[\int_0^x U(\xi)\, d\xi \right] dx,$$

and since, according to (14), $\int_0^N U(x)d_x = 0$, then

$$|T| \leqslant \left| \int_0^x U(\xi)\, d\xi \right|_{max} \int_0^N |f''(x)|\, dx = \left| \int_0^x U(\xi)\, d\xi \right|_{max} |f'(N) - f'(0)|, \tag{24a}$$

in which $\left| \int_0^x U(\xi)\, d\xi \right|_{max}$ is the maximum modulus of the integral under the absolute value signs.

Let us now specialize this estimate to the very interesting, practical case when $\phi(x) = e^{-i\nu x}$, where ν is a real number, less than π in absolute value (this incurs no loss of generality).

Formula (23) with the remainder term is

$$\sum_{k=1}^{N-1} f(k)\, e^{-i\nu k} = \frac{\nu^3 \cos\frac{\nu}{2}}{8\sin^3\frac{\nu}{2}} \left\{ \int_0^N e^{-i\nu x} f(x)\, dx + \frac{1}{i\nu}\left(1 - \frac{4e^{i\nu/2}\sin^2\frac{\nu}{2}}{\nu^2\cos\frac{\nu}{2}}\right) e^{-i\nu N} f(N) - \right.$$

$$\left. - \frac{1}{i\nu}\left(1 - \frac{4e^{-i\nu/2}\sin^2\frac{\nu}{2}}{\nu^2\cos\frac{\nu}{2}}\right) f(0) - \frac{1}{\nu^2}\left(1 - \frac{2}{\nu}\operatorname{tg}\frac{\nu}{2}\right)[f'(N)\, e^{-i\nu N} - f'(0)] \right\} -$$

$$- i \frac{\nu^2 \cos\frac{\nu}{2}}{8\sin\frac{\nu}{2}} T,$$

wherein the remainder term is given by

$$R = -i \frac{\nu^2\cos\frac{\nu}{2}}{8\sin^3\frac{\nu}{2}} T.$$

We define the function $U(\xi)$ entering into (24a). In accordance with (15a) we write

$$U(\xi) = \int_0^\xi e^{-i\nu\xi'} d\xi' + (\xi - k + 1)\, a_{k-1} + \sum_{s=0}^{k-2} a_s + C$$

and, taking (19) and (20) into account, we obtain after some algebraic manipulations

$$U(\xi) = i\frac{1}{\nu} e^{-i\nu(k-1)} \left\{ e^{-i\nu\Delta} - \frac{\operatorname{tg}\frac{\nu}{2}}{\nu}\left[2 - 4i\Delta e^{-i\nu/2}\sin\frac{\nu}{2}\right]\right\}, \tag{25}$$

where $\Delta = \xi - k + 1$, with $0 \leq \Delta < 1$.

The expression in brackets is denoted by $P(\Delta) = P_R + iP_i$. Then

$$P_R(\Delta) = \cos\nu\Delta - \frac{\operatorname{tg}\frac{\nu}{2}}{\nu}\left[2 - 4\Delta\sin^2\frac{\nu}{2}\right],$$

$$P_i(\Delta) = -\sin\nu\Delta + \frac{\operatorname{tg}\frac{\nu}{2}}{\nu}2\Delta\sin\nu.$$

Now form the expressions

$$S_R = \int_0^\Delta P_R(\Delta)\,d\Delta = \frac{1}{\nu}\sin\nu\Delta - \frac{\operatorname{tg}\frac{\nu}{2}}{\nu}\left[2\Delta - 2\Delta^2\sin^2\frac{\nu}{2}\right],$$

$$S_i = \int_0^\Delta P_i(\Delta)\,d\Delta = \frac{1}{\nu}(\cos\nu\Delta - 1) + \frac{\operatorname{tg}\frac{\nu}{2}}{\nu}\Delta^2\sin\nu.$$

We can now estimate S_R and S_i by writing

$$|S_R| = \frac{1}{\nu}\left|\sin\nu\Delta - 2\Delta\operatorname{tg}\frac{\nu}{2}\left(1 - \Delta\sin^2\frac{\nu}{2}\right)\right| = \frac{1}{\nu}\sin\nu\Delta\left(1 - \frac{\Delta\nu}{\sin\nu\Delta}\cdot\frac{\sin\nu}{\nu}\right) +$$

$$+ 2\Delta(1-\Delta)\operatorname{tg}\frac{\nu}{2}\sin^2\frac{\nu}{2}\Big|,$$

$$|S_i| = \left|\frac{1}{\nu}(\cos\nu\Delta - 1) + \frac{\operatorname{tg}\frac{\nu}{2}}{\nu}\Delta^2\sin\nu\right| = \frac{2\sin^2\frac{\nu\Delta}{2}}{\nu}\left|1 - \frac{\sin^2\frac{\nu}{2}}{\left(\frac{\nu}{2}\right)^2}\cdot\frac{\left(\frac{\nu\Delta}{2}\right)^2}{\sin^2\frac{\nu\Delta}{2}}\right|.$$

Now consider

$$1 - \frac{\Delta\nu}{\sin\Delta\nu}\cdot\frac{\sin\nu}{\nu} = f_1(\nu,\Delta), \qquad 1 - \frac{\sin^2\frac{\nu}{2}}{\left(\frac{\nu}{2}\right)^2}\cdot\frac{\left(\frac{\nu\Delta}{2}\right)^2}{\sin^2\frac{\nu\Delta}{2}} = f_2(\nu,\Delta),$$

For $\Delta = 1$,

$$f_1(\nu, 1) = f_2(\nu, 1) = 0,$$

and for $\Delta = 0$,

$$f_1(\nu, 0) = 1 - \frac{\sin\nu}{\nu}, \qquad f_2(\nu, 0) = 1 - \frac{\sin^2\frac{\nu}{2}}{\left(\frac{\nu}{2}\right)^2}.$$

Considering that $f_1(\nu,\Delta)$ and $f_2(\nu,\Delta)$ are monotonic functions which attain a maximum for $\Delta = 0$, we can write

$$|S_R| \leqslant \frac{\sin\frac{\nu}{2}}{\nu}\left|2\left(1 - \frac{\sin\nu}{\nu}\right) + \frac{1}{2}\operatorname{tg}\frac{\nu}{2}\sin\frac{\nu}{2}\right|.$$

Here use has been made of the fact that $\sin \nu\Delta = 2 \sin \frac{\nu\Delta}{2} \cos \frac{\nu\Delta}{2} \leqslant 2 \sin \frac{\nu\Delta}{2} \leqslant 2 \sin \frac{\nu}{2}$, and also that $\Delta(1-\Delta)$ has a maximum of 1/4 at $\Delta = 1/2$.

For $|S_i|$ we obtain the following estimate

$$|S_i| \leqslant \frac{2\sin^2\frac{\nu}{2}}{\nu}\left|1 - \frac{\sin^2\frac{\nu}{2}}{\left(\frac{\nu}{2}\right)^2}\right|.$$

and we can now write

$$|R| = \frac{\nu^2 \cos\frac{\nu}{2}}{8\sin^3\frac{\nu}{2}}|T| \leqslant \frac{\nu\cos\frac{\nu}{2}}{8\sin^3\frac{\nu}{2}}[|S_R| + |S_i|] =$$
$$= \frac{1}{16}\left|1 - 4\,\mathrm{ctg}\,\frac{\nu}{2}\left[1 - \frac{\sin\nu}{\nu} + \sin\frac{\nu}{2}\left(1 - \frac{\sin^2\frac{\nu}{2}}{\left(\frac{\nu}{2}\right)^2}\right)\right]\right| |f''(0) - f''(N)|. \tag{26}$$

Formula (26) gives an estimate for the remainder term and consequently for the error involved in approximating a sum using (23).

4. CONCLUSION

The problem of plane wave diffraction by an array, whose solution was given above, has been considered previously (see, e.g., [2-5]). However, the indicated papers only carried out the solution for special cases of the incident wave (normal or oblique, with the $\underset{\sim}{E}$ vector parallel to the wires of the array), and an expression for the current in the wires was obtained in the form of an infinite sum. In the present paper, the case of arbitrary incidence was considered, onto an array whose period is not small compared to a wavelength. An expression for the current I_o was obtained in finite form. In the case of small wire separations, the method of averaged boundary conditions can be applied to solve the problem[3].

In order to evaluate the accuracy of formula (12), the reflection coefficient was evaluated numerically using (12) and compared with the corresponding values computed from formulas derived in [2]; for an array whose

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[3]See, e.g. [5].

period is 0.6λ, with wire radii $r_o = 0.002\lambda$, at oblique incidence (α=0), results obtained for the absolute value $|R_o|$ of the reflection coefficient are presented in the following table:

δ°;	Formula (12)	Formula from [2]
0	0.195	0.199
15	0.203	0.208
30	0.209	0.215

Thus the numerical agreement is seen to be good.

Formula (1), finally can also be applied to the solution of other problems in diffraction theory, and is useful for numerical purposes as well.

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