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An analysis of the propagation of electromagnetic waves along a perfectly conducting wire parallel to a plane interface between dissimilar materials is presented. Particular emphasis is placed on determination of the significance of the first order Fourier components of current on the wire compared with the zero order current which has no azimuthal dependence around the wire.

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Introduction

Wave propagation along a perfectly conducting cylindrical wire parallel to a plane interface between two dissimilar materials has been an area of theoretical and practical interest for several decades. During this time the problem has been treated in varying degrees of approximation by many investigators. Among the more relevant to this work are the treatments of Wait[1] and Chang and Olsen[2]. Wait postulated a primary longitudinal Hertz potential which generated a cylindrically symmetric TM field surrounding the wire. He then expanded the field into a spectrum of plane waves and scattered these plane waves from the interface to obtain the total electric field. Requiring the longitudinal component of this total electric field to be zero on the surface of the wire he obtained an equation for the modal propagation constants. Chang and Olsen studied this modal equation in detail and showed the existence of not only the usual transmission line mode but also a so-called "earth-attached" or "fast-wave" mode. However, all of this work was done in the thin-wire approximation. That is, the only current accounted for on the wire was the angular average longitudinal electric current. If the wire is not thin and/or is less than a fraction of a wavelength away from the interface, one would expect that the longitudinal current would have some azimuthal variation and that there would be some azimuthally directed currents. Thus, one is led to inquire as to the conditions under which these angularly dependent currents can be neglected in formulating the modal equation.

In addressing ourselves to the question posed above, we formulate the problem in general and then express the modal equation retaining only the zero-order and first-order terms in the azimuthal variation. Then by comparing the first-order terms with the zero-order ones, we obtain a criterion for determining the error incurred in retaining only the zero order terms. Some work in this regard was carried out by Grinberg and Bonshtedt[3] but their results appear to be limited to cases where the propagation constant is unmodified by the presence of the first order currents. That is, they have assumed conditions such that the zero-order equations de-couple from the higher order ones. The approach presented here takes full account of the coupling.

Formulation

The geometry to be analyzed is shown in Figure 1. The wire is infinite in the z-direction and all field quantities are assumed to vary as $e^{ik_1\alpha z} e^{-i\omega t}$ where $k_1 = \omega\sqrt{\mu_o\varepsilon_1}$. Similarly $k_2 = \omega\sqrt{\mu_o\varepsilon_2}$ and may be complex to account for loss in medium 2. A cylindrical coordinate system is shown with its axis coincident with the axis of the wire. We postulate an infinite sum of two dimensional electric and magnetic multipoles of unknown amplitudes located on the axis of the wire. Expressing the field of each multipole as a spectrum of plane waves and scattering each plane wave from the interface one could obtain an expression for the total tangential electric field exterior to the wire. Then, requiring the tangential field components to be zero on the wire surface one would obtain an infinite set of simultaneous linear equations for the amplitude of the multipoles. Being homogeneous, these equations require for non-trivial solution that the determinant of the coefficients be zero. This requirement would yield the modal equation.

Before proceeding with this formulation, however, we first divide the modes into three categories of which only one will be studied. The categories are: those with no zero-order magnetic multipole field, those with no zero

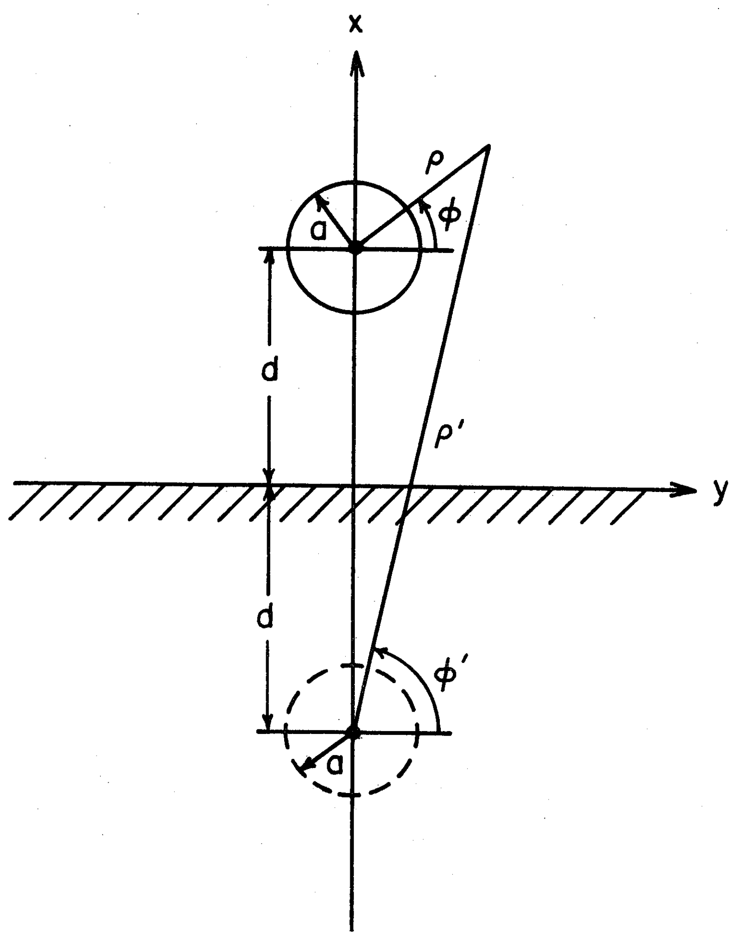

Fig. 1 Geometry of the problem

order (angularly independent) electric multipole field, and hybrid modes with both zero-order multipole fields. In this treatment we will concern ourselves only with the modes of the first category because among them are the modes studied in the thin-wire approximation by previous investigators. Moreover, we will treat only modes which are symmetric about the x-z plane. Thus, we postulate the presence of a zero-order electric multipole and first order electric and magnetic multipoles and assume that all higher order multipoles have negligible amplitude.

The postulated multipoles will be represented by their Hertz vector potentials as follows. The zero-order electric multipole potential is

$$\vec{\pi}_o^e = \frac{1}{k_1^2} \hat{a}_z \, H_o^{(1)}(\zeta k_1 \rho) \tag{1}$$

where

$$\zeta = (1 - \alpha^2)^{\frac{1}{2}} , \quad \text{and } \mathrm{Im}(\zeta) > 0$$

Now, we note that one may generate from this potential the potential for the first-order electric multipole by merely differentiating with respect to the distance d from the multipole to the interface and dividing by ζk_1. That is,

$$\vec{\pi}_1^e = \frac{1}{\zeta k_1} \frac{\partial \pi_o^e}{\partial d} = \frac{1}{k_1^2} \hat{a}_z H_1^{(1)}(\zeta k_1 \rho) \cos\phi \tag{2}$$

If there were to be a zero-order magnetic multipole, its potential would be

$$\vec{\pi}_o^m = \frac{1}{k_1^2} \sqrt{\frac{\varepsilon_1}{\mu_o}} \, \hat{a}_z H_o^{(1)}(\zeta k_1 \rho) \tag{3}$$

Now, the symmetric first order magnetic multipole field corresponds to the antisymmetric first order potential. This potential can be obtained from

(3) by taking the negative of its derivative with respect to y and dividing by $k_1\zeta$. Thus

$$\vec{\pi}_1^m = \frac{1}{k_1\zeta}\frac{\partial}{\partial y}\pi_o^m = \frac{1}{k_1^2}\sqrt{\frac{\varepsilon_1}{\mu_o}}\,\hat{a}_z H_1^{(1)}(\zeta k_1\rho)\sin\phi \tag{4}$$

Next, each of the potentials (1), (2), and (4) will be used as a primary source a distance, d, from the interface and in each case the components of the total exterior electric fields tangent to the wire surface will be derived.

Considering first the zero-order electric multipole field, we write its potential as a plane wave spectrum in the form

$$\vec{\pi}_o^e = \frac{1}{k_1^2}\hat{a}_z\frac{1}{i\pi}\int_{-\infty}^{\infty}\frac{1}{u_1}e^{-u_1|k_1x-D|}e^{-i\lambda k_1 y}d\lambda \tag{5}$$

where $u_1 = [\lambda^2-\zeta^2]^{\frac{1}{2}}$ and $D = k_1 d$. Now, following a procedure similar to that outlined by Wait[1] we introduce transmitted and reflected Hertzian potentials of electric and magnetic types as follows.

$$\vec{\pi}_{oR}^e = \hat{a}_z\frac{1}{i\pi k_1^2}\int_{-\infty}^{\infty}\frac{R(\lambda)}{u_1}e^{-u_1(k_1x+D)}e^{-i\lambda k_1 y}d\lambda \qquad \text{for } x > 0 \tag{6}$$

$$\vec{\pi}_{oM}^e = \hat{a}_z\frac{1}{i\pi k_1^2}\int_{-\infty}^{\infty}\frac{M(\lambda)}{u_1}e^{-u_1(k_1x+D)}e^{-i\lambda k_1 y}d\lambda \qquad \text{for } x > 0 \tag{7}$$

$$\vec{\pi}_{oT}^e = \hat{a}_z\frac{1}{i\pi k_1^2}\int_{-\infty}^{\infty}\frac{T(\lambda)}{u_1}e^{-u_1 D}e^{u_2k_1x}e^{-i\lambda k_1 y}d\lambda \qquad \text{for } x < 0 \tag{8}$$

$$\vec{\pi}_{oN}^e = \hat{a}_z\frac{1}{i\pi k1^2}\int_{-\infty}^{\infty}\frac{N(\lambda)}{u_1}e^{-u_1 D}e^{u_2k_1x}e^{-i\lambda k_1 y}d\lambda \qquad \text{for } x < 0 \tag{9}$$

where $u_2 = [\lambda^2+\alpha^2-n^2]^{1/2}$, $n = k_2/k_1$ and $\vec{\pi}^e_m$ and $\vec{\pi}^e_N$ are magnetic Hertz vectors

Now, computing the tangential components of electric and magnetic fields from these potentials and requiring continuity at x=0 we obtain,

$$1+R(\lambda) = \frac{2}{\zeta^2}\frac{\lambda^2-u_1u_2}{u_2+n^2u_1}\quad u_1 \tag{10}$$

$$M(\lambda) = \frac{-2i\alpha(n^2-1)}{\zeta^2}\sqrt{\frac{\epsilon_1}{\mu_o}}\left[\frac{\lambda u_1}{(u_1+u_2)(u_2+n^2u_1)}\right] \tag{11}$$

Thus, the z component of the total electric field for x > 0 is,

$$E^e_{zo} = \zeta^2H_o^{(1)}(\zeta k_1\rho) - \zeta^2H_o^{(1)}(\zeta k_1\rho') + \frac{2}{i\pi}\int_{-\infty}^{\infty}\frac{\lambda^2-u_1u_2}{u_2+n^2u_1}e^{-u_1(k_1x+D)}e^{-ik_1\lambda y}d\lambda \tag{12}$$

and the ϕ component is,

$$E^e_{\phi o} = \frac{i\alpha k_1}{\rho}\frac{\partial}{\partial\phi}\left(\pi^e_o + \pi^e_{oR}\right) - i\omega\mu_o\frac{\partial}{\partial\rho}\pi^e_{oM}$$

$$= -\frac{i\alpha}{k_1\rho}\frac{\partial}{\partial\phi}H_o^{(1)}(\zeta k_1\rho')$$

$$+ \frac{2\alpha}{\pi\zeta^2k_1\rho}\frac{\partial}{\partial\phi}\int_{-\infty}^{\infty}\frac{\lambda^2-u_1u_2}{u_2+n^2u_1}e^{-u_1(k_1x+D)}e^{-ik_1\lambda y}d\lambda$$

$$+ \frac{2i\alpha(n^2-1)}{\pi\zeta^2k_1}\frac{\partial}{\partial\rho}\int_{-\infty}^{\infty}\frac{\lambda}{(u_1+u_2)(u_2+n^2u_1)}e^{-u_1(k_1x+D)}e^{-ik_1\lambda y}d\lambda \tag{13}$$

where $\rho' = [(x+d)^2+y^2]^{1/2}$, and $D = k_1d$.

Turning next to the first order electric potential (2), recalling that $\vec{\pi}^e_1 = \frac{1}{\zeta k_1}\frac{\partial\vec{\pi}^e_o}{\partial d}$, and noting the linearity of the reflection process we see that $\vec{\pi}^e_{1R} = \frac{1}{\zeta k_1}\frac{\partial\vec{\pi}^e_{oR}}{\partial d}$ and $\vec{\pi}^e_{1M} = \frac{1}{\zeta k_1}\frac{\partial\vec{\pi}^e_{oM}}{\partial d}$. Thus, $E^e_{z1} = \frac{1}{\zeta k_1}\frac{\partial E^e_{zo}}{\partial d}$ or

$$E^e_{z1} = \zeta^2 H^{(1)}_1(\zeta k_1\rho)\cos\phi + \zeta^2 H^{(1)}_1(\zeta k_1\rho')\cos\phi'$$

$$- \frac{2}{i\pi\zeta}\int_{-\infty}^{\infty} u_1 \frac{\lambda - u_1u_2}{u_2+n^2u_1} e^{-u_1(k_1x+D)} e^{-i\lambda k_1 y} d\lambda \tag{14}$$

where ρ' and ϕ' are cylindrical coordinates referred to $x = -d$, $y = 0$; i.e.,

$$\cos\phi' = \frac{x+d}{\rho'} = \frac{x+d}{[(x+d)^2+y^2]^{1/2}} \tag{15}$$

and

$$\sin\phi' = \frac{y}{\rho'} = \frac{y}{[(x+d)^2+y^2]^{1/2}} \tag{16}$$

Similarly, by <u>first</u> computing π^e_1, π^e_{1R}, and π^e_{1M} and then substituting in,

$$E^e_{\phi 1} = \frac{i\alpha k_1}{\rho}\frac{\partial}{\partial\phi}(\pi^e_1 + \pi^e_{1R}) - i\omega\mu_o \frac{\partial}{\partial\rho}\pi^e_{1M} \tag{17}$$

we obtain,

$$E^e_{\phi 1} = -\frac{i\alpha}{k_1\rho} H^{(1)}_1(\zeta k_1\rho)\sin\phi + \frac{i\alpha}{k_1\rho}\frac{\partial}{\partial\phi}[H^{(1)}_1(\zeta k_1\rho')\cos\phi']$$

$$- \frac{2\alpha}{\pi\zeta^3}\frac{1}{k_1\rho}\frac{\partial}{\partial\phi}\int_{\infty}^{\infty} u_1\frac{\lambda^2-u_1u_2}{u_2+n^2u_1} e^{-u_1(k_1x+D)} e^{-i\lambda k_1 y} d\lambda$$

$$-\frac{2i\alpha(n^2-1)}{\pi\zeta^3 k_1}\frac{\partial}{\partial\rho}\int_{-\infty}^{\infty}\frac{\lambda u_1}{(u_1+u_2)(u_2+n^2u_1)} e^{-u_1(k_1x+D)} e^{-i\lambda k_1 y} d\lambda \tag{18}$$

In order to obtain fields due to a first-order magnetic multipole, we begin with a zero-order magnetic multipole in anticipation of obtaining the desired result by differentiation. Thus we proceed to express the zero-order potential in terms of its plane wave spectrum as,

$$\vec{\pi}_o^m = \frac{1}{k_1^2} \hat{a}_z \sqrt{\frac{\varepsilon_1}{\mu_o}} \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{1}{u_1} e^{-u_1 |k_1 x - D|} e^{-i\lambda k_1 y} d\lambda \tag{19}$$

As before we postulate the following scattered potentials.

$$\vec{\pi}_{oR}^m = \frac{1}{k_1^2 i\pi} \sqrt{\frac{\varepsilon_1}{\mu_o}} \int_{-\infty}^{\infty} \frac{R^*(\lambda)}{u_1} e^{-u_1 (k_1 x + D)} e^{-i\lambda k_1 y} d\lambda \qquad \text{for } x > 0 \tag{20}$$

$$\vec{\pi}_{oM}^m = \frac{1}{k_1^2 i\pi} \sqrt{\frac{\varepsilon_1}{\mu_o}} \int_{-\infty}^{\infty} \frac{M^*(\lambda)}{u_1} e^{-u_1 (k_1 x + D)} e^{-i\lambda k_1 y} d\lambda \qquad \text{for } x > 0 \tag{21}$$

$$\vec{\pi}_{oT}^m = \frac{1}{k_1^2 i\pi} \sqrt{\frac{\varepsilon_1}{\mu_o}} \int_{-\infty}^{\infty} \frac{T^*(\lambda)}{u_1} e^{-u_1 D} e^{u_2 k_1 x} e^{-i\lambda k_1 y} d\lambda \qquad \text{for } x < 0 \tag{22}$$

$$\vec{\pi}_{oN}^m = \frac{1}{k_1^2 i\pi} \sqrt{\frac{\varepsilon_1}{\mu_o}} \int_{-\infty}^{\infty} \frac{N^*(\lambda)}{u_1} e^{-u_1 D} e^{u_2 k_1 x} e^{-i\lambda k_1 y} d\lambda \qquad \text{for } x < 0 \tag{23}$$

where we emphasize that $\vec{\pi}_{oM}^m$ and $\vec{\pi}_{oN}^m$ are <u>electric</u> Hertz potentials. Imposing continuity of tangential fields at the interface yields,

$$1 + R^*(\lambda) = \frac{2}{\zeta^2} \frac{\lambda^2 (u_1 + n^2 u_2) - u_1 u_2 (u_2 + n^2 u_1)}{(u_1 + u_2)(u_2 + n^2 u_1)} u_1 \tag{24}$$

$$M^*(\lambda) = \frac{2i\alpha(n^2-1)}{\zeta^2} \sqrt{\frac{\mu_o}{\varepsilon_1}} \left[\frac{\lambda u_1}{(u_1 + u_2)(u_2 + n^2 u_1)} \right] \tag{25}$$

Thus, the z component of the total electric field for x > 0 is generated by π_{oM}^m . In particular,

$$E_{zo}^m = \frac{2\alpha(n^2-1)}{\pi} \int_{-\infty}^{\infty} \frac{\lambda}{(u_1 + u_2)(u_2 + n^2 u_1)} e^{-u_1 (k_1 x + D)} e^{-i\lambda k_1 y} d\lambda \tag{26}$$

and by differentiating with respect to y and dividing by k_1 we obtain,

$$E^m_{z1} = \frac{-2i\alpha(n^2-1)}{\pi\zeta} \int_{-\infty}^{\infty} \frac{\lambda^2}{(u_1+u_2)(u_2+n^2u_1)} e^{-u_1(k_1x+D)} e^{-i\lambda k_1 y} d\lambda \tag{27}$$

as the z-component of the total electric field for $x > 0$ due to a first order magnetic multipole source. Likewise,

$$E_{\phi 1}^{m} = \frac{i\alpha k_1}{\rho} \frac{\partial}{\partial\phi} \pi^m_{1M} - i\omega\mu_o \frac{\partial}{\partial\rho} (\pi^m_1 + \pi^m_{1R}) \tag{28}$$

That is,

$$E^m_{\phi 1} = \frac{2\alpha^2(n^2-1)}{\pi\zeta^3 k_1\rho} \frac{\partial}{\partial\phi} \int_{-\infty}^{\infty} \frac{\lambda^2}{(u_1+u_2)(u_2+n^2u_1)} e^{-u_1(k_1x+D)} e^{-i\lambda k_1 y} d\lambda$$

$$-i\zeta H_1^{(1)'}(\zeta k_1\rho)\sin\phi + \frac{i}{k_1}\frac{\partial}{\partial\rho} H_1^{(1)}(\zeta k_1\rho')\sin\phi'$$

$$- \frac{2}{i\pi\zeta^3 k_1} \frac{\partial}{\partial\rho} \int_{-\infty}^{\infty} \lambda \left[\frac{\lambda^2(u_1+n^2u_2)-u_1u_2(u_2+n^2u_1)}{(u_1+u_2)(u_2+n^2u_1)}\right] e^{-u_1(k_1x+D)} e^{-i\lambda k_1 y} d\lambda \tag{29}$$

The fields of all higher order multipoles will be neglected.

The boundary conditions at the surface of the perfectly conducting wire $\rho = a$ imply that,

$$A_o E^e_{zo} + A_1 E^e_{z1} + M_1 E^m_{z1} = 0 \tag{30}$$

$$A_o E^e_{\phi o} + A_1 E^e_{\phi 1} + M_1 E^m_{\phi 1} = 0 \tag{31}$$

on the wire. In these equations A_o, A_1, and M_1 are unknown amplitudes of the zero order electric, first order electric, and first order magnetic multipoles, respectively. Each multipole field evaluated on the wire surface is now written in Fourier components. For example,

$$E^{e}_{zo} = E^{e}_{z\,oo} + E^{e}_{zo1}\cos\phi + E^{e}_{zo2}\cos 2\phi + \ldots$$

$$E^{m}_{\phi 1} = E^{m}_{\phi 11}\sin\phi + E^{m}_{\phi 12}\sin 2\phi + \ldots \qquad (32)$$

$$\vdots$$

Equating the coefficients of ϕ independent, $\cos\phi$, and $\sin\phi$ terms yield a homogeneous set of three simultaneous linear equations. Thus, we require that the determinant of the coefficients be zero; that is,

$$\begin{vmatrix} E^{e}_{zoo} & E^{e}_{z1o} & E^{m}_{z1o} \\ E^{e}_{zo1} & E^{e}_{z11} & E^{m}_{z11} \\ E^{e}_{\phi o1} & E^{e}_{\phi 11} & E^{m}_{\phi 11} \end{vmatrix} = 0 \qquad (33)$$

This represents the modal equation for α up to first order Fourier components. It remains to find explicit expressions for the Fourier components of the fields, i.e., the matrix elements. Had we neglected the first order terms the equation would have read $E^{e}_{zoo}=0$ which is equivalent in the thin wire approximation to the result of Wait[1] and Chang and Olsen[2] .

Representation of the Matrix Elements

Recall that all of the integrands involved here contain the factor $e^{-u_1(k_1x+D)}e^{-i\lambda k_1 y}$. This factor can be conveniently represented in the following form.

$$e^{-u_1(k_1x+D)}e^{-i\lambda k_1 y} = e^{-2u_1D}\sum_{m=-\infty}^{\infty} J_m(\zeta k_1\rho)e^{im(\phi-\phi_o+\pi/2)} \qquad (34)$$

where $\phi_o = -\sin^{-1}\frac{1}{\zeta}$ or $\phi_o = \cos^{-1}\frac{iu_1}{\zeta}$. For our purposes, a convenient form of this relation is

$$e^{-u_1(k_1x+D)}e^{-i\lambda k_1 y} = e^{-2u_1D}\{J_o(\zeta k_1\rho)+2\sum_{m=1}^{\infty} J_m(\zeta k_1\rho) \times [\cos m\phi_o \cos m\phi + \sin m\phi_o \sin m\phi]\} \qquad (35)$$

Upon substitution of this form in the integrands it is found expedient to define two fundamental integrals.

$$I_1(\alpha;D) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{e^{-2u_1D}}{u_1(u_1+u_2)}\, d\lambda \tag{36}$$

$$I_2(\alpha;D) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{e^{-2u_1D}}{u_1(u_2+n^2u_1)}\, d\lambda \tag{37}$$

Now, all of the integrals resulting from substitution (35) may be expressed in terms of these two fundamental integrals and their derivatives with respect to D.

The expressions with which we are concerned also contain $H_o^{(1)}(\zeta k_1\rho')$ which we desire to write in terms of ρ and ϕ. This is accomplished by the addition theorem,

$$H_n^{(1)}(\zeta k_1\rho')e^{in\phi'} = \sum_{m=-\infty}^{\infty} H_m^{(1)}(2\zeta D)J_{m+n}(\zeta k_1\rho)e^{i(m+n)\phi} \tag{38}$$

which implies that,

$$H_o^{(1)}(\zeta k_1\rho') = H_o^{(1)}(2\zeta D)J_o(\zeta k_1\rho) + 2\sum_{m=1}^{\infty} H_m^{(1)}(2\zeta D)J_m(\zeta k_1\rho)\cos m\phi \tag{39}$$

$$H_1^{(1)}(\zeta k_1\rho')\cos\phi' = -H_o^{(1)'}(2\zeta D)J_o(\zeta k_1\rho) - 2\sum_{m=1}^{\infty} H_m^{(1)'}(2\zeta D)J_m(\zeta k_1\rho)\cos m\phi \tag{40}$$

$$H_1^{(1)}(\zeta k_1\rho')\sin\phi' = -\frac{1}{\zeta D}\sum_{m=1}^{\infty} m H_m^{(1)}(2\zeta D)J_m(\zeta k_1\rho)\sin m\phi \tag{41}$$

Substitution of (35), (39), (40), and (41), and $\rho = a$, into (12), (13), (14), (18), (27), and (29) yields the following expressions for the matrix elements.

$$E^e_{zoo} = \zeta^2 H_o^{(1)}(\zeta k_1 a) - [\zeta^2 H_o^{(1)}(2\zeta D) + I_3']J_o(\zeta k_1 a) \qquad (42)$$

$$E^e_{z01} = -[2\zeta^2 H_1^{(1)}(2\zeta D) + \frac{1}{i\zeta} I_3'']J_1(\zeta k_1 a) \qquad (43)$$

$$E^e_{\phi o1} = \frac{2i\alpha}{k_1 a}[H_1^{(1)}(2\zeta D) + \frac{1}{2i\zeta^3} I_3'']J_1(\zeta k_1 a) + \frac{\alpha}{\zeta^2}[I_4'' + 4\zeta^2 I_4]J_1'(\zeta k_1 a) \qquad (44)$$

$$E^e_{z10} = -[\zeta^2 H_o^{(1)'}(2\zeta D) + \frac{1}{2\zeta} I_3'']J_o(\zeta k_1 a) \qquad (45)$$

$$E^e_{z11} = \zeta^2 H_1^{(1)}(\zeta k_1 a) - 2\zeta^2[H_1^{(1)'}(2\zeta D) + \frac{1}{4i\zeta^4} I_3''']J_1(\zeta k_1 a) \qquad (46)$$

$$E^e_{\phi 11} = -\frac{i\alpha}{k_1 a} H_1^{(1)}(\zeta k_1 a) + \frac{2i\alpha}{k_1 a}[H_1^{(1)'}(2\zeta D) + \frac{1}{4i\zeta^4} I_3''']J_1(\zeta k_1 a)$$

$$+ \frac{\alpha}{2\zeta^3}[I_4''' + 4\zeta^2 I_4']J_1'(\zeta k_1 a) \qquad (47)$$

$$E^m_{z10} = \frac{\alpha}{2\zeta}[I_4'' + 4\zeta^2 I_4]\, J_o(\zeta k_1 a) \qquad (48)$$

$$E^m_{z11} = -\frac{i\alpha}{2\zeta^2}[I_4''' + 4\zeta^2 I_4']J_1(\zeta k_1 a) \qquad (49)$$

$$E^m_{\phi 11} = -i\zeta H_1^{(1)'}(\zeta k_1 a) + \frac{i}{D} H_1^{(1)}(2\zeta D)J_1'(\zeta k_1 a) - \frac{\alpha^2}{2\zeta^4 k_1 a}[I_4''' + 4\zeta^2 I_4']J_1(\zeta k_1 a)$$

$$+ \left[\frac{n^2 - \alpha^2}{2\zeta^3}(I_4''' + 4\zeta^2 I_4') - \frac{n^2}{2\zeta^3}(I_3''' + 4\zeta^2 I_3')\right] J_1'(\zeta k_1 a) \qquad (50)$$

where $I_3 = I_1 - \alpha^2 I_2$, $I_4 = I_1 - I_2$, and primes on the I symbols denote differentiation with respect to D. It is now easy to see that as $k_1 a$ approaches zero, E^e_{zoo}, E^e_{z11}, $E^e_{\phi 11}$, and $E^m_{\phi 11}$ predominate and equation (33), the modal equation, becomes

$$E^e_{zoo}\, E^e_{z11}\, E^m_{\phi 11} = 0 \qquad (51)$$

which is certainly satisfied when $E^{e}_{zoo} = 0$; as required by Wait[1] and by Chang and Olsen[2]. It remains to obtain approximate expressions for the fundamental integrals, I_1 and I_2, in order to arrive at a criterion for validity of the thin wire approximation. We further note that all the I_j's and their derivatives vanish when the earth becomes perfectly conducting. It is then not difficult to show that the only acceptable solution is $\zeta^2 = 0$ or $\alpha = \pm 1$. The result then reduces to the well-known result of a two-wire transmission line.

Proximity effect for a small height

We now proceed to discuss the proximity effect as the thin-wire approaches the earth surface. In particular, we are interested in the case when the height of the wire is much less than both the free-space wavelength and the skin-depth of earth so that the fundamental integrals (36) and (37), and their higher derivatives can be evaluated in a manner similar to that used by Chang and Wait[5]. Provided that $|\zeta| D \ll 1$ and $|n| D \ll 1$, the following approximate expressions for $I_1'(\alpha;D)$ and $I_2'(\alpha;D)$ hold:

$$I_1'(\alpha;D) \cong -H_o^{(1)}(2\zeta D) - \frac{i2}{\pi}\hat{I}'_{10}, \tag{52}$$

$$I_2'(\alpha;D) \cong -\left(\frac{2}{1+n^2}\right) H_o^{(1)}(2\zeta D) + \frac{i2}{\pi}\hat{I}'_{20} \tag{53}$$

where $\hat{I}'_{10}$ and $\hat{I}'_{20}$ are independent of D and given by

$$I'_{10}(\alpha) = \frac{\zeta_n^2}{n^2-1}\,\ell n\,\frac{\zeta}{\zeta_n} + \frac{1}{2} \tag{54}$$

$$I'_{20}(\alpha) = -\left(\frac{2}{1+n^2}\right)\left\{\ell n\,\frac{\zeta}{\zeta_n} + \frac{in^2}{\lambda_p(1+n^2)^{\frac{1}{2}}}\left[\ell n\,\frac{n^2-i\lambda_p(1+n^2)^{\frac{1}{2}}}{1-i\lambda_p(1+n^2)^{\frac{1}{2}}} + \ell n\,\frac{\zeta}{\zeta_n}\right]\right\} \tag{55}$$

and $\lambda_p = (\alpha^2-\alpha_p^2)^{\frac{1}{2}}$ and $\alpha_p = n(n^2+1)^{-\frac{1}{2}}$.

Successive differentiation with respect to D then gives the higher derivatives of $I_1^{(j)}$ and $I_2^{(j)}$; j = 2,3. On the other hand, the integration on D on both sides of (52) and (53) yields the result of

$$I_1(\alpha;D) = I_{10}(\alpha) - D\{H_o^{(1)}(2\zeta D) - \frac{i2}{\pi}[1+I'_{10}(\alpha)]\} \tag{56}$$

$$I_2(\alpha;D) = I_{20}(\alpha) - D\{(\frac{2}{1+n^2})[H_o^{(1)}(2\zeta D) - \frac{i2}{\pi}] - \frac{i2}{\pi} I'_{20}\} \tag{57}$$

where $I_{10}(\alpha)$ and $I_{20}(\alpha)$ are the values of I_1 and I_2 evaluated at D = 0. Thus, if now we substitute (52) - (57) into (42) - (50), while retaining only the leading terms in the small-argument expansion in D and A, the following approximate expressions for all the relevant field components are obtained:

$$E^e_{z00} \simeq \frac{i2}{\pi}[\zeta^2 \ln \frac{A}{2D} + \frac{(\zeta^2+\zeta_n^2)}{n^2+1}(\ln\zeta D + \gamma - \frac{i\pi}{2}) - I'_{30}]; \tag{58}$$

where $\gamma = 0.577216 \cdots$ is the Euler's constant, $A = k_1 a$ and

$I'_{30} = I'_{10} - \alpha^2 I'_{20}$,

$$E^e_{z01} \simeq -\frac{i2}{\pi}(\frac{A}{2D})(\zeta^2 - i\frac{\zeta^2+\zeta_n^2}{n^2+1}) \tag{59}$$

$$E^e_{\phi 01} \simeq -\frac{i2}{\pi}(\frac{i\alpha}{2\zeta^2 D})\left[\zeta^2 - i(\frac{2\zeta_n^2}{n^2+1})\right] \tag{60}$$

$$E^e_{z10} \simeq \frac{i2\alpha^2}{\pi}(\frac{n^2-1}{n^2+1})(\frac{1}{2\zeta D}) , \tag{61}$$

$$E^e_{z11} \simeq -\frac{i2}{\pi}(\frac{1}{\zeta A})\left[\zeta^2 + (\frac{A}{2D})^2(\zeta^2 - i\frac{\zeta^2+\zeta_n^2}{n^2+1})\right], \tag{62}$$

$$E^e_{\phi 11} \simeq -\frac{2\alpha}{\pi A^2\zeta^3}\{\zeta^2 + (\frac{A}{2D})^2\left[\zeta^2 - i(\frac{2\zeta_n^2}{n^2+1})\right]\} \tag{63}$$

$$E^m_{z10} \simeq -\frac{i2\alpha}{\pi}\left(\frac{n^2-1}{n^2+1}\right)\left(\frac{1}{2\zeta D}\right) , \tag{64}$$

$$E^m_{z11} \simeq \frac{2\alpha}{\pi}\left(\frac{n^2-1}{n^2+1}\right)\left(\frac{A}{2D}\right)^2\left(\frac{1}{\zeta A}\right) , \tag{65}$$

$$E^m_{\phi 11} \simeq \frac{2}{\pi A^2 \zeta^3}\left\{\zeta^2 + \left(\frac{A}{2D}\right)^2\left[\zeta^2 - i\left(\frac{2\zeta_n^2}{n^2+1}\right)\right]\right\} \tag{66}$$

With these approximate expressions, the original model equation as given by the determinant in (33) now takes up a much simplified form of

$$E^e_{z00} + \frac{i2}{\pi}\delta(\alpha) = 0 , \tag{67}$$

or

$$-\zeta^2 \ln \frac{2D}{A} + \frac{(\zeta^2+\zeta_n^2)}{n^2+1}(\ln \zeta D + \gamma - i\,\pi/2) - I'_{30} + \delta(\alpha) = 0 , \tag{68}$$

where

$$\delta(\alpha) = \alpha^2\left(\frac{A}{2D}\right)^2 \left(\frac{n^2-1}{n^2+1}\right)\left[\left(\frac{A}{2D}\right)^2 + \frac{\zeta^2}{\zeta^2 - i2\zeta_n^2/(n^2+1)}\right]^{-1} \tag{69}$$

and the expression for $I'_{30} = I'_{10} - \alpha^2 I'_{20}$ is given by (54) and (55). Except for the term $\delta(\alpha)$, (67) and (68) now correspond exactly to the approximate modal equation derived by Chang and Wait[5] using the so-called "thin-wire" approximation in which the longitudinal current is assumed to be angularly uniform and the azimuthal current neglected. Denoting α_o as the solution of $E^e_{z00}(\alpha_o) = 0$, one then concludes that the condition $|\delta(\alpha_o)| << |\alpha_o \partial_\alpha E^e_{z00}(\alpha_o)|$ must be imposed. We note that a suitable solution of $E^e_{z00}(\alpha_o) = 0$ is known in Chang and Wait[5] to be

$$\zeta_o^2 = (1-\alpha_o^2) \sim |\ln nD|\ [\ln 2D/A]^{-1}$$

provided that the condition $|n^2| \gg \zeta_o^2 \gg |n^2|^{-1}$ holds. Thus, $\delta(\alpha_o)$ is generally of the order of $(A/2D)^2 \ell n(2D/A)$. We further note that the magnitude of first order electric and magnetic multipoles are related to the zero-order electric multipole by (30)-(32). Using the simplified expressions in (58) - (66) and also neglecting terms of order $(A/2D)^2$, it is not difficult to show that

$$\frac{A_1}{A_o} \cong \left(\frac{A^2}{2\zeta D}\right)\left(\zeta^2 - i\,\frac{\zeta^2+\zeta_n^2}{n^2+1}\right), \tag{70}$$

and

$$\frac{M_1}{A_o} \simeq \left(\frac{A^2\alpha}{2\zeta D}\right)\left(2\zeta^2 - i\,\frac{\zeta^2+\zeta_n^2}{n^2+1}\right). \tag{71}$$

To determine the non-uniformity of the surface current, one merely needs to compute the dominant contribution to the current density at the wire surface. Now since the uniform current density J_{szo} is proportional to $A_o H_o'(\zeta A)$ and the cosinusoidal current J_{sz1} is proportional to $A_1 H_1'(\zeta A)$, we obtain

$$\left|\frac{J_{sz1}}{J_{szo}}\right| \propto \left(\frac{A}{2D}\right) \ell n \left(\frac{2D}{A}\right) \tag{72}$$

which means the perturbation of the current is usually larger than that of the propagation constant.

Equation (69) and the condition $|\delta(\alpha_o)| \ll |\alpha_o \partial_\alpha E^e_{z00}(\alpha_o)|$ give the explicit dependence of thin-wire approximation on the radius-to-height ratio. In that derivation however, we have not taken into the account the fact that each term in the small (ζD) expansion individually can be very large because of the different functional behavior in α. In particular, it has been shown that, as a result of the residue contribution from a pair of poles in the integral given in (37), both I_3 and I_4 possess an inverse square-root singularity with the branch points located at $\alpha_p = \pm n/(n^2+1)^{\frac{1}{2}}$ in the complex α-plane. These branch cuts are associated with that part of the continuous spectrum which radiates along the earth surface. Thus, in the neighborhood of $\alpha \approx \alpha_p$, we have from (37) and the definition of I_3 and I_4

$$I_3 \cong \alpha^2 I_4; \quad I_4 = \frac{2n^2}{n^4-1} \quad (\alpha_p^2 - \alpha^2)^{-\frac{1}{2}} \, e^{-i2D(n^2-1)^{-\frac{1}{2}}} \tag{73}$$

with the higher-order derivatives $I_3^{(m)}$ and $I_4^{(m)}$ for $m > 0$ given as

$$I_{3,4}^{(m)} = \left[\frac{-2i}{(n^2-1)^{\frac{1}{2}}}\right]^m I_{3,4} \tag{74}$$

Provided that $(A/2D)$ is small, we can again find the approximate field expression from (42)-(50). A similar analysis then leads to the conclusion that $|\delta(\alpha)|$ is negligible whenever $|A^2/n^3| << |(\alpha^2-\alpha_p^2)^{\frac{1}{2}}|$. Thus except for a small region near $\alpha = 1$, specified by $|(\alpha^2-1)| << (A/2D)^2$ according to (69), and an even smaller region near $\alpha = \alpha_p$, the thin-wire approximation is indeed well justified.

Corresponding results for a large height

In the case when the height is greater than several skin-depths of earth (i.e., $|n|^2 D^2 >> 1$), the fundamental integrals (36) and (37) can be approximated asymptotically in a manner similar to that used by Chang and Olsen. (In fact their $W(\alpha)$ is in their approximation just equal to our

$i\pi n^2 I_2$.) Beginning with I_1 we write

$$I_1 \approx \frac{1}{i\pi(n^2-1)} \int_{-\infty}^{\infty} \frac{u_1+in}{u_1} e^{-2u_1 D} d\lambda$$

$$= \frac{-1}{i\pi(n^2-1)} \left\{ \left(\frac{\partial}{\partial w} - in\right) \int_{-\infty}^{\infty} \frac{e^{-wu_1}}{u_1} d\lambda \right\}_{w=2D}$$

$$= \frac{1}{n^2-1} [\zeta H_1^{(1)}(2\zeta D) + inH_o^{(1)}(2\zeta D)] \tag{75}$$

where u_2 has been approximated by $-in$ because most of the contribution to the integral occurs where

$$u_2 = [\lambda^2 + \alpha^2 - n^2]^{\frac{1}{2}} \approx -in[1 - \frac{\lambda^2+\alpha^2}{2n^2}] \tag{76}$$

and $\left|\frac{\lambda^2+\alpha^2}{2n^2}\right| << 1$. As indicated by Chang and Olsen, a better approximation is $u_2 \approx \frac{-in^2}{(n^2+1)^{\frac{1}{2}}}$.

An approximation for I_2 can be obtained directly from Chang and Olsen as mentioned above. However, we present here the derivation of a more approximate but simpler expression. First, we cast I_2 in the form,

$$I_2 = \frac{1}{n^2\zeta} \int_{2\zeta D}^{\infty} H_o^{(1)}(x) e^{i\frac{x}{n\zeta}} e^{-2i D/n} dx \tag{77}$$

An approximation for large $2\zeta D$ may now be obtained using the large argument asymptotic expression for $H_o^{(1)}(x)$. That is,

$$I_2 \approx \frac{1}{n^2\zeta}\sqrt{\frac{2}{\pi}} \int_{2\zeta D}^{\infty} \frac{1}{\sqrt{x}}\, e^{i(1+\frac{1}{n\zeta})x} \quad e^{-2iD/n}\, dx$$

$$= \frac{\sqrt{2}}{n^2\zeta\sqrt{1+\frac{1}{n\zeta}}} \quad \operatorname{erfc}\left[\left(\frac{1-i}{\sqrt{2}}\right)\sqrt{2D\zeta\left(1+\frac{1}{n\zeta}\right)}\,\right] e^{-2i\,D/n} \tag{78}$$

Now, using the large argument asymptotic form for the complementary error function we obtain:

$$I_2 \approx \sqrt{\frac{i}{\pi}} \; \frac{1}{(\zeta n)^2\left(1+\frac{1}{n\zeta}\right)} \; \frac{e^{2i\zeta D}}{\sqrt{\zeta D}} \tag{79}$$

To the same order in $1/\zeta D$ we have

$$I_1 \approx \sqrt{\frac{i}{\pi}}\left(\frac{n-\zeta}{n^2-1}\right)\frac{e^{2i\zeta D}}{\sqrt{\zeta D}} \tag{80}$$

To this order of approximation $\frac{\partial}{\partial D} \rightarrow 2i\zeta$ so that the derivatives of the I's in the matrix elements may all be replaced by factors of $2i\zeta$. Use of (79) and (80) will result in a lowest order approximation for large ζD. If ζD is not very large one must obtain more accurate expressions for I_1 and I_2 as was done by Chang and Olsen. For purposes of this treatment, however, it will be assumed that ζD is sufficiently large that our asymptotic forms are appropriate.

The approximations obtained above will now be substituted in (42)-(50) and the amplitudes A_1 and M_1 will be found in terms of A_o and parameters. For notational convenience we define,

$$G = \frac{e^{2i\zeta D}}{\sqrt{i\pi\zeta D}} \tag{81}$$

Thus,

$$I_1 \approx -i\left(\frac{n-\zeta}{n^2-1}\right) G \tag{82}$$

$$I_2 \approx -i\left(\frac{1/(n\zeta)^2}{1+1/(n\zeta)}\right) G \tag{83}$$

and

$$I_3 \approx -i\left[\frac{n-\zeta}{n^2-1} - \frac{(\alpha/n\zeta)^2}{1+1/(n\zeta)}\right] G = q_3 G \tag{84}$$

$$I_4 \approx -i\left[\frac{n-\zeta}{n^2-1} - \frac{(1/n\zeta)^2}{1+1/(n\zeta)}\right] G = q_4 G \tag{85}$$

Retaining only constant and linear terms in the series for the Bessel functions $J_n(\zeta k_1 \rho)$ and evaluating at $\rho = a$; i.e., $k_1\rho = A$, we have for large ζD and small ζA,

$$E^e_{z00} \approx \zeta^2 H_o^{(1)}(\zeta A) - (\zeta^2+2i\zeta q_3)G \tag{86}$$

$$E^e_{z01} \approx i\zeta A(\zeta^2-2\zeta q_3)G \tag{87}$$

$$E^e_{\phi 01} \approx \alpha(\zeta - 2q_3)G \tag{88}$$

$$E^e_{z10} \approx -i\zeta(\zeta + 2iq_3)G \tag{89}$$

$$E^e_{z11} \approx \zeta^2 H_1^{(1)}(\zeta A) - \zeta^2(\zeta-2q_3)AG \tag{90}$$

$$E^{e}_{\phi 11} \approx -\frac{i\alpha}{A} H_1^{(1)}(\zeta A) + i\alpha(\zeta - 2q_3)G \tag{91}$$

$$E^{m}_{z10} \approx 0 \tag{92}$$

$$E^{m}_{z11} \approx 0 \tag{93}$$

$$E^{m}_{\phi 11} \approx -i\zeta H_1^{(1)}(\zeta A) \tag{94}$$

Using these approximations to the matrix elements in (33), one then obtains $E^{e}_{z00} + \delta(\alpha) = 0$, where

$$\delta(\alpha) = -\left(\frac{A}{2D}\right) (\zeta A)(\zeta + i2q_3)(\zeta - 2q_3)\, e^{i4\zeta D} \tag{95}$$

The condition $|\delta(\alpha)|$ is small requires a small radius-to-height ratio in addition to a small $|\zeta A|$. Letting ζ_o be such that $E^{e}_{z00}(\zeta_o) = 0$, and defining $\Delta\zeta$ to be $\zeta_1 - \zeta_o$ where ζ_1 is the value of ζ which satisfies (33) in this approximation we find in agreement with the general theory of small perturbations that

$$\left|\frac{\Delta\zeta}{\zeta_o}\right| \approx \frac{\pi A^2}{2\zeta_o D \, \ln\left(\frac{\zeta A}{2}\right)} \; \frac{(\alpha_o/n\,\zeta_o)^4}{(1 + 1/n\,\zeta_o)^2} \tag{96}$$

and,

$$|A_1| \approx \frac{\sqrt{\pi}}{\zeta_o} \frac{(\zeta_o A)^2}{\sqrt{\zeta_o D}} \; \frac{(\alpha_o/n\zeta_o)^2}{(1 + 1/n\,\zeta_o)} \; |A_o| \; ; \; |M_1| \approx 0 \tag{97}$$

so that the ratio of the first-order current component to the uniform current is given by

$$\left|\frac{J_{sz1}}{J_{sz0}}\right| \simeq \pi^{\frac{1}{2}} \frac{A}{(\zeta_o D)^{\frac{1}{2}}} \; \frac{(\alpha_o/n\,\zeta_o)^2}{1 + 1/(n\zeta_o)} \tag{98}$$

Now, since $\zeta_o D$ is the large parameter, one might say the currents are perturbed but the propagation constant is not. The conclusion we derived for a large height is therefore consistent with the case of a small height.

Conclusion

The work presented was primarily concerned with obtaining a measure of the significance of the first-order components of the currents on the wire compared with the zero-order on the uniform current. Using a multipole expansion method, we derived a formal expression for the modal equation in the form of a determinant, i.e. (33). Validity of the so-called "thin-wire" approximation was discussed for two cases; one corresponds to the case of a vanishing height, $|n| D << 1$ and the other for height greater than several wavelengths of waves in free-space. In both cases the amplitude of the first order current becomes more significant as the height decreases before the propagation constant is significantly modified.

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