A Note on the Radiation Principle*

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In the formulation of the diffraction problem of sinusoidal waves for the equation

$$\Delta u + k^2 u = 0 \tag{1}$$

(where k is real and greater than zero) the radiation condition at infinity is applied, requiring that the field at infinity has the character of an outgoing wave. If, for example, the time dependence is determined by the factor $e^{i\omega t}$, then according to this condition the fundamental solution multiplied by $e^{i\omega t}$ must have the form

$$\frac{e^{i(\omega t - kr)}}{r} \tag{2}$$

The object of this note is to show that this requirement is not universal, and that for certain media (in accordance with physical principles) it must be replaced by the requirement that the field at infinity has the character of an incoming wave.[1] In such cases, the fundamental solution must, on the contrary, have the form

$$\frac{e^{i(\omega t + kr)}}{r} \tag{3}$$

A solution of the form (2) is valid for the case when (1) is obtained from the wave equation

$$\Delta v = \frac{1}{c^2}\ddot{v} \tag{4}$$

as a result of the substitution $v(x, y, z, t) = u(x, y, z, t)e^{i\omega t}$. Then $k^2 = (\omega/c)^2$.

However, equation (1) can also be obtained, for example, from the equation

$$\Delta \ddot{v} = \frac{1}{a^2} v \tag{5}$$

with the help of the same substitution. In this case,

$$k^2 = \frac{1}{(\omega a)^2}$$

In order to express the physical sense of equation (5), we note that in the one-dimensional case a discrete model for it is the network of capacitances and self-inductances shown in Fig. 1, while for equation (4) the model is a circuit of a different kind (Fig. 2).

*This is an English translation by E. F. Kuester of a Russian paper which originally appeared in *Zhurnal Tekhnicheskoi Fiziki*, vol. 21, pp. 940-942, 1951. A number of typographical errors in the original have been corrected, and annotations made, by the translator. These are indicated by footnotes in square brackets.

[1] [From the context, it is clear that the author means that the phase of the field is incoming.]

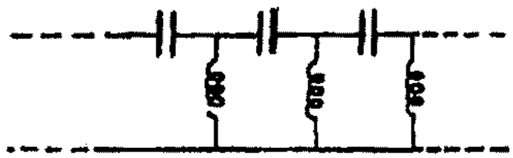

Figure 1

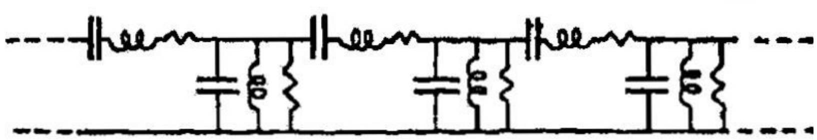

Figure 2

It is known that the correct solution for the oscillations established by a source in an infinite region can be obtained as the limit

$$e^{i\omega t}u(x,y,z) = \lim_{t_0 \to -\infty} \ddot{v}(x,y,z,t;t_0)$$

where $\ddot{v}(x,y,z,t;t_0)$ is the solution to the wave equation in the case when for $t < t_0$ the entire region is at rest, and at the instant $t = t_0$ a radiator begins to produce oscillations with time-dependence $e^{i\omega t}$.

To obtain the result we are interested in, we can restrict our attention to the one-dimensional case. Consider the solutions of equations (4) and (5) in the region $x > 0$ and for $t > t_0$ subject to the conditions: at $t = t_0$, $v = \dot{v} = 0$; and for $t > t_0$ and $x = 0$, $v = e^{i\omega t}$.

The unique solutions of these problems are as follows: for equation (4)

$$v(x,t;t_0) = \begin{cases} e^{i(\omega t - \frac{\omega}{c}x)} & \text{for} \quad x < c(t-t_0) \\ 0 & \text{for} \quad x > c(t-t_0) \end{cases}$$

and for equation (5)[2]

$$v(x,t;t_0) = e^{i(\omega t + \frac{x}{a\omega})} - e^{i\omega t}\int_{\infty}^{2\sqrt{\frac{x(t-t_0)}{a}}} e^{-\frac{i\omega a\xi^2}{4x}} J_1(\xi)\, d\xi$$

For $t_0 \to -\infty$ we obtain the solution of equation (1). In the first case

$$e^{i\omega t}u(x) = e^{i(\omega t - kx)} \qquad \left(k = \frac{\omega}{c}\right)$$

[2][Corrected by the translator. This solution holds for $t > t_0$, while v must equal zero for $t < t_0$. Because of the step function behavior of the source at $t = t_0$, it seems possible that extra terms containing step and/or delta functions might be needed here. A solution of equation (5) with a slightly different switched-on source had previously been given by H. C. Pocklington, "Growth of a wave-group when the group-velocity is negative," *Nature*, vol. 71, pp. 607-608, 1905.]

and in the second case (as a result of the vanishing of the integral as $t_0 \to -\infty$[3])

$$e^{i\omega t}u(x) = e^{i(\omega t + kx)} \qquad \left(k = \frac{1}{a\omega}\right)$$

We see that in the second case, in contrast with the first, the wave propagates not from the radiator (located at $x = 0$) to infinity, but proceeds from infinity towards the radiator. This result is also valid for the two- and three-dimensional versions of equation (5). It can be said that the medium has a positive phase velocity in the first case, but a negative one in the second.

In the formulation of the radiation condition for the problem of diffraction of a sinusoidal wave it is usually assumed [sometimes tacitly, if no reference is made to equation (4)] that the phase velocity is positive. In fact however, as shown by the example considered here, the opposite may be true. But at the same time, the energy propagated is always positive, i. e., it moves in the direction away from the source.

Mandel'shtam [1] indicated a class of real media in which the group and phase velocities could have opposite signs in certain frequency ranges. In these cases the phase velocity is negative in the sense we have indicated above.

References

[1] L. I. Mandel'shtam, "Group velocity in a crystal lattice [Russian]," *Zhurn. Eksp. Teor. Fiz.*, vol. 15, pp. 475-478, 1945.

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[3] [Corrected by the translator.]