Lectures on Certain Problems in the Theory of Oscillations*

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Abstract

[By the translator] This lecture covers the problems of energy velocity and its relation to group velocity. The effect of negative group velocity on the familiar problems of reflection and refraction is considered. This was Prof. Mandel'shtam's last lecture; he died on November 27, 1944.

Lecture 4

Last time, we explained in detail the question of the concept of velocity in a dispersive medium. We saw that in the presence of dispersion this concept becomes ambiguous. Besides the phase velocity $v = \omega/k$, at which a harmonic wave of frequency ω is propagated, we can introduce the concept of the group velocity $u = d\omega/dk$, connected with the presence of a train and in general a slowly varying amplitude. The applicability of this concept of group velocity is subject to a number of conditions, and in particular it is not suitable when the medium is strongly absorptive. However, if the absorption is sufficiently small, it can still be used. In addition to u and v, we have also talked about the velocity of a wavefront and the signal velocity.

The introduction of these concepts supposes that the superposition principle is valid, and that a harmonic wave is propagated without undergoing any change. The former means that the equation governing the propagation of the wave is linear; the latter that this equation is homogeneous. Of course, if k is given for all ω, i. e., we have given to us the equation for the dispersion law, then the group velocity can be calculated immediately. However, this does not mean that the result of such a calculation has a definite meaning for all cases without exception. The conditions under which u really characterizes the propagation were explained in detail last time.

*This is an English translation by E. F. Kuester of one of Mandel'shtam's lectures which originally appeared in *Polnoe Sobraniye Trudov*, tom 5 (Leningrad: Izdat. Akademiya Nauk SSSR, 1950, pp. 461-467); also in his *Lektsii po Optike, Teorii Otnositel'nosti i Kvantovoi Mekhanike* (Moscow: Nauka, 1972, pp. 431-437).

We now consider one more important question: how is *energy* propagated in the medium? Suppose a sinusoidal wave extending from $-\infty$ to ∞ is propagating in the medium (Fig. 11).

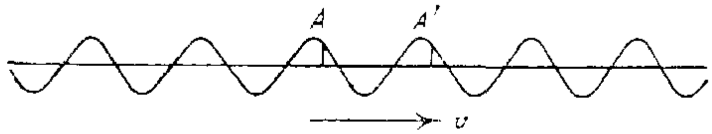

Fig. 11

Its phase A is propagated with the phase velocity. The energy is proportional to the square of the amplitude, but since the amplitude (the maximum displacement) moves at the phase velocity, one would think that the energy would also be propagated at this same velocity. But such a definition of the energy velocity is vacuous. It does not satisfy a fundamental requirement connected with energy propagation. In a sinusoidal wave there is no *motion* of energy; it is not transferred from point A to point A'. A sinusoidal infinite wave can be represented by a model consisting of an array of identical but uncoupled pendulums (Reynolds[1]). In this chain of pendulums one can form such a sequence of phases that the form of the oscillations will exactly correspond to a travelling sinusoidal wave, however no transfer of energy takes place here at all. Rayleigh noted in connection with this model that "to call the Reynolds pendulums a medium is to pay them a compliment."[2] Therefore, in identifying the energy velocity with the phase velocity, we fail to obtain the main thing—the flow of energy from one part of space to another. In an arbitrary volume through which a sinusoidal wave passes, the energy will remain constant at all times.

So then, what definition of energy transport velocity could be considered more suitable? We split the wave by a surface perpendicular to the direction of propagation. If the average amount of energy per unit time in a unit area on the right side of this surface increases by $\bar{S}$, while on the left side it decreases by the same amount (or conversely), while the mean energy density near the area under consideration is $\bar{E}$, then it is natural to consider

$$U = \frac{\bar{S}}{\bar{E}}$$

as the energy transport velocity. This definition has a simple hydrodynamic analogy: the quantity of fluid flowing through a unit area per second (oriented normally, i. e., such that the amount of fluid flow is maximized), divided by the fluid density gives the velocity of the flow. But there is a fundamental difference here. In hydrodynamics we can *localize* a particle of the fluid (we can mark it) and therefore we can speak of the velocity of the flow even in the case when this

[1] [Translator's note: O. Reynolds, "On the rate of progression of groups of waves and the rate at which energy is transmitted by waves," *Nature*, vol. 16, pp. 343-344 (1877).]

[2] [Translator's note: The exact quote is "it is only by compliment that it [Reynolds' system of pendulums] is regarded as a single system.": Lord Rayleigh, "On iso-periodic systems," *Phil. Mag.*, vol. 46, pp. 567-569 (1898).]

flow is stationary and homogeneous over all space. In the electromagnetic field, there is no such localization. Thus in the case of an unlimited sinusoidal wave our definition loses meaning: here it is not proper to speak of such an increase or decrease of energy on both sides of the surface. The change in the amount of electromagnetic energy in an arbitrary volume of the medium can be stated only when we have a limited "sinusoid", i. e., a *train* or *group* of waves. Then the velocity U is valid, and will characterize how fast the energy flows through the surface.

Can we then find a connection between the energy transport velocity U and the group velocity u?

M. A. Leontovich has proved an extremely general theorem, valid for arbitrary waves. If a Lagrangian function, dependent on quantities characterizing the state of the medium (strength of the electromagnetic field, displacement of a particle of the fluid) is a quadratic form of these quantities and of their first derivatives with respect to spatial coordinates and time, then in those cases where the group velocity has meaning at all, it coincides with the energy transport velocity U, i. e., we then have

$$\bar{S} = \bar{E}u$$

The very general assumptions about the existence and form of the Lagrangian function that were made in order to prove this theorem nonetheless impose known limitations on the properties of the medium: there can be no absorption or such phenomena as rotation of the plane of polarization. On the other hand, for the concept of group velocity to make sense, the spectrum of the wave group must, as we have seen, be narrow enough for the given dispersion.

Let all these conditions be satisfied, and consequently the energy be transported at the group velocity. But we know that the group velocity can be negative. This means that the group (and the energy) are moving in a direction opposite to that of the phase of the wave. Can such a thing happen in real life?

In 1904, Lamb[3] proposed a certain artificial mechanical model of a one-dimensional "medium", in which the group velocity can be negative. Apparently, he himself did not think his example could have any physical application. But as it turns out, there exist entirely real media in which for certain frequency ranges the phase and group velocities are in fact directed towards each other. This happens in the so-called "optical" branch of the acoustic spectrum of a crystal lattice, considered by M. Born.[4] The possibility of similar phenomena permits such seemingly well-known things as reflection and refraction of plane waves at a plane interface between two nonabsorbing media to be approached from a rather different viewpoint. The consequences of these phenomena, in which the choice of group velocity is usually not mentioned, depend critically upon its sign.

[3] [Translator's note: H. Lamb, "On group-velocity," *Proc. London Math. Soc.*, ser. 2, vol. 1, pp. 473-479, 1904.]

[4] [See L. I. Mandel'shtam, *Polnoe Sobraniye Trudov*, tom 2 (Leningrad: Izdat. Akademiya Nauk SSSR, 1947, paper 52] [Translator's Note: L. I. Mandel'shtam, "Group velocity in a crystal lattice," [Russian] *Zh. Eksp. Teor. Fiz.*, vol. 15, pp. 475-478, 1945.]

In fact, how does the idea of the derivation of the Fresnel formula usually proceed?

Consider a sinusoidal plane wave, incident at an angle φ to the plane interface $y = 0$:

$$E_{\mathrm{inc}} = e^{i[\omega t - k(x \sin\varphi + y \cos\varphi)]}$$

and the two waves which result from it—reflected:

$$E_{\mathrm{refl}} = e^{i[\omega t - k(x \sin\varphi' - y \cos\varphi')]}$$

and refracted:

$$E_{\mathrm{refr}} = e^{i[\omega t - k_1(x \sin\varphi_1 + y \cos\varphi_1)]}$$

At the plane $y = 0$ these waves must satisfy the so-called boundary conditions. For an elastic body these conditions are the continuity of stress and displacement on both sides of the boundary. In electromagnetic problems the tangential components of the intensities and the normal components of the inductions must be continuous at the plane interface. It is easy to show that these boundary conditions cannot be satisfied with only a reflected wave (or only a refracted wave). Conversely, in the presence of both waves the conditions can always be satisfied. By the way, however, it does not follow at all that there must be only *three* waves and no more: the boundary conditions allow for the presence of another, fourth, wave, travelling at an angle $\pi - \varphi$ in the second medium. Usually it is tacitly assumed that this wave is absent, i. e., it is postulated that there is only *one* wave propagating in the second medium.

From the boundary conditions there immediately follow the law of reflection

$$\sin\varphi = \sin\varphi' \quad \text{or} \quad \varphi = \varphi'$$

and the law of refraction

$$k \sin\varphi = k_1 \sin\varphi_1$$

However, the latter equation is satisfied both by the angle φ_1 and by the angle $\pi - \varphi_1$. The wave in the second medium corresponding to φ_1 propagates in a direction away from the interface (Fig. 12).

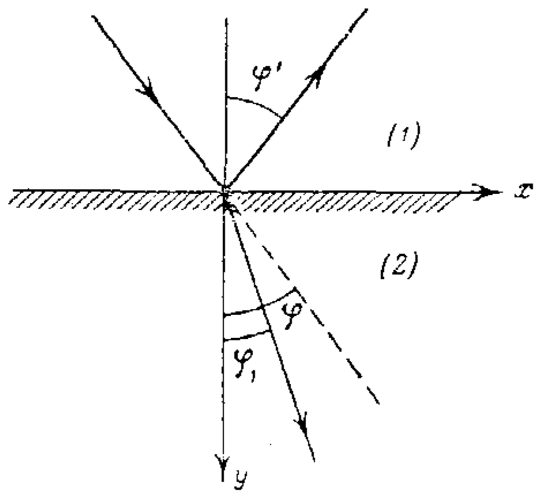

Fig. 12

But the wave corresponding to $\pi - \varphi_1$ propagates in a direction towards the interface (Fig. 13).

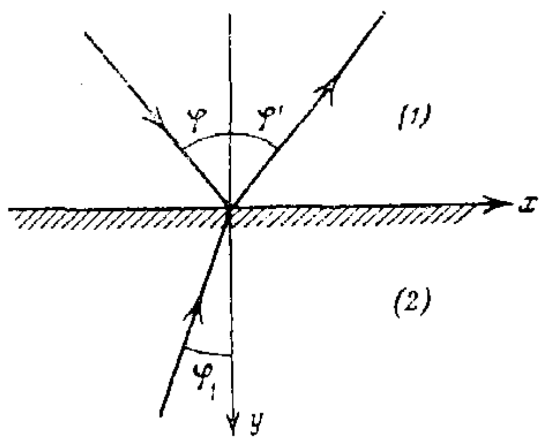

Fig. 13

It is taken to be self-evident that the second wave is not possible, since the light is incident from the first medium towards the second, but it means that in the second medium *energy* must flow away from the interface. But where is the energy going here? In fact, the direction of wave propagation is determined by its *phase* velocity, while energy is transported at the *group* velocity. Therefore we postulate here a logical jump, which feels that way only because we are used to the directions of phase and energy propagation being the same. If such is the case, i. e., if the group velocity is positive, then all these things are found to be true. If on the other hand we have the case of negative group velocity—a case as I already said which is completely real—then everything changes. Requiring as before that the energy in the second medium *flows away* from the interface, we then find that the phase must approach this interface, and consequently the direction of propagation of the refracted wave will make an angle of $\pi - \varphi_1$ with the normal. There is nothing out of the ordinary in such a construction, nor anything remarkable in it in the end, because again phase velocity says nothing about the direction of energy flow.

The problems which we have worked out are very general—they are problems of the *propagation* of oscillations. As I already emphasized, they relate to oscillations of quite diverse types. I should point out that in essence this is a *geometry* of oscillatory motion, not connected with this or that concrete physical object. But it is true that the propagation of energy already brings us outside this circle of ideas somewhat, since it is a question of dynamics.

We turn now to other, likewise very general questions of the geometry (or kinematics) of oscillations. The question before us is now not the propagation but the behavior, so to speak, of the wave process *at a given point*. Here also there arise analogous and very essential questions, for which besides touching on oscillations of very different origins, so that here again we can and must make an effort to select those features which are common to the widest possible class of phenomena. First of all, here again the *physics* requires us (and this is typical) to answer questions about *approximately sinusoidal* oscillations. To show what kinds of problems arise here, it is most convenient to consider some examples.

In any radio transmission, via telephone, telegraph, it is important to us that the transmission is represented not by a sinusoidal oscillation, but by a carrier *marked* by the transmitting process—a sound oscillation, dashes and dots, or an image-facsimile signal. Here it is typical that there is a *slow* perturbation of the true sinusoidal nature of the signal: to the high-frequency oscillation $y = A \cos \omega t$ there is introduced a variation of the amplitude (or phase) at the rate of our speech or telegraph signal, i. e., at frequencies considerably smaller than ω. Such oscillations are called *modulated*. Why is it appropriate to separate them?

First of all, it can be noted that they arise in all branches of physics and engineering where we deal with oscillations in general. Modulation of oscillations is used in musical instruments (vibrato, in particular), where we also encounter combinational dispersion, in ultrasound, in nonuniform sea swells, etc. Furthermore, the nearness of modulated oscillations to harmonic ones means that their theory is relatively simple, and a number of problems can be worked out in considerably more detail than they could if the oscillation had an arbitrary form. All these questions and a general exposition of the theory of modulated oscillations has been carried out in the monograph of S. M. Rytov.[5]

A special case of modulation is so-called *amplitude* modulation, in which the oscillation is endowed with a "varying amplitude"

$$y = B[1 + kf(t)] \cos \omega t$$

in which, as mentioned before, the function $f(t)$ varies quite slowly compared to the oscillation frequency ω. In radiotelephony the variation of $f(t)$ is carried out at audio frequencies while ω is a high radio frequency. If we take the special case $f(t) = \cos \Omega t$, then this means that $\Omega \ll \omega$. Such an oscillation can be written in either of two ways:

$$\begin{aligned} y &= B[1 + k \cos \Omega t] \cos \omega t \\ &= B \left[\cos \omega t + \frac{k}{2} \cos(\omega + \Omega)t + \frac{k}{2} \cos(\omega - \Omega)t \right] \end{aligned}$$

This is a mathematical identity, i. e., two perfectly equivalent representations for one and the same process. The question now concerns the *action* of such oscillations at the *receiver*, i. e., at a device acting as an ordinary *harmonic* analyzer. You see how essential a role is played in all statements of problems involving modulated oscillations by the properties of the receiving apparatus, and how ignoring this has led and still leads to various statements that are largely devoid of meaning. Thus, for example, Fleming asserted that "really" there are not three harmonic oscillations in the signal y, rather there is only *one* oscillation of frequency ω, *but* of variable amplitude. Hence he deduced that it was not necessary to extend the transmitted radio signal to a specific frequency interval. The question at hand, therefore, deals with extremely fundamental practical issues.[6]

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[5] [He had in mind the dissertation "Modulated oscillations and waves," *Trudy FIAN*, vol. 2, p. 41, 1938.]

[6] [Here ends the last lecture of L. I. Mandel'shtam. In his abstracts, he would lay out plans for the problems to be considered in future lectures. On one of these abstracts was written the following:

"We have considered nearly sinusoidal oscillations in the sense of their *propagation*. Now we will look at them in the sense of their *action at a given point*. What does 'action' mean? What are the problems here? — The action on a receiver, analyzer. Experiments: I) Tuning fork a acts on a resonant tuning fork b; a_1 does not act on b; $a + a_1$ acts on b just like a did, but *to the ear* it sounds completely different: beating. II) a_1 does not act on b, but when interrupted, does act. III) Frequency meter. What kind of receivers do we deal with in physical problems? These problems are on one hand deeply theoretical (white light), and on the other, quite practical (discussion of Fleming, frequency modulation)."]