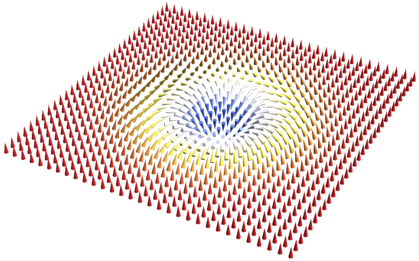
Summary of Lecture 2

- Chern Number $$C = \frac{1}{2\pi}\int_{\mathrm{BZ}} \mathrm{d}^2 q\, \Omega_q$$

  Berry curvature $$\Omega_q = -\mathrm{i}\nabla_q \times \langle u_q | \nabla_q u_q \rangle \cdot \hat{z}$$



2-band models

- Harper Model

- Haldane Model $\rightarrow$ Low energy theory

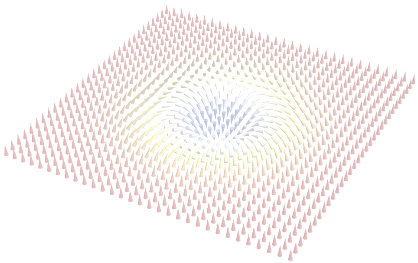
- Optical Flux Lattices $\Leftrightarrow$ Haldane Model in reciprocal space

  $$\hat{H} = \frac{\hat{p}^2}{2m}\hat{\mathbb{1}} + \hat{V}(\underline{r})$$

Lecture 3

- Chern Number $C = \frac{1}{2\pi}\int_{\mathrm{BZ}} \mathrm{d}^2 q \, \Omega_q$

  Berry curvature $\Omega_q = -\mathrm{i}\nabla_q \times \langle u_q | \nabla_q u_q \rangle \cdot \hat{z}$



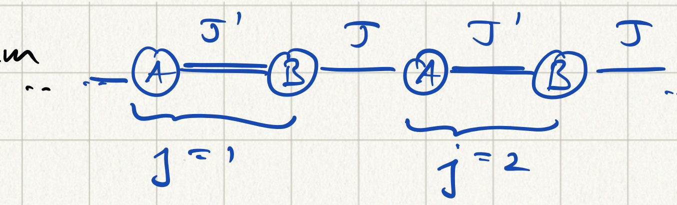
2-band models

- Physical consequences

- Symmetry Protected Topological Insulators and Superconductors & Beyond

- Far-from-equilibrium dynamics

SSH Model: From Lattice to Continuum



$$E\,\psi_j^A = -J'_j\,\psi_j^B - J\,\psi_{j-1}^B$$
$$E\,\psi_j^B = -J'_j\,\psi_j^A - J\,\psi_{j+1}^A$$

$$M_j = 1 - \frac{J'_j}{J} \simeq 0$$

gap closing at $qa = \pi$

$$\psi_j^A = (-1)^j\,\tilde\psi^A(x = ja) \longleftarrow$$
$$\psi_j^B = (-1)^j\,\tilde\psi^B(x = ja)$$

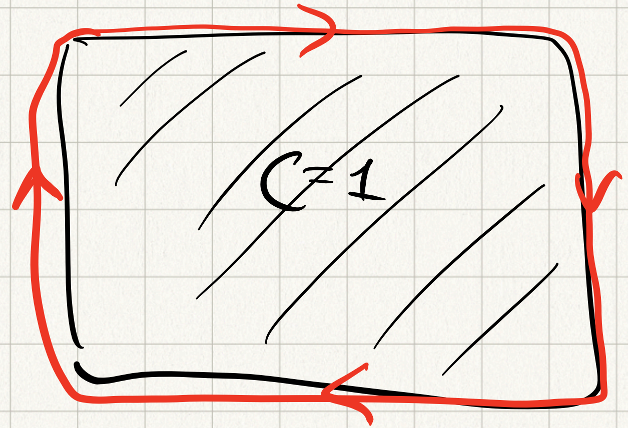
smooth continuous functions (smooth on scale a)

$$E\,\tilde\psi^A(x) = -J'(x)\,\tilde\psi^B(x) + J\left(\tilde\psi^B - a\,\partial_x\,\tilde\psi^B\right)$$
$$E\,\tilde\psi^B(x) = -J'(x)\,\tilde\psi^A(x) + J\left(\tilde\psi^A + a\,\partial_x\,\tilde\psi^B\right)$$

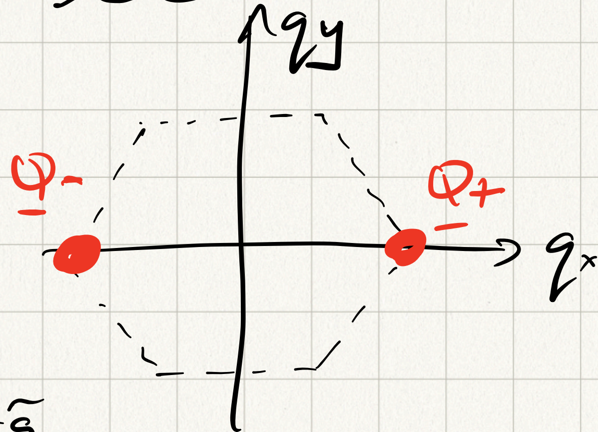
$$\Rightarrow \frac{E}{J}\begin{pmatrix}\tilde\psi^A\\ \tilde\psi^B\end{pmatrix} = \underbrace{\begin{pmatrix}0 & M(x) - a\,\partial_x\\ M(x) + a\,\partial_x & 0\end{pmatrix}}_{\hat H = M(x)\,\hat\sigma_x - i a\,\hat\sigma_y\,\partial_x}\begin{pmatrix}\tilde\psi^A\\ \tilde\psi^B\end{pmatrix}$$

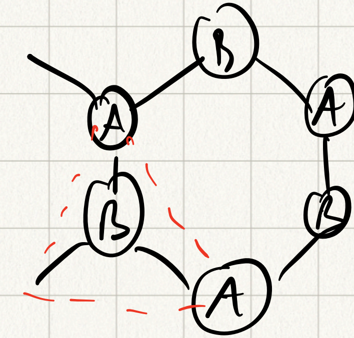
Physical Consequences of Chern Bands

1) Gapless edge state



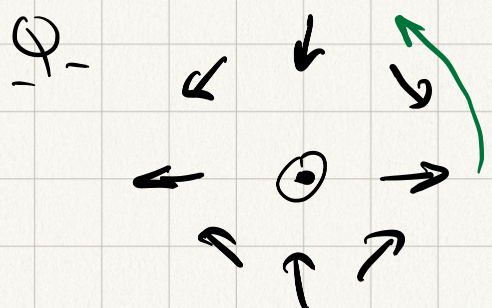
Example: Haldane Model (Lecture 2)

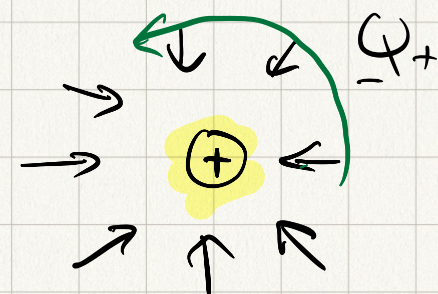




$$\underline{q} \simeq \underline{Q}_\pm + \underline{\tilde{q}}$$

$$H^\pm_{\underline{\tilde{q}}} \simeq \hbar v \left[\pm \tilde{q}_x \hat{\sigma}_x + \tilde{q}_y \hat{\sigma}_y \right] + \underline{\underline{h_z^\pm \hat{\sigma}_z}} + \underline{\underline{\Delta \hat{\sigma}_z}}$$





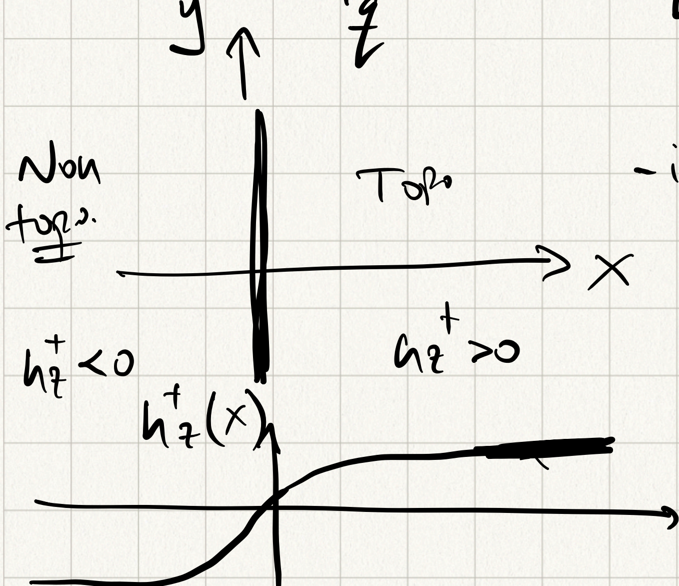
$h_z^- < 0$

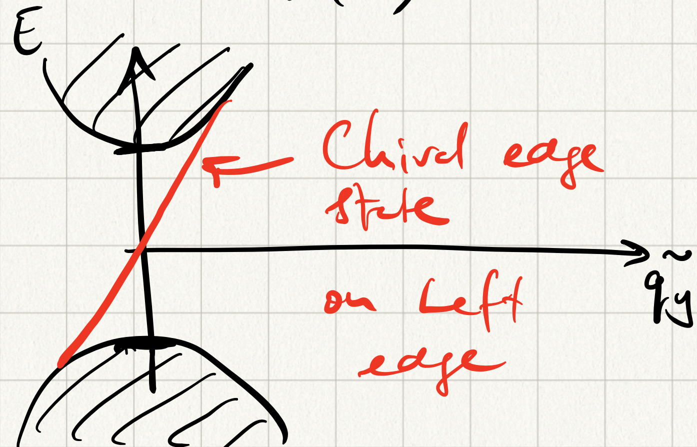
$h_z^+ > 0$

Continuum Theory for Haldane model near Q^+

$$H^+_{\tilde{q}} \simeq \hbar v \left[+\tilde{q}_x \hat{\sigma}_x + \tilde{q}_y \hat{\sigma}_y \right] + h_z^+ \hat{\sigma}_z$$

$\tilde{q}_x \to -ia\partial_x$ $\qquad$ $h_z^+ \to h_z^+(x)$

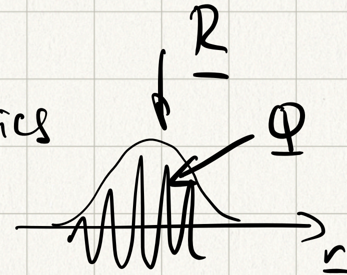


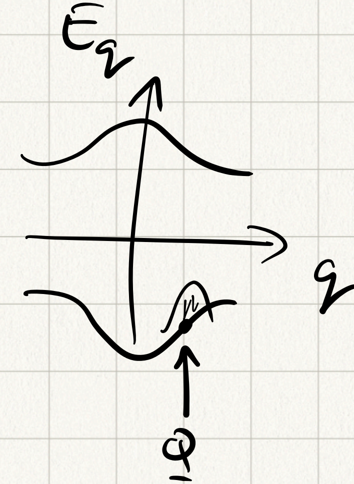


Exercise: Show that $E_{\text{edge}}(\tilde{q}_y) = \hbar v \tilde{q}_y$

2) Anomalous Velocity

Semiclassical Dynamics





$$\frac{d\underline{R}}{dt} = \frac{1}{\hbar}\frac{dE_Q}{d\underline{Q}} + \underline{\Omega}_B \times \dot{\underline{Q}}$$

Berry curvature.

$$\hbar \frac{d\underline{Q}}{dt} = \underline{E}^{ext} + \underbrace{\dot{\underline{R}} \times \underline{B}^{ext}}_{\text{Lorentz}}$$

Xiao, Chang + Niu
RMP 2010

3) Quantized Hall Response (Band insulator of fermions)

one particle per state

$$\underline{I} = \sum_{\underline{q}} {}^{n} \left(\frac{1}{\hbar} \frac{\partial E_q}{\partial \underline{q}} + \underline{\omega}_B \times \frac{\underline{\mathcal{E}}^{ext}}{\hbar} \right)$$

$$= \frac{A}{(2\pi)^2} \int d^2q \left(\underbrace{\frac{1}{\hbar} \frac{\partial E_q}{\partial \underline{q}}}_{0} + \underline{\omega}_B \times \frac{\underline{\mathcal{E}}^{ext}}{\hbar} \right)$$

$$j = \frac{\underline{\underline{I}}}{A} = \frac{1}{(2\pi)^2} \frac{1}{\hbar} \int d^2q \, \underline{\omega}_B \times \underline{\mathcal{E}}^{ext}$$

$$\boxed{j = \frac{1}{h} C \, n \, \hat{\underline{z}} \times \underline{\mathcal{E}}^{ext}} \qquad \sigma_{xy} = C/h$$

4 A.C. Response

[Tran et al Science Advances 2017]

$\sigma_{xy}(\omega = 0)$

$$j_x = \sigma_{xy} \, \mathcal{E}_y^{ext}$$

$$\underline{\mathcal{E}}^{ext}(t) = \mathcal{E}_0 \left(\cos\omega t, \pm \sin\omega t\right)$$

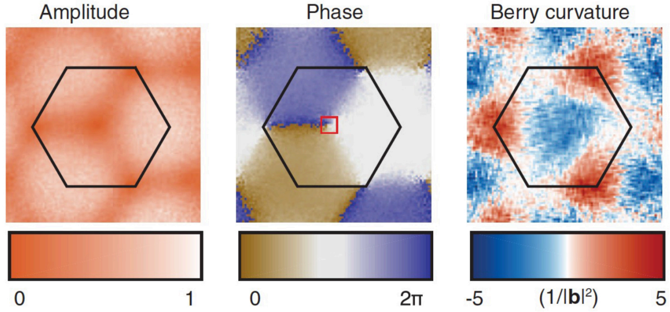
$\Rightarrow$ Absorbed Power $\quad P_{\pm}(\omega)$

$$\underline{\sigma_{xy}(\omega = 0)} = \frac{2}{\pi \, \mathcal{E}_0 \, A} \int_0^\infty d\omega \, \frac{1}{\omega} \left[P_+(\omega) - P_-(\omega) \right]$$

Measurements of Chern bands

- Band mapping & Berry curvature

[N. Fläschner *et al.* [Hamburg] Science **352**, 1091 (2016)]
[W. Sun *et al.* [USTC] Phys. Rev. Lett. **121**, 150401 (2018)]



$$A_i = i \langle u | \partial_{x_i} | u \rangle$$

$$\Omega_{ij} = \partial_{x_i} A_j - \partial_{x_j} A_i$$

- Edge states in synthetic dimensions

[Mancini *et al.* [LENS] Science **349**, 1510 (2015); Stuhl *et al.* [NIST], Science **349**, 1514 (2015)]

- Anomalous velocity

[Jotzu *et al.* [ETH], Nature **515**, 237 (2014)]

- Quantized transport of bosons: Chern number

$$\bar{n}_q = \text{constant}$$

[M. Aidelsburger *et al.* [Munich], Nat. Phys. 2015]

- Quantized circular dichroism

[L. Asteria *et al.* [Hamburg], Nat. Phys. **15**, 449 (2019)]

Symmetry Protected Topological Insulators and Superconductors

“Ten-fold way” for free fermions

[Chiu, Teo, Schnyder & Ryu, RMP 2016]

[Time-reversal, particle-hole & chiral symmetries]

	symmetries			spatial dimension							
Class	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	$+$	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	$+$	$+$	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	$+$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	$-$	$+$	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	$-$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	$-$	$-$	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	$-$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	$+$	$-$	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Beyond the ten-fold way

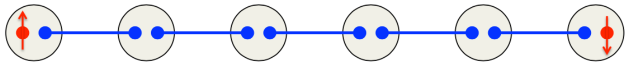
- Crystalline symmetries, higher order topological phases…

[Y. Ando & L. Fu, Ann. Rev. Condens. Matter Phys. 2015] [S. Parameswaran & Y. Wan, Physics Viewpoint 2017]

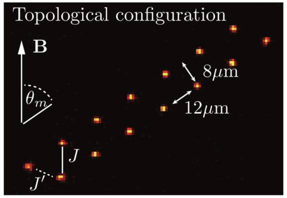
- Interacting symmetry-protected topological phases

[T. Senthil, Ann. Rev. Cond. Matt. Phys. (2015)]

e.g. Haldane phase of spin-1 chain



[Image: Wierschem & Sengupta, Mod Phys Lett B (2015)]



Rydberg systems: Hardcore bosons on SSH lattice

[S. de Léséleuc *et al.* [Paris], Science 2019]

- "Intrinsic Topological Order": particle fractionalization

e.g. Fractional quantum Hall effect

[FQHE: New developments, Eds. B. Halperin & J. Jain, World Scientific (2020)]

➡ Dynamical Effects

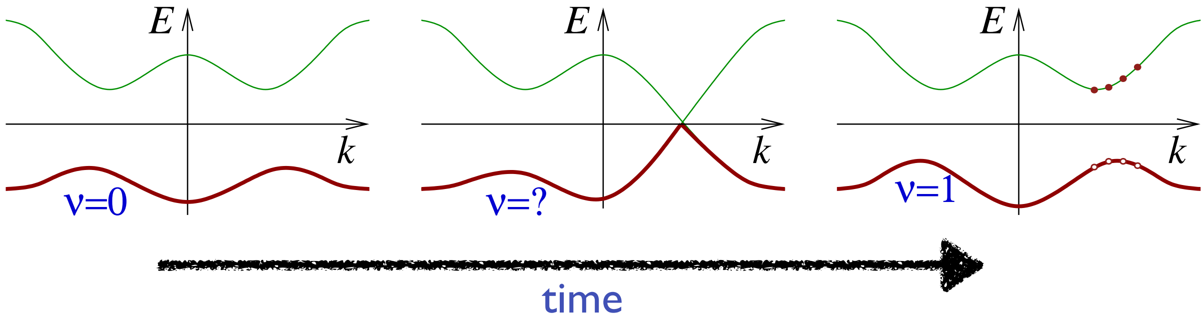
Lecture 3: Dynamical Effects in Topological Systems

- ~~Adiabatic pumping~~
- ~~Floquet topology~~
- Far-from-equilibrium dynamics

Far-from-equilibrium dynamics

Unitary evolution: $$|\Psi(t)\rangle = \mathcal{T}\exp\left[-i\int_0^t \hat{H}(t')dt'\right]|\Psi(0)\rangle$$

e.g. dynamical change in band topology



- Preparation of topological phases?

- Is there a topological classification of *non-equilibrium* many-body states?

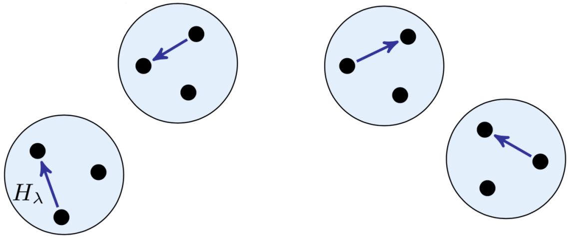
[⇒ stability of boundary modes in non-equilibrium settings]

[Focus on free fermions, but applies also for interacting systems]

Non-Equilibrium Topological Classification

<u>Equilibrium:</u>
adiabatically connected under symmetric Hamiltonian

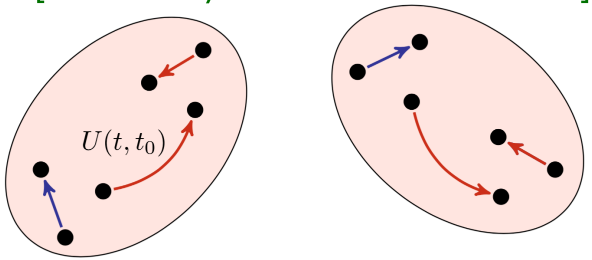
$$|\Psi_1\rangle \xrightarrow{H_\lambda} |\Psi_2\rangle$$



<u>Non-equilibrium:</u>
connected by (finite-time) unitary evolution governed by a symmetric Hamiltonian

$$|\Psi_1\rangle \xrightarrow{e^{-\mathrm{i}Ht}} |\Psi_2\rangle$$

[Max McGinley & NRC, PRB 2019; PRR 2019]



Non-Equilibrium Topological Classification

- Free-fermion topological insulators / superconductors

  [Altland-Zirnbauer: Time-reversal, charge-conjugation & sublattice symmetries]

[Max McGinley & NRC, PRB 2019]

Class	Symmetries			Spatial dimension d							
	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z} \to 0$	0	$\mathbb{Z} \to 0$	0	$\mathbb{Z} \to 0$	0	$\mathbb{Z} \to 0$
AI	$+$	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2 \to 0$	$\mathbb{Z}_2 \to 0$
BDI	$+$	$+$	1	$\mathbb{Z}_2$	$\mathbb{Z} \to \mathbb{Z}_2$	0	0	0	$2\mathbb{Z} \to 0$	0	$\mathbb{Z}_2 \to 0$
D	0	$+$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	$-$	$+$	1	0	$\mathbb{Z}_2 \to 0$	$\mathbb{Z}_2 \to 0$	$\mathbb{Z} \to 0$	0	0	0	$2\mathbb{Z} \to 0$
AII	$-$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2 \to 0$	$\mathbb{Z}_2 \to 0$	$\mathbb{Z}$	0	0	0
CII	$-$	$-$	1	0	$2\mathbb{Z} \to 0$	0	$\mathbb{Z}_2 \to 0$	$\mathbb{Z}_2$	$\mathbb{Z} \to \mathbb{Z}_2$	0	0
C	0	$-$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	$+$	$-$	1	0	0	0	$2\mathbb{Z} \to 0$	0	$\mathbb{Z}_2 \to 0$	$\mathbb{Z}_2 \to 0$	$\mathbb{Z} \to 0$

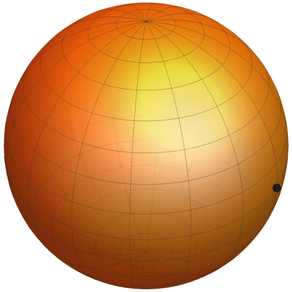
- Interacting symmetry-protected topological phases [Max McGinley & NRC, PRR 2019]

Non-Equilibrium Topological Classification: Physical Consequences

[Max McGinley & NRC, PRB 2019; PRR 2019]

1) Preparation of topological states

Determines which states can be quickly interconverted dynamically by symmetry-respecting Hamiltonians

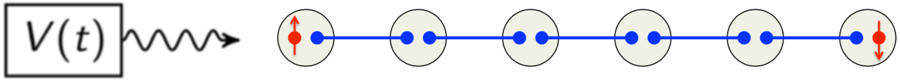


e.g. Su-Schrieffer-Heeger model (class BDI in 1D): $\mathbb{Z} \rightarrow \mathbb{Z}_2$

Topological "invariant" can be changed by even integers

2) Stability of "topologically protected" surface states

External symmetry-respecting noise



[Image: Wierschem & Sengupta, Mod Phys Lett B (2015)]

Determines which symmetry-protected topological quantum registers decohere

Relevant also to topological protection in open quantum systems

[Lectures of Sebastian Diehl]

Overview

- Topological Phases of Matter
  - Topological band theory
  - Symmetry protection
  - Strong Interactions

- Dynamical Effects
  - Adiabatic pumping
  - Floquet topology
  - Far-from-equilibrium systems

Review article: NRC, Jean Dalibard & Ian Spielman, RMP **91**, 015005 (2019)

Work with: Marcello Caio, Joe Bhaseen, Gunnar Möller, Max McGinley