

Recap of Lecture I

- Introduction to Topological Band theory

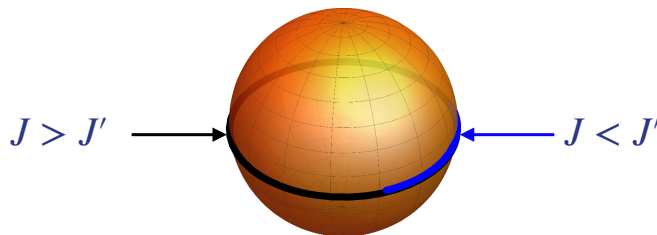
- ▶ Zak phase for 1D bands $\phi_{\text{Zak}} = \int_{\text{BZ}} i \langle u_q | \partial_q | u_q \rangle dq$

- ▶ For “chiral” symmetry $\hat{\sigma}_z \hat{H} = -\hat{H} \hat{\sigma}_z$ and $\hat{H}_q = \hat{H}_{q+G}$

$$\phi_{\text{Zak}} = -\pi \times N_w$$

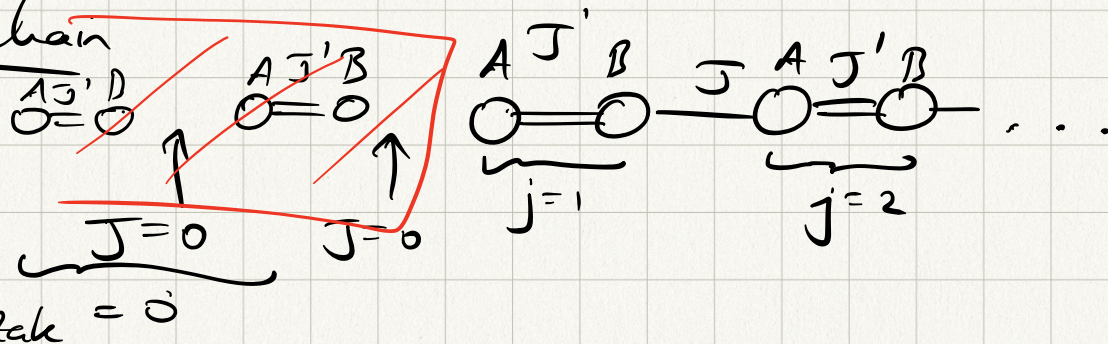
e.g. Su-Schrieffer-Heeger model with

$$\hat{H}_q = - \begin{pmatrix} 0 & J' + J e^{-iqa} \\ J' + J e^{iqa} & 0 \end{pmatrix}$$



- ▶ *Symmetry protected* topological state, has gapless edge state at $E = 0$

Semi-infinite chain



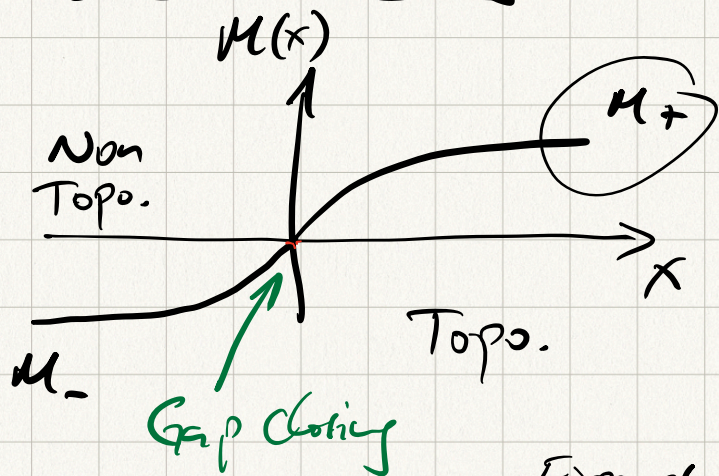
Ex: Show there is an $E=0$ state

$$\begin{pmatrix} \psi_j^A \\ \psi_j^B \end{pmatrix} \propto \begin{pmatrix} (-J'/J)^j \\ 0 \end{pmatrix}$$

which is normalisable for $J'/J < 1$

- Zero energy edge state in topological phase.
- $E=0$ exactly, provided symmetry is respected

Continuum Theory: Let $M(x) \equiv 1 - J'/J$ vary smoothly in x



$$H_g = \begin{pmatrix} 0 & J' + J e^{-iqa} \\ J & 0 \end{pmatrix}$$

$$h_x = J' + J \cos qa$$

$$h_y = J \sin qa$$

Expand $\hat{H}_g$ around $qa = \pi + \tilde{q}a$

$$\frac{\hat{H}_g}{J} \simeq \tilde{q}a \hat{\sigma}_y + M(x) \hat{\sigma}_x$$

Semiclassical $\Rightarrow$ $= -ia\partial_x \hat{\sigma}_y + M(x) \hat{\sigma}_x$

Jackiw-Rebbi (1976)

$E=0$ state: $|4\rangle = \begin{pmatrix} e^{-\int_0^x M(x') dx'/a} e^{i\pi x/a} \\ 0 \end{pmatrix}$

Today

- ▶ 2D Chern bands
- ▶ Symmetry Protected Topological Insulators and Superconductors
- ▶ Strong Interactions

Berry Phase (Berry 1984)

$$\hat{H}(\underline{x}) |\psi^{(n)}(\underline{x})\rangle = E^{(n)}(\underline{x}) |\psi^{(n)}(\underline{x})\rangle$$

Adiabatic evolution, $\underline{x}(t)$, over a cycle $t=0 \dots T$
 $\underline{x}(0) = \underline{x}(T)$

$$|\psi^{(n)}(\underline{x})\rangle \rightarrow |\psi^{(n)}(\underline{x})\rangle e^{-\frac{i}{\hbar} \int_0^T E^{(n)}(\underline{x}(t')) dt'} e^{i \gamma_{\text{geo}}^{(n)}}$$

$$\gamma_{\text{geo}}^{(n)} = \oint \underline{A}^{(n)} \cdot d\underline{x} \Rightarrow \underline{A}^{(n)} = i \langle \psi^{(n)} | \frac{\partial}{\partial \underline{x}} | \psi^{(n)} \rangle$$

$$\gamma_{\text{geo}}^{(n)} = \int \nabla \times \underline{A}^{(n)} \cdot d\underline{s} \Rightarrow \underline{\omega}^{(n)} = \nabla \times \underline{A}^{(n)}$$

Berry curvature.

Topological Bands

$$: \quad \underline{x} \rightarrow \underline{q}$$

$$|\psi^{(n)}(\underline{x})\rangle \rightarrow |u_{\underline{q}}^{(n)}\rangle$$

$$\underline{A}_{\underline{q}} = i \langle u_{\underline{q}}^{(n)} | \frac{\partial}{\partial \underline{q}} | u_{\underline{q}}^{(n)} \rangle$$

1D

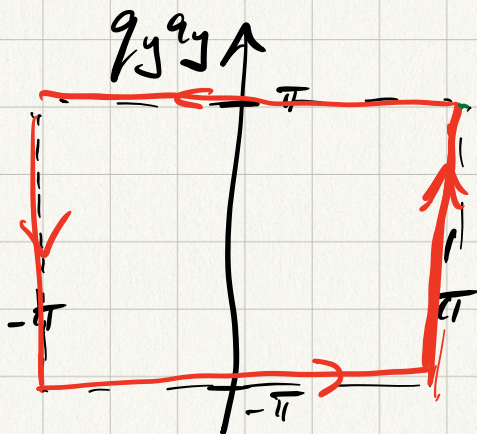
$$\phi_{\text{Zak}} = \int_{\text{BZ}} \underline{A}_{\underline{q}} d\underline{q}$$

2D

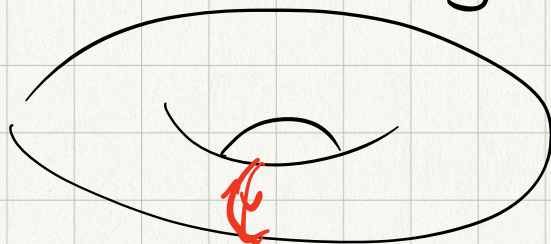
Chern Number

$$C = \frac{1}{2\pi} \int \nabla_{\underline{q}} \times \underline{A}_{\underline{q}} d^2 \underline{q}.$$

2D Chern Number



$$\int_{BZ} \nabla \times \underline{A}_g d\mathbf{g}^2 = \oint_{\text{Boundary of } BZ} \underline{A}_g \cdot d\mathbf{g}$$



No: $\underline{A}_g$ can have unphysical point-like singularities
gauge-dependent $|u_g\rangle \rightarrow e^{-i\chi_g} |u_g\rangle$

$$\underline{A}_g \rightarrow \underline{A}_g + \nabla_g \chi_g$$

C = net number of 2π vortices in BZ .

Time Reversal Symmetry

TRS if equations of motion are invariant under

$$t \rightarrow -t \quad p \rightarrow -p \quad \underline{r} \rightarrow \underline{r}$$

$$\frac{d\underline{r}}{dt} = \frac{\partial H}{\partial \underline{p}} \quad \frac{d\underline{p}}{dt} = -\frac{\partial H}{\partial \underline{r}}$$

$$\text{i.e. } H(\underline{r}, \underline{p}) = H(\underline{r}, -\underline{p})$$

Q.M. (spinless particle) : $\hat{H} = \hat{H}^*$

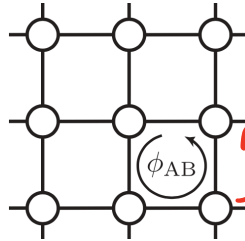
$$\underline{r}^* = \underline{r}$$

$$\underline{p}^* = \left(\frac{\hbar}{i} \frac{\partial}{\partial \underline{r}} \right)^* = -\underline{p}$$

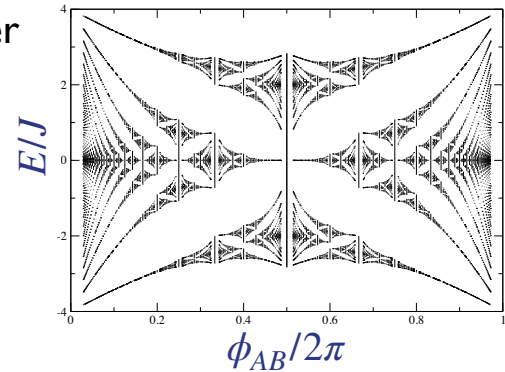
Ex: If $\hat{H} = \hat{H}^*$, show that $\underline{\psi}(\underline{q}) = -\underline{\psi}(-\underline{q})$
 $\Rightarrow C = \frac{1}{2\pi} \int \underline{\psi} \cdot d\underline{\Sigma} = 0$

Models

- Harper Model
[Proc. Phys. Soc. A 1955]



Hofstadter spectrum
[PRB 1976]



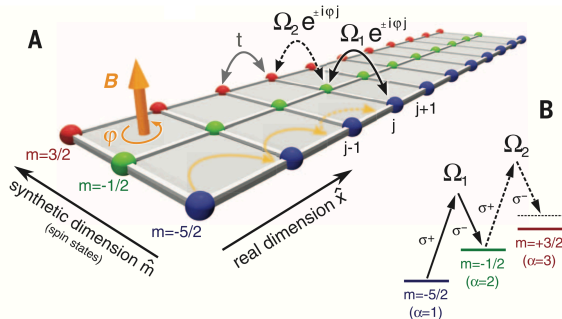
- laser-assisted tunnelling in optical superlattice

$$\phi_{AB} = \pi/2 \quad [\text{M. Aidelsburger et al. [Munich], Phys. Rev. Lett. 111, 185301 (2013)}]$$

$$\phi_{AB} = \pi \quad [\text{H. Miyake et al. [MIT], Phys. Rev. Lett. 111, 185302 (2013)}]$$

QG microscope ladder [M. E. Tai et al. [Harvard] Nature **546**, 519 (2017)]

- spin coupling in synthetic dimensions

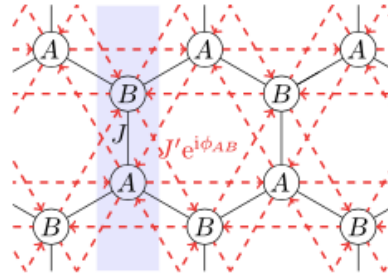


[M. Mancini et al. [LENS], Science **349**, 1510 (2015)]

[B. Stuhl et al. [JQI], Science **349**, 1514 (2015)]

Models

- Haldane Model



- ▶ off-resonant shaking of honeycomb (brick-wall) lattice

[Jotzu et al. [ETH], Nature **515**, 237 (2014)]

- ▶ 2-lattice-site analogue (resonant coupling)

[N. Fläschner et al. [Hamburg] Science **352**, 1091 (2016)]

- Spin-orbit coupled lattices

$$\hat{H} = -\frac{\hbar^2 \nabla^2}{2m} \hat{1} + \hat{V}(\mathbf{r})$$

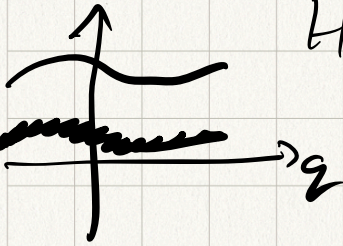
- ▶ deep lattices (tight binding)

[W. Sun et al. [USTC] Phys. Rev. Lett. **121**, 150401 (2018)]

- ▶ shallow lattices, with flat bands

[NRC, PRL (2011); NRC & J. Dalibard, PRL **110**, 185301 (2013)]

2 Band Models (eg. Haldane)

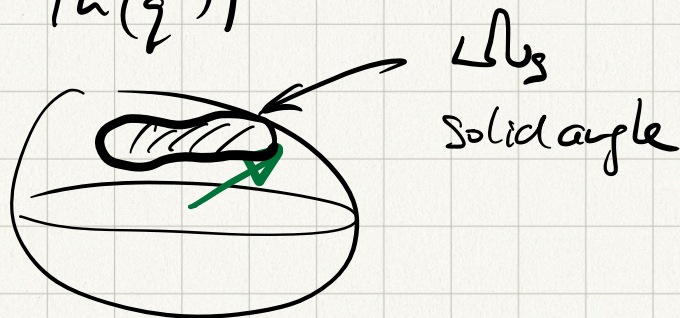
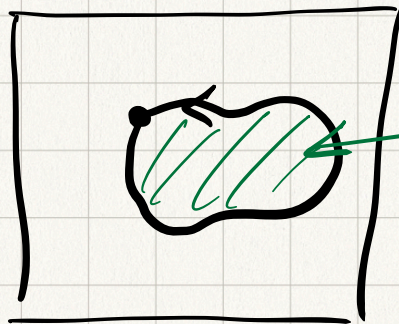


$$\hat{H}_q = - \begin{pmatrix} h_z & h_x - i h_y \\ h_x + i h_y & -h_z \end{pmatrix} = -\vec{h}(q) \cdot \vec{\sigma}_2$$

$$\epsilon^\pm(q) = \pm |\vec{h}(q)|$$

$\rightarrow \underline{|u_q^- \rangle}$

Bz



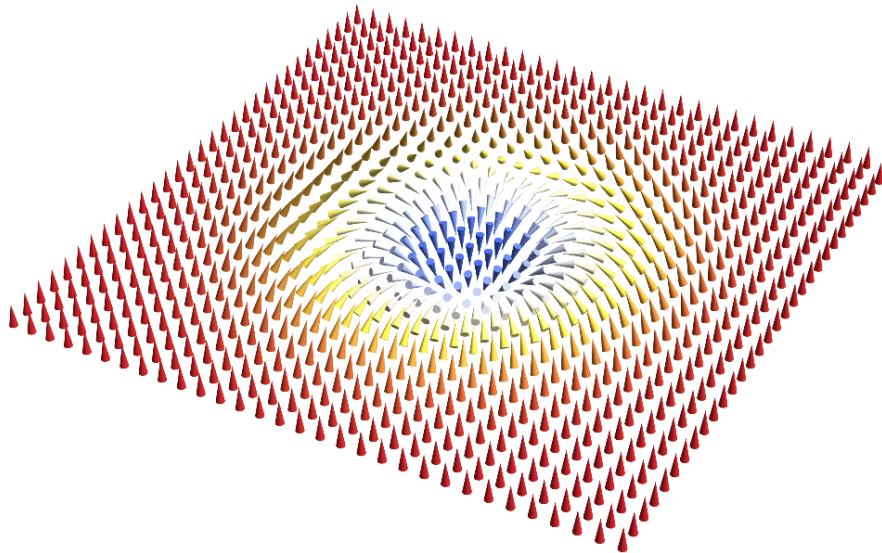
$$\oint \underline{A}_q \cdot d\underline{q}$$

$$= \int \nabla \times \underline{A}_q \cdot d\underline{S} = \frac{1}{2} \Omega_B$$

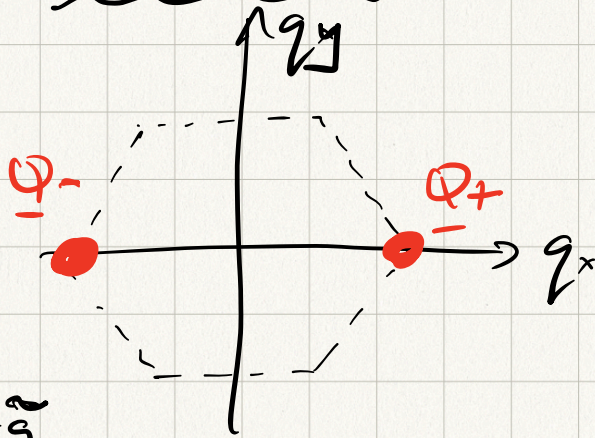
$$C \equiv \frac{1}{2\pi} \int \nabla \times \underline{A}_q \cdot d\underline{S} = \frac{1}{4\pi} \Omega_B = \text{No. times the sphere is wrapped.}$$

Chern number as 2D winding number

Two band model: $|u_q\rangle = \begin{pmatrix} \cos(\theta_q/2) \\ \sin(\theta_q/2)e^{i\phi_q} \end{pmatrix}$

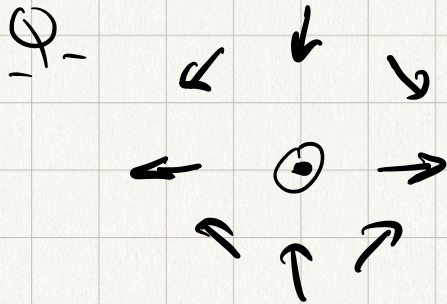


Example: Haldane Model

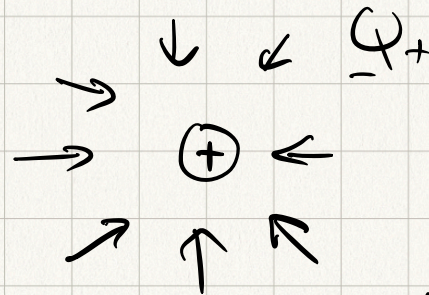


$$\mathbf{q} \approx \mathbf{Q}_\pm + \tilde{\mathbf{q}}$$

$$H_{\tilde{\mathbf{q}}}^\pm \approx t v \left[\pm \tilde{q}_x \hat{\sigma}_x + \tilde{q}_y \hat{\sigma}_y \right] + \underline{h_z^\pm} \hat{\sigma}_z$$



$$h_z^- < 0$$



$$h_z^+ > 0$$

Aside:

Optical Flux Lattices
Spin-orbit coupled Lattices

[NRC, 2011]

$$\hat{H} = \frac{\hat{p}^2}{2m} \hat{1} + \hat{V}(\underline{r})$$



$m \rightarrow \infty$

Adiabatic Motion in Real Space.

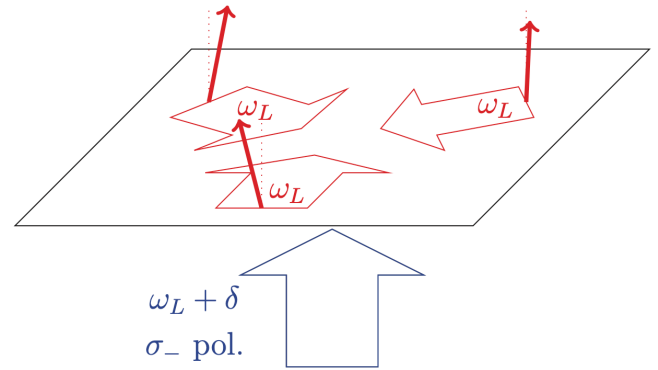
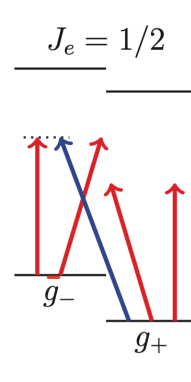
$$\hat{V}(\underline{r}) |\psi^{(n)}(\underline{r})\rangle = E^{(n)}(\underline{r}) |\psi^{(n)}(\underline{r})\rangle$$

$$\underline{A}^{(n)}(\underline{r}) = i \langle \psi^{(n)} | \underline{\nabla} | \psi^{(n)} \rangle$$

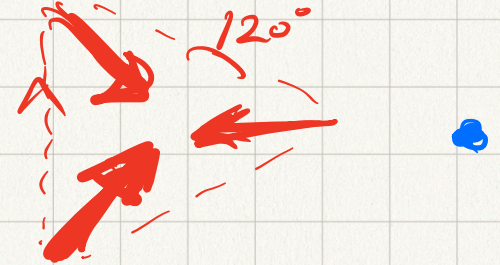
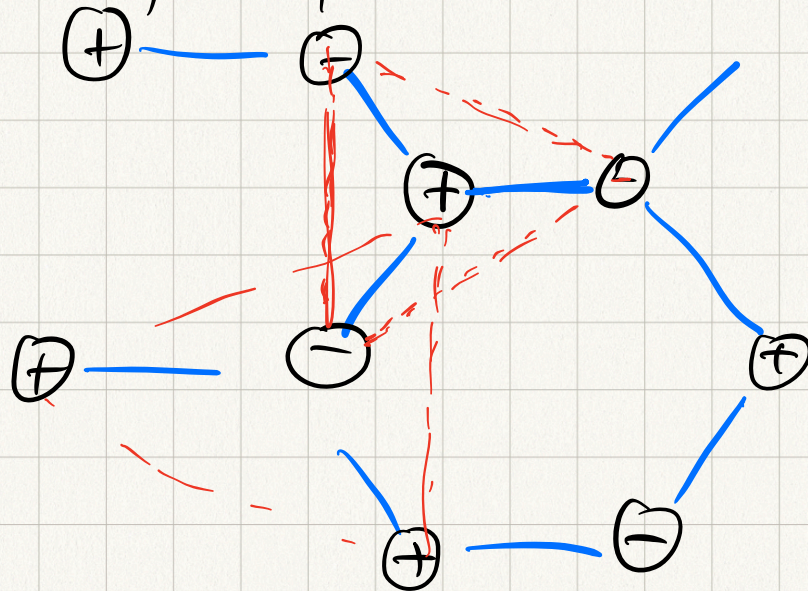
Aharonov Bohm effect

$\Rightarrow$ Magnetic field

$$\underline{B}^{(n)} = \underline{\nabla} \times \underline{A}^{(n)}$$



Reciprocal space

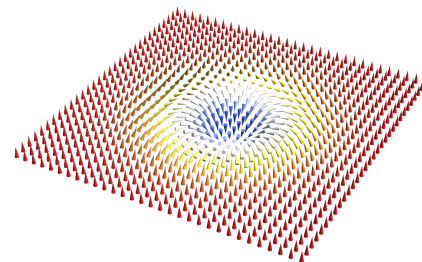


- Haldane Model in reciprocal space
 - Real Space Berry Curvature (Magnetic field)
- $C = 1 \Rightarrow 1 \text{ Flux per unit cell.}$

Summary of Lecture 2

- Chern Number $C = \frac{1}{2\pi} \int_{\text{BZ}} d^2q \Omega_q$

Berry curvature $\Omega_q = -i \nabla_q \times \langle u_q | \nabla_q u_q \rangle \cdot \hat{z}$



2-band models

Physical consequences

- chiral edge states
- anomalous velocity
- quantized Hall transport
- circular dichroism

Symmetry Protected Topological Insulators and Superconductors & Beyond

