Topological Phases of Matter and Dynamics

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Overview

- Topological Phases of Matter
  - Topological band theory
  - Symmetry protection
  - Strong Interactions

- Dynamical Effects
  - Adiabatic pumping
  - Floquet topology
  - Far-from-equilibrium systems

Review article: NRC, Jean Dalibard & Ian Spielman, RMP **91**, 015005 (2019)

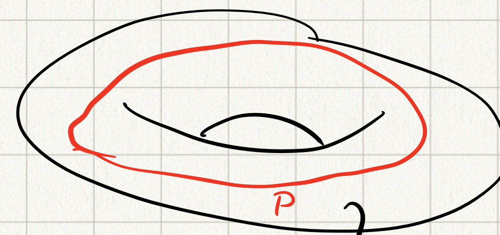
Work with: Marcello Caio, Joe Bhaseen, Gunnar Möller, Max McGinley

Topological Band Theory

Hasan & Kane RMP 2010
Qi & Zhang RMP 2011

i) Topological Invariants

- Superfluid in ring



$\psi(\underline{r}) = \sqrt{\rho}\, e^{i\phi(\underline{r})}$

Winding number:

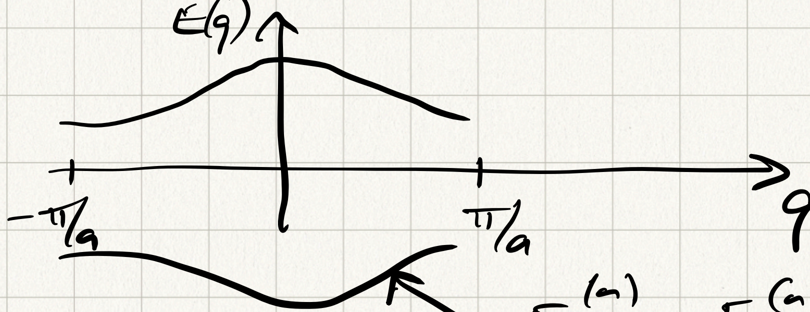
$$N_w = \frac{1}{2\pi}\oint_P \underline{\nabla}\phi \cdot d\underline{\ell}$$

$= $ integer "topological invariant"

"Smooth deformation" if $\rho \neq 0$ on path P

Physical consequences: Metastable persistent current.

(ii) Bloch Bands   e.g. $V(x) = V(x+a)$



Bloch wavenumber

$$E_q^{(n)} = E_{q+\underline{G}}^{(n)}$$

$\underline{G} \in$ Reciprocal Lattice vectors.

e.g. here: $G = \frac{2\pi}{a} \times$ integer

Bloch states.

$$\psi_q^{(n)}(\underline{r}) = e^{i q \cdot \underline{r}} \underbrace{u_q^{(n)}(\underline{r})}_{\text{periodic function}}$$

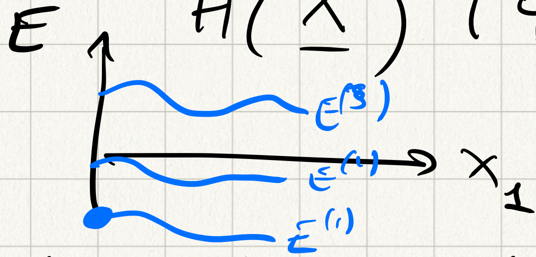
Topological invariants characterise how $u_q^{(n)}(\underline{r})$ varies across the Brillouin zone.

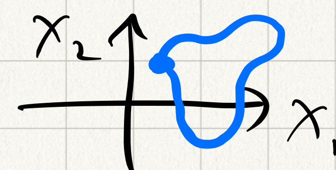
"Smooth deformation": no gap closing.

Physical consequences: Gapless edge states (+ quantized response)

Berry Phase [Berry 1984]

$$\hat{H}(\underline{X})\,|\psi^{(n)}(\underline{x})\rangle = E^{(n)}(\underline{x})\,|\psi^{(n)}(\underline{x})\rangle$$





Adiabatic evolution, $\underline{X}(t)$, over a cycle $t = 0 \ldots T$

$\underline{X}(0) = \underline{X}(T)$

$$|\psi^{(n)}(\underline{x})\rangle \rightarrow |\psi^{(n)}(\underline{x})\rangle\, e^{-\frac{i}{\hbar}\int_0^T E^{(n)}(\underline{X}(t'))\,dt'}\; e^{i\gamma^{(n)}_{geo}}$$

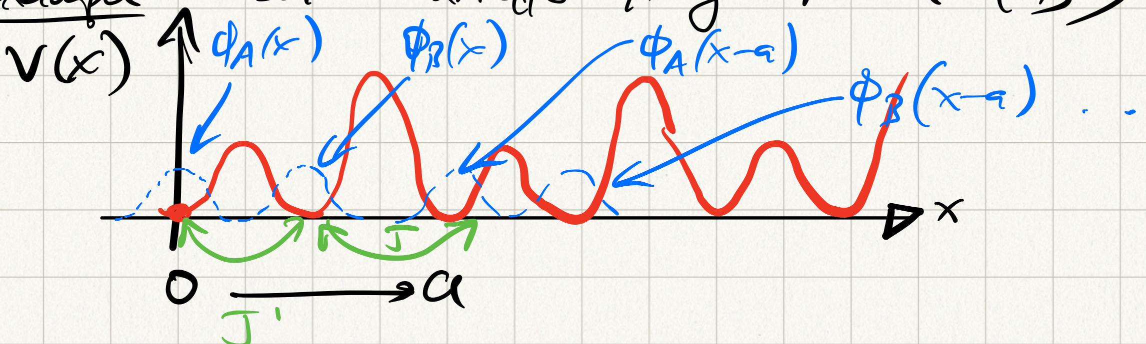
$$\gamma^{(n)}_{geo} = \oint \underline{A}^{(n)} \cdot d\underline{X} \;\Rightarrow\; \underline{A}^{(n)} = i\langle\psi^{(n)}|\frac{\partial}{\partial \underline{X}}|\psi^{(n)}\rangle$$

Berry connection

$$\gamma^{(n)}_{geo} = \int \underline{\nabla}\times\underline{A}^{(n)} \cdot d\underline{S} \;\Rightarrow\; \underline{\Omega}^{(n)} = \underline{\nabla}\times\underline{A}^{(n)}$$

Berry curvature.

Example: Su-Schrieffer-Heeger Model (1D)



Tight binding model

$$|\psi\rangle = \sum_j \psi_j^A |A_j\rangle + \psi_j^B |B_j\rangle$$

$$\langle x|A_j\rangle = \phi_A(x) \qquad \langle x|B_j\rangle = \phi_B(x)$$

Bloch wave: $$|\psi_q^{\cancel{(n)}}\rangle = \sum_j e^{iqaj}\left[u_q^A |A_j\rangle + u_q^B |B_j\rangle\right]$$

$$\hat{H} = -J' \sum_j \left[|B_j\rangle\langle A_j| + |A_j\rangle\langle B_j| \right] - J \sum_j \left[|A_{j-1}\rangle\langle B_j| + |B_j\rangle\langle A_{j-1}| \right]$$

J'_j

$$\hat{H}_q \begin{pmatrix} u_q^A \\ u_q^B \end{pmatrix} = E_q \begin{pmatrix} u_q^A \\ u_q^B \end{pmatrix}$$

$$\hat{H}_q = \begin{pmatrix} 0 & -J' - Je^{-iqa} \\ -J' - Je^{+iqa} & 0 \end{pmatrix}$$

$$\hat{H}_q = \hat{H}_{q+\frac{2\pi}{a}}$$

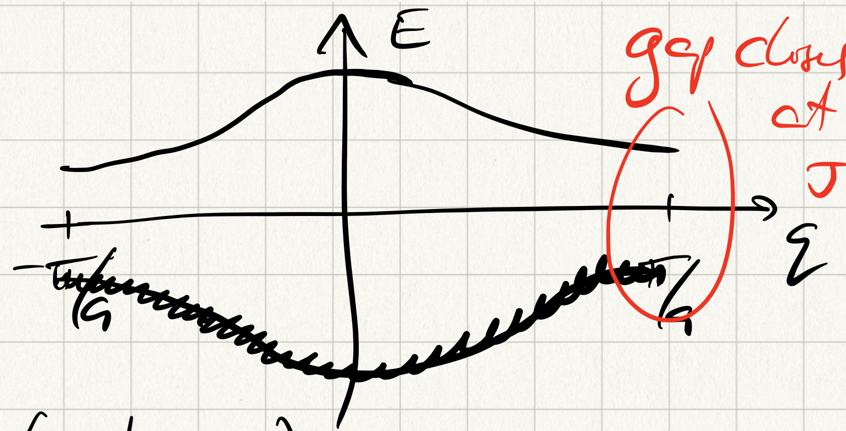
$$\hat{H}_q = -\vec{h}(q) \cdot \hat{\vec{\sigma}} = -\begin{pmatrix} 0 & h_x - ih_y \\ h_x + ih_y & 0 \end{pmatrix}$$

$\uparrow$

Pauli Matrices

$$h_x + ih_y = J' + Je^{iqa} = |\vec{h}| e^{i\phi_q}$$

$$E_q^{\pm} = \pm |\vec{h}|$$



$$\begin{pmatrix} u_q^A \\ v_q^B \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mp e^{i\phi_q} \end{pmatrix}$$

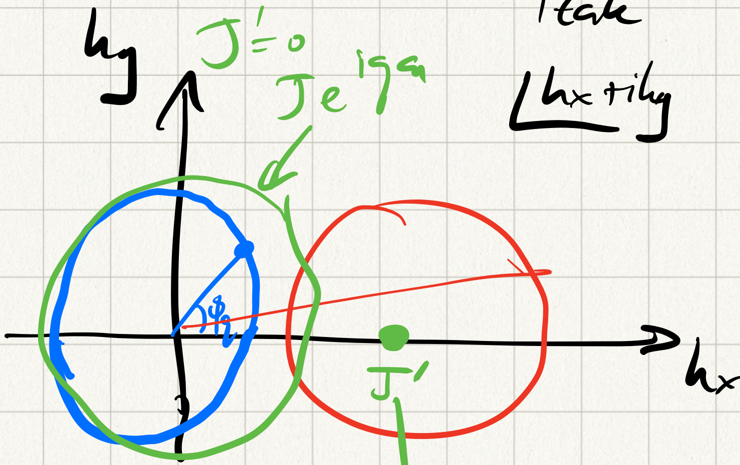
one fermion per unit cell : Band insulator for $J \neq J'$

Topological Invariant from Zak phase. [Zak, 1989]

$$\phi_{Zak} = \int_{BZ,\,-\pi/a}^{\pi/a} \underbrace{i \langle u_q | \frac{\partial}{\partial q} | u_q \rangle}_{\text{Berry connection}} \, dq$$

$$\hat{H}_q = \hat{H}_{q+\frac{2\pi}{a}}$$

Here: SSH Model



$$\phi_{Zak} = -\frac{1}{2}\underbrace{\int_{BZ} \frac{d\phi_q}{dq}\,dq}_{2\pi \times N_W}.$$

$\angle h_x + i h_y$

$h_x + i h_y = |\vec{h}|\, e^{i\phi_q}$

$h_x + i h_y = J' + \underline{\underline{J e^{iqa}}}$

$\phi_{Zak} = -\pi$

$J' < J$

"topological"

$J' > J$

$\phi_{Zak} = 0$

non-topological.

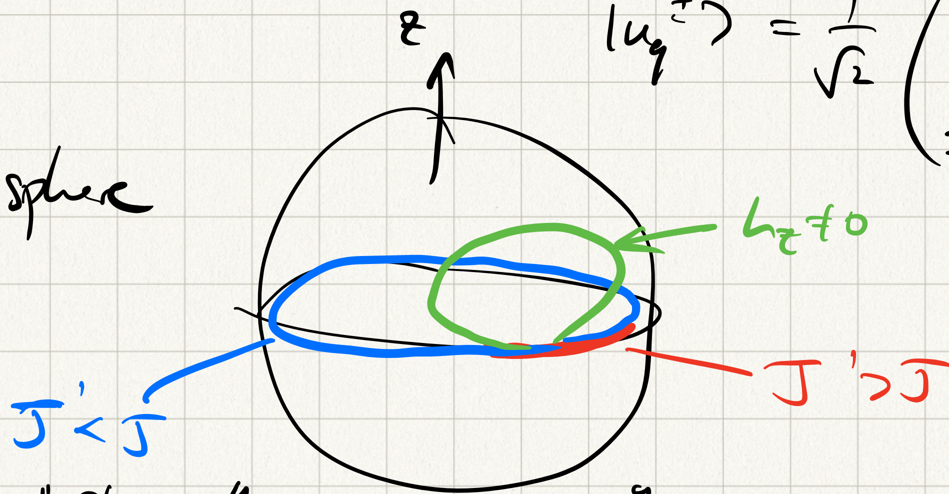
Cannot be smoothly connected without closing the gap ($J = J'$)

Symmetry Protection

$$\hat{H}_q = -\vec{h}(q)\cdot\hat{\vec{\sigma}} \quad \leftarrow h_z = 0$$

$$|u_q^{\pm}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp e^{i\phi_q} \end{pmatrix}$$

Bloch sphere



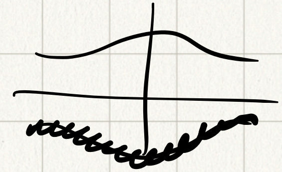
$\underline{h_z = 0}$: "Chiral" or "sublattice" symmetry

$$\hat{\sigma}_z \hat{H} = -\hat{H}\hat{\sigma}_z$$

Broken by energy offsets: $\Delta\varepsilon\, \hat{\sigma}_z$

"Symmetry Protected" topological phase.

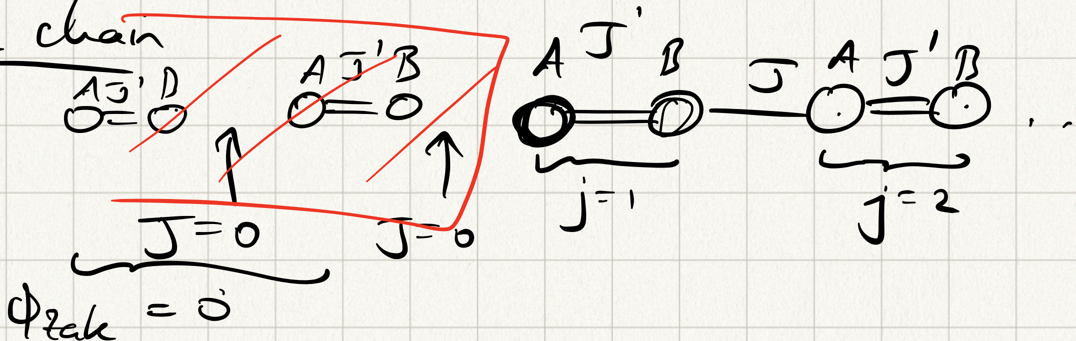
Physical Consequences: Edge States.



Band insulator

- Gap in bulk
- Has a gapless surface state in topological phase

Semi-infinite chain



$$E\psi_j^A = \begin{cases} -J'\psi_j^B - J\psi_{j-1}^B & j>1 \\ -J'\psi_j^B & j=1 \end{cases}$$

$$E\psi_j^B = -J'\psi_j^A - J\psi_{j+1}^A \qquad j\geq 1$$

Ex: Show there is an $E=0$ state

$$\begin{pmatrix} \psi_j^A \\ \psi_j^B \end{pmatrix} \propto \begin{pmatrix} \left(-J'/J\right)^{j} \\ 0 \end{pmatrix}$$

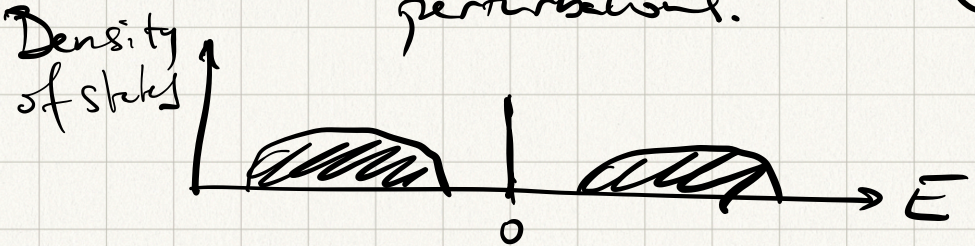
which is normalisable for $J'/J < 1$

- Zero energy edge state in topological phase.
- $E=0$ exactly, provided symmetry is respected.

Ex : For $\underbrace{\hat{\sigma}_z \hat{H} = -\hat{H} \hat{\sigma}_z}$ show that:

i) The energy spectrum is symmetric under $E \to -E$

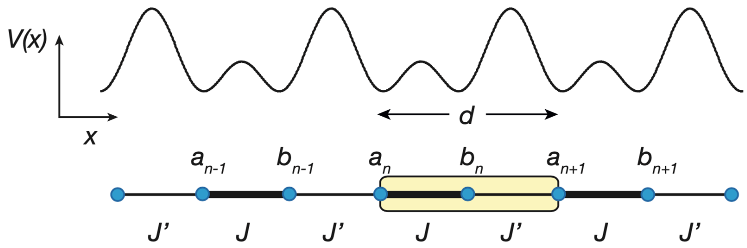
$\Rightarrow$ Edge state is robust to symmetry preserving perturbations.



ii) A state of energy $E=0$ is also an eigenstate of $\hat{\sigma}_z$.

SSH Model: Realizations

- Optical superlattices



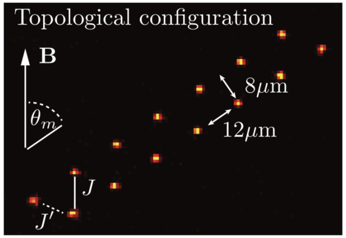
  - Interferometric detection of Zak phase (difference) [M. Atala *et al.* [Munich], Nat. Phys. (2013)]
  - Adiabatic pumping (see later)

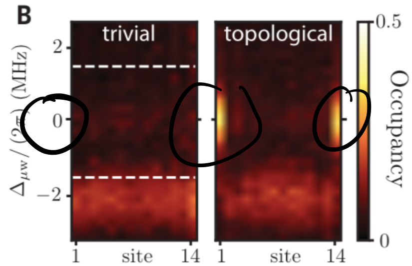
[S. Nakajima *et al.* [Kyoto] Nat. Phys. (2016); M. Lohse *et al.* [Munich] Nat. Phys. (2016);H.-I. Lu *et al.* [JQI], PRL (2016)]

- Momentum space lattice $p_n = 2n\hbar k \quad (n = 0,1,...20)$
  - Spectral resolution of edge state [E. Meier, F. An & B. Gadway [Illinois], Nat. Commun. (2016)]

Related (see later)

- Rydberg atoms in tweezer arrays (hard core bosons, SPT phase)
  - Spectral resolution of edge state





[S. de Léséleuc *et al.* [Paris], Science 2019]

Summary of Lecture 1

- Introduction to Topological Band theory

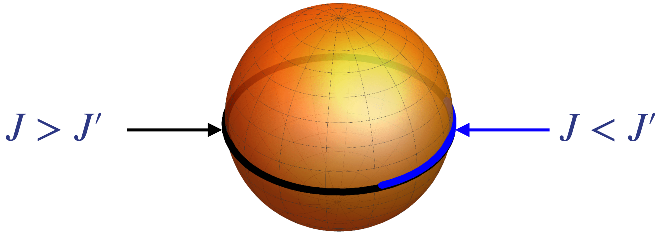
  - Zak phase for 1D bands $\phi_{\mathrm{Zak}} = \int_{\mathrm{BZ}} \mathrm{i}\langle u_q | \partial_q | u_q \rangle \mathrm{d}q$

  - For sublattice symmetry $\hat{\sigma}_z \hat{H} = -\hat{H}\hat{\sigma}_z$ and $\hat{H}_q = \hat{H}_{q+G}$

$$\phi_{\mathrm{Zak}} = -\pi \times N_{\mathrm{w}}$$

e.g. Su-Shrieffer-Heeger model with

$$\hat{H}_q = -\begin{pmatrix} 0 & J' + J\mathrm{e}^{-\mathrm{i}qa} \\ J' + J\mathrm{e}^{\mathrm{i}qa} & 0 \end{pmatrix}$$



  - *Symmetry protected* topological state, has gapless edge state at $E = 0$