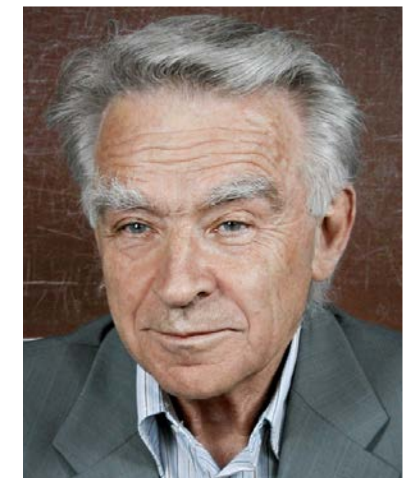
Boulder Summer School 2021
July 5-30, 2021
Boulder, USA (via zoom)

Institute for Theoretical Physics University of Cologne

Göran Lindblad

Lindblad meets Keldysh: phase transitions, topology, and measurements



Leonid W. Keldysh

Sebastian Diehl

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Outline

Keldysh theory general: A. Kamenev, *Field theory or non-equilibrium systems*, Cambridge University Press

Review Lecture I: L. Sieberer, M. Buchhold, SD, *Keldysh Field Theory for Driven Open Quantum Systems*, Reports on Progress in Physics (2016)

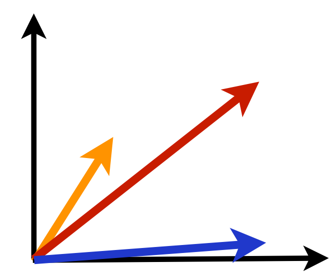
Lecture I: Theoretical background & non-equilibrium phases

- From the quantum master equation to the Keldysh functional integral
  - construction
  - semiclassical limit, connection to exciton-polariton systems
  - “what is non-equilibrium about it?”

$$\partial_t \rho = -i[H,\rho] + \mathcal{L}[\rho]$$

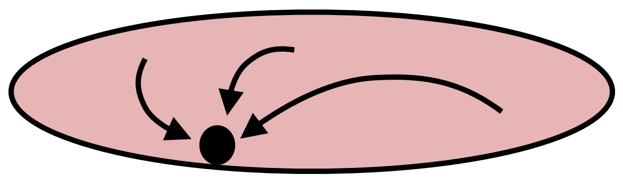
$$\mathrm{e}^{\mathrm{i}\Gamma[\Phi]} = \int \mathcal{D}\delta\Phi \mathrm{e}^{\mathrm{i}S_M[\Phi+\delta\Phi]}$$

- Stationary states of driven open quantum systems
  - fate of BKT physics out of equilibrium
  - phase transition driven by non-equilibrium drive



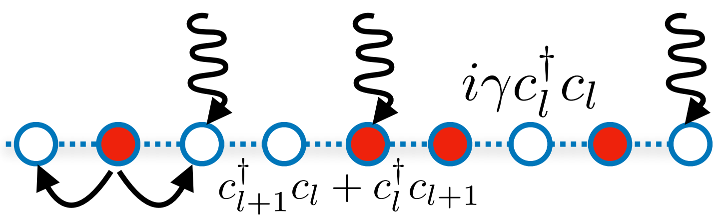
Lecture II: Symmetry and topology out of equilibrium

- Topological states induced by dissipation
- Dynamics: topological field theory out of equilibrium
- Dynamical symmetry classification in- and out-of-equilibrium

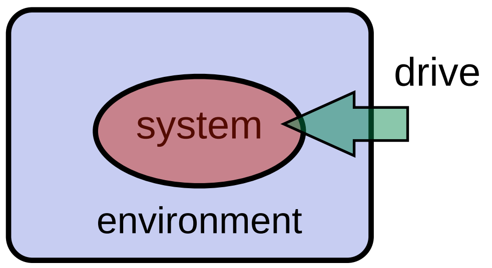


Lecture III: Lindblad-Keldysh 2.0: Measurement induced transitions

- Background: weak continuous measurements
- Measurement induced phase transitions of fermions
- Replica Keldysh field theory approach



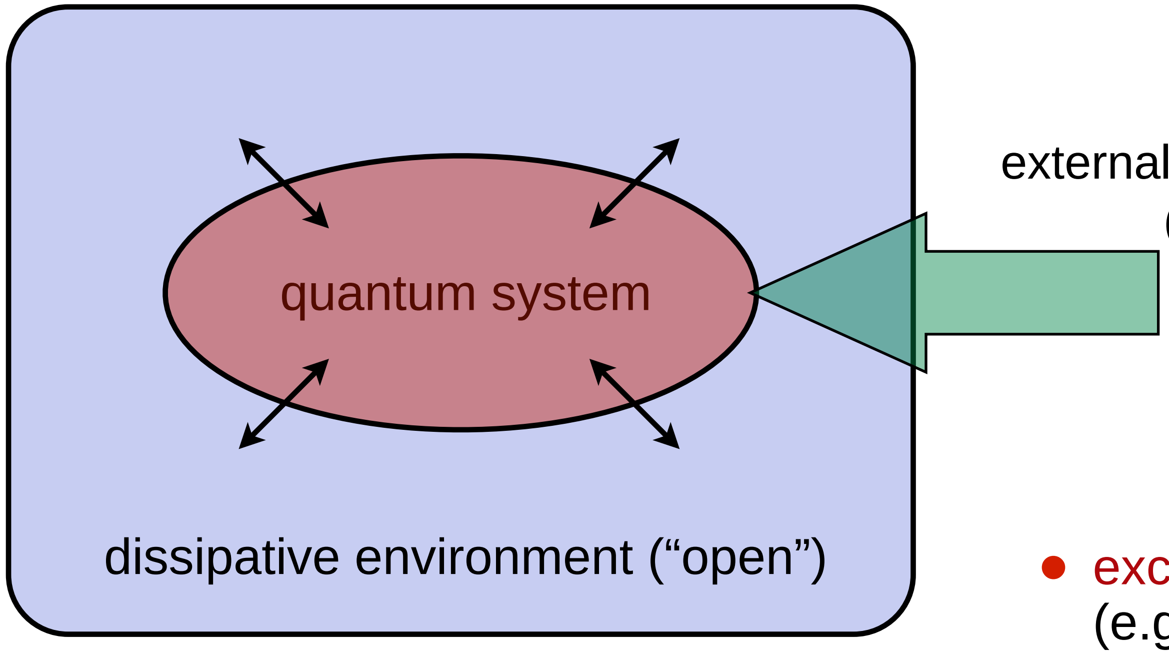
Lindblad quantum master equation:
From few to many degrees of freedom



$$\partial_t \rho = -i[H,\rho] + \kappa \sum_i L_i \rho L_i^\dagger - \tfrac{1}{2}\{L_i^\dagger L_i, \rho\}$$

What is a driven open quantum system?

- quantum Optics:



- exchange between system and bath (e.g. energy, entropy, particle number)

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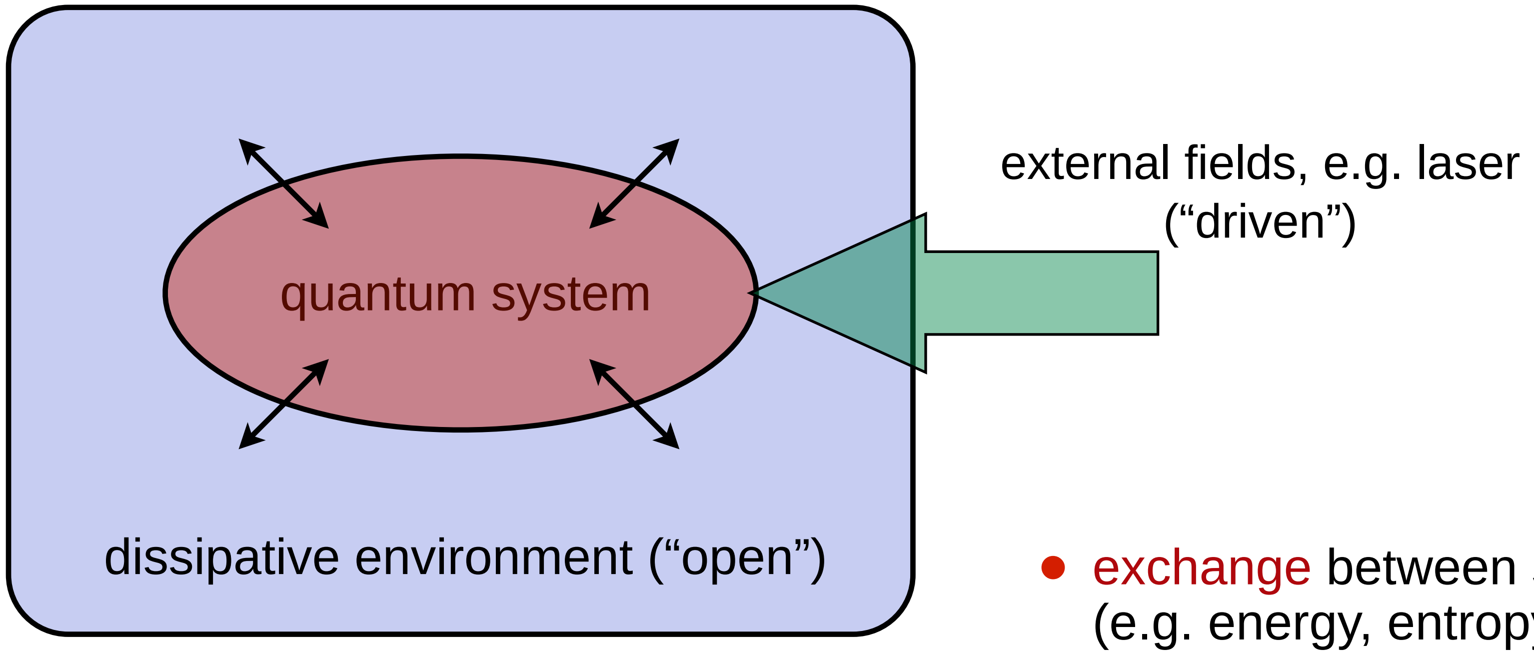
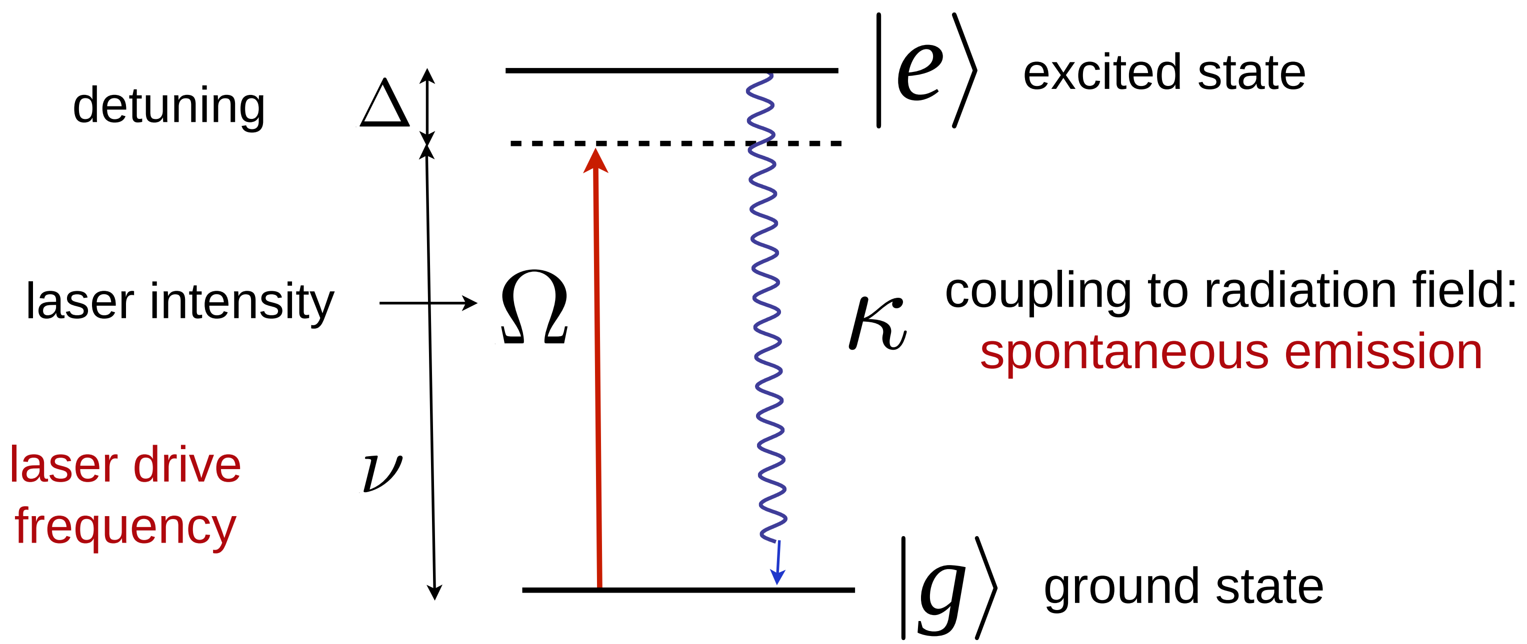


- exchange between system and bath (e.g. energy, entropy, particle number)

- example: laser driven atom coupled to the radiation field (two-level system)



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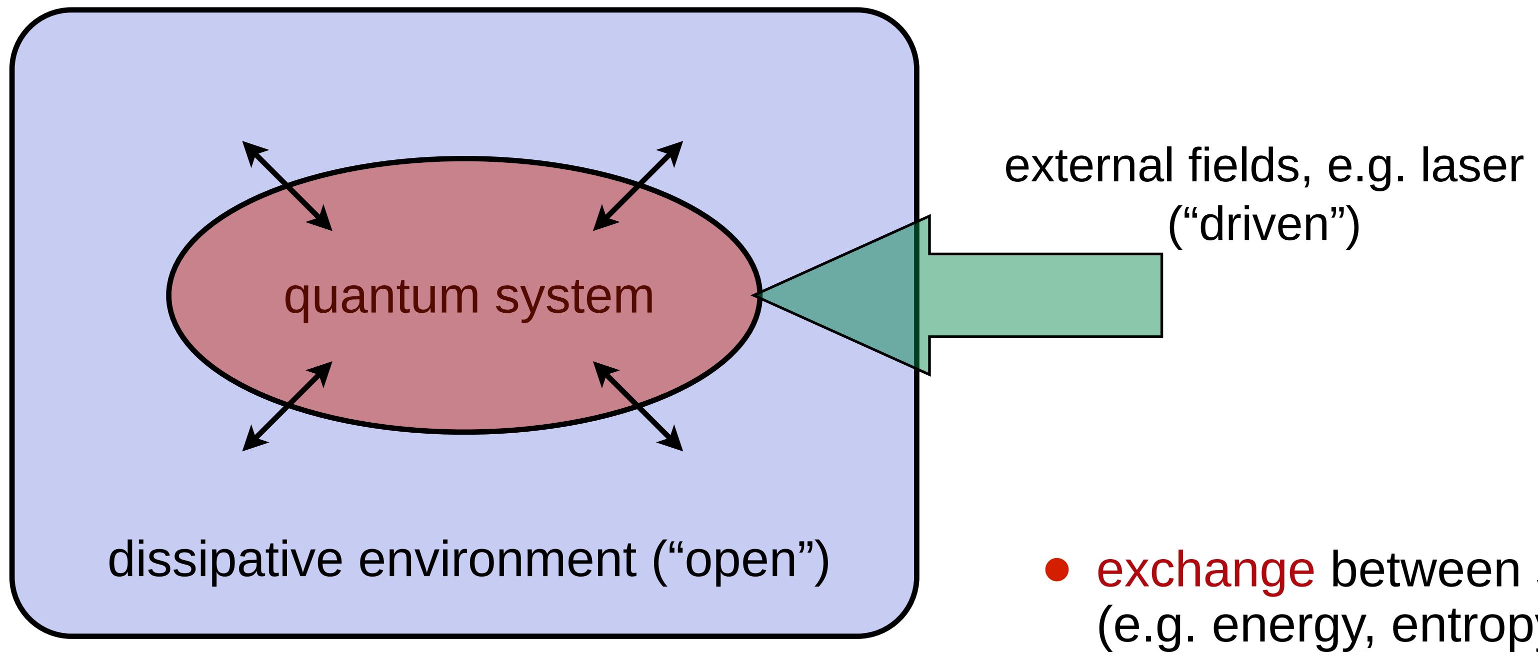

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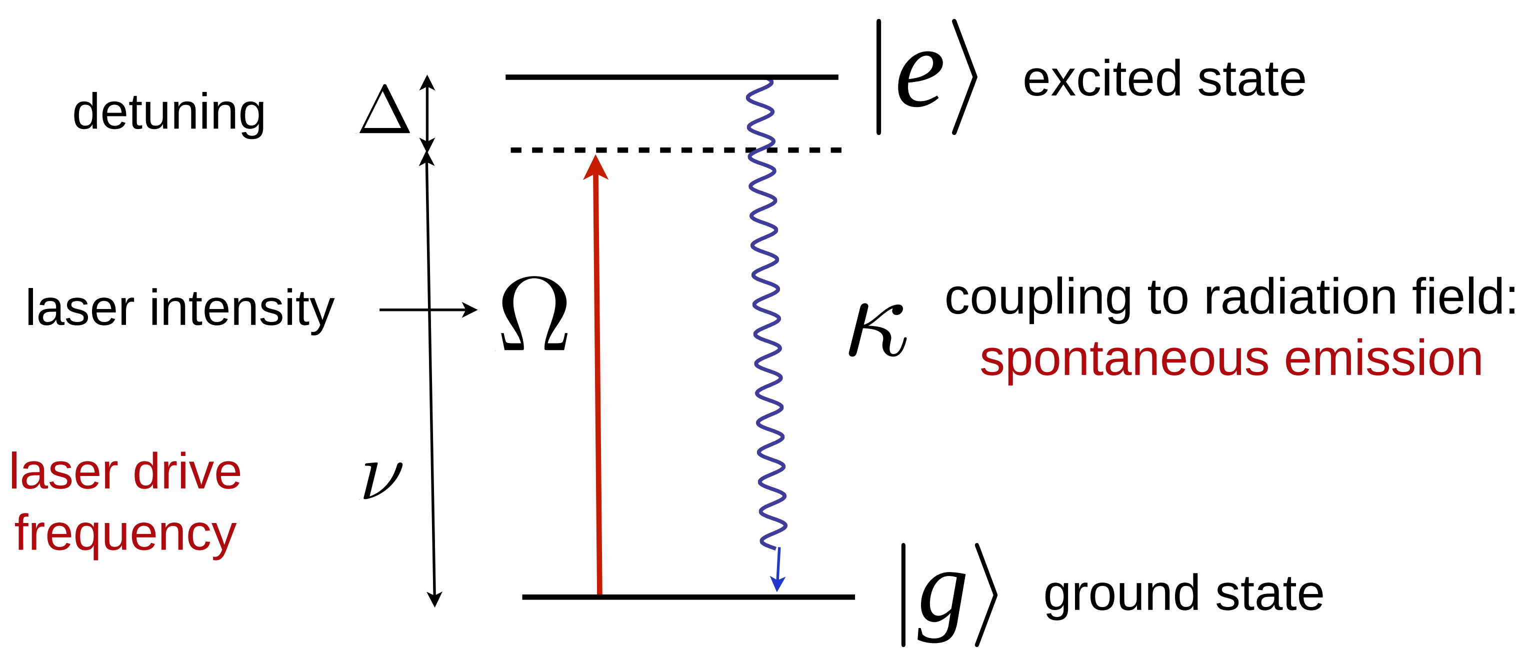


- simple fact: drive essential to access upper level

- implications:

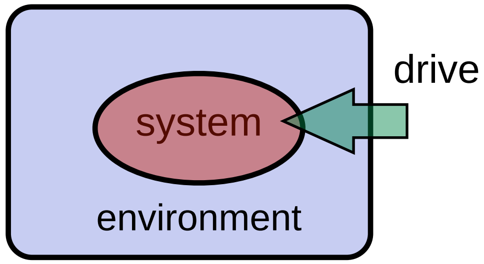
> - no guarantee for detailed balance
> - no obedience of the second law of thermodynamics (state purification)

Driven open quantum systems: microscopic description

- quantum master equation

Lindblad operators

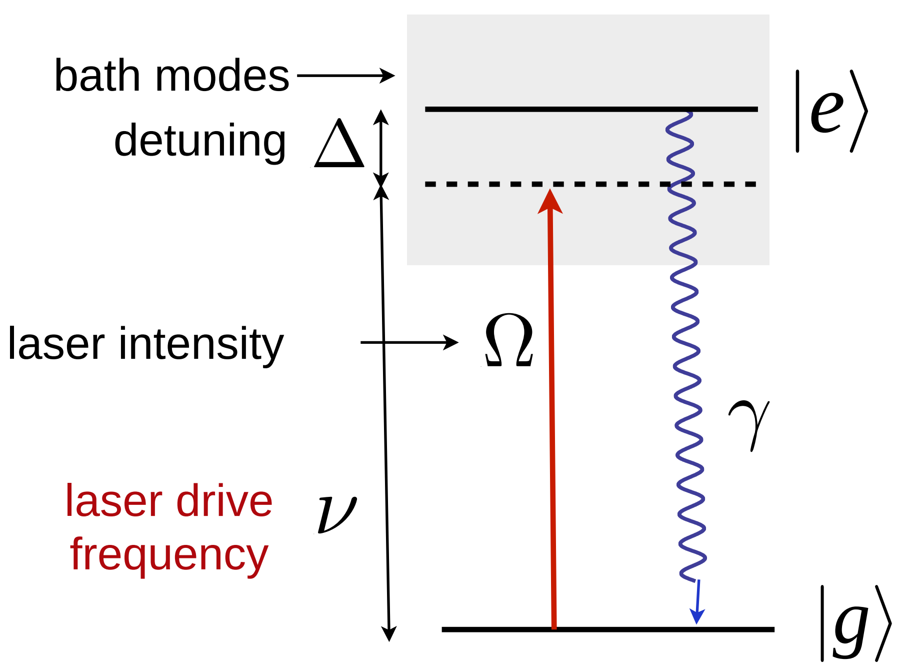
$$\partial_t \hat\rho = \underbrace{-i[\hat H, \hat\rho]}_{\text{coherent evolution}} + \sum_i \underbrace{\gamma_i [2\hat L_i \hat\rho \hat L_i^\dagger - \hat L_i^\dagger \hat L_i \hat\rho - \hat\rho \hat L_i^\dagger \hat L_i]}_{\text{driven-dissipative evolution}}$$



$$\equiv \hat{\mathcal{L}}[\hat\rho]$$ Lindbladian; also: Liouvillian

- derivation from system-bath setting: second order time dependent perturbation theory see later, and appendix

- example: two-level system



- starting point: system-bath setting

$$\hat H_{\mathrm{t}} = \hat H + \hat H_{\mathrm{b}} + \hat H_{\text{s-b}}$$

$$\hat H_{\mathrm{b}} = \sum_\mu \epsilon_\mu \hat b_\mu^\dagger \hat b_\mu$$

$$\hat H_{\text{s-b}} = \sum_\mu g_\mu e^{-i\nu t}\, \hat L\, \hat b_\mu^\dagger + \mathrm{h.\ c.}$$

(for single L)

- 3 approximations:
  - Born: weak system-bath coupling (2nd order pert. th.)
  - Markov: constant (in frequency) bath spectral density
  - rotating wave: drive ν biggest scale

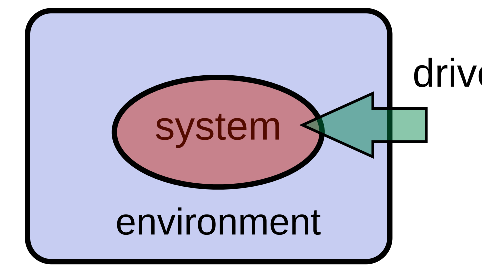
$$H = (|e\rangle, |g\rangle) \begin{pmatrix} \Delta & \Omega \\ \Omega & 0 \end{pmatrix} \begin{pmatrix} \langle e| \\ \langle g| \end{pmatrix} \qquad \hat L = |g\rangle\langle e| = \sigma^- \qquad g_\mu \sim \Omega$$

Driven open quantum systems: microscopic description

- quantum master equation

Lindblad operators

$$\partial_t \hat{\rho} = \underbrace{-i[\hat{H},\hat{\rho}]}_{\text{coherent evolution}} + \sum_i \underbrace{\gamma_i[2\hat{L}_i\hat{\rho}\hat{L}_i^\dagger - \hat{L}_i^\dagger\hat{L}_i\hat{\rho} - \hat{\rho}\hat{L}_i^\dagger\hat{L}_i]}_{\text{driven-dissipative evolution}}$$



$$\equiv \hat{\mathcal{L}}[\hat{\rho}]$$ Lindbladian; also: Liouvillian

- derivation from ‘symmetry’ (i.e. implementing key physical requirements)
  - Lindbladian defines a dynamical map $\hat{\rho}(t+\Delta t) = \hat{\rho}(t) + \Delta t \cdot \hat{\mathcal{L}}[\hat{\rho}]$
  - with properties
    - Hermiticity: $\hat{\rho}(t)^\dagger = \hat{\rho}(t) \implies \hat{\rho}^\dagger(t+\Delta t) = \hat{\rho}(t+\Delta t)$ since $\hat{\mathcal{L}}[\hat{\rho}]^\dagger = \hat{\mathcal{L}}[\hat{\rho}]$
    - complete positivity: $\hat{\rho}(t) \geq 0 \implies \hat{\rho}(t+\Delta t) \geq 0$
    - trace preservation / probability conservation $\partial_t \mathrm{tr}\hat{\rho}(t) = 0$ since $\mathrm{tr}\hat{\mathcal{L}}[\hat{\rho}] = 0$

➡ up to a unitary transformation (above: diagonal form in index i), $\hat{\mathcal{L}}[\hat{\rho}]$ is the most general time-local generator with these properties

G. Lindblad, Commun. Math. Phys. (1976)

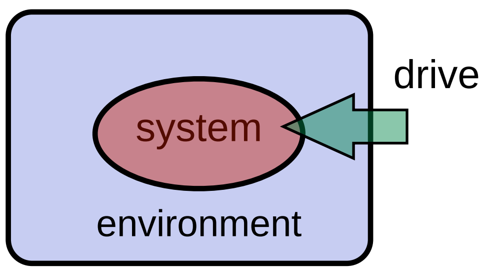
Nielsen & Chuang, Chap. 8

Driven open quantum systems: microscopic description

- quantum master equation

Lindblad operators

$$\partial_t \hat\rho = \underbrace{-i[\hat H, \hat\rho]}_{\text{coherent evolution}} + \sum_i \underbrace{\gamma_i [2\hat L_i \hat\rho \hat L_i^\dagger - \hat L_i^\dagger \hat L_i \hat\rho - \hat\rho \hat L_i^\dagger \hat L_i]}_{\text{driven-dissipative evolution}}$$



$$\equiv \hat{\mathcal{L}}[\hat\rho]$$ Lindbladian; also: Liouvillian

- interpretation:

$$\partial_t \hat\rho = -i(\hat H - \sum_i \gamma_i \hat L_i^\dagger \hat L_i)\hat\rho + \mathrm{h.c.} + 2\sum_i \gamma_i \hat L_i \hat\rho \hat L_i^\dagger$$

energy decay (dissipation) ensures probability conservation (fluctuation)

$$\text{“}E - i\Gamma\text{”}$$

$$\partial_t \mathrm{tr}\hat\rho(t) = 0$$

Driven open quantum systems: microscopic description

- quantum master equation

Lindblad operators

$$\partial_t \hat\rho = \underbrace{-i[\hat H, \hat\rho]}_{\text{coherent evolution}} + \sum_i \underbrace{\gamma_i [2\hat L_i \hat\rho \hat L_i^\dagger - \hat L_i^\dagger \hat L_i \hat\rho - \hat\rho \hat L_i^\dagger \hat L_i]}_{\text{driven-dissipative evolution}}$$

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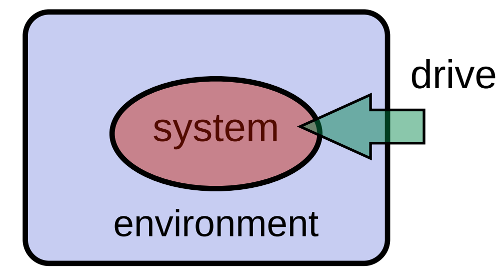


- So far: few degrees of freedom in the “system”
- Question: What if we replace few by many degrees of freedom?

➡ The interface of quantum optics and many-body physics

➡ Quantum Optics:
coherent and driven-dissipative dynamics on equal footing

➡ Many-Body Physics:
continuum of spatial degrees of freedom

➡ Statistical Mechanics:
physics at the largest distances

microphysics → macrophysics

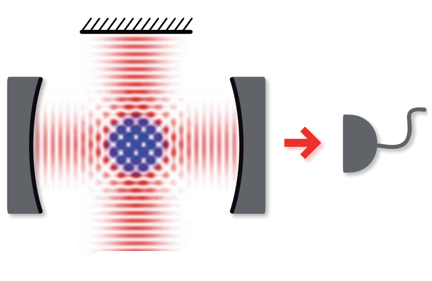
The interface of quantum optics and many-body physics

➡ Quantum Optics ➡ Many-Body Physics ➡ Statistical Mechanics

microphysics → macrophysics

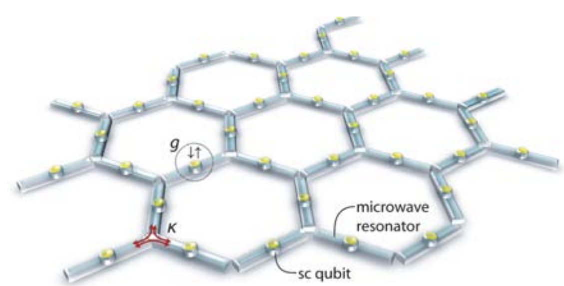
- The experimental platforms:

Atoms



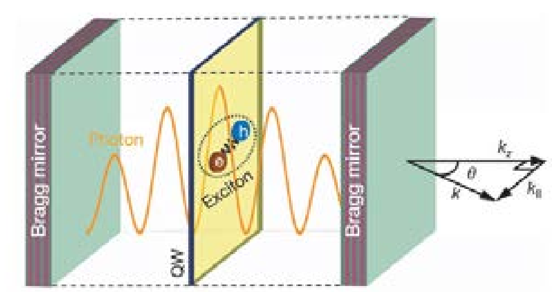
Bose-Einstein condensate in a cavity
Baumann et al., Nature 2010

Light



Microcavity arrays
Houck, Türeci, Koch, Nat. Phys. 2012

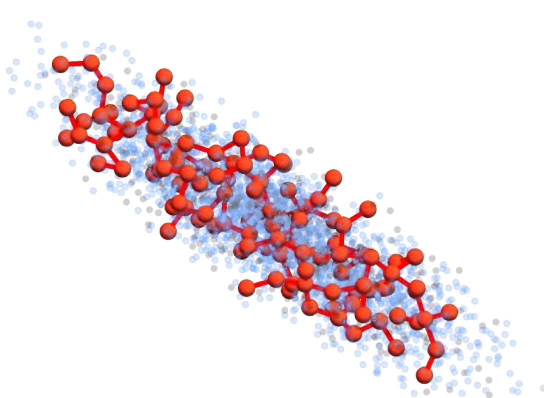
Solids



Exciton-polariton condensates
Kasprzak et al., Nature 2006

and more:

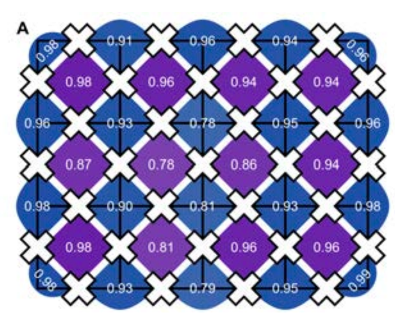
- polar molecules
- nano-mechanics
- photon BECs



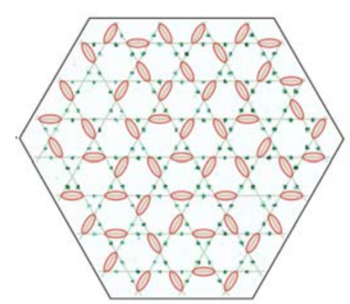
driven-dissipative Rydberg gases

S. Helmrich, A. Arias, G. Lochhead, M. Buchhold, SD, S. Whitlock, Nature (2020); T. Wintermantel, … SD, S. Whitlock, Nat. Comm. (2021)

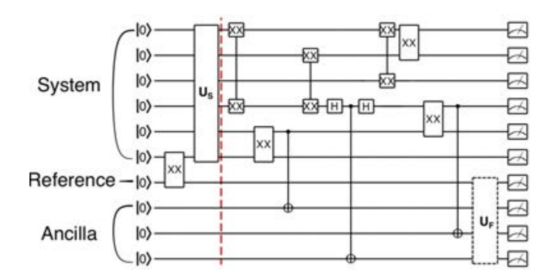
Quantum devices / NISQ Platforms



superconducting circuits
K. Satzinger et al. arxiv (2021)



Rydberg tweezers
G. Semeghini et al. arxiv (2021)
Lectures by A. Browaeys!



trapped ions
C. Noel et al. arxiv (2021)

The interface of quantum optics and many-body physics

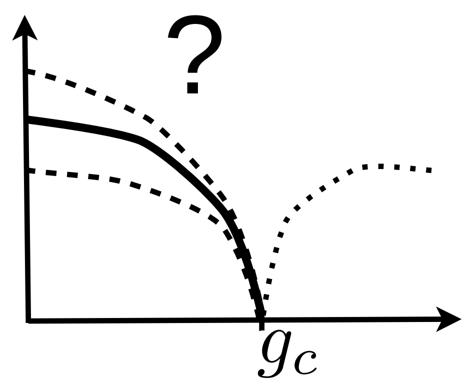
➡ Quantum Optics ➡ Many-Body Physics ➡ Statistical Mechanics

microphysics → macrophysics

Microscopic "Thermodynamic" Long wavelength

- Questions and challenges to theory: physics at various length scales

Novel universal phenomena ?



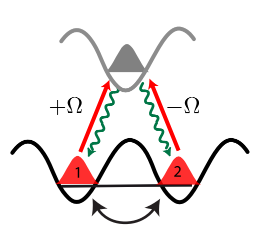
Efficient theoretical tools ?

$$Z[J] = \int \mathcal{D}\varphi \, e^{i(S[\varphi] + \int J\varphi)}$$

perform the transition form micro-to macrophysics:

quantum field theory out of equilibrium

Experimental platforms ?

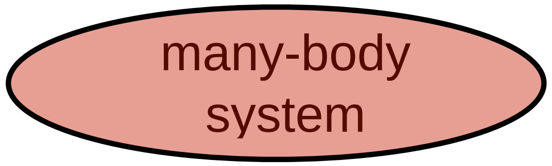


cold atoms, light-driven semiconductors, microcavity arrays, trapped ions ...

A workhorse model: Lindbladian formulation

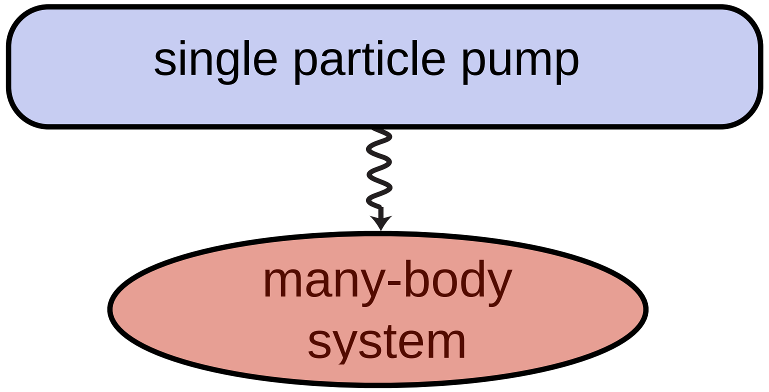
- generic microscopic many-body model: ‘Lindblad ϕ^4 theory’

$$\partial_t \rho = -i[H, \rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho]$$



$$H = \int_{\mathbf{x}} \hat{\phi}^{\dagger}_{\mathbf{x}} \left(\frac{\triangle}{2M} - \mu \right) \hat{\phi}_{\mathbf{x}} + \frac{\lambda}{2} (\hat{\phi}^{\dagger}_{\mathbf{x}} \hat{\phi}_{\mathbf{x}})^2$$

A workhorse model: Lindbladian formulation



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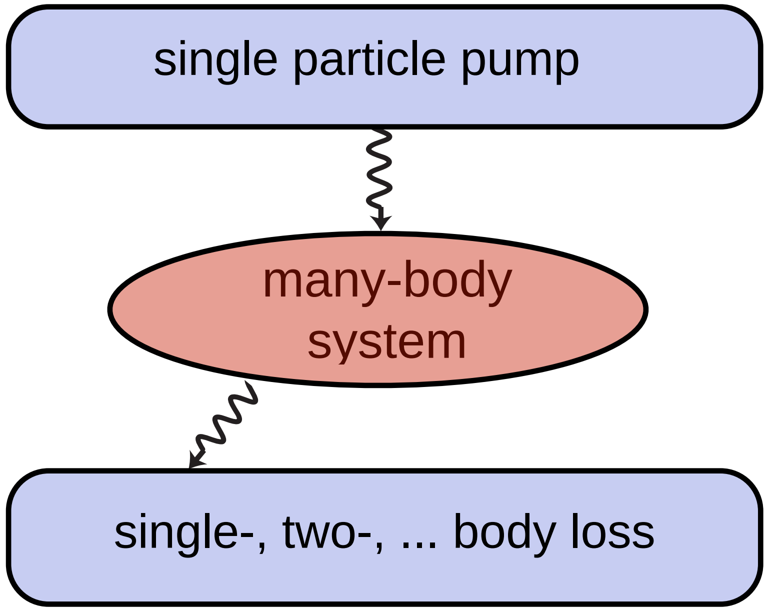
$$\mathcal{D}[\rho] = \gamma_p \int_{\mathbf{x}} [\hat{\phi}^\dagger_{\mathbf{x}} \rho \hat{\phi}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}_{\mathbf{x}} \hat{\phi}^\dagger_{\mathbf{x}}, \rho\}] \quad +$$

single particle pump

A workhorse model: Lindbladian formulation

- generic microscopic many-body model: 'Lindblad ϕ^4 theory'



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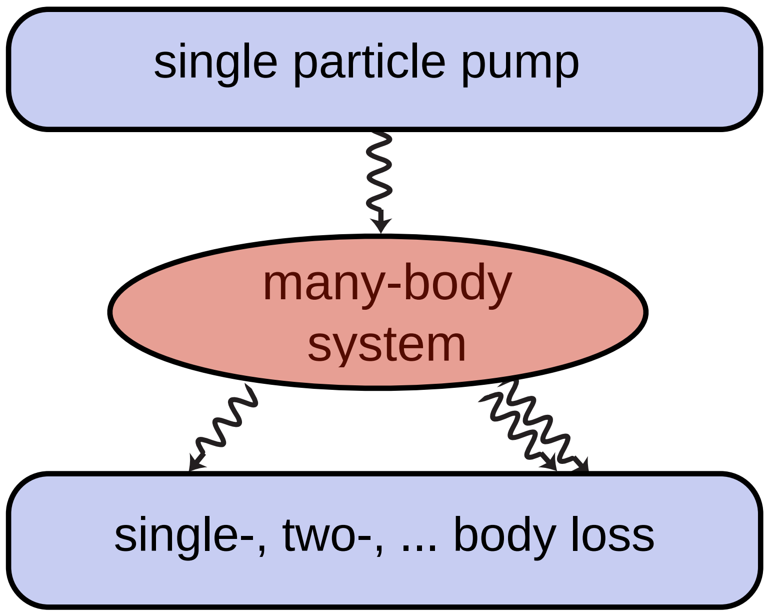
A workhorse model: Lindbladian formulation

- generic microscopic many-body model: ‘Lindblad ϕ^4 theory’



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single particle pump

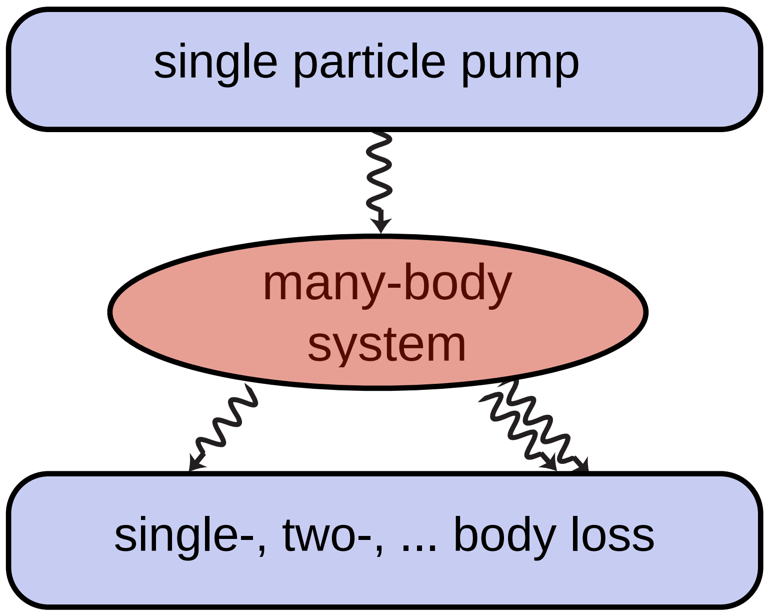
single particle loss

$$\kappa \int_{\mathbf{x}} [\hat{\phi}^2_{\mathbf{x}} \rho \, \hat{\phi}^{\dagger\,2}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^{\dagger\,2}_{\mathbf{x}} \hat{\phi}^2_{\mathbf{x}}, \rho\}]$$

two particle loss

A workhorse model: Lindbladian formulation

- generic microscopic many-body model: ‘Lindblad ϕ^4 theory’



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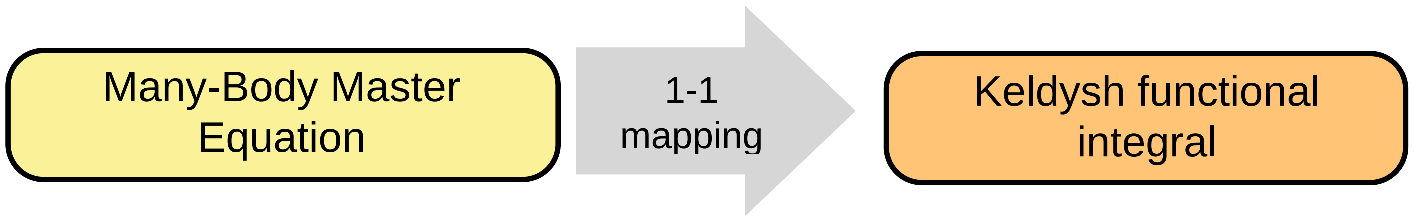
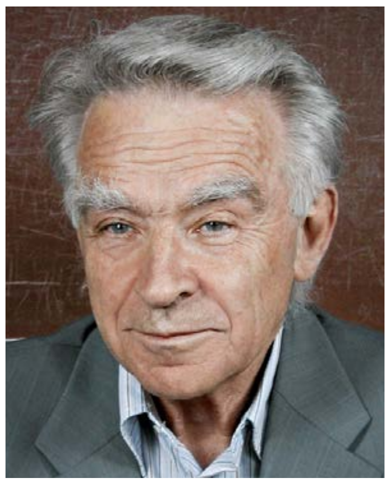
$$\underbrace{\kappa \int_{\mathbf{x}} [\hat{\phi}^2_{\mathbf{x}} \rho \, \hat{\phi}^{\dagger\,2}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^{\dagger\,2}_{\mathbf{x}} \hat{\phi}^2_{\mathbf{x}}, \rho\}]}_{\text{two particle loss}}$$

- plan:
  - translate to Lindblad-Keldysh functional integral



  - how does this model relate e.g. to exciton-polariton systems? (semiclassical limit)
  - ‘what is non-equilibrium about it’?
  - how to extract the phase structure?



Keldysh functional integral for stationary states of driven open quantum systems

- Construction from quantum master equation
- Semiclassical limit
- “What is non-equilibrium about it?”

Many-Body Master Equation $\xrightarrow{\text{1-1 mapping}}$ Keldysh functional integral

$$\partial_t \hat\rho = -i(\hat H - \sum_i \gamma_i \hat L_i^\dagger \hat L_i)\hat\rho + \text{h.c.} + 2\sum_i \gamma_i \hat L_i \hat\rho \hat L_i^\dagger$$

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) e^{\mathrm{i}(S_M[\Phi_+, \Phi_-]}$$

Keldysh functional integrals: Why?

- Feynman's formulation of quantum mechanics

REVIEWS OF

MODERN PHYSICS

VOLUME 20, NUMBER 2 APRIL, 1948

Space-Time Approach to Non-Relativistic Quantum Mechanics

R. P. FEYNMAN

Cornell University, Ithaca, New York

Non-relativistic quantum mechanics is formulated here in a different way. It is, however, mathematically equivalent to the familiar formulation. In quantum mechanics the probability of an event which can happen in several different ways is the absolute square of a sum of complex contributions, one from each alternative way. The probability that a particle will be found to have a path $x(t)$ lying somewhere within a region of space time is the square of a sum of contributions, one from each path in the region. The contribution from a single path is postulated to be an exponential whose (imaginary) phase is the classical action (in units of $\hbar$) for the path in question. The total contribution from all paths reaching x, t from the past is the wave function $\psi(x, t)$. This is shown to satisfy Schroedinger's equation. The relation to matrix and operator algebra is discussed. Applications are indicated, in particular to eliminate the coordinates of the field oscillators from the equations of quantum electrodynamics.

1. INTRODUCTION

IT is a curious historical fact that modern quantum mechanics began with two quite different mathematical formulations: the differential equation of Schroedinger, and the matrix algebra of Heisenberg. The two, apparently dissimilar approaches, were proved to be mathematically equivalent. These two points of view were destined to complement one another and to be ultimately synthesized in Dirac's transformation theory.

This paper will describe what is essentially a classical action[3] to quantum mechanics. A probability amplitude is associated with an entire motion of a particle as a function of time, rather than simply with a position of the particle at a particular time.

The formulation is mathematically equivalent to the more usual formulations. There are, therefore, no fundamentally new results. However, there is a pleasure in recognizing old things from a new point of view. Also, there are problems for which the new point of view offers a distinct advantage. For example, if two systems

- Useful language for systems with many degrees of freedom

- general: powerful techniques
  - diagrammatic perturbation theory;
  - collective variables;
  - renormalization group

Keldysh functional integrals: Why?

- Feynman's formulation of quantum mechanics

REVIEWS OF

MODERN PHYSICS

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Space-Time Approach to Non-Relativistic Quantum Mechanics

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Non-relativistic quantum mechanics is formulated here in a different way. It is, however, mathematically equivalent to the familiar formulation. In quantum mechanics the probability of an event which can happen in several different ways is the absolute square of a sum of complex contributions, one from each alternative way. The probability that a particle will be found to have a path $x(t)$ lying somewhere within a region of space time is the square of a sum of contributions, one from each path in the region. The contribution from a single path is postulated to be an exponential whose (imaginary) phase is the classical action (in units of $\hbar$) for the path in question. The total contribution from all paths reaching x, t from the past is the wave function $\psi(x, t)$. This is shown to satisfy Schroedinger's equation. The relation to matrix and operator algebra is discussed. Applications are indicated, in particular to eliminate the coordinates of the field oscillators from the equations of quantum electrodynamics.

1. INTRODUCTION

It is a curious historical fact that modern quantum mechanics began with two quite different mathematical formulations: the differential equation of Schroedinger, and the matrix algebra of Heisenberg. The two, apparently dissimilar approaches, were proved to be mathematically equivalent. These two points of view were destined to complement one another and to be ultimately synthesized in Dirac's transformation theory.

This paper will describe what is essentially a classical action[3] to quantum mechanics. A probability amplitude is associated with an entire motion of a particle as a function of time, rather than simply with a position of the particle at a particular time.

The formulation is mathematically equivalent to the more usual formulations. There are, therefore, no fundamentally new results. However, there is a pleasure in recognizing old things from a new point of view. Also, there are problems for which the new point of view offers a distinct advantage. For example, if two systems

- Useful language for systems with many degrees of freedom
  - general: powerful techniques
    - diagrammatic perturbation theory;
    - collective variables;
    - renormalization group
  - non-equilibrium Keldysh
    - closer to the real-time formulations of quantum mechanics
    - yields directly observable quantities (responses and correlations)
    - indispensable for many systems:
      - disorder
      - dissipation

      infinite harmonic baths!
    - open the powerful toolbox of quantum field theory for many-body non-equilibrium situations

Keldysh functional integral

more details: L. Sieberer, M. Buchhold, SD, *Keldysh Field Theory for Driven Open Quantum Systems*, Reports on Progress in Physics (2016)

- The basic idea in three steps:

$$\hbar = 1$$

$$U(t, t_0) = e^{-iH(t-t_0)}$$

1. Schrödinger equation: evolving a state vector

$$i\partial_t |\psi\rangle(t) = H|\psi\rangle(t) \quad \Rightarrow |\psi\rangle(t) = U(t, t_0)|\psi\rangle(t_0)$$

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$$\partial_t\rho(t) = -i[H, \rho(t)] \quad \Rightarrow \rho(t) = U(t, t_0)\rho(t_0)U^\dagger(t, t_0)$$

   - identical for pure (factorizable) states $\rho = |\psi\rangle\langle\psi|$

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3. The same is true for the Master Equation:

$$\partial_t \rho = -i[H,\rho] + \kappa \sum_i L_i \rho L_i^\dagger - \tfrac{1}{2}\{L_i^\dagger L_i, \rho\} \;\equiv \mathcal{L}[\rho]$$

$$\Rightarrow \rho(t) = e^{\mathcal{L}(t-t_0)}\rho(t_0)$$

Keldysh functional integral (bosons)

fermions: see appendix!

1. Functional integral idea:

   ➡ "Trotterization" of time interval and insertion of coherent states: $e^{iH(t-t_0)} = \lim_{N\to\infty} (1 + i\delta_t H)^N$

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- one time step

coherent states (bosons):

$$a|\phi\rangle = \phi|\phi\rangle$$

$$\langle\phi'|\phi\rangle = e^{\phi'^*\phi}$$

$$\mathbf{1} = \int \frac{d\phi^* d\phi}{\pi} e^{-\phi^*\phi} |\phi\rangle\langle\phi|$$

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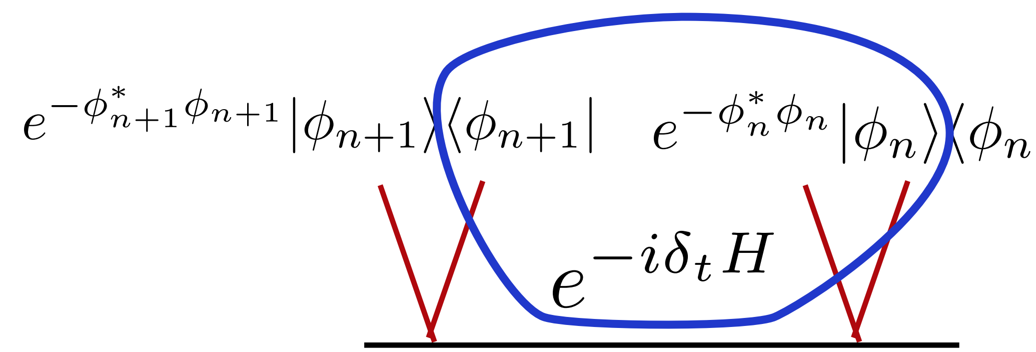

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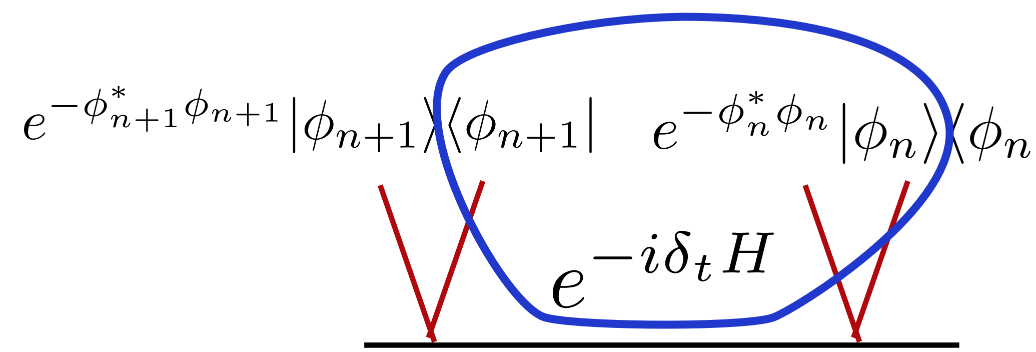
- one time step

$$e^{-\phi_n^*\phi_n} \langle \phi_{n+1} | e^{-i\delta_t H[a^\dagger, a]} | \phi_n \rangle$$



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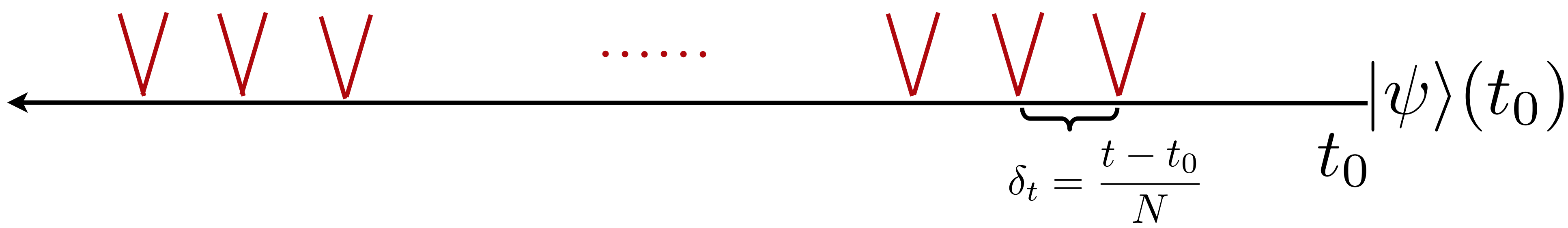
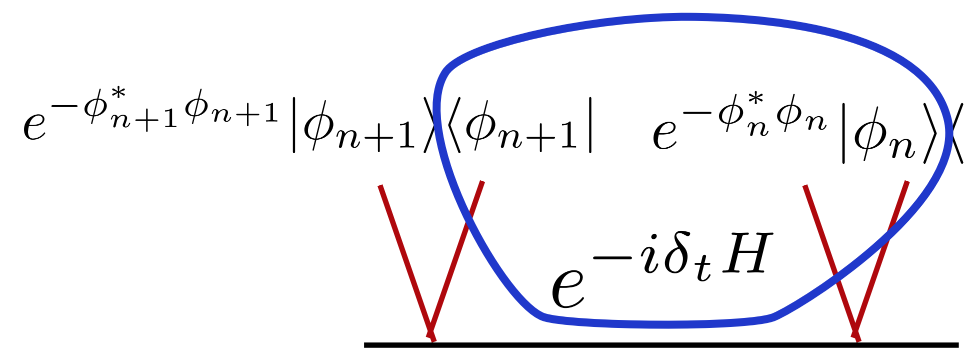


- one time step



$$e^{-\phi^*_n\phi_n}\langle\phi_{n+1}|e^{-i\delta_t H[a^\dagger,a]}|\phi_n\rangle$$

$$\approx e^{-\phi^*_n\phi_n}\langle\phi_{n+1}|1-i\delta_t H[a^\dagger,a]|\phi_n\rangle$$

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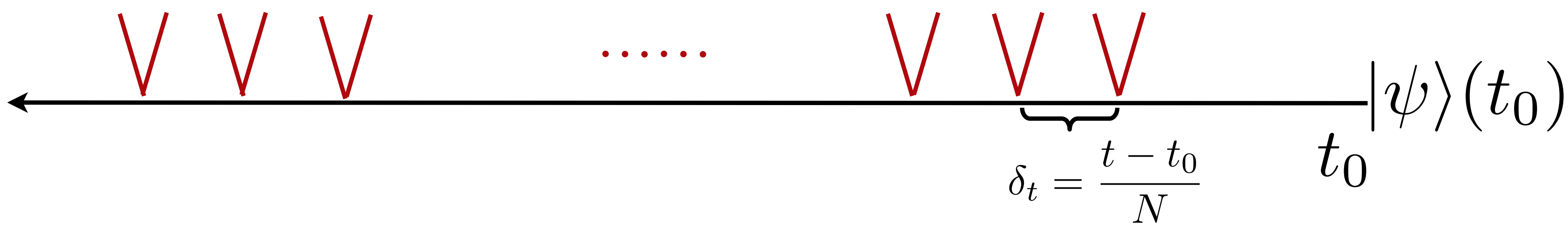

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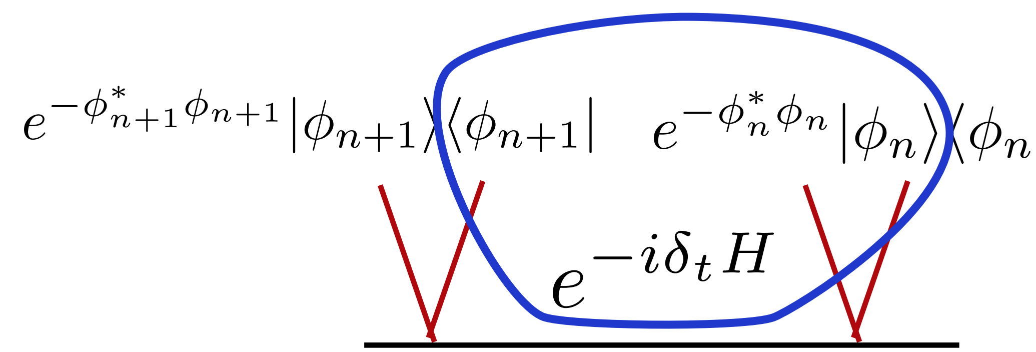

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$$\approx e^{-\phi^*_n\phi_n}\langle\phi_{n+1}|1 - i\delta_t H[a^\dagger,a]|\phi_n\rangle$$

H normally ordered

$$= e^{-\phi^*_n\phi_n} e^{+\phi^*_{n+1}\phi_n}(1 - i\delta_t H[\phi^*_{n+1},\phi_n])$$

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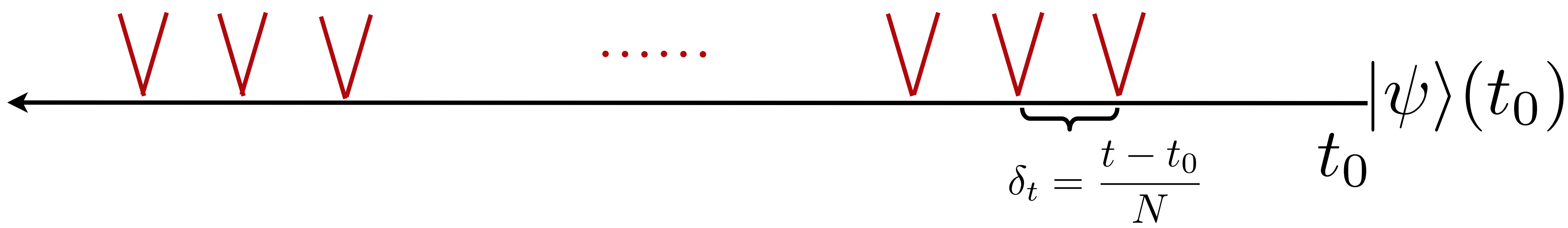

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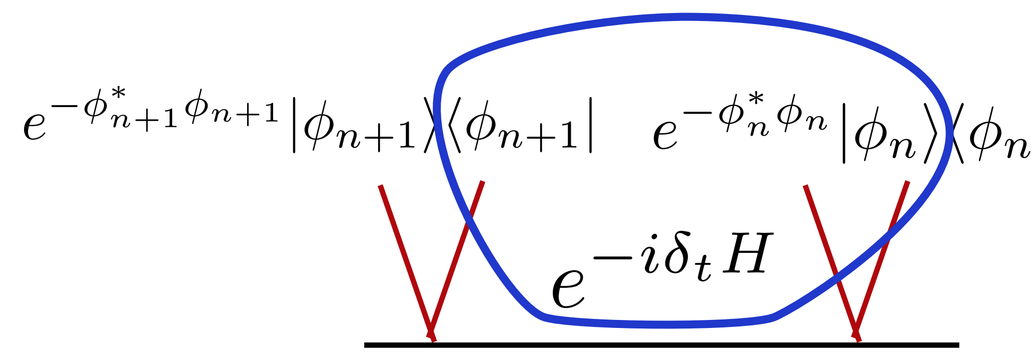

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$$\approx e^{i\delta_t[-i\frac{(\phi_{n+1}^*-\phi_n^*)}{\delta_t}\phi_n - H[\phi_{n+1}^*,\phi_n]]}$$

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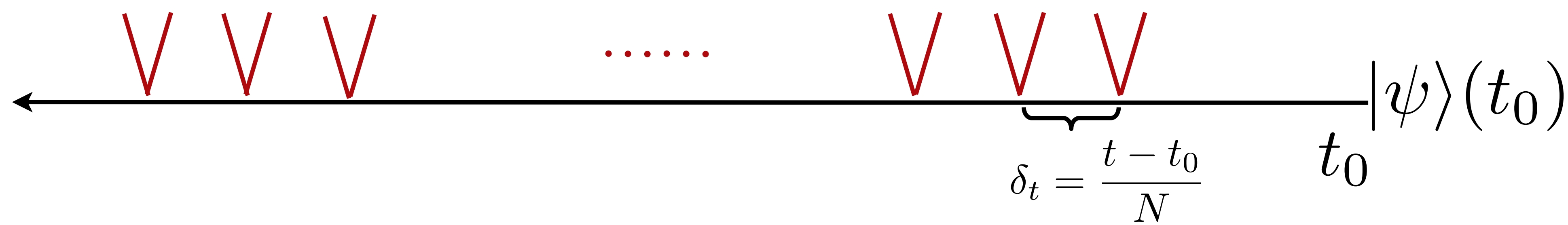
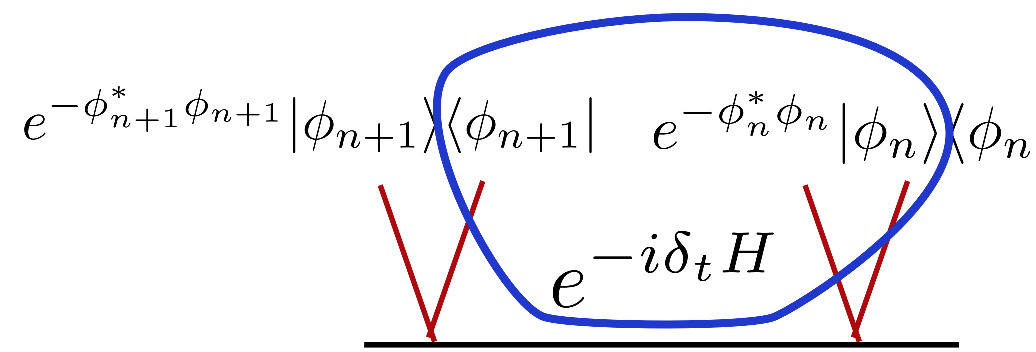


- one time step



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continuum limit

$$dt \qquad -i\partial_t\phi^*(t)\cdot\phi(t) \qquad H[\phi^*(t),\phi(t)]$$

coherent states (bosons):

$$a|\phi\rangle = \phi|\phi\rangle$$

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Keldysh functional integral

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➡ “Trotterization” of time interval and insertion of coherent states: $e^{iH(t-t_0)} = \lim_{N\to\infty}(1+i\delta_t H)^N$

- many time steps

$$\underbrace{\int \prod_t \frac{d\phi^*(t)d\phi(t)}{\pi}}_{=:\int \mathcal{D}(\phi^*,\phi)} e^{i\int_{t_0}^{t_f} dt[-i\partial_t\phi^*(t)\cdot\phi(t) - H[\phi^*(t),\phi(t)]]}$$

functional integral measure

- Discussion

➡ operator H -> complex, time dependent functional H

➡ time evolution from overlap of neighbouring states

➡ no reference to single particle or many-body Hamiltonian, lattice or continuum!

➡ single set of degrees of freedom for vector evolution

Keldysh functional integral

2. Schrödinger vs. Heisenberg-von Neumann

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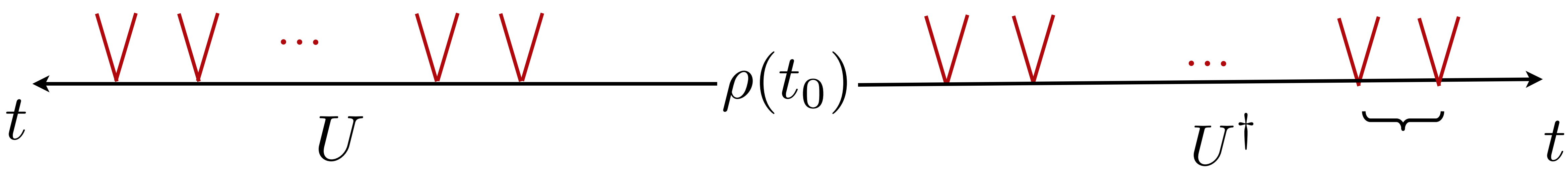
- Second case: “Trotterization” on both sides:



$$e^{iH(t-t_0)} = \lim_{N\to\infty}(1+i\delta_t H)^N \qquad \delta_t = \frac{t-t_0}{N}$$

➡ two sets of degrees of freedom for matrix evolution

Keldysh functional integral

3. Schrödinger vs. Quantum Master

$$U(t,t_0) = e^{-iH(t-t_0)}$$

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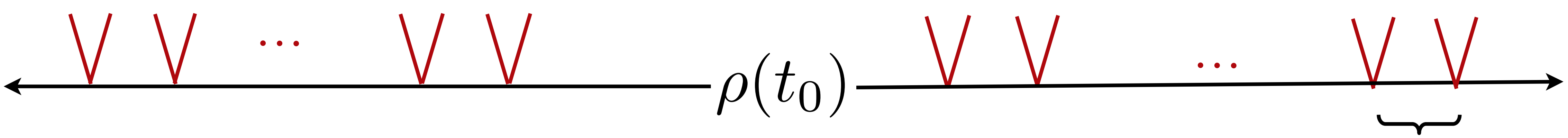
$$\partial_t\rho = -i[H,\rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho] \qquad \Rightarrow \rho(t) = e^{\mathcal{L}(t-t_0)}\rho(t_0)$$

- Identical program for Liouville generator of dynamics (left and right action on density matrix)



$$\rho(t) = e^{(t-t_0)\mathcal{L}}\ \rho_0 = \lim_{N\to\infty}\left(1+\delta_t\mathcal{L}\right)^N \rho_0 \qquad\qquad \delta_t = \frac{t-t_0}{N}$$

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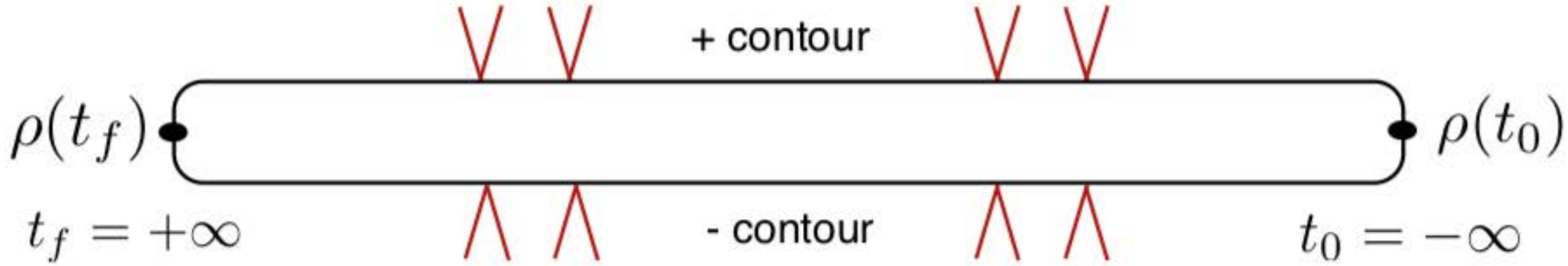
- final step: Keldysh "partition function"

$$Z = \mathrm{tr}\rho(t) = \mathrm{tr}\rho(t_0) = 1$$



$t_0 \to -\infty, t_f \to +\infty$

information on all stages;
stationarity reached
(boundary conditions
irrelevant)

Keldysh functional integral: Final result

- quantum master equation: $$\partial_t \rho = -i[H,\rho] + \mathcal{D}[\rho]$$
$$= -i(H\rho - \rho H) + \kappa \sum_i (L_i \rho L_i^\dagger - \tfrac{1}{2} L_i^\dagger L_i \rho - \tfrac{1}{2}\rho L_i^\dagger L_i)$$

- equivalent Keldysh functional integral:

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) e^{\mathrm{i}(S_M[\Phi_+,\Phi_-]}$$

$$\Phi_\pm = \begin{pmatrix} \phi_\pm \\ \phi_\pm^* \end{pmatrix}$$

$$S_M[\Phi_+,\Phi_-] = \int dt (\phi_+^* i\partial_t \phi_+ - \phi_-^* i\partial_t \phi_- - i\mathcal{L}[\Phi_+,\Phi_-])$$

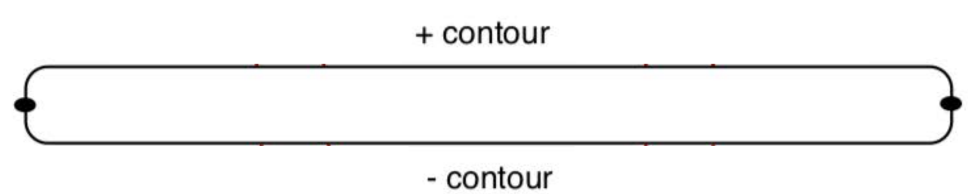
$$\boxed{\mathcal{L}[\Phi_+,\Phi_-] = -i\,(H_+ - H_-) - \kappa \sum_i \left(L_{i,+} L_{i,-}^\dagger - \tfrac{1}{2} L_{i,+}^\dagger L_{i,+} - \tfrac{1}{2} L_{i,-}^\dagger L_{i,-} \right)}$$

$$H_\pm = H(\Phi_\pm) \text{ etc.}$$

➡ recognize Lindblad structure

➡ simple translation table (for contour normal ordered Lindbladian)

- operator right of density matrix -> - contour
- operator left of density matrix -> + contour



Keldysh functional integral: Probability conservation / "causality"

- quantum master equation: $\partial_t \rho = -i[H,\rho] + \mathcal{D}[\rho]$

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$H_\pm = H(\Phi_\pm)$ etc.

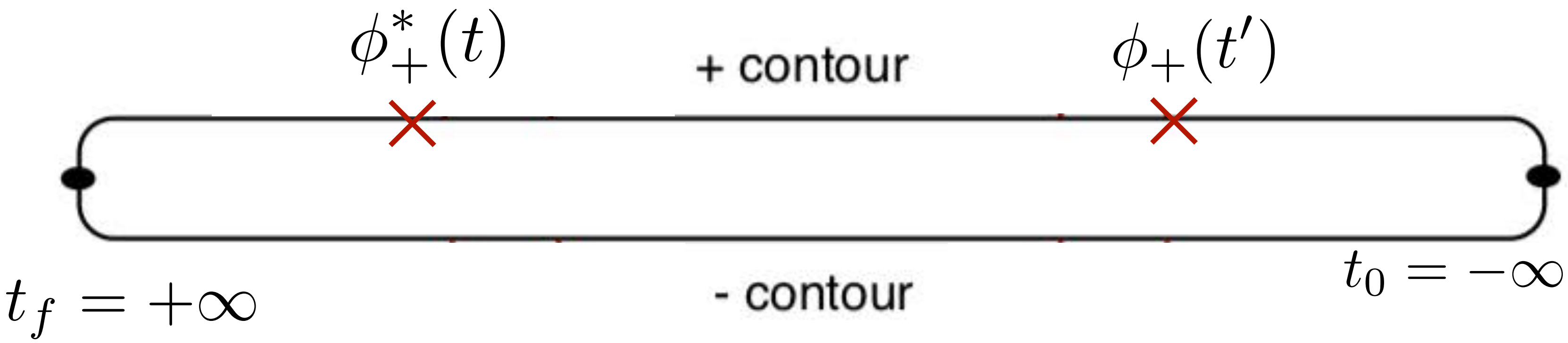
- trace / probability preservation:
- QME: $\partial_t \mathrm{tr}\rho = \mathrm{tr}\big(-i(H\rho - \rho H) + \kappa \sum_i (L_i \rho L_i^\dagger - \tfrac{1}{2} L_i^\dagger L_i \rho - \tfrac{1}{2} \rho L_i^\dagger L_i)\big) = 0$
- Keldysh: $Z = \mathrm{tr}\rho(t) = 1$

cyclicity

- mnemonic: taking trace = ignoring contour order: $\Phi_+ = \Phi_- \Rightarrow \boxed{S_M[\Phi_+, \Phi_-] = 0}$

Physical observables

- correlation functions: field insertions on the contour



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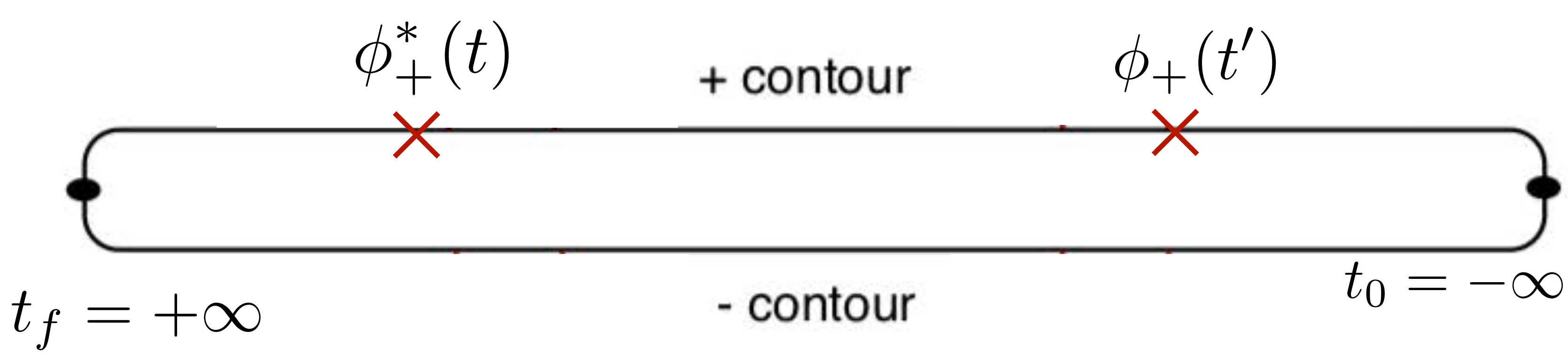

- compute them:
  - introduce sources (cf. Stat Mech)

$$Z[j_+, j_-] = \langle e^{\mathrm{i}\int (j_+\phi_+^* - j_-\phi_-^* + c.c.)} \rangle$$

$$Z = \mathrm{Tr}(1 \cdot \rho) = \langle 1 \rangle$$

$$Z[0,0] = \langle 1 \rangle = 1$$

normalization

Physical observables

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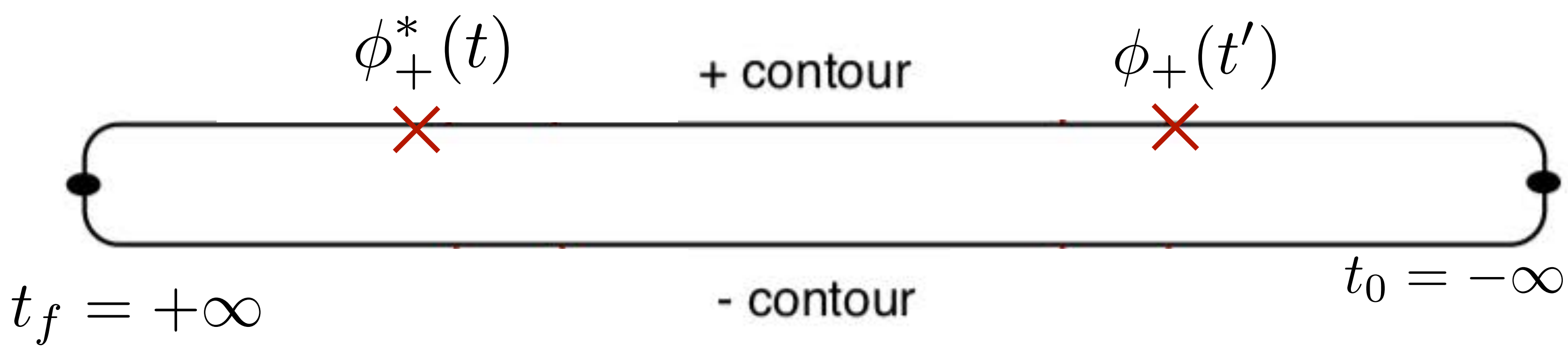

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normalization

  - take variational derivative; example above:

$$\langle \phi_+^*(t)\phi_-(t') \rangle = \frac{\delta^2 Z}{\delta j_+(t)\delta j_+^*(t')}$$

➡ interpretation?

➡ there is a more intuitive basis to do computations

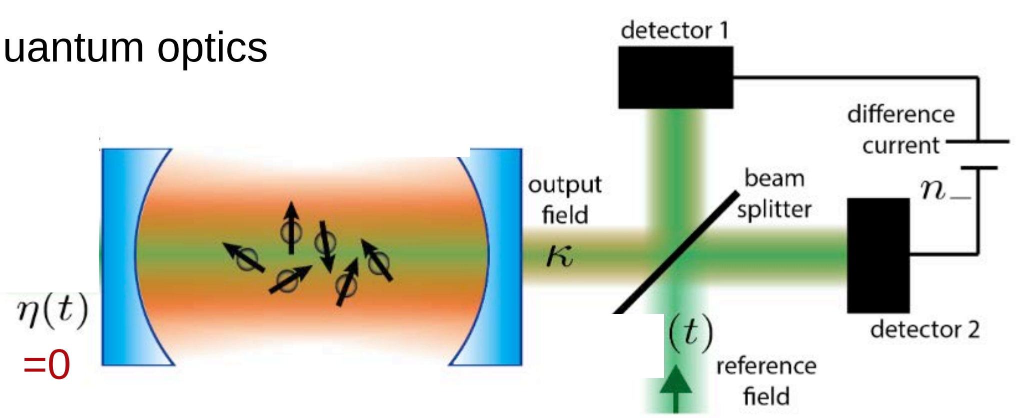
Correlation vs. response functions: Physics

more details: L. Sieberer, M. Buchhold, SD, Reports on Progress in Physics (2016)

- two basic types of experiments:
  - correlation measurements: study without disturbing

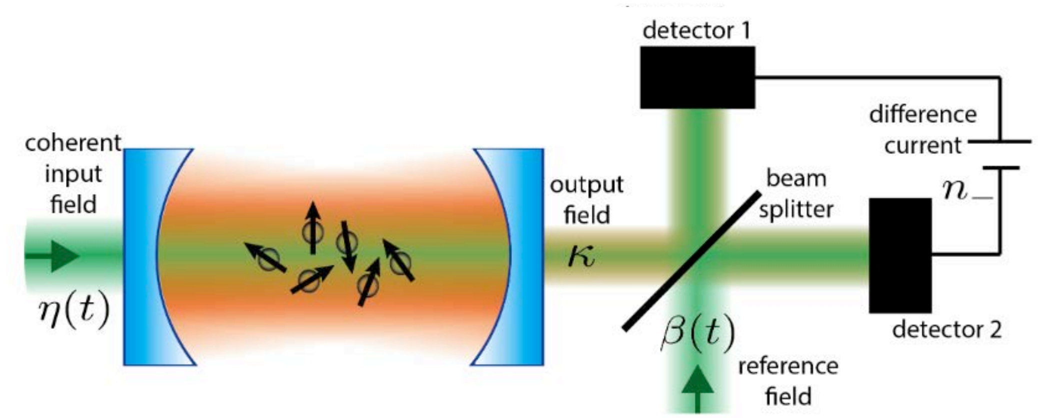
- (linear) response measurements: probe system with (weak) external fields

eg. quantum optics



e.g. photon quadrature component at vacuum input field

(or: $g^{(1)}(\tau)$)



e.g. coherent input field

in homodyne detection: retarded response of quadrature components

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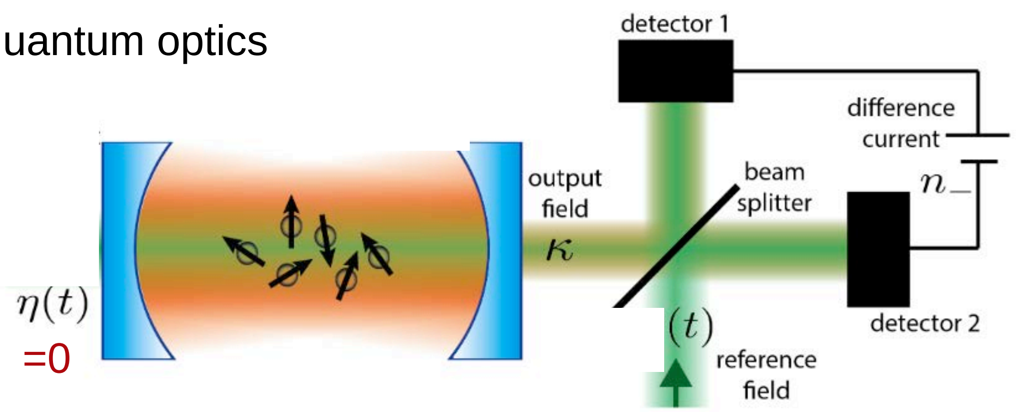

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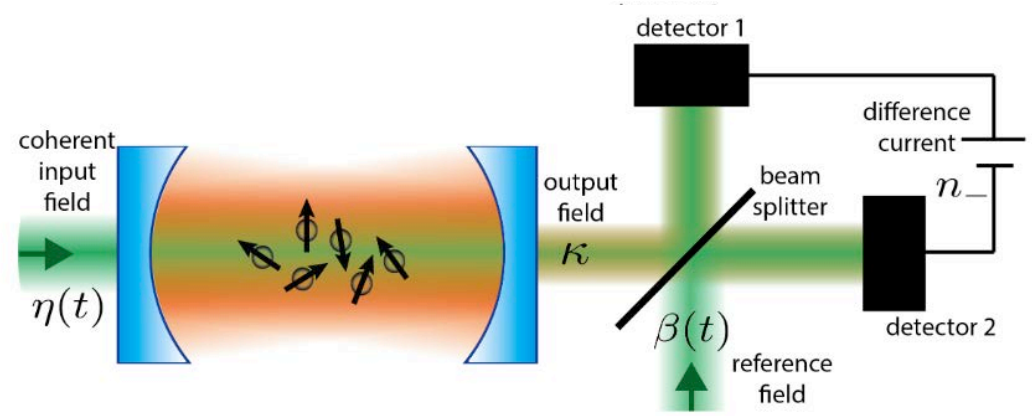

e.g. coherent input field

in homodyne detection: retarded response of quadrature components

- directly delivered in the functional framework via basis transformation: Keldysh rotation

$$\begin{pmatrix}\phi_c\\ \phi_q\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_+ + \phi_-\\ \phi_+ - \phi_-\end{pmatrix}$$

"classical field": center-of-mass coordinate

"quantum field": relative coordinate

- classical field can acquire finite expectation value (e.g. lasing, Bose condensation)
- quantum / noise field cannot
- probability preservation: $\boxed{S_M[\Phi_c, \Phi_q = 0] = 0 \quad \forall \Phi_c}$

exercise: verify relation to operator formalism!

Correlation vs. response functions: Calculation

more details: L. Sieberer, M. Buchhold, SD, Reports on Progress in Physics (2016)

- partition function in new basis

$$Z[j] = \langle e^{\mathrm{i}\int (j_+\phi_+^* - j_-\phi_-^* + c.c.)}\rangle = \langle e^{\mathrm{i}\int (j_c\phi_q^* + j_q\phi_c^* + c.c.)}\rangle$$

- order parameter / occupation field:

$$\langle\phi_c(t,\mathbf{x})\rangle = -i\frac{\delta Z[j]}{\delta j_q^*(t,\mathbf{x})}\Big|_{j=0}$$

q,c appear as conjugate pairs for the source

homodyne detection: vacuum input

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q,c appear as conjugate pairs for the source

homodyne detection: vacuum input

- single particle response: how does the field react to external perturbations?

relation to operator formalism (once and for all)

response to coherent field

t = t'

$$G^R(t-t',\mathbf{x}-\mathbf{x}') = i\frac{\delta^2 Z}{\delta j_q^*(t,\mathbf{x})\delta j_c(t',\mathbf{x}')}\Big|_{j=0} = -i\langle\phi_c(t,\mathbf{x})\phi_q^*(t',\mathbf{x}')\rangle \overset{\downarrow}{=} -i\theta(t-t')\langle[\hat{\phi}(t,\mathbf{x}),\hat{\phi}^\dagger(t',\mathbf{x}')]\rangle \overset{\downarrow}{=} 1$$

Correlation vs. response functions: Calculation

exercise: verify relation to operator formalism!

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- single particle correlations: how are states occupied?

t = t', x=x'

$$G^K(t-t',\mathbf{x}-\mathbf{x}') = i\frac{\delta^2 Z}{\delta j_q^*(t,\mathbf{x})\delta j_q(t',\mathbf{x}')}\Big|_{j=0} = -i\langle\phi_c(t,\mathbf{x})\phi_c^*(t',\mathbf{x}')\rangle = -i\langle\{\hat{\phi}(t,\mathbf{x}),\hat{\phi}^\dagger(t',\mathbf{x}')\}\rangle = 2\langle\hat{n}(\mathbf{x})\rangle + 1$$

time and space translation invariance assumed

Correlation vs. response functions: Calculation

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time and space translation invariance assumed

- total Green's function

$$G = \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix}$$

$$G^A = (G^R)^\dagger, \qquad (G^K)^\dagger = -G^K$$

Hermitian conjugates

anti-Hermitian

Correlation vs. response: relation to Keldysh action

exercise: verify calculations on this page!

more details: L. Sieberer, M. Buchhold, SD, Reports on Progress in Physics (2016)

- by example: master equation for decaying cavity

$$\partial_t \rho = -i[\omega_0 \hat{a}^\dagger \hat{a}, \rho] + \kappa(2\hat{a}\rho\hat{a}^\dagger - \{\hat{a}^\dagger\hat{a}, \rho\})$$

- action:

$$S = \int dt (a_{cl}^*, a_q^*) \begin{pmatrix} 0 & \overbrace{i\partial_t - \omega_0 - i\kappa}^{} \\ i\partial_t - \omega_0 + i\kappa & 2i\kappa \end{pmatrix} \begin{pmatrix} a_{cl} \\ a_q \end{pmatrix}$$

time domain

$$a_\nu(t)$$

$$\equiv P^A(\omega)$$

$$= \int \frac{d\omega}{2\pi} (a_{cl}^*, a_q^*) \underbrace{\begin{pmatrix} 0 & \omega - \omega_0 - i\kappa \\ \underset{\equiv P^R(\omega)}{\omega - \omega_0 + i\kappa} & \underset{\equiv P^K}{2i\kappa} \end{pmatrix}}_{G^{-1}(\omega)} \begin{pmatrix} a_{cl} \\ a_q \end{pmatrix}$$

frequency domain

$$a_\nu(\omega)$$

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- action:

time domain $a_\nu(t)$

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- partition function: completion of the square

$$Z[j_c, j_q] = \langle e^{i\int \frac{d\omega}{2\pi}(j_c^* a_q + j_q^* a_c + \mathrm{h.c.})} \rangle = e^{i\int \frac{d\omega}{2\pi}(j_q^*, j_c^*) G(\omega) \begin{pmatrix} j_q \\ j_c \end{pmatrix}}$$

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- single particle Green's function

$$\begin{pmatrix} \langle \phi_c(\omega)\phi^*_c(\omega')\rangle & \langle \phi_c(\omega)\phi^*_q(\omega')\rangle \\ \langle \phi_q(\omega)\phi^*_c(\omega')\rangle & \langle \phi_q(\omega)\phi^*_q(\omega')\rangle \end{pmatrix} = - \begin{pmatrix} \frac{\delta^2 Z}{\delta j^*_q(\omega)\delta j_q(\omega')} & \frac{\delta^2 Z}{\delta j^*_q(\omega)\delta j_c(\omega')} \\ \frac{\delta^2 Z}{\delta j^*_c(\omega)\delta j_q(\omega')} & \frac{\delta^2 Z}{\delta j^*_c(\omega)\delta j_c(\omega')} \end{pmatrix}\Bigg|_{j=0} = iG(\omega)\delta(\omega - \omega')$$

Correlation vs. response: relation to Keldysh action

exercise: verify calculations on this page!

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- action:

$$S = \int dt (a_{cl}^*, a_q^*) \begin{pmatrix} 0 & i\partial_t - \omega_0 - i\kappa \\ i\partial_t - \omega_0 + i\kappa & 2i\kappa \end{pmatrix} \begin{pmatrix} a_{cl} \\ a_q \end{pmatrix}$$

time domain $a_\nu(t)$

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- summary in matrix components (valid beyond example):

$$\boxed{G^{-1} = \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \Longrightarrow G = \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix} \qquad G^{R/A} = [P^{R/A}]^{-1} \qquad G^K = -G^R P^K G^A}$$

action matrix kernel

single particle Green's function

Correlation vs. response: relation to Keldysh action

exercise: verify calculations on this page!

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time domain

$$a_\nu(t)$$

$$= \int \frac{d\omega}{2\pi} (a_{cl}^*, a_q^*) \begin{pmatrix} 0 & \omega - \omega_0 - i\kappa \\ \omega - \omega_0 + i\kappa & 2i\kappa \end{pmatrix} \begin{pmatrix} a_{cl} \\ a_q \end{pmatrix}$$

frequency domain

$$a_\nu(\omega)$$

- observables from the Green's functions:

response

  - decay of single-particle response: $$G^R(t-t') = \int_\omega e^{i\omega(t-t')} G^R(\omega) = \theta(t-t') e^{i\omega(t-t')} e^{-\kappa(t-t')}$$
  - Lorentzian spectral density: $$A(\omega) = \mathrm{Im} G^R(\omega) = \frac{2\kappa}{(\omega-\omega_0)^2 + \kappa^2}$$

Correlation vs. response: relation to Keldysh action

exercise: verify calculations on this page!

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time domain $a_\nu(t)$

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frequency domain $a_\nu(\omega)$

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response

- decay of single-particle response: $G^R(t-t') = \int_\omega e^{i\omega(t-t')} G^R(\omega) = \theta(t-t') e^{i\omega(t-t')} e^{-\kappa(t-t')}$
- Lorentzian spectral density: $A(\omega) = \mathrm{Im} G^R(\omega) = \dfrac{2\kappa}{(\omega-\omega_0)^2 + \kappa^2}$

correlation

- cavity mode occupation in stationary state $2\langle \hat{n}(t)\rangle + 1 = \langle \hat{a}^\dagger(t)\hat{a}(t) + \hat{a}(t)\hat{a}^\dagger(t)\rangle = iG^K(t-t) = i\int_\omega e^{i\omega(t-t)} G^K(\omega) = 1$

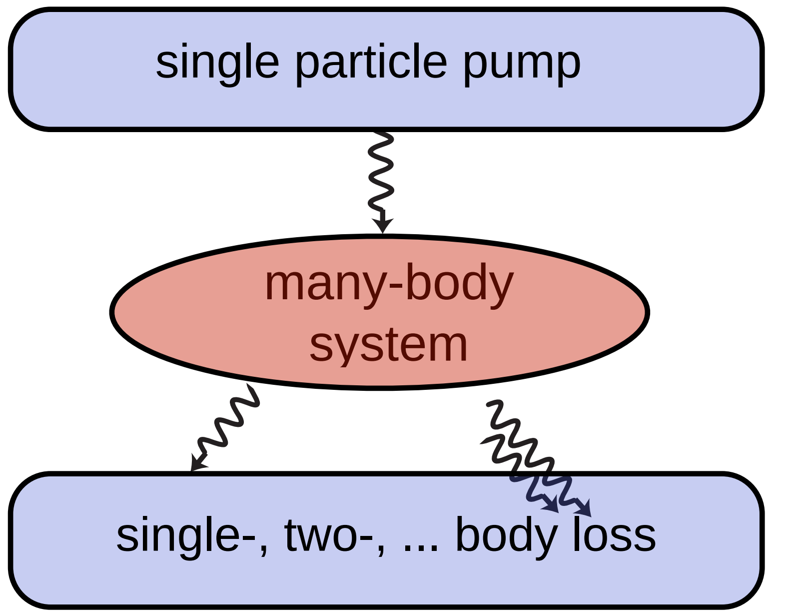
$\langle \hat{n}(t\to\infty)\rangle = 0$ $(t \to \infty)$

➡ correlation / statistical properties: G^K

➡ response / spectral properties: G^R

Back to many-body model: Workhorse Lindbladian

- generic microscopic many-body model:



$$\partial_t \rho = -i[H,\rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho]$$

$$H = \int_{\mathbf{x}} \hat{\phi}^\dagger_{\mathbf{x}} \left(\frac{\triangle}{2M} - \mu\right) \hat{\phi}_{\mathbf{x}} + \frac{\lambda}{2}(\hat{\phi}^\dagger_{\mathbf{x}} \hat{\phi}_{\mathbf{x}})^2$$

kinetic energy — two-particle interaction

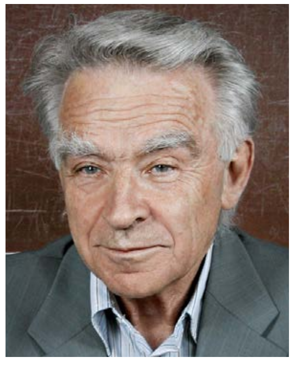
$$\mathcal{D}[\rho] = \gamma_p \int_{\mathbf{x}} [\hat{\phi}^\dagger_{\mathbf{x}} \rho \hat{\phi}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}_{\mathbf{x}} \hat{\phi}^\dagger_{\mathbf{x}}, \rho\}] \quad + \quad \gamma_l \int_{\mathbf{x}} [\hat{\phi}_{\mathbf{x}} \rho \hat{\phi}^\dagger_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^\dagger_{\mathbf{x}} \hat{\phi}_{\mathbf{x}}, \rho\}] \quad +$$

single particle pump — single particle loss

$$\kappa \int_{\mathbf{x}} [\hat{\phi}^2_{\mathbf{x}} \rho \hat{\phi}^{\dagger\,2}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^{\dagger\,2}_{\mathbf{x}} \hat{\phi}^2_{\mathbf{x}}, \rho\}]$$

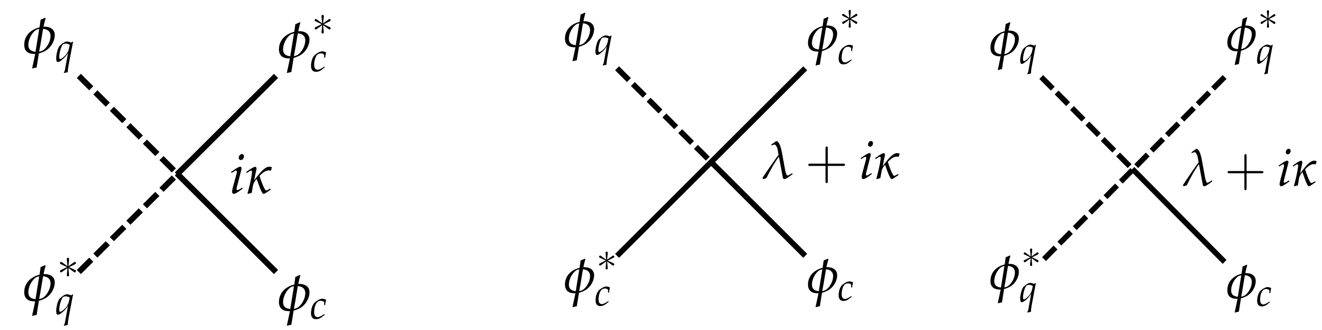
two-particle loss

Many-Body Master Equation → 1-1 mapping → Keldysh functional integral



Many-body model: Workhorse Lindblad-Keldysh action

$$\mathcal{S} = \int_{t,\mathbf{x}} \left\{ \left(\phi_c^*, \phi_q^*\right) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} + 2i\kappa \phi_c^* \phi_c \phi_q^* \phi_q \; - \tfrac{1}{2} \left[(\lambda + i\kappa) \left(\phi_c^{*2} \phi_c \phi_q + \phi_q^{*2} \phi_c \phi_q \right) + c.c. \right] \right\}$$



- Gaussian sector: inverse Green's function
  - retarded/advanced $\quad P^R(\omega, \mathbf{q}) = \omega - \mathbf{q}^2 - \mu + i\left(\gamma_l - \gamma_p\right)/2$

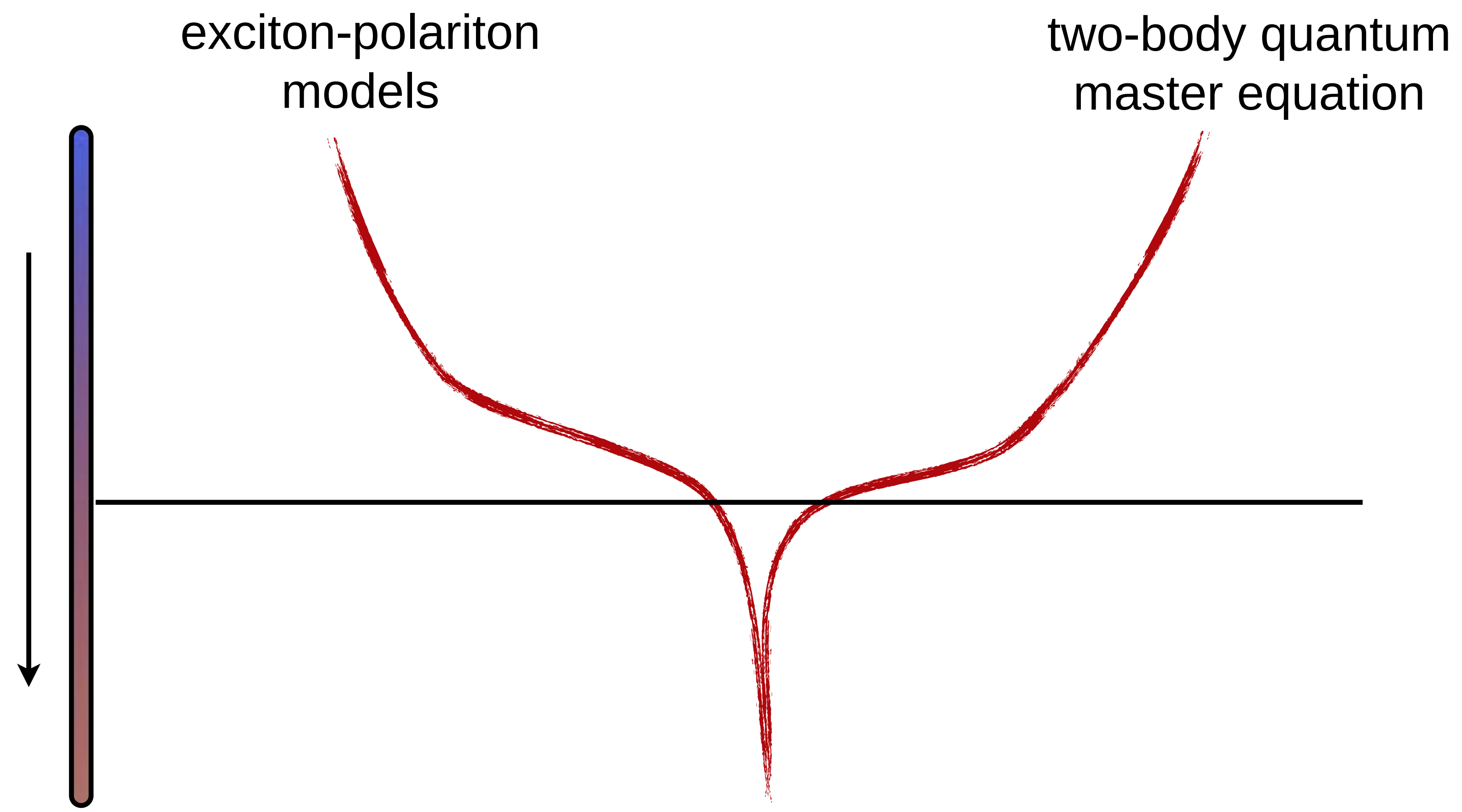
difference: distance from a phase transition

  - Keldysh component $\quad P^K = i\left(\gamma_l + \gamma_p\right)$

sum: noise of loss and pumping add up

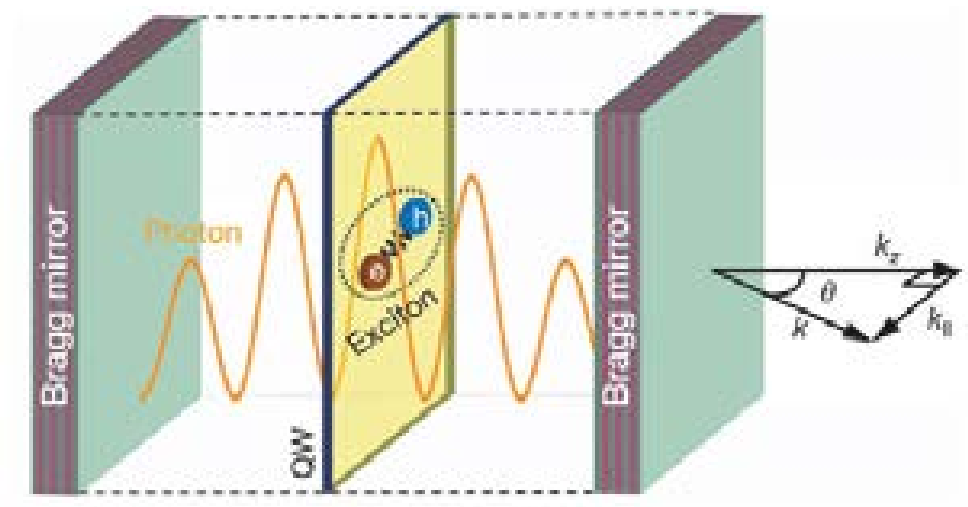
- now: simplifications in the semiclassical limit:
  - sharp argument close to a critical point
  - provides intuition for a frequency regime $\omega \ll \gamma = \gamma_l + \gamma_p$

Semi-classical limit and Langevin equations

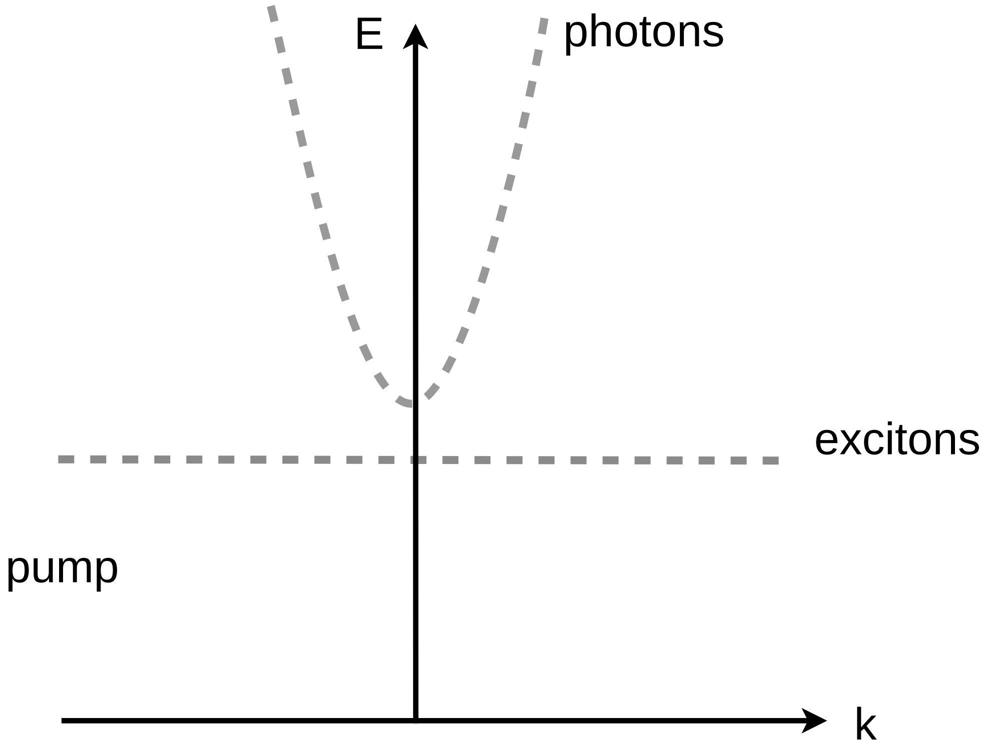


Intermezzo: Exciton-polariton systems

Kasprzak et al., Nature 2006



Imamoglu et al., PRA 1996



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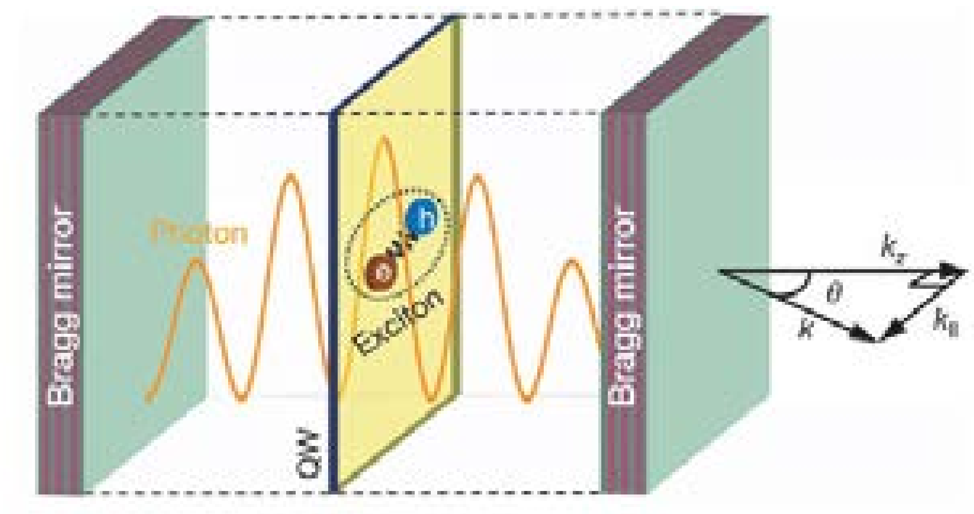

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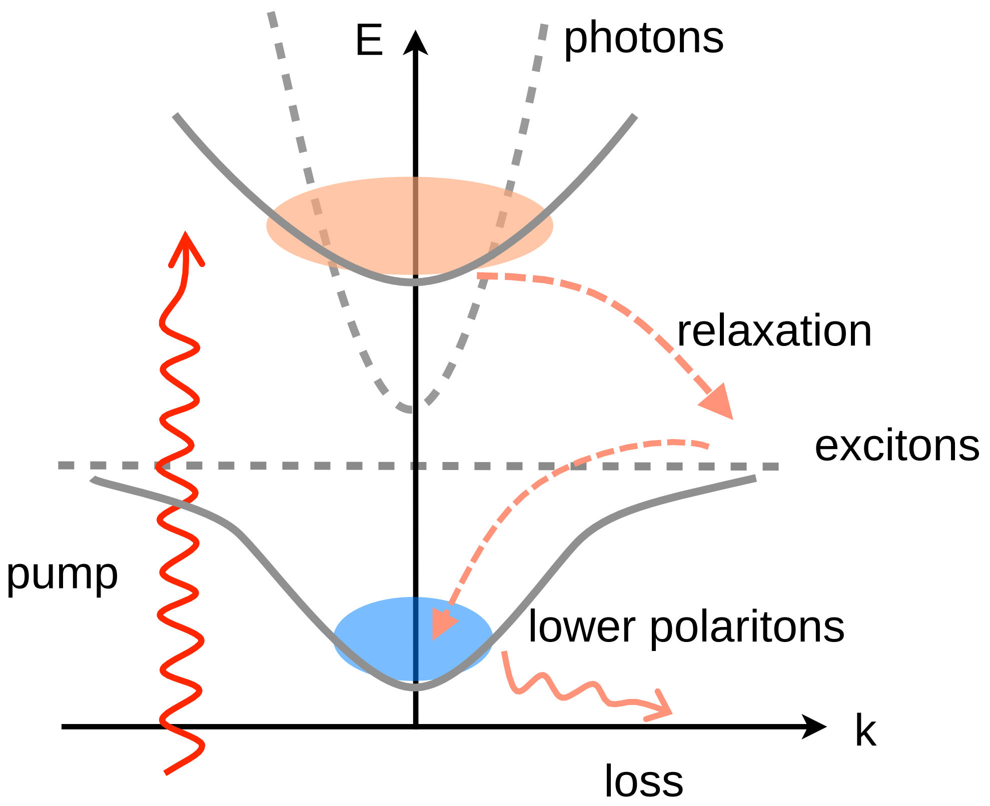

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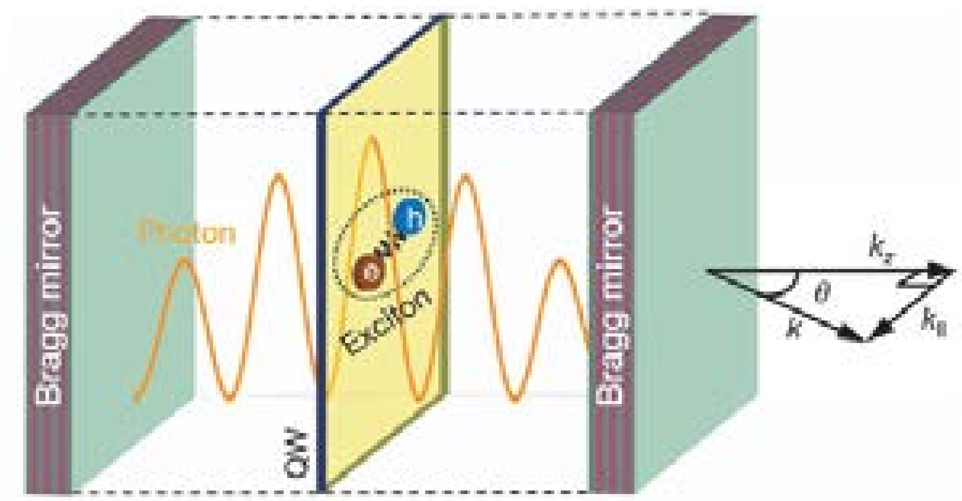

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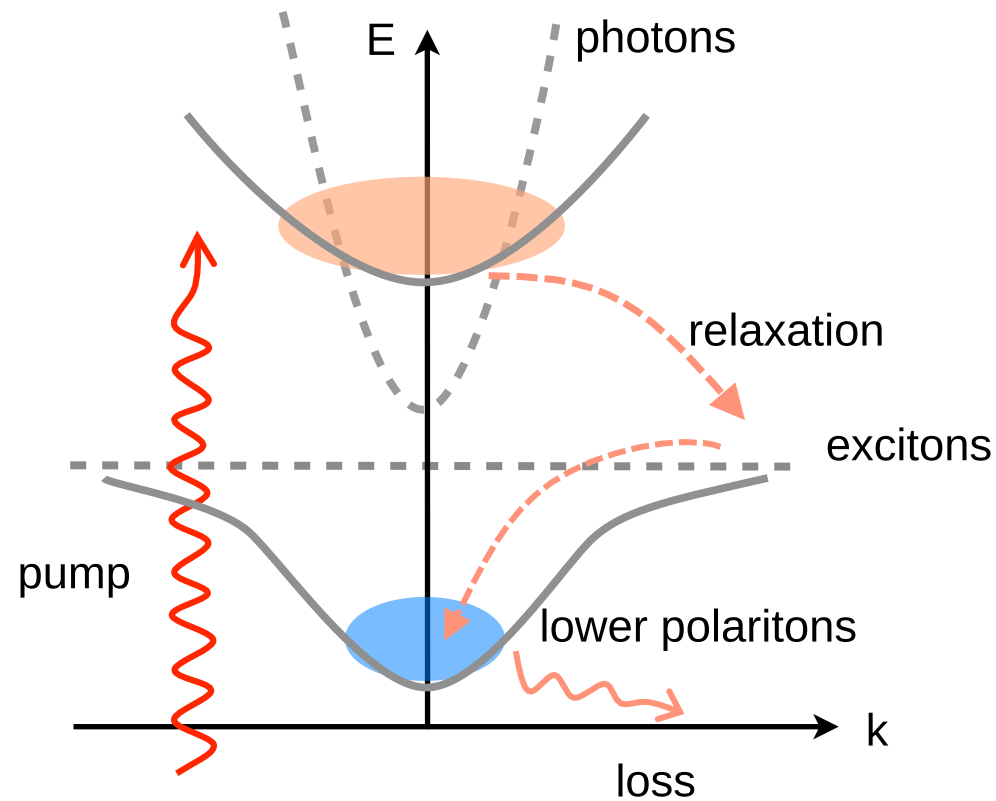


- phenomenological description: stochastic driven-dissipative Gross-Pitaevskii-Eq

$$i\partial_t \phi = \left[-\frac{\nabla^2}{2m} - \mu + i(\gamma_p - \gamma_l) + (\lambda - i\kappa)\,|\phi|^2 \right] \phi + \zeta$$

propagation

pump & loss rates

elastic collisions

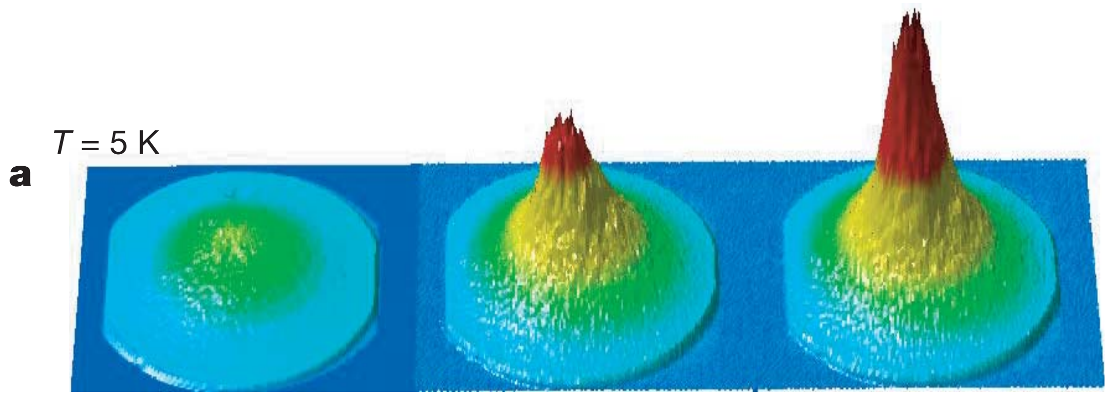
two-body loss

$$\langle \zeta^*(t,\mathbf{x})\zeta(t',\mathbf{x}')\rangle = \gamma\delta(t-t')\delta(\mathbf{x}-\mathbf{x}')$$

Szymanska, Keeling, Littlewood PRL (04, 06); PRB (07));
Wouters, Carusotto PRL (07,10)

Intermezzo: Exciton-polariton systems

- Bose condensation seen despite non-equilibrium conditions



Kasprzak et al., Nature 2006

- stochastic driven-dissipative Gross-Pitaevskii-Eq

$$i\partial_t \phi = \left[-\frac{\nabla^2}{2m} - \mu + i(\gamma_p - \gamma_l) + (\lambda - i\kappa) |\phi|^2 \right] \phi + \zeta$$

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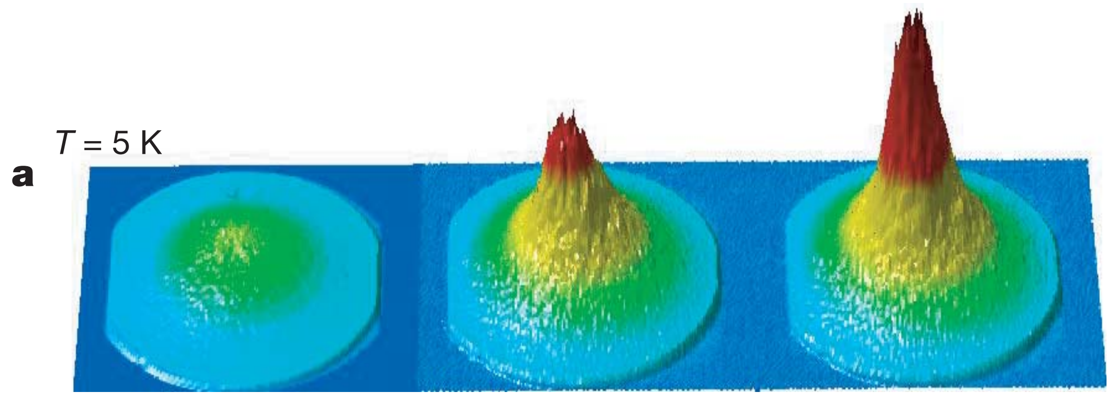


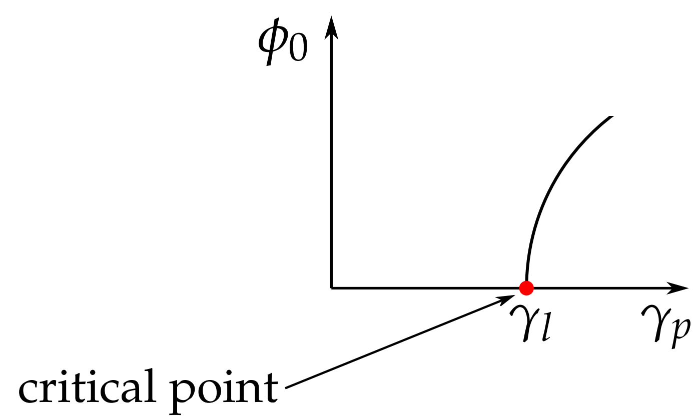
Kasprzak et al., Nature 2006

- stochastic driven-dissipative Gross-Pitaevskii-Eq

$$\cancel{i\partial_t \phi} = \left[-\cancel{\frac{\nabla^2}{2m}} - \mu + i(\gamma_p - \gamma_l) + (\lambda - i\kappa)\left|\phi\right|^2\right]\phi + \cancel{\zeta}$$

Szymanska, Keeling, Littlewood PRL (04, 06); PRB (07)); Wouters, Carusotto PRL (07,10)

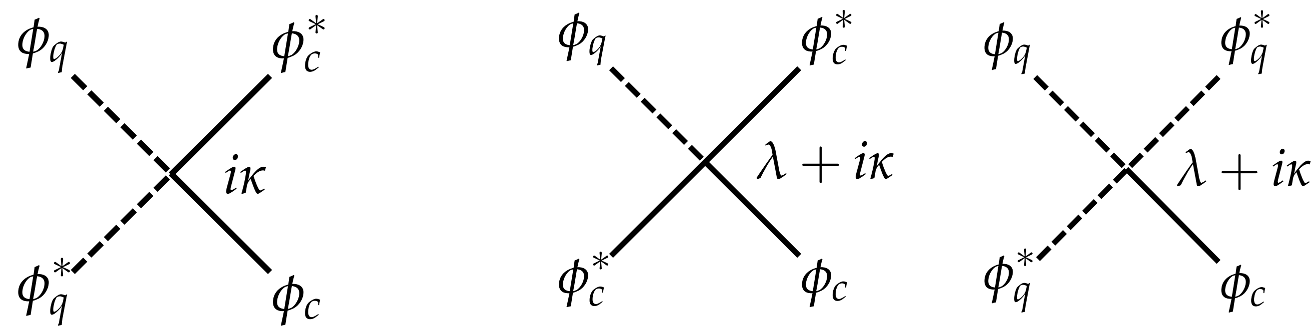
- mean field
  - neglect noise
  - homogeneous solution $\phi(\mathbf{x}, t) = \phi_0$



- naively, just as Bose condensation in equilibrium!
- Q1: How does this model relate to the Lindbladian and Lindblad-Keldysh field theory?
- Q2: What is “non-equilibrium” about it?

Semiclassical limit of Lindblad-Keldysh action: power counting

$$\mathcal{S} = \int_{t,\mathbf{x}} \left\{ (\phi_c^*, \phi_q^*) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} + 2i\kappa\phi_c^*\phi_c\phi_q^*\phi_q - \tfrac{1}{2}\left[(\lambda + i\kappa)\left(\phi_c^{*2}\phi_c\phi_q + \phi_q^{*2}\phi_c\phi_q\right) + c.c.\right] \right\}$$



- Gaussian sector close to a critical point:
  - retarded/advanced $$P^R(\omega, \mathbf{q}) = \omega - \mathbf{q}^2 \overbrace{- \mu + i\left(\gamma_l - \gamma_p\right)/2}^{\to 0} \sim q^2$$
  - Keldysh component $$P^K = i\left(\gamma_l + \gamma_p\right) \sim q^0$$

---

* for fields in real space. Confirm this: take the critical Gaussian action in real space/time; count exercise

$\partial_x \sim k, \partial_t \sim k^2, d^dx \sim k^{-d}, dt \sim k^{-2},$ why? $\phi_{c/q}(x) \sim k^{[\phi_{c/q}(x)]}$

What do we get for fields in momentum space? $\phi_{c/q}(\omega, \mathbf{q}) = \int d^dx dt e^{-i(\omega t - \mathbf{q}\mathbf{x})}\phi_{c/q}(t, \mathbf{x})$

What are the canonical dimensions of the quartic vertices?

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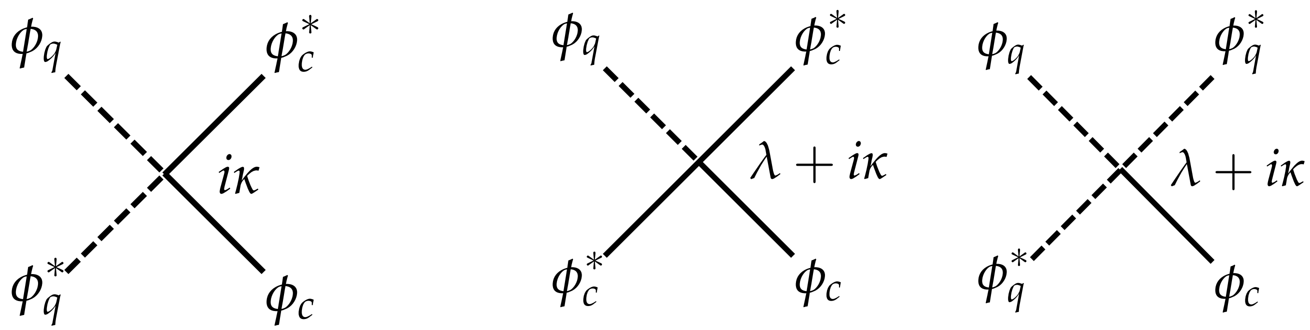

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  - Keldysh component $\quad P^K = i\left(\gamma_l + \gamma_p\right) \sim q^0$
- canonical field dimensions*: $\quad \boxed{[\phi_c] = \frac{d-2}{2} < [\phi_q] = \frac{d+2}{2}}$
  - action is dimensionless: phase e^{iS} in the functional integral
  - quadratic/Gaussian sector: scaling dimensions of inverse Green's function known
  - intuitive: high order local couplings not relevant at large distances

---

* for fields in real space. Confirm this: take the critical Gaussian action in real space/time; count $\partial_x \sim k, \partial_t \sim k^2, d^dx \sim k^{-d}, dt \sim k^{-2}$, why? $\phi_{c/q}(x) \sim k^{[\phi_{c/q}(x)]}$ — exercise

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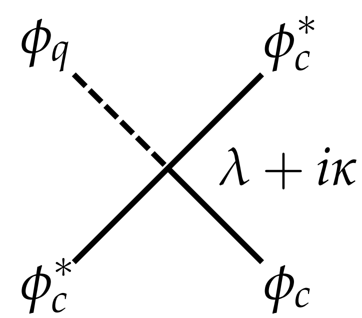

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➡ local vertices with more than two quantum fields are irrelevant in the RG sense in d > 2

- ➡ note preservance of probability in semiclassical limit $S_M[\Phi_c, \Phi_q = 0] = 0 \quad \forall \Phi_c$
- ➡ massive diagrammatic simplification
- ➡ to be seen now: result identical to phenomenological models of exciton-polariton condensates (Wouters and Carusotto PRL 06; Szymanska, Keeling, Littlewood PRL 04)

Semiclassical limit: MSR action & Langevin equation

Martin, Siggia, Rose, PRA (1973); Janssen, Z. Phys. B (1976); DeDominicis, J. Phys. (1976)

- Keldysh integral after power counting $Z = \int \mathcal{D}[\phi_c, \phi_c^*, \phi_q, \phi_q^*] \, e^{iS[\phi_c, \phi_c^*, \phi_q, \phi_q^*]}$
- with Martin-Siggia-Rose (MSR) action

$$S = \int_{t,\mathbf{x}} \left\{ \phi_q^* \frac{\delta \bar{S}[\phi_c]}{\delta \phi_c^*} + c.c. + i2\gamma \phi_q^* \phi_q \right\}$$

➡ phi_q only up to quadratic order

$$\bar{S} = \int_{t,\mathbf{x}} \{\phi_c^* i\partial_t \phi_c - \mathcal{H}_c + i\mathcal{H}_d\} \qquad \mathcal{H}_\alpha = r_\alpha |\phi_c|^2 + K_\alpha |\nabla \phi_c|^2 + \lambda_\alpha |\phi_c^* \phi_c|^4, \quad \alpha = c, d$$

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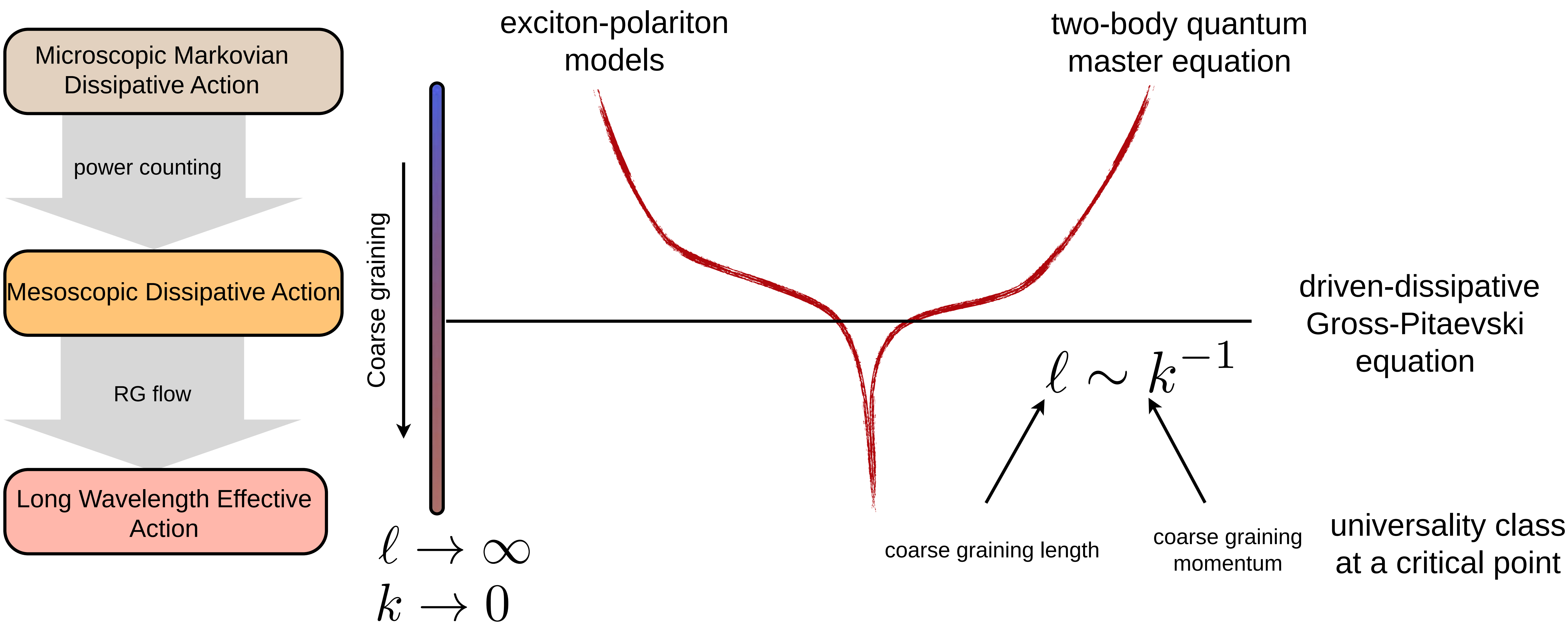
- linear in phi_q: Fourier representation of delta-functional

$$Z = \int \mathcal{D}[\xi, \xi^*]\, e^{-\frac{1}{2\gamma} \int_{t,\mathbf{x}} \xi^* \xi} \int \mathcal{D}[\phi_c, \phi_c^*] \delta \left(\boxed{i\partial_t \phi_c - \frac{\delta \mathcal{H}_c}{\delta \phi_c^*} + i \frac{\delta \mathcal{H}_d}{\delta \phi_c^*} - \xi} \right) \delta\,(c.c.)$$

➡ noise averaging ➡ at each instant of time: ➡ driven-dissipative Gross-Pitaevski equation

Semiclassical limit and exciton-polariton model

- example of “weak” universality

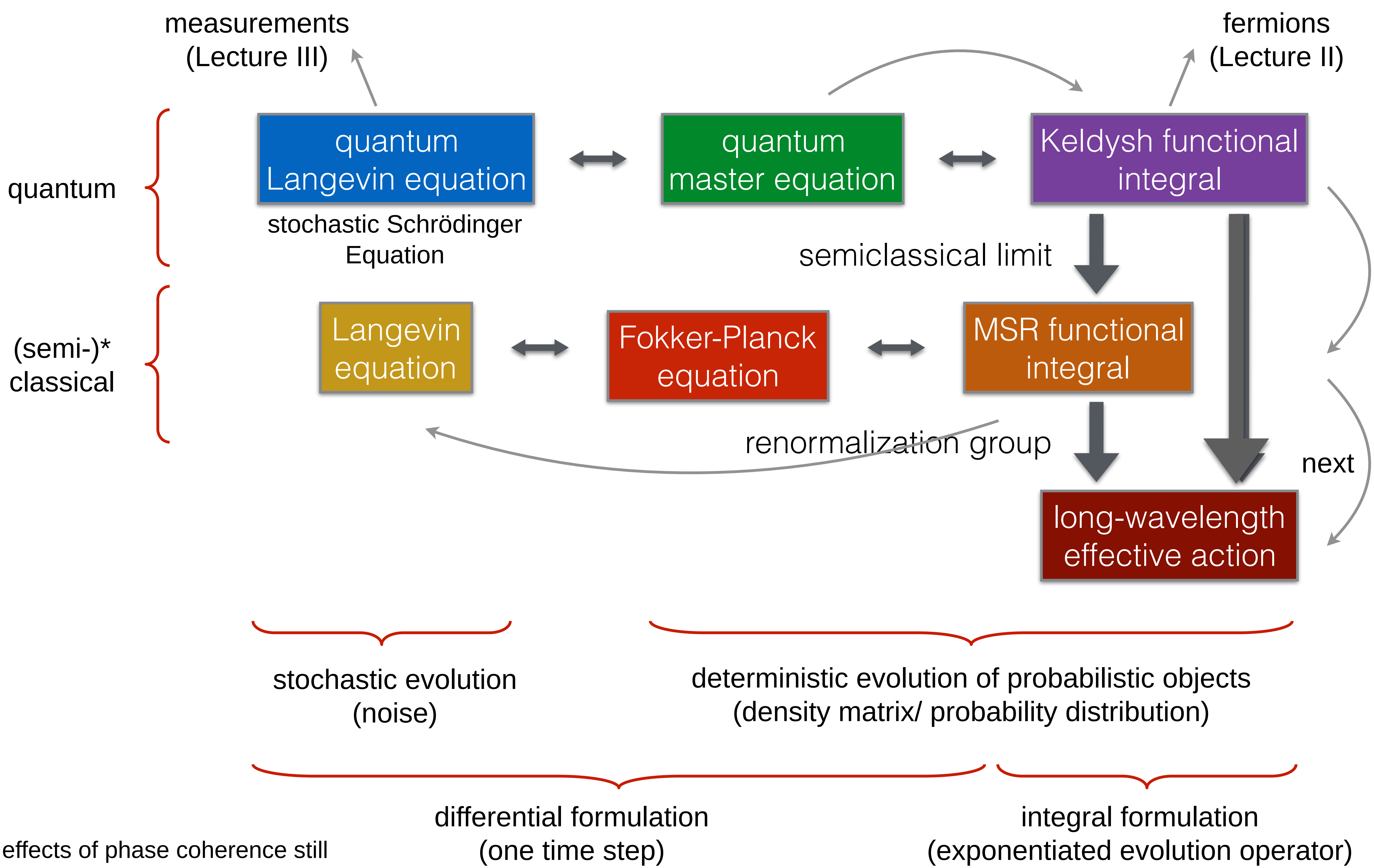


➡ many microscopic models collapse to an effective low energy model

➡ form dictated by microscopic symmetries

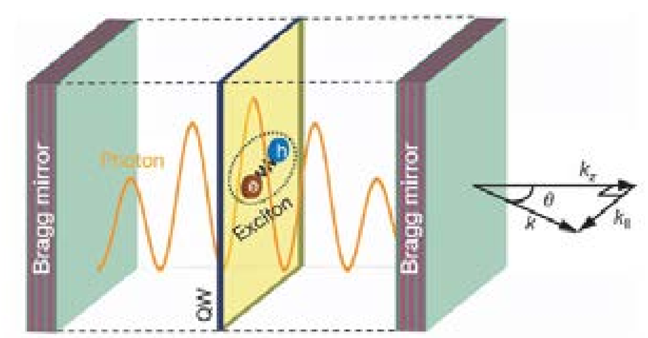
➡ longer wavelength behavior to be determined by calculation

Overview: Langevin equations, master equation, Keldysh integral

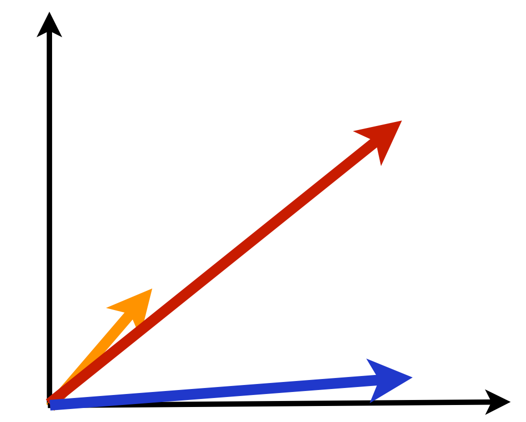


* effects of phase coherence still present (cf. BEC as classical wave)

“What is non-equilibrium about it?”



$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+,\Phi_-]}$$



“What is non-equilibrium about it?”

- different notions of ‘non-equilibrium’

Time evolution

- ➡ time translation invariance broken (e.g. thermalization, Floquet.. -> Lectures by D. Huse, N. Cooper, T. Esslinger!)

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Stationary states (considered here)

➡ flux equilibria

- not in static observables:

$$\rho = e^{-\beta H}/\mathrm{tr} e^{-\beta H}$$

➡ any positive semidefinite Hermitian operator can be written like this

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- dynamical observables, e.g.:

$$\langle \psi^\dagger(t)\psi(0)\rangle \qquad\qquad \psi(t) = e^{iHt}\psi e^{-iHt}$$

➡ non-equilibrium conditions are encoded in the generator of dynamics

➡ thermal equilibrium realized if generator of dynamics coincides with statistical weight

➡ otherwise must expect non-equilibrium conditions (Lindbladian)

“What is non-equilibrium about it?”

more details: see appendix!

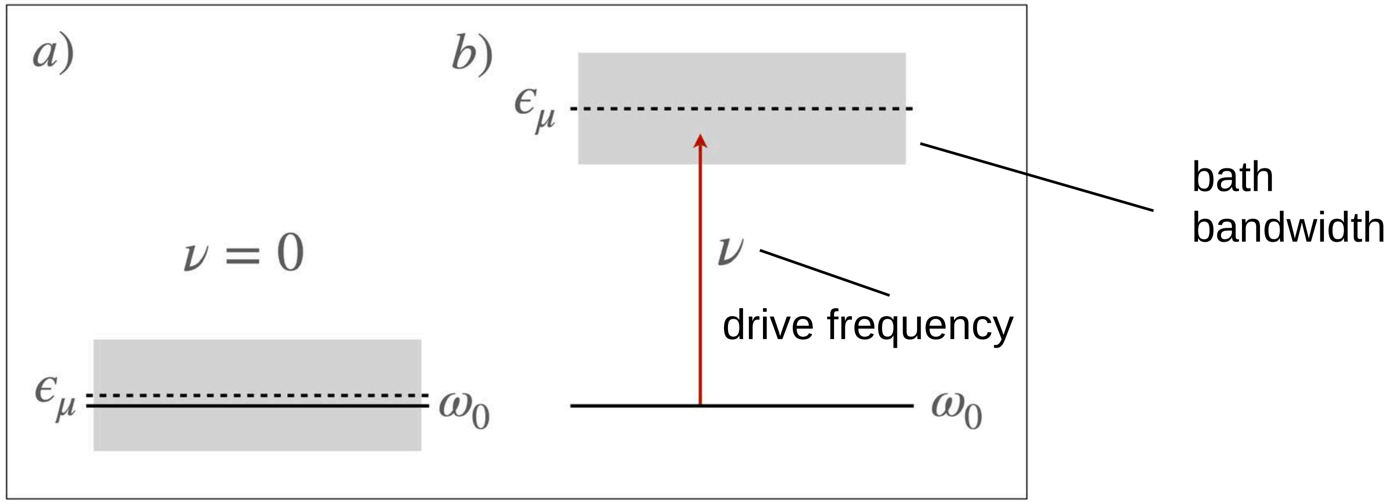
- non-equilibrium stationary states:
  - open system: is it the coupling to a bath —> irreversibility?
    - ➡ no, can be compatible with thermal equilibrium (Caldeira-Leggett Models)

“What is non-equilibrium about it?”

more details: see appendix!

- non-equilibrium stationary states:
  - open system: is it the coupling to a bath —> irreversibility?
    - ➡ no, can be compatible with thermal equilibrium (Caldeira-Leggett Models)
  - driven & open system: it is in the way how we couple to a bath:

$$\underbrace{\hat{H}_{\mathrm{t}} = \hat{H}}_{\text{system}} + \hat{H}_{\mathrm{int}} + \hat{H}_{\mathrm{b}} \qquad \underbrace{\hat{H}_{\mathrm{b}} = \sum_{\mu} \epsilon_{\mu} b^{\dagger}_{\mu} b_{\mu}}_{\text{bath}} \qquad \underbrace{\hat{H}_{\mathrm{int}} = \sum_{\mu} g_{\mu} e^{-i\nu t} L b^{\dagger}_{\mu} + \mathrm{h.\ c.}}_{\text{system-bath}}$$



equilibrium

$$\nu = 0 \ll g_{\mu} \ll \omega_0 \approx \epsilon_{\mu}$$

non-equilibrium

$$g_{\mu} \ll \omega_0 \ll \nu \approx \epsilon_{\mu}$$

- the state of the bath is fixed, distribution function $n^{\mathrm{b}}_{B/F}(\omega)$
- singles out a frame of reference —> drive scale cannot be removed
  - ➡ driven open nature incompatible with thermal equilibrium

“What is non-equilibrium about it?”

- more formally: quantum master equation

$$\underbrace{\partial_t \rho = -i[H, \rho]}_{\Longrightarrow S_H} + \underbrace{\mathcal{D}[\rho]}_{\Longrightarrow S_{\mathcal{D}}}$$

- equivalent Keldysh functional integral: $Z = \int \mathcal{D}\phi_{\pm} e^{i(S_H[\phi_{\pm}] + S_{\mathcal{D}}[\phi_{\pm}])}$

- equilibrium dynamics microscopically generated by a time-independent (undriven) Hamiltonian alone

$$S_{\mathcal{D}} = 0$$

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➡ symmetry of Keldysh action under discrete transformation

L. Sieberer, A Chiocchetta, U. Täuber, A. Gambassi, SD PRB (2015); F. Haehl, R. Loganayagam, M. Rangamani, JHEP (2016); M. Crossley, P. Glorioso, H. Liu, JHEP (2016)

$$\boxed{\mathcal{T}_\beta : \quad \phi_\pm(t,\mathbf{x}) \to \phi_\pm(-t + i\beta/2, \mathbf{x}), \quad i \to -i} \qquad \mathcal{T}_\beta^2 = 1 \qquad \beta = 1/T$$

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- associated “Ward identities” are equilibrium quantum fluctuation-dissipation relations of arbitrary order

e.g. single-particle sector

$$\boxed{G^K(\omega,\mathbf{q}) = (2n_B(\omega/T)+1)[G^R(\omega,\mathbf{q}) - G^A(\omega,\mathbf{q})]}$$

correlations — Bose distribution — responses

any order <=> detailed balance
<=> global thermal equilibrium

➡ the Lindbladian ($S_{\mathcal{D}}$) violates this symmetry and therefore detailed balance explicitly

➡ intuition: underlying is a (rapidly) driven system with no energy conservation

Equilibrium symmetry: some details

L. Sieberer, A Chiocchetta, U. Täuber, A. Gambassi, SD PRB (2015)

- Undriven system: equilibrium dynamics generated by a time-independent Hamiltonian

➡ symmetry of Schwinger-Keldysh action under discrete transformation

$$\mathcal{T}_\beta : \quad \phi_\pm(t,\mathbf{x}) \to \phi_\pm(-t+i\beta/2,\mathbf{x}), \quad i \to -i \qquad \mathcal{T}_\beta^2 = 1 \qquad \beta = 1/T$$

- symmetry: invariance of $Z = \int \mathcal{D}\phi_\pm e^{i(S_H[\phi_\pm]+S_\mathcal{D}[\phi_\pm])}$

- implies for correlation functions

$$\langle \mathcal{O}[\phi_\pm]\rangle = \langle \mathcal{T}_\beta(\mathcal{O}[\phi_\pm])\rangle \qquad \langle \mathcal{O}[\phi_\pm]\rangle = \int \mathcal{D}\phi_\pm \mathcal{O}[\phi_\pm] e^{iS[\phi_\pm]}$$

- physical consequence: Fluctuation-dissipation relations, of any order, e.g. single particle sector:

$$\boxed{G^K(\omega,\mathbf{q}) = (2n_B(\omega/T)+1)[G^R(\omega,\mathbf{q}) - G^A(\omega,\mathbf{q})]}$$

any order <=> detailed balance
<=> global thermal equilibrium

correlations Bose distribution responses

- connection to operator formalism: compact functional formulation of Kubo-Martin-Schwinger boundary condition: for any two operators A,B,

$$\langle A(t)B(t')\rangle = \langle B(t'-i\beta)A(t)\rangle. \qquad \langle \mathcal{O}\rangle = \mathrm{tr}(\mathcal{O}\rho)$$

- reason: $A(t) = e^{iHt}Ae^{-iHt}, \rho = e^{-\beta H}/\mathrm{tr}e^{-\beta H}$
  $\Rightarrow A(t)\rho = \rho A(t-i\beta)$ & cyclic invariance

Equilibrium symmetry: Semiclassical limit

- Undriven system: equilibrium dynamics generated by a time-independent Hamiltonian

➡ symmetry of Schwinger-Keldysh action under discrete transformation

$$\mathcal{T}_\beta : \quad \phi_\pm(t,\mathbf{x}) \to \phi_\pm(-t + i\beta/2, \mathbf{x}), \quad i \to -i \qquad \mathcal{T}_\beta^2 = 1 \qquad \beta = 1/T$$

$$= e^{\pm i \frac{\beta}{2}\partial_t}\phi_\pm(-t,\mathbf{x})$$

- semiclassical limit: T large => $e^{\pm i \frac{\beta}{2}\partial_t} \approx 1 \pm i\frac{\beta}{2}\partial_t$

- action on the fields:

irrelevant by power counting

$$\mathcal{T}_\beta \phi_c(t,\mathbf{x}) = \phi_c^*(-t,\mathbf{x}) + \frac{i}{2T}\partial_t \cancel{\phi_q^*(-t,\mathbf{x})},$$

$$\mathcal{T}_\beta \phi_q(t,\mathbf{x}) = \phi_q^*(-t,\mathbf{x}) + \frac{i}{2T}\partial_t \phi_c^*(-t,\mathbf{x})$$

reproduces classical result

H. K. Janssen (1976); C. Aron et al, J Stat. Mech (2011)

➡ obtain geometric interpretation of the equilibrium symmetry

Geometric interpretation: equilibrium vs. non-equilibrium dynamics

- couplings spanning the Keldysh action lie in the complex plane

$$\underbrace{\partial_t \rho = -i[H,\rho]}_{\Leftrightarrow\, S_H} + \underbrace{\mathcal{D}[\rho]}_{\Leftrightarrow\, S_{\mathcal{D}}}$$

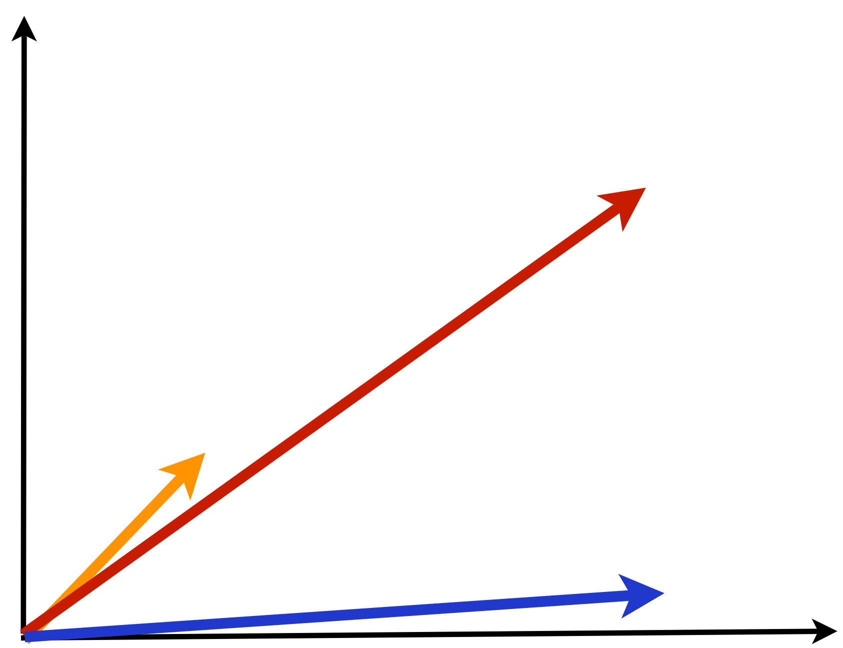
$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_H[\Phi_+,\Phi_-] + S_{\mathcal{D}}[\Phi_+,\Phi_-])}$$

& semiclassical limit: higher terms in ϕ_q irrelevant

Im

incoherent/ irrev. dynamics

$\Leftrightarrow S_{\mathcal{D}}$



Re

coherent/ reversible dynamics

$\Leftrightarrow S_H$

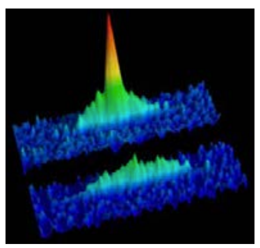
example: two-body processes λ

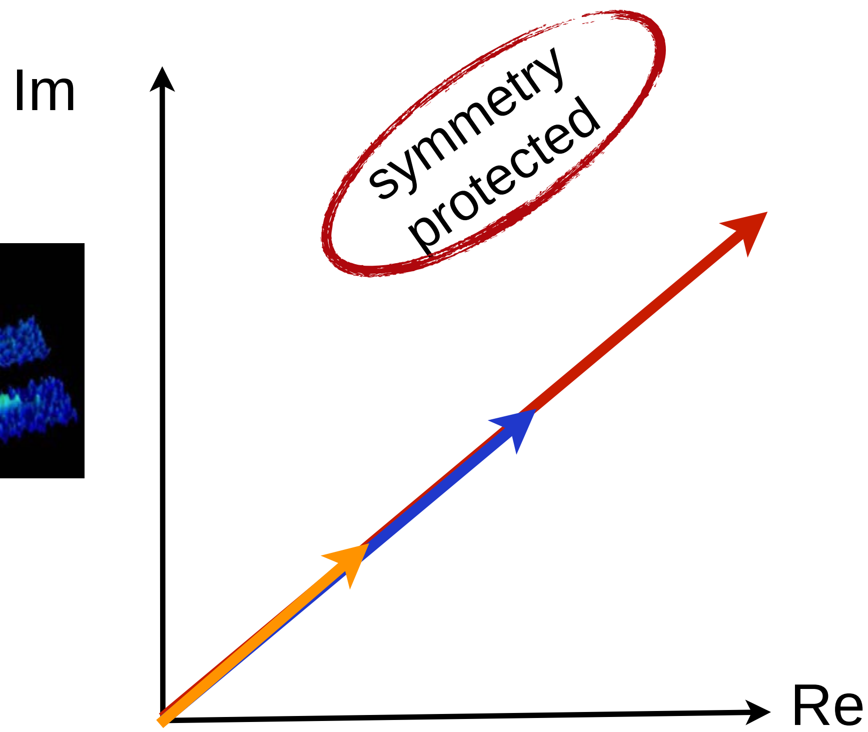
$\mathrm{Re}\lambda$ elastic two-body collisions

$\mathrm{Im}\lambda$ inelastic two-body losses

Geometric interpretation: equilibrium vs. non-equilibrium dynamics

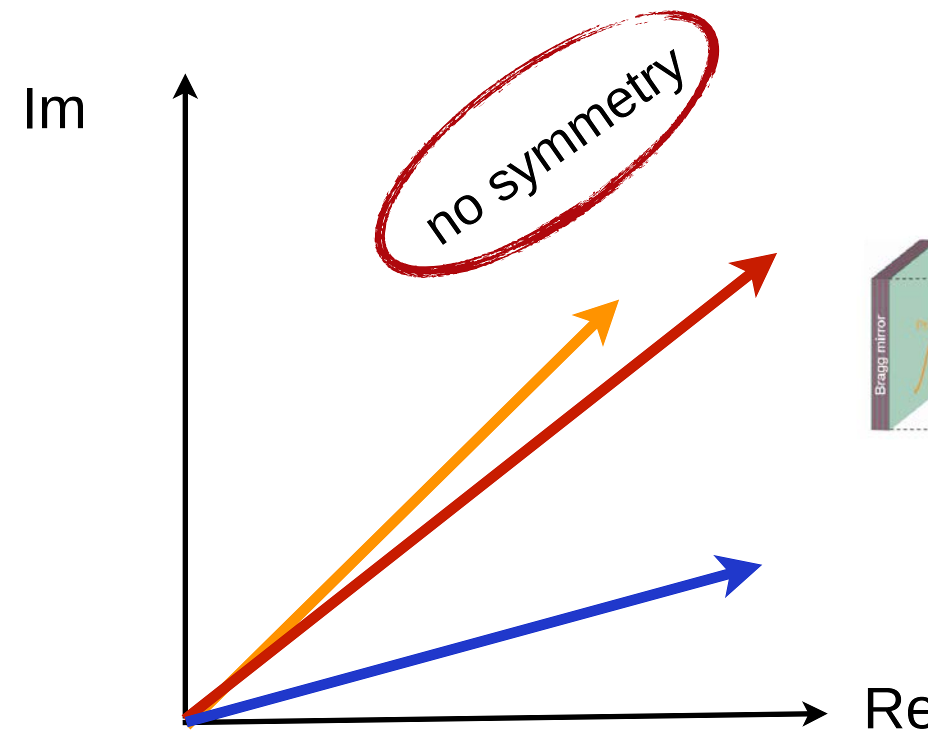
equilibrium dynamics

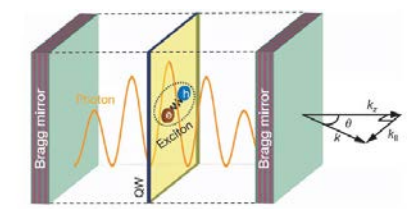




- coherent and dissipative dynamics may occur simultaneously
- but they are not independent

non-equilibrium dynamics

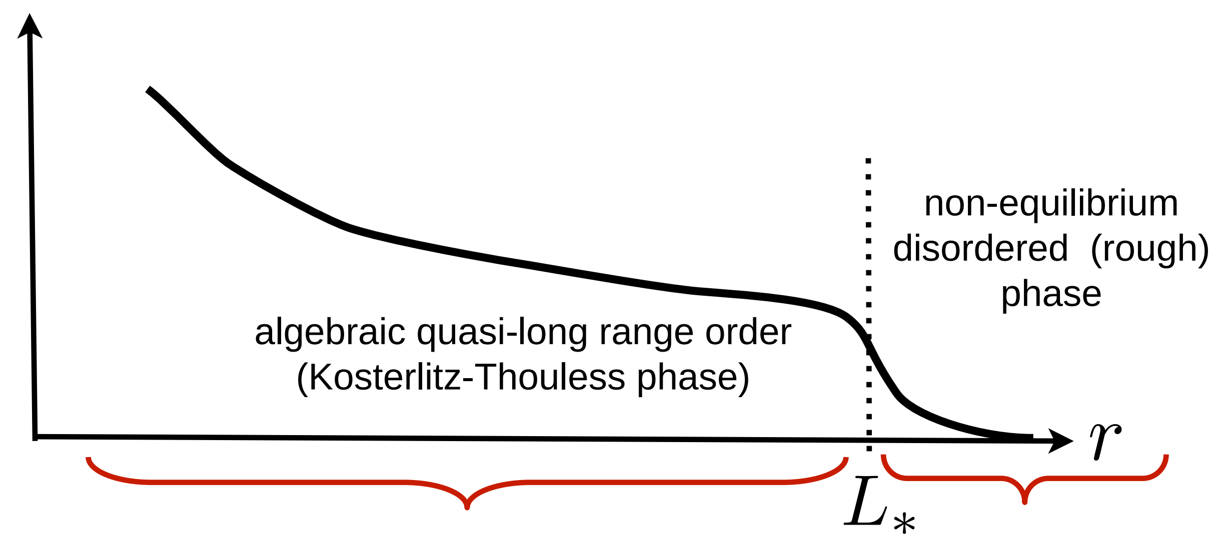




- coherent and dissipative dynamics do occur simultaneously
- they result from different dynamical resources

➡ what are the physical consequences of the spread in the complex plane?

Application:
Fate of BKT physics in
driven open quantum systems



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G. Wachtel, L. Sieberer, SD, E. Altman, PRB (2016)
L. Sieberer, G. Wachtel, E. Altman, SD, PRB (2016)

L. He, L. Sieberer, E. Altman, SD, PRB (2015)
L. He, L. Sieberer, SD, PRL (2017)

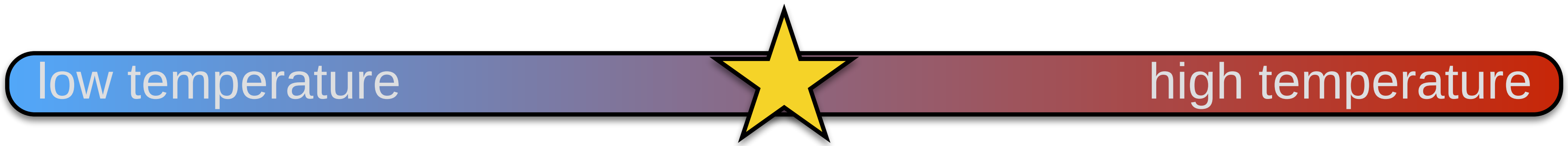
Microscopic
Quantum Optics

“Thermodynamic”
Many-body physics

Long wavelength
Statistical mechanics

Phase transitions in two dimensions

- continuous symmetry U(1): no spontaneous symmetry breaking, but a phase transition



- correlations

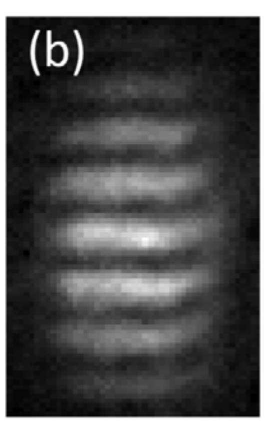
$$\langle \phi^*(\mathbf{x})\phi(0)\rangle \sim r^{-\frac{1}{2\pi K}} \qquad\qquad \sim e^{-r/\xi}$$

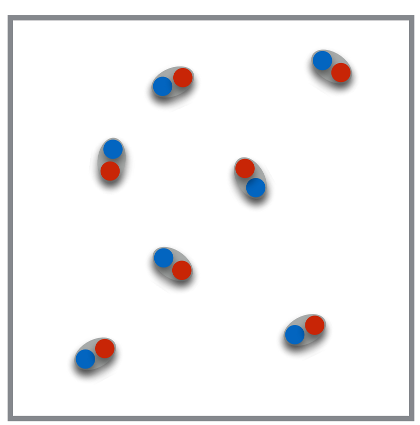
- responses: superfluidity

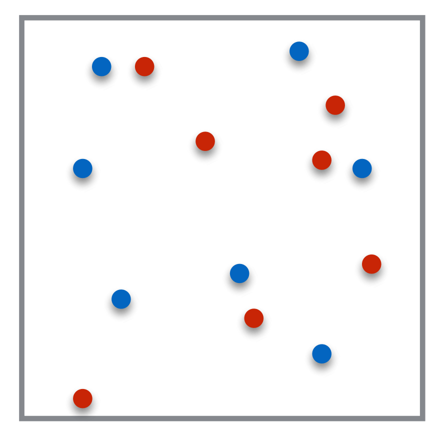
$$\rho_s \neq 0 \qquad\qquad \rho_s = 0$$

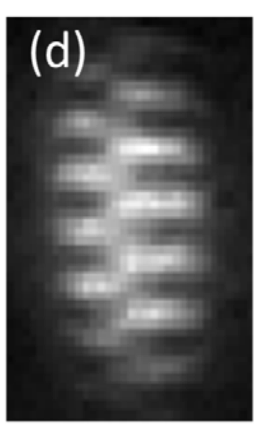
- BKT transition: unbinding of vortex-antivortex pairs

J. M. Kosterlitz, D. J. Thouless J. Phys. C (1973)





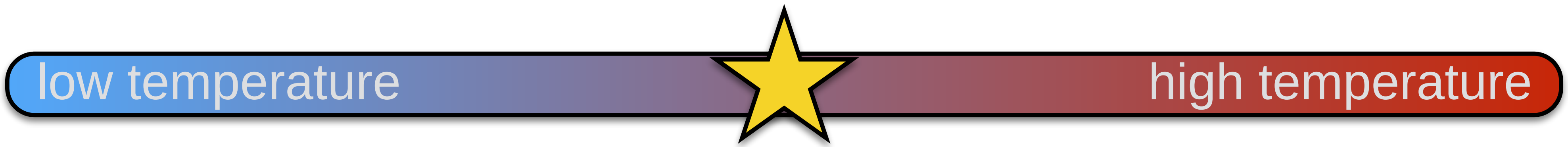




matter wave interferometry:
Z. Hadzibabic et al. Nature (2006)

… fate in driven open condensates?

Short reminder: Algebraic correlations



- correlations

$$\langle \phi^*(\mathbf{x})\phi(0)\rangle \sim r^{-\frac{1}{2\pi K}} \qquad\qquad \sim e^{-r/\xi}$$

- physical reason: gapless spin wave fluctuations

  - spin wave action $\quad S = \frac{K}{2}\int d^2x (\nabla\theta)^2 = \frac{K}{2}\int \frac{d^2q}{(2\pi)^2}\, q^2\, \theta(-\mathbf{q})\theta(\mathbf{q})$

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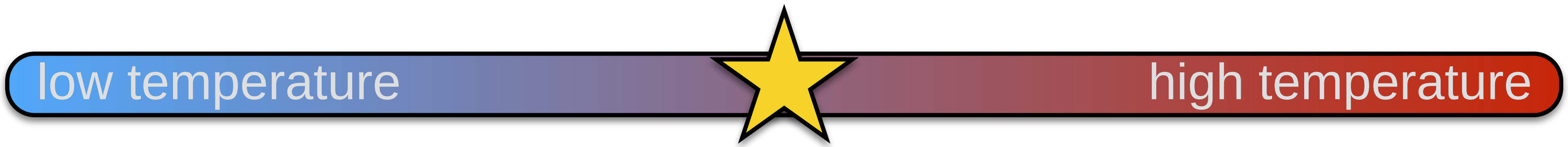

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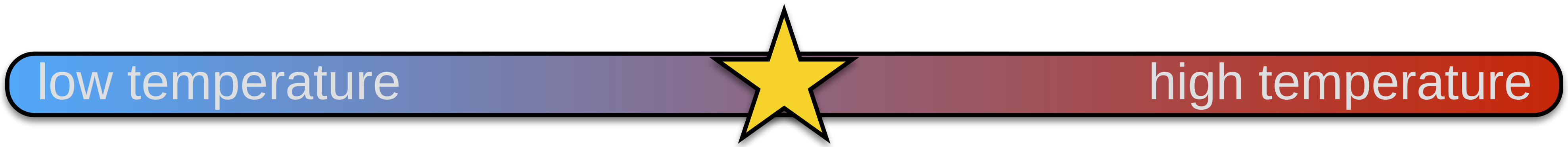

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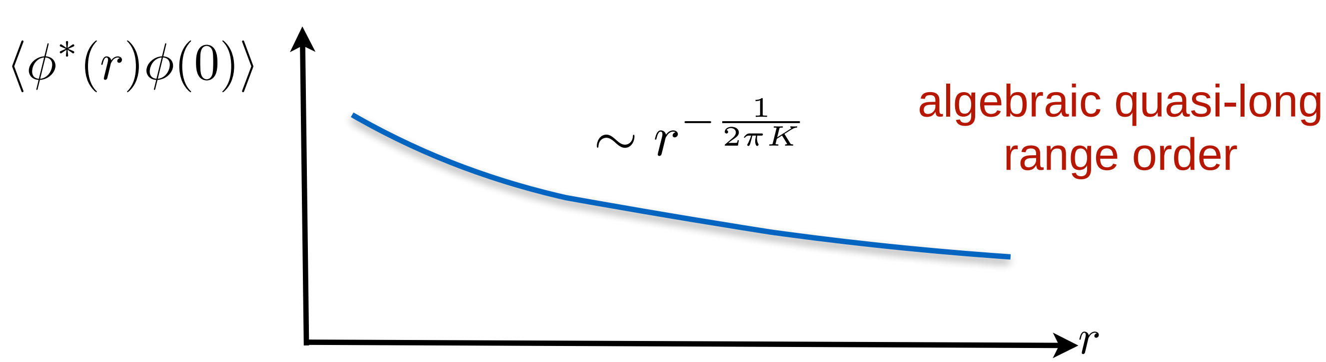
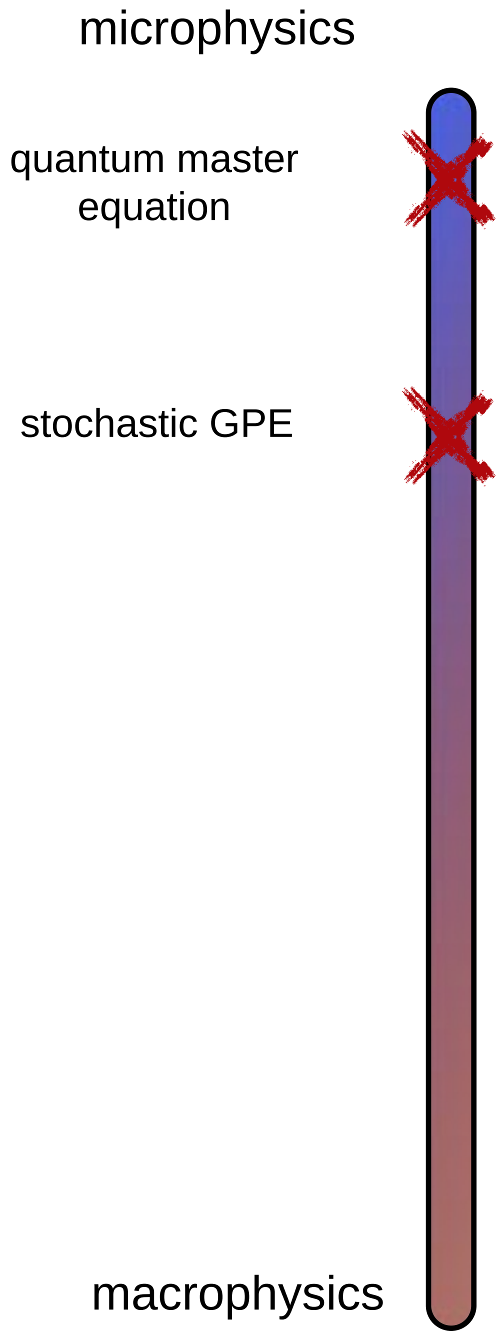


Description: Effective model

- mesoscopic starting point: driven-dissipative stochastic Gross-Pitaevski equation

$$i\partial_t \phi = \left[-\frac{\nabla^2}{2m} - \mu + i(\gamma_p - \gamma_l) + (g - i\kappa)\,|\phi|^2 \right] \phi + \zeta$$

$$\phi = \rho e^{i\theta}$$



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- effective low frequency dynamics

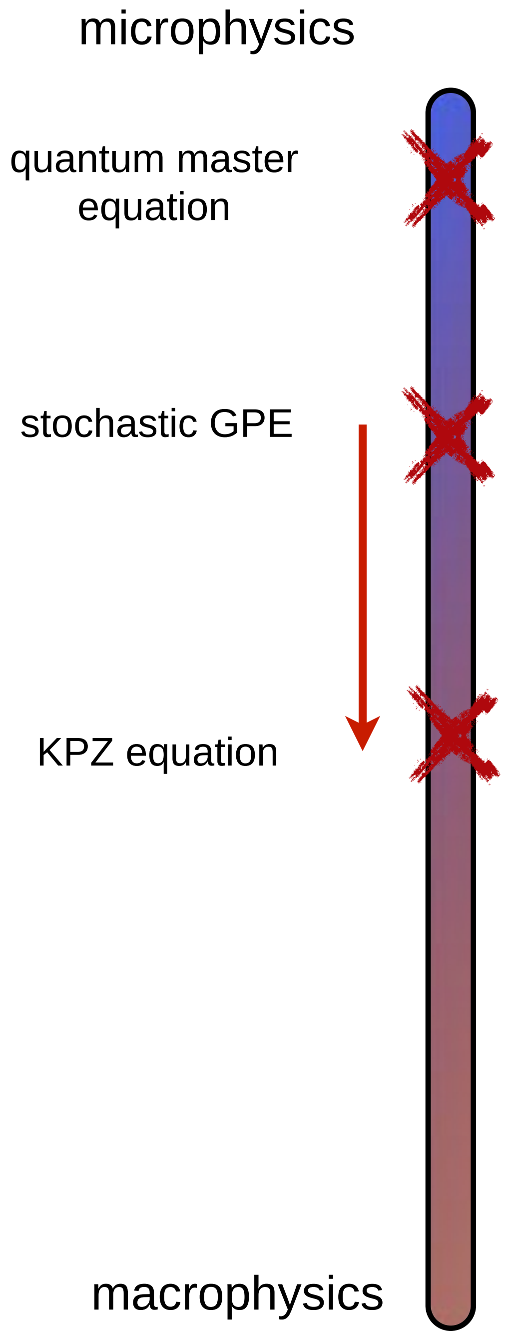
see also: G. Grinstein et al., PRL (1993)

$$\partial_t \theta = D\nabla^2\theta + \lambda(\nabla\theta)^2 + \xi$$

phase diffusion   phase nonlinearity   Markov noise

form of the KPZ equation

Kardar, Parisi, Zhang, PRL (1986)



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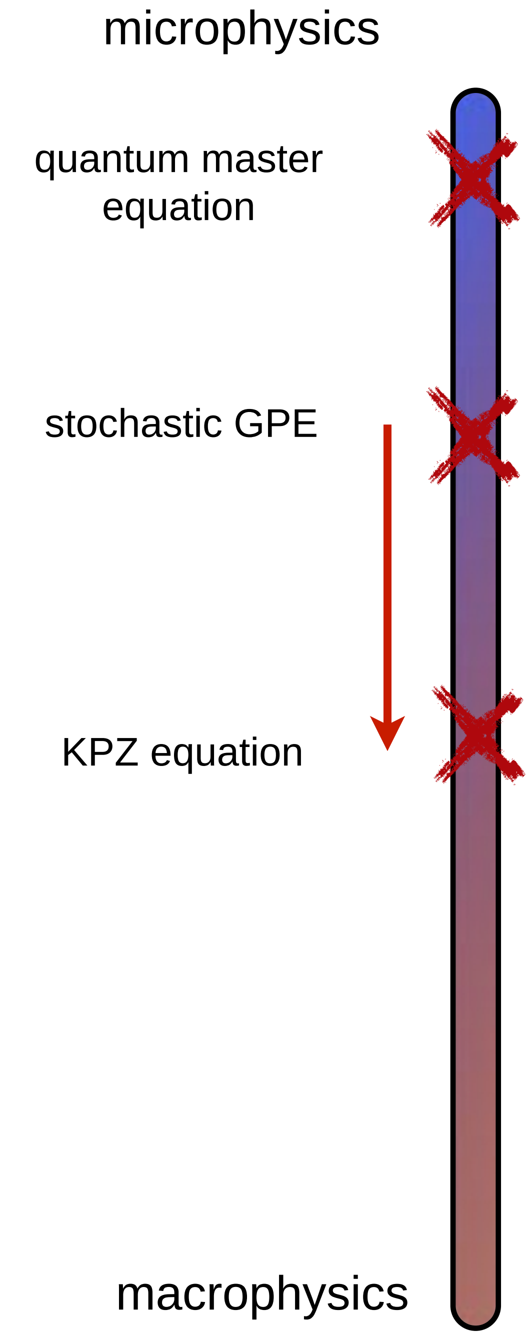

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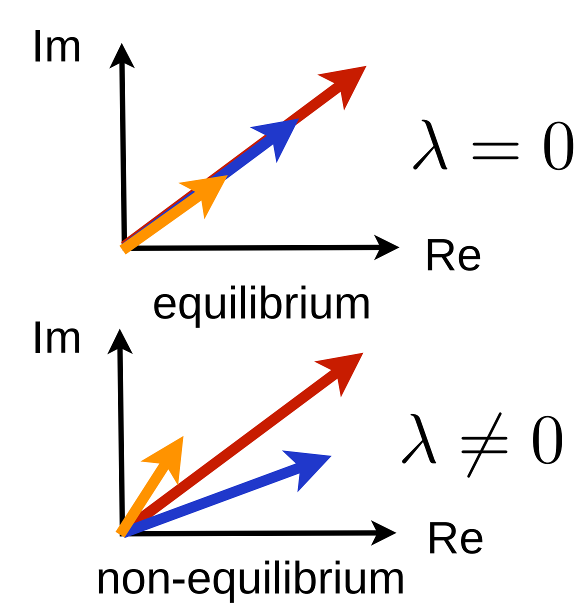
phase diffusion | phase nonlinearity | Markov noise

form of the KPZ equation

Kardar, Parisi, Zhang, PRL (1986)

- meaning: non-linear spin wave mode
- nonlinearity: single-parameter measure of non-equilibrium strength (ruled out in equilibrium by symmetry)



KPZ equation: A paradigm of non-equilibrium stat mech

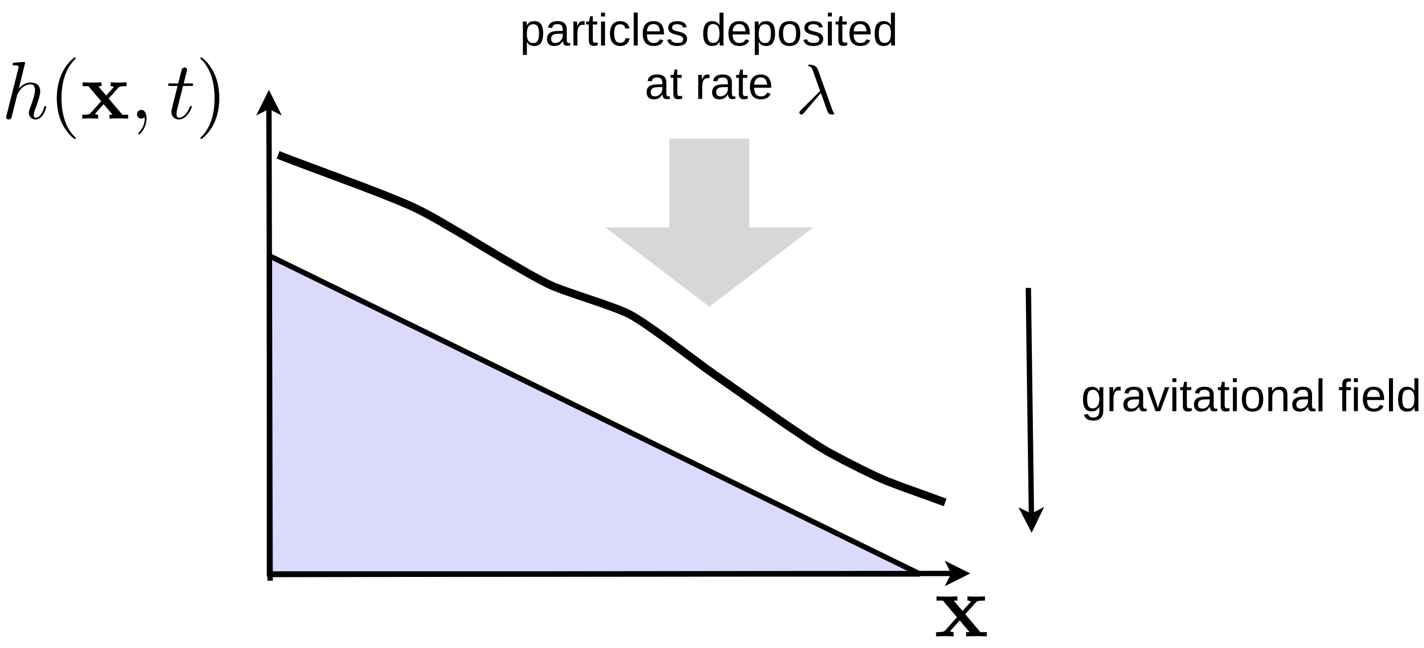
- originally: describes stochastic roughening of surface height $h(\mathbf{x},t)$

$$\partial_t h = D\nabla^2 h + \lambda(\nabla h)^2 + \xi$$

smoothens　　nonlinear growth　　noise

Kardar, Parisi, Zhang, PRL (1986)
Review: Krug, Adv. Phys. (1997)

- simplest physical scenario



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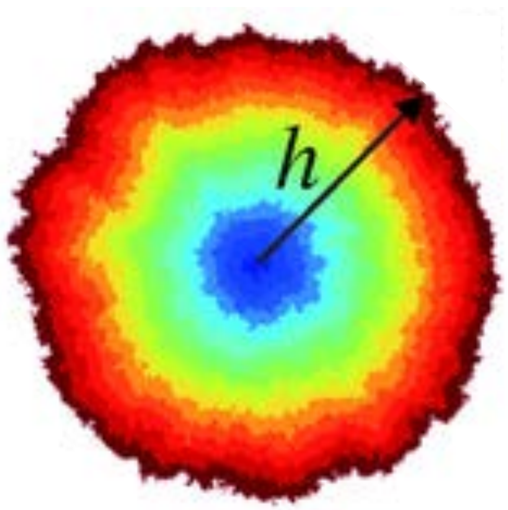
- multiple physical contexts



defect growth in liquid crystals

drive: electric field

from Takeuchi et al., Scientific Reports (2011)

bacterial colony growth

drive: sugar

Wakita et al., J. Phys. Jpn. Soc. (1997)

burning paper

drive: oxygen

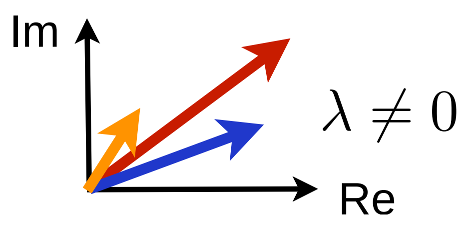
Maunuksela et al., PRL (1997)

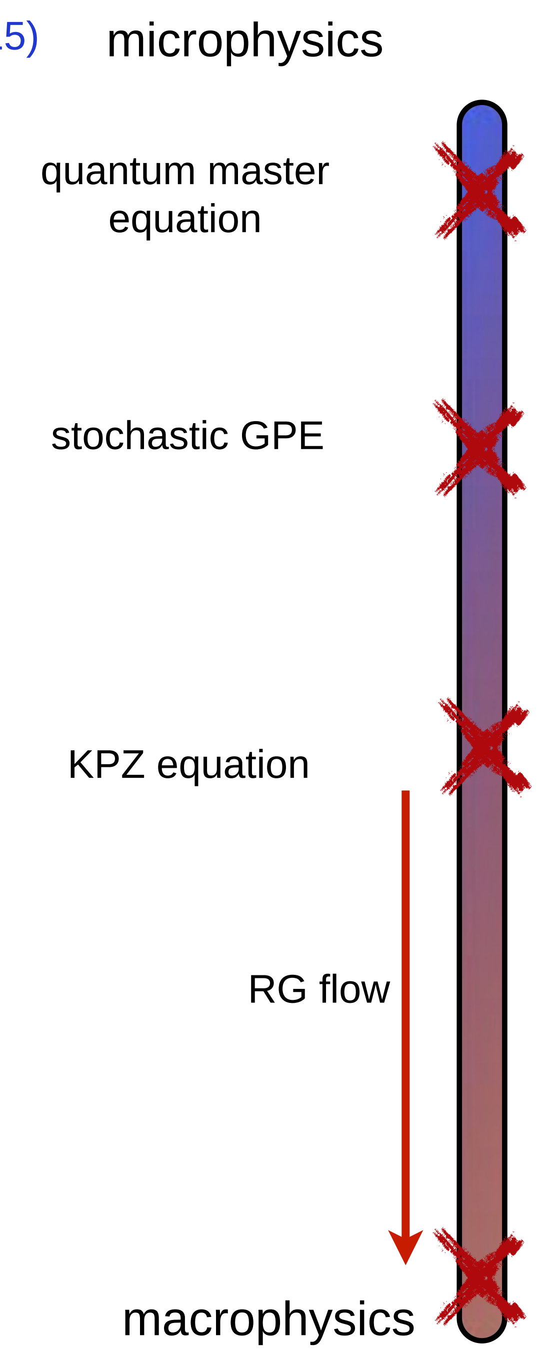
E. Altman, L. Sieberer, L. Chen, SD, J. Toner, PRX (2015)

Physical implication I: Smooth KPZ fluctuations

- How important are non-equilibrium conditions at large distance?

- behavior of non-equilibrium strength under coarse graining (RG flow)





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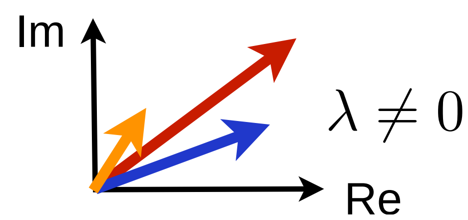

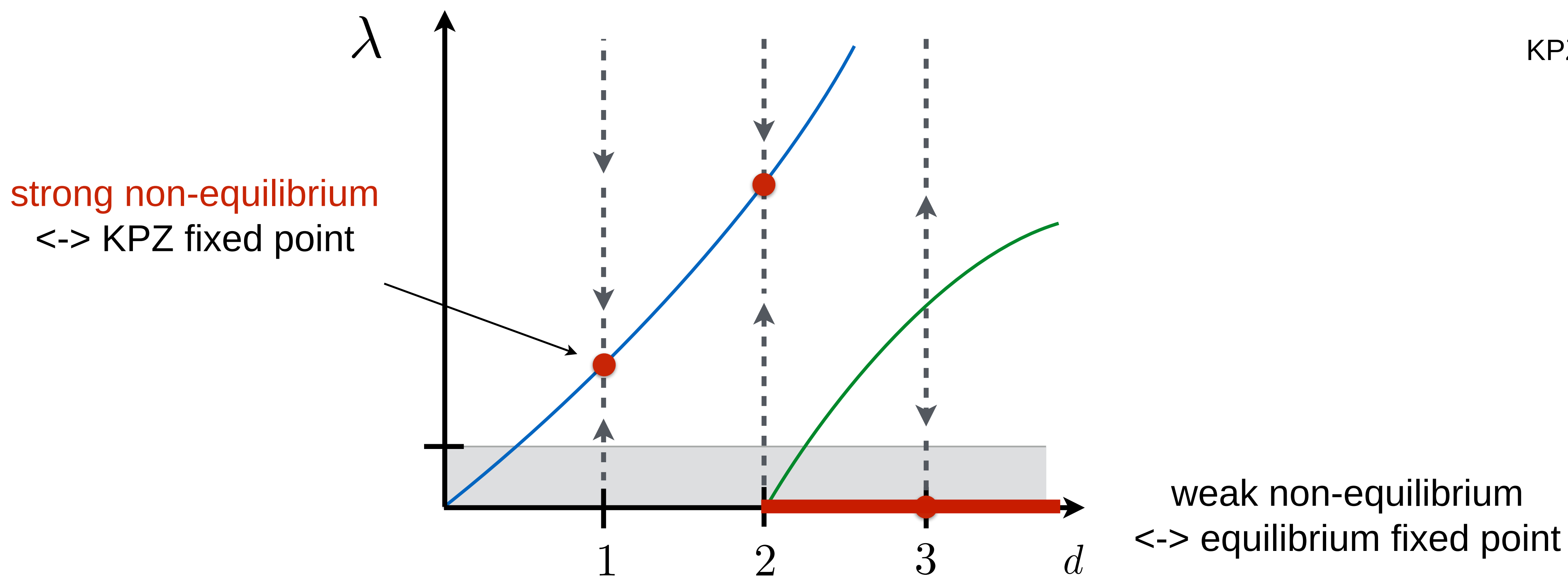

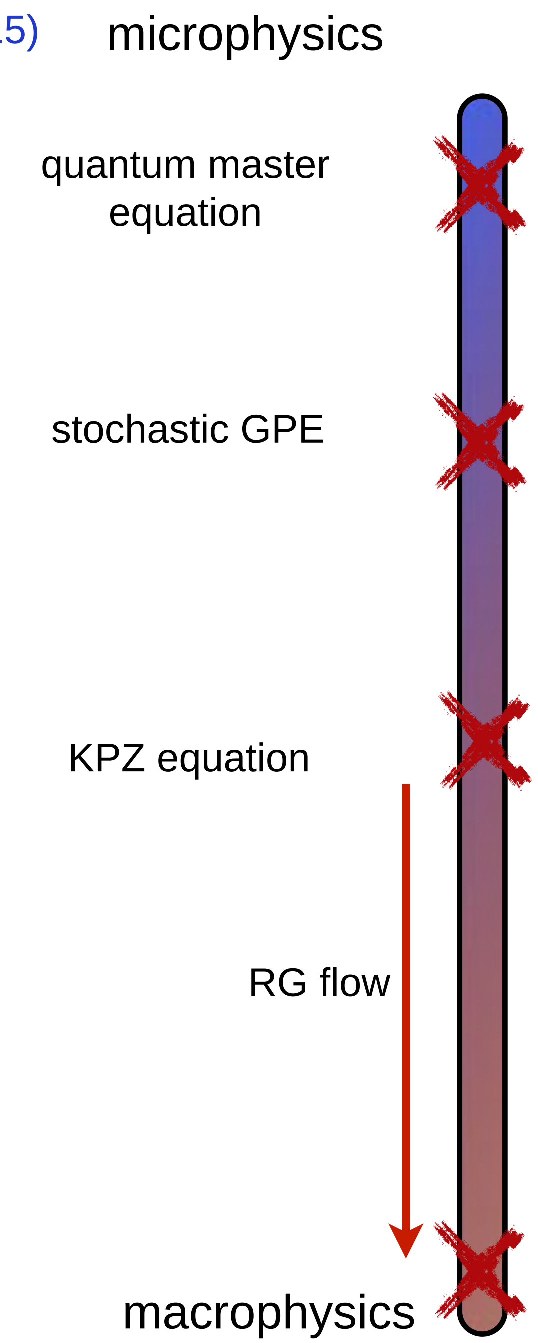

- general trend: non-equilibrium effects in systems with soft mode are
  - enhanced in d = 1,2
  - softened in d = 3 (below a threshold)

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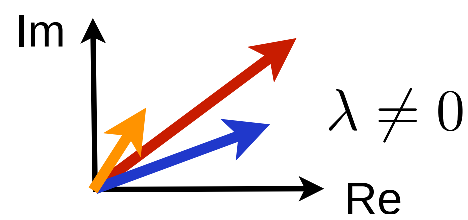
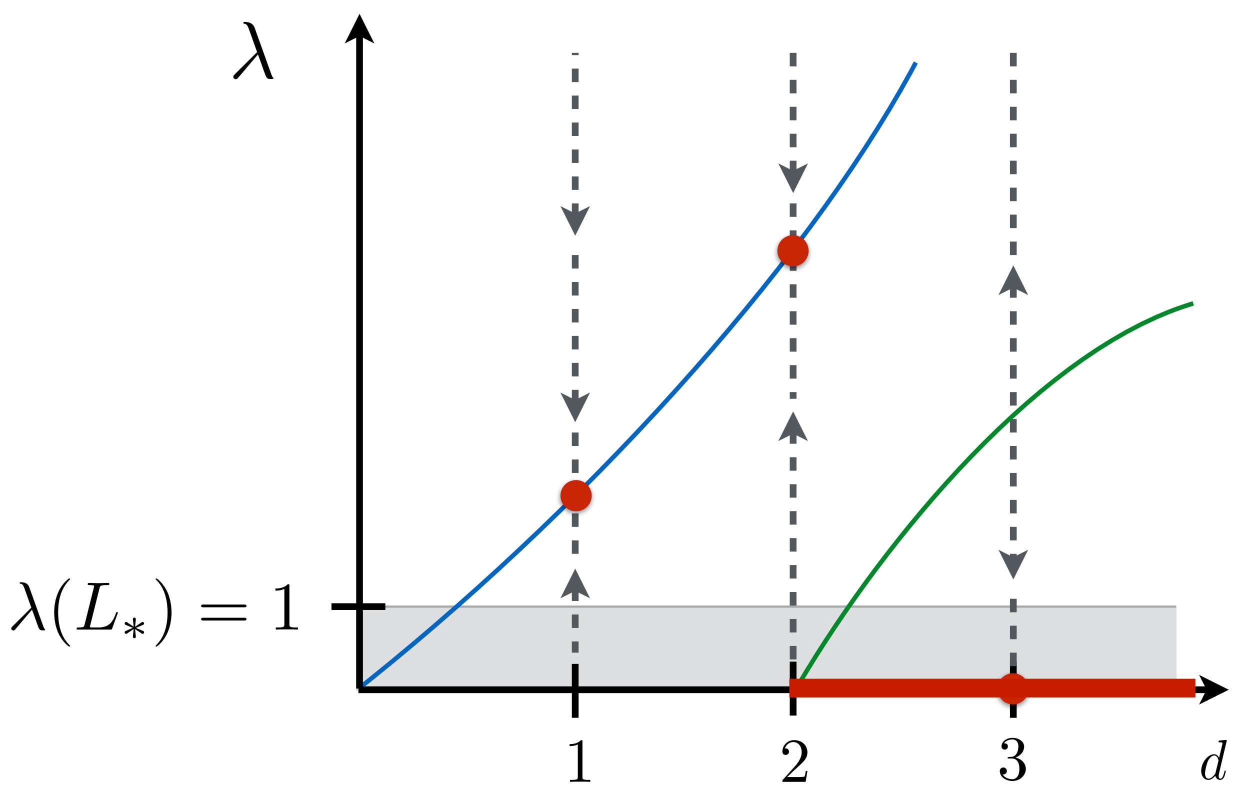
- behavior of non-equilibrium strength under coarse graining (RG flow)

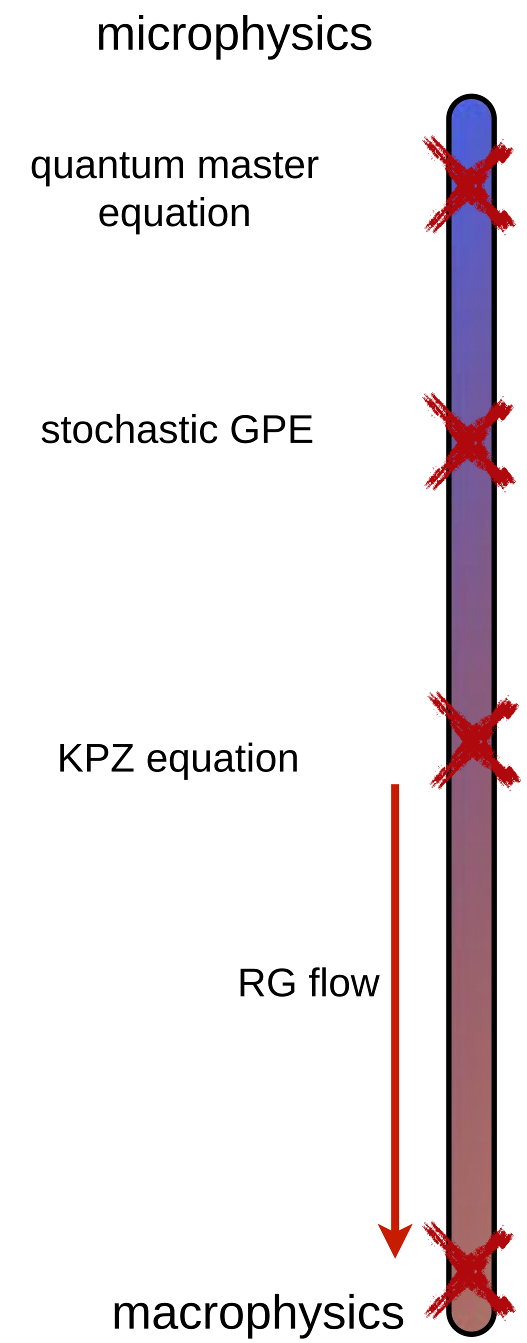




- 2D: implication: a length scale is generated

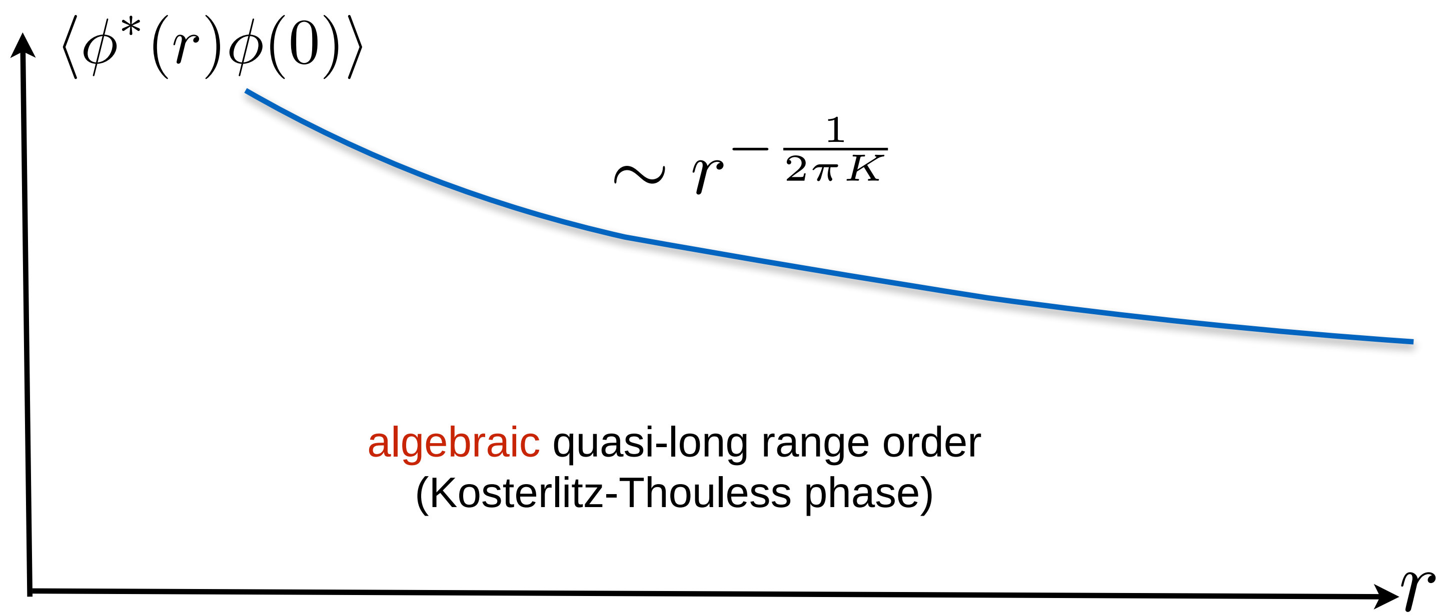
$$L_* = a_0 e^{\frac{16\pi D^3}{\lambda^2 \Delta}}$$

microscopic (healing) length



Physical implications I: Absence of algebraic order

- generated length scale distinguishes two regimes: $L_* = a_0 e^{\frac{16\pi D^3}{\lambda^2 \Delta}}$



equilibrium fixed point relevant

Physical implications I: Absence of algebraic order

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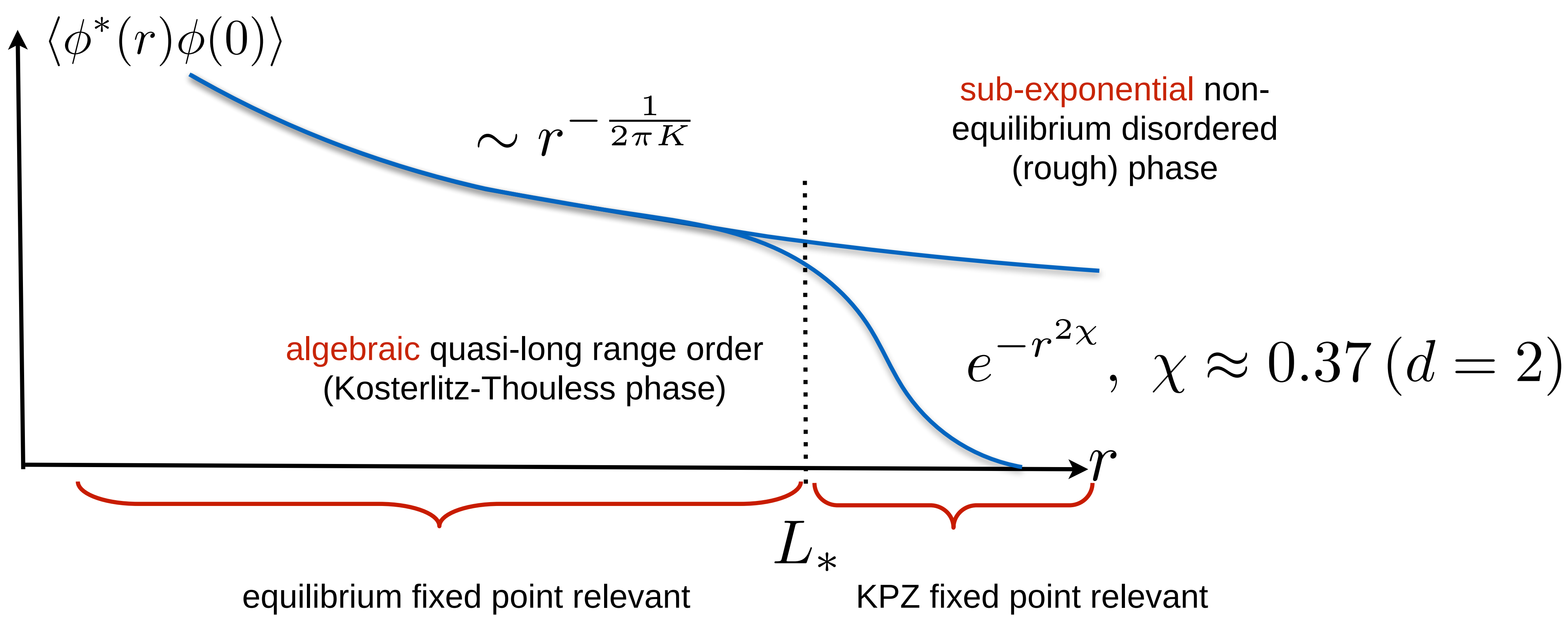

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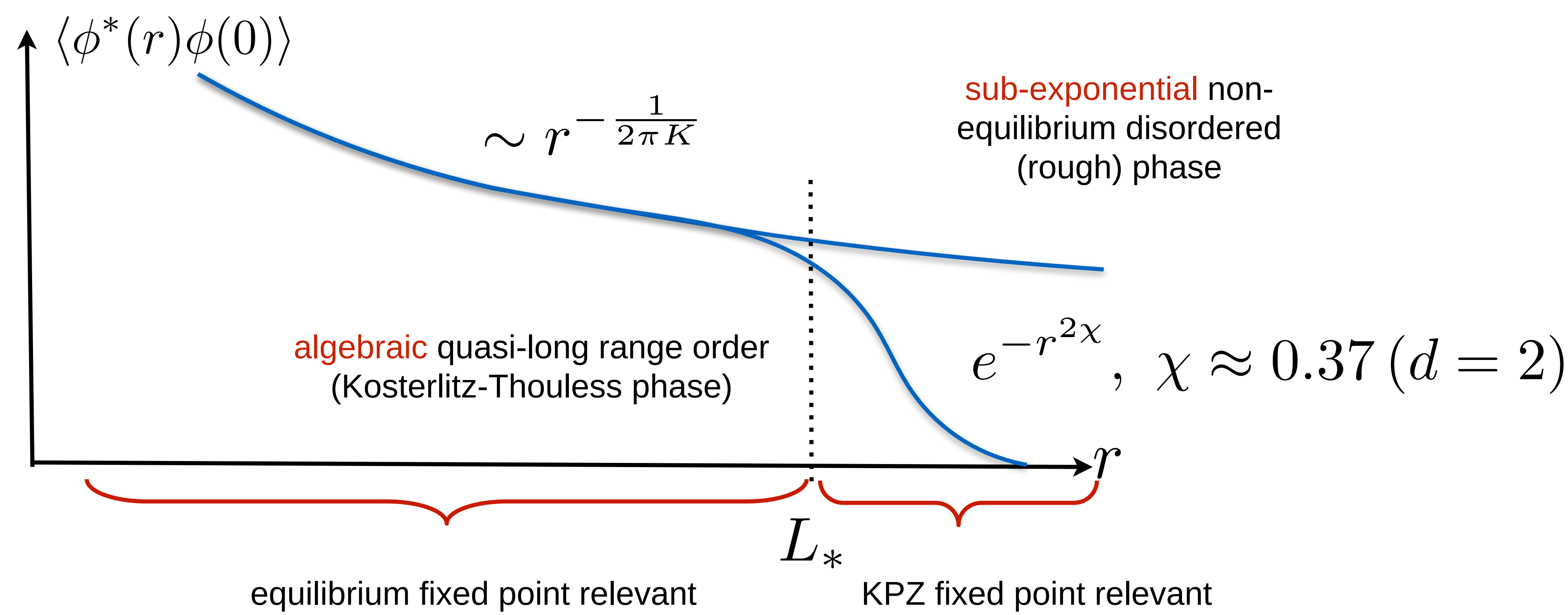
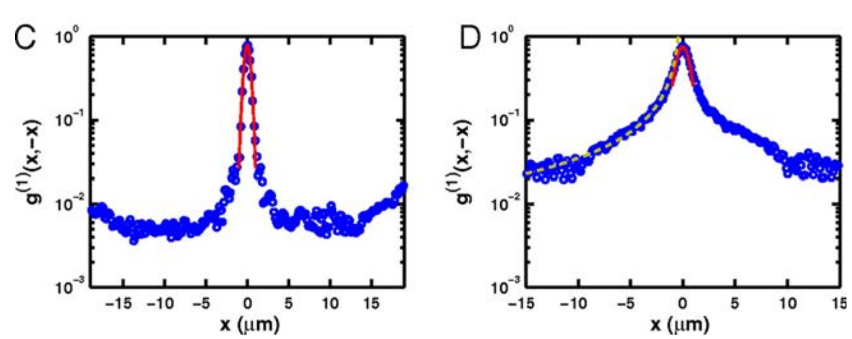


➡ algebraic order absent in any two-dimensional driven open system at the largest distances
➡ but crossover scale exponentially large for small deviations from equilibrium
➡ were precursors already observed in 2D experiment? (announced: 1D, Bloch group (Paris))

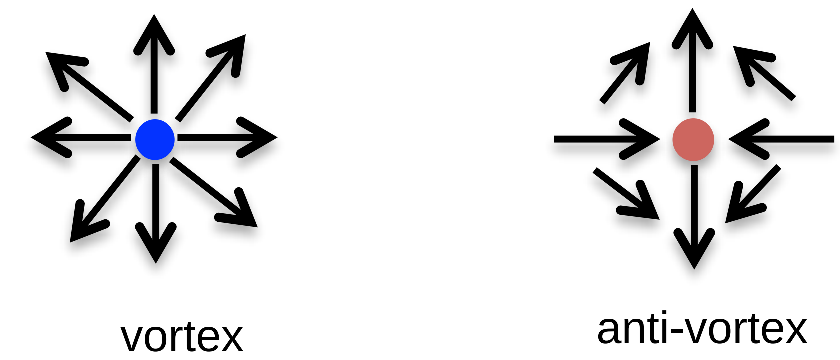


Roumpos et al., PNAS (2012)

Physical implications II: Non-equilibrium Kosterlitz-Thouless

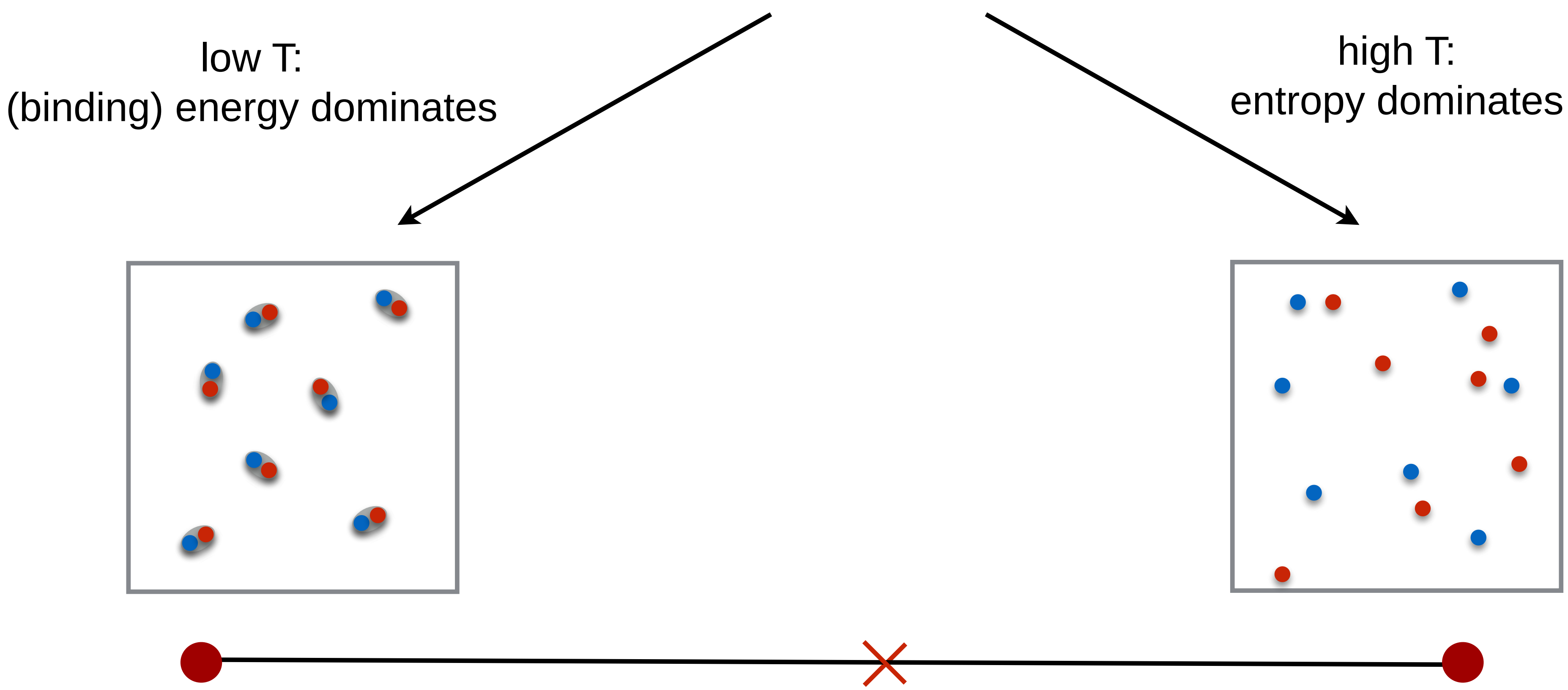
- KPZ equation for phase variable

$$\partial_t \theta = D\nabla^2\theta + \lambda(\nabla\theta)^2 + \xi$$



- compact nature of phase allows for vortex defects in 2D!
- key ingredient of Kosterlitz-Thouless transition

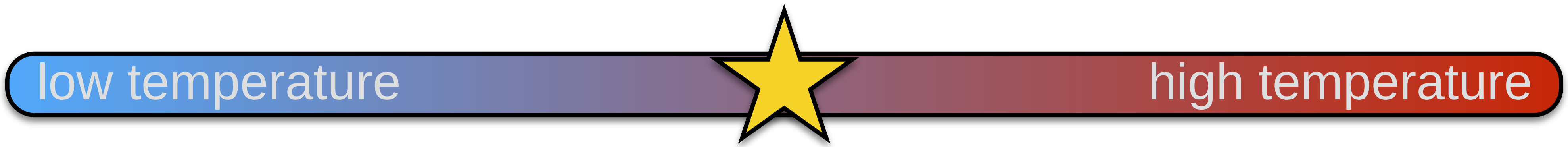
$$F = E - TS$$

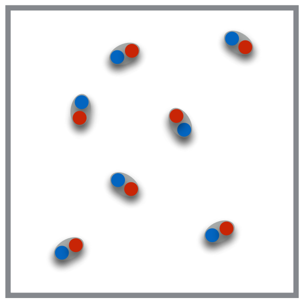


➡ so far, compactness of phase variable ignored

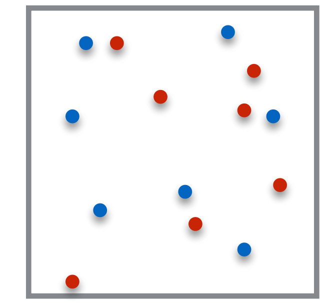
➡ how is this scenario modified in the driven system?

Mini-review: BKT transition





- BKT transition: unbinding of vortex-antivortex pairs
  J. M. Kosterlitz, D. J. Thouless J. Phys. C (1973)



- Single vortex picture: balance of energy (deterministic) and entropy (statistic)
  - Low T: vortices and antivortices bound in neutral pairs (irrelevant at long distance)
  - Q: when is it favorable (free energy) minimum to have unbound vortices?

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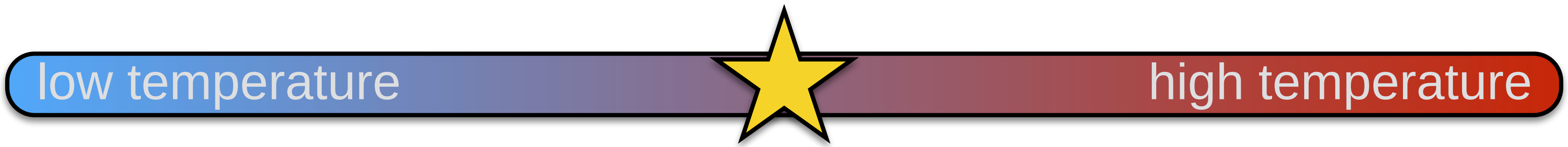

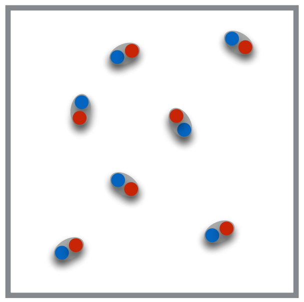

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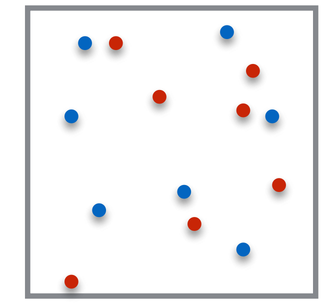

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- energy of single free vortex:

vortex configuration: mapping $(r, \varphi) \to \theta(r, \varphi) = \varphi \qquad \Longrightarrow \nabla\theta = \frac{1}{r}\hat{e}_\phi \partial_\varphi \theta(r,\varphi) = \frac{1}{r}\hat{e}_\phi$

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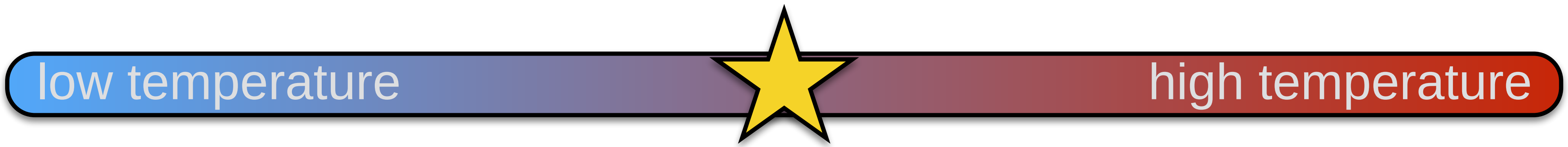

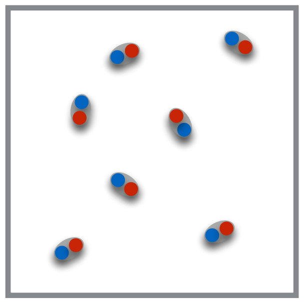

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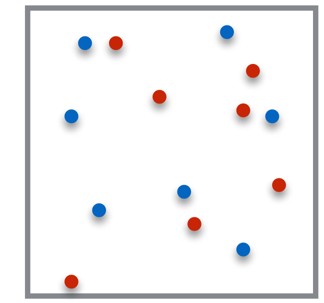

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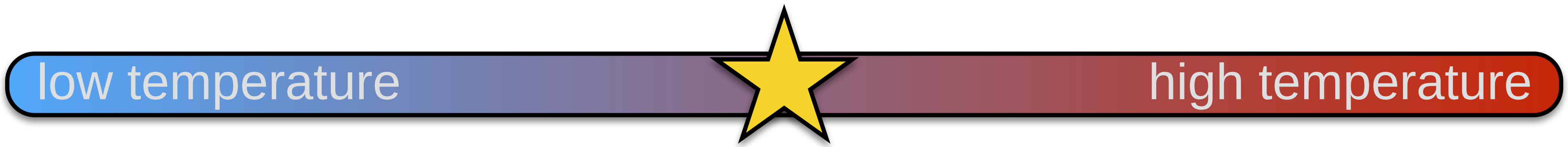

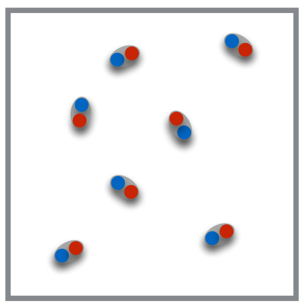

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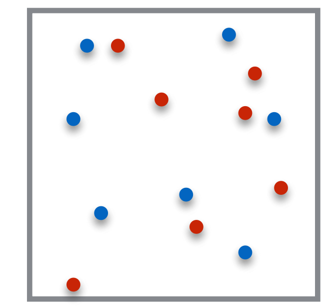

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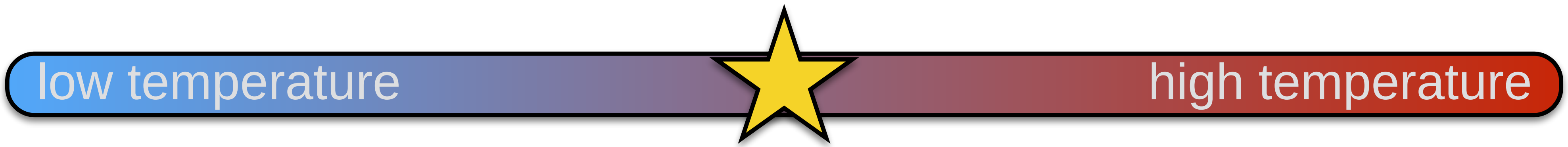
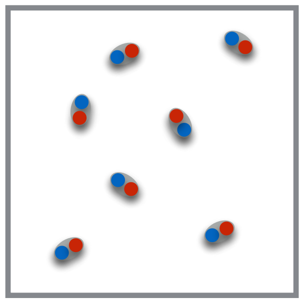
- free energy $\quad F = E - TS = (\pi K - 2k_B T)\log(L/a)$

- vortex proliferation above KT critical temperature $\quad T_{KT} = \dfrac{\pi K}{2k_B}$

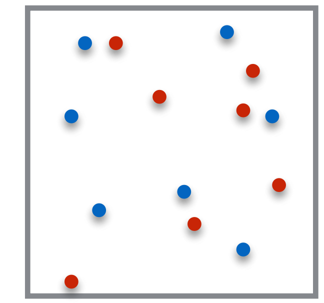
Mini-review: BKT transition





- BKT transition: unbinding of vortex-antivortex pairs

  J. M. Kosterlitz, D. J. Thouless J. Phys. C (1973)



- Single vortex picture: balance of energy (deterministic) and entropy (statistic)
  - Low T: vortices and antivortices bound in neutral pairs (irrelevant at long distance)
  - Q: when is it favorable (free energy) minimum to have unbound vortices?

- energy of single free vortex:

  vortex configuration: mapping $(r,\varphi)\to\theta(r,\varphi)=\varphi \qquad \Longrightarrow \nabla\theta=\frac{1}{r}\hat{e}_\phi\partial_\varphi\theta(r,\varphi)=\frac{1}{r}\hat{e}_\phi$

$$\Longrightarrow\ E=\frac{K}{2}\int d^2x(\nabla\theta)^2=\pi K\int_a^L drr\frac{1}{r^2}=\pi K\log(L/a)$$

- entropy: sum all equally probable possibility of placing vortices in 2D plane at minimal distance a:

$$S=-k_B\sum_i p_i\log p_i=k_B\log(L/a)^2$$

- free energy $\qquad F=E-TS=(\pi K-2k_BT)\log(L/a)$

- vortex proliferation above KT critical temperature $\qquad T_{KT}=\dfrac{\pi K}{2k_B}$

➡ but out of equilibrium: no free energy at hand!

➡ field theory approach (analogous Kosterlitz' real space RG for vortices)

J. M. Kosterlitz, J. Phys. C (1974)

Field theoretical duality approach: Phase compactness

G. Wachtel, L. Sieberer, SD, E. Altman, PRB (2016)
L. Sieberer, G. Wachtel, E. Altman, SD, PRB (2016)

- wait a second — we ignored a fundamental symmetry of polaritons so far

$$\phi(t, \mathbf{x}) = \rho(t, \mathbf{x}) e^{i\theta(t, \mathbf{x})}$$

- phase compactness = local discrete gauge invariance under

$$\theta_{t,\mathbf{x}} \mapsto \theta_{t,\mathbf{x}} + 2\pi n_{t,\mathbf{x}}$$

➡ how to teach to the KPZ equation? lattice regularization and discrete stochastic update

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➡ how to teach to the KPZ equation? lattice regularization and discrete stochastic update

$$\theta_{t+\epsilon,\mathbf{x}} = \theta_{t,\mathbf{x}} + \epsilon\left(\mathcal{L}[\theta]_{t,\mathbf{x}} + \eta_{t,\mathbf{x}}\right) + 2\pi n_{t,\mathbf{x}}$$

lattice regularized deterministic term

$$\boxed{Z = \sum_{\{\tilde{n}_{t,\mathbf{x}}\}} \int \mathcal{D}[\theta] e^{iS[\theta,\tilde{n}]}}$$

stochastic difference equation $\Leftrightarrow$ discrete noise MSR functional integral

non-equilibrium lattice gauge theory (manifestly gauge invariant)

Duality approach: Emergent electrodynamics

- discrete gauge invariant dynamical functional integral $S = \sum_{t,\mathbf{x}} \tilde{n}_{t,\mathbf{x}} \left[-\Delta_t \theta_{t,\mathbf{x}} + \epsilon \left(\mathcal{L}[\theta]_{t,\mathbf{x}} + i \Delta \tilde{n}_{t,\mathbf{x}} \right) \right]$

$$\boxed{Z = \sum_{\{\tilde{n}_{t,\mathbf{x}}\}} \int \mathcal{D}[\theta] e^{iS[\theta,\tilde{n}]}}$$

$$\Updownarrow$$

- dual description in terms of gauge theory: noisy electrodynamics in the presence of charges/ currents (vortices)

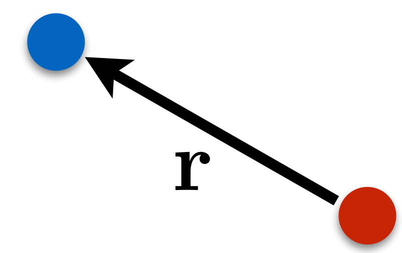
$$Z \propto \underbrace{\sum_{\substack{\{n_{vX}, \tilde{n}_{vX}, \\ \mathbf{J}_{vX}, \tilde{\mathbf{J}}_{vX}\}}}}_{\text{vortex density and current}} \int \underbrace{\mathcal{D}[\phi, \tilde{\phi}, \mathbf{A}, \tilde{\mathbf{A}}]}_{\text{smooth spin wave fluctuations (equivalent KPZ equation)}} e^{iS[\phi,\tilde{\phi},\mathbf{A},\tilde{\mathbf{A}},n_v,\tilde{n}_v,\mathbf{J}_v,\tilde{\mathbf{J}}_v]}$$

- key impact of non-equilibrium conditions: non-linear electrodynamics
- integrate out (perturbatively) smooth fluctuations to get effective theory for vortex interactions

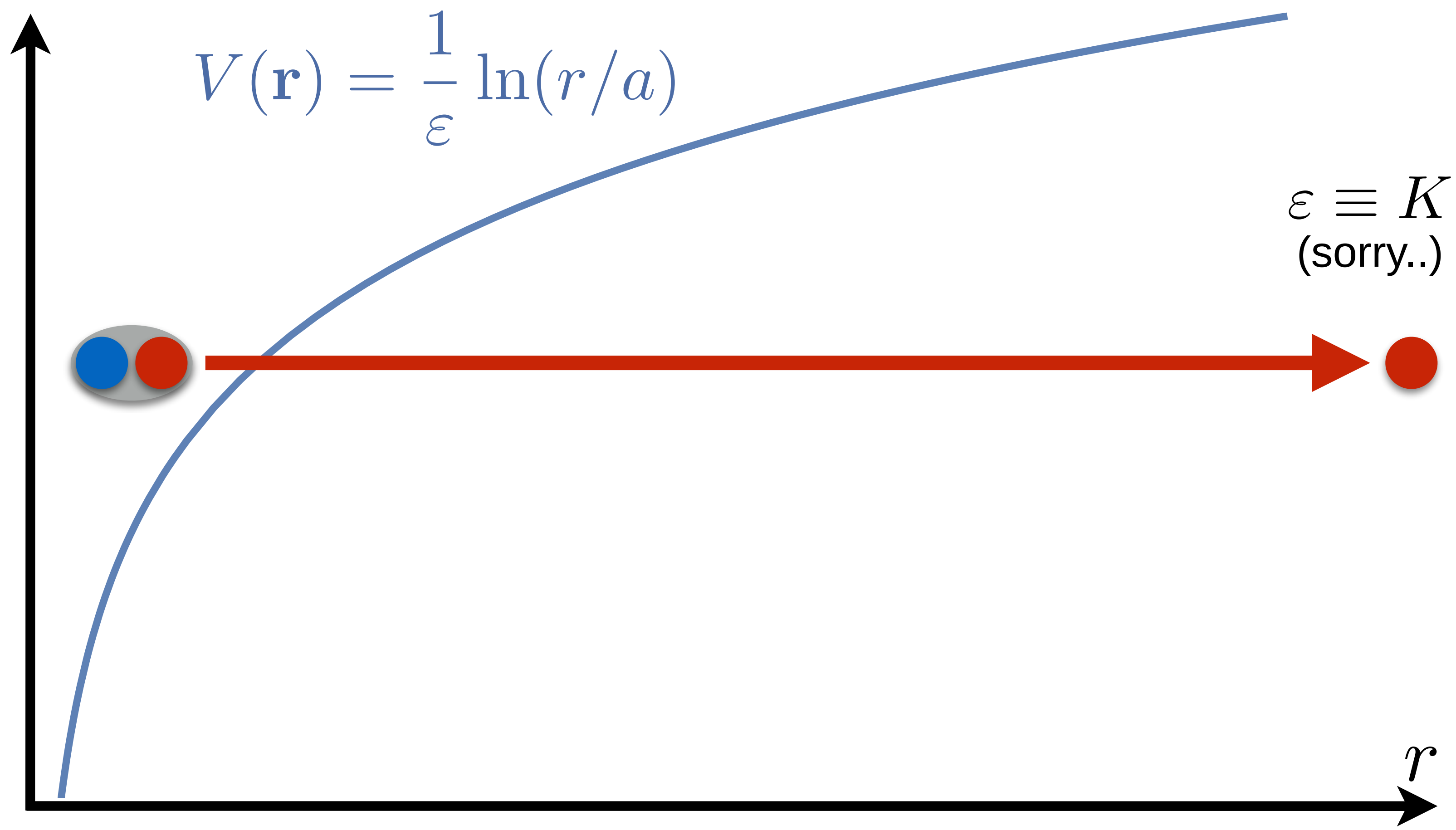
Effective theory for a single vortex-antivortex pair

- equation of motion for a single vortex-antivortex pair

$$\frac{d\mathbf{r}}{dt} = -\mu \nabla V(r) + \boldsymbol{\xi}$$



equilibrium: Coulomb potential (2D)



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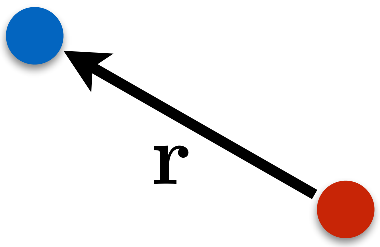

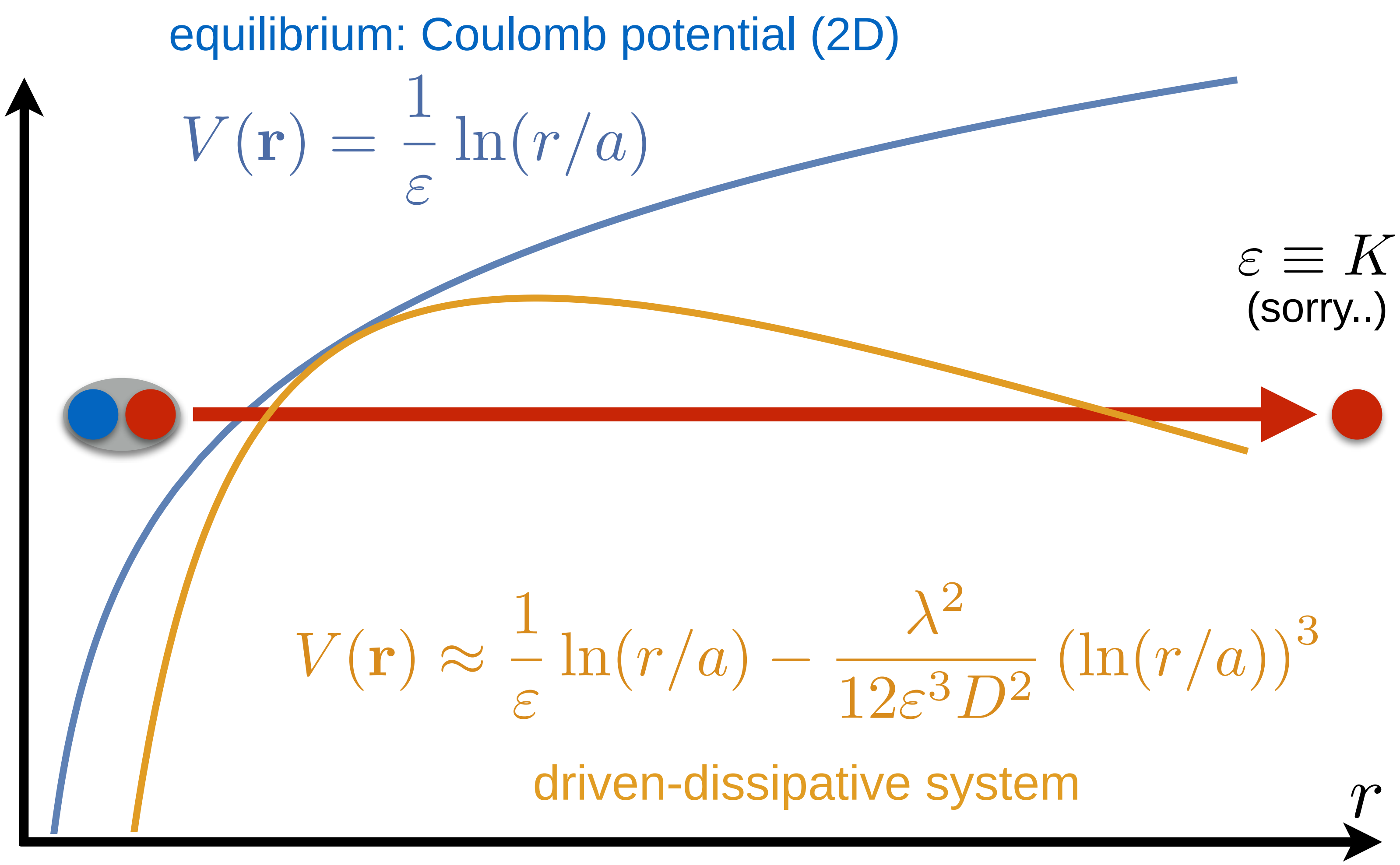

➡ noise-activated unbinding for a single pair (at exp small rate)

Effective theory for a single vortex-antivortex pair

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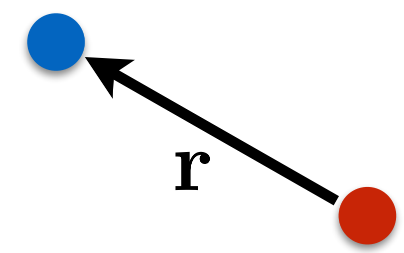
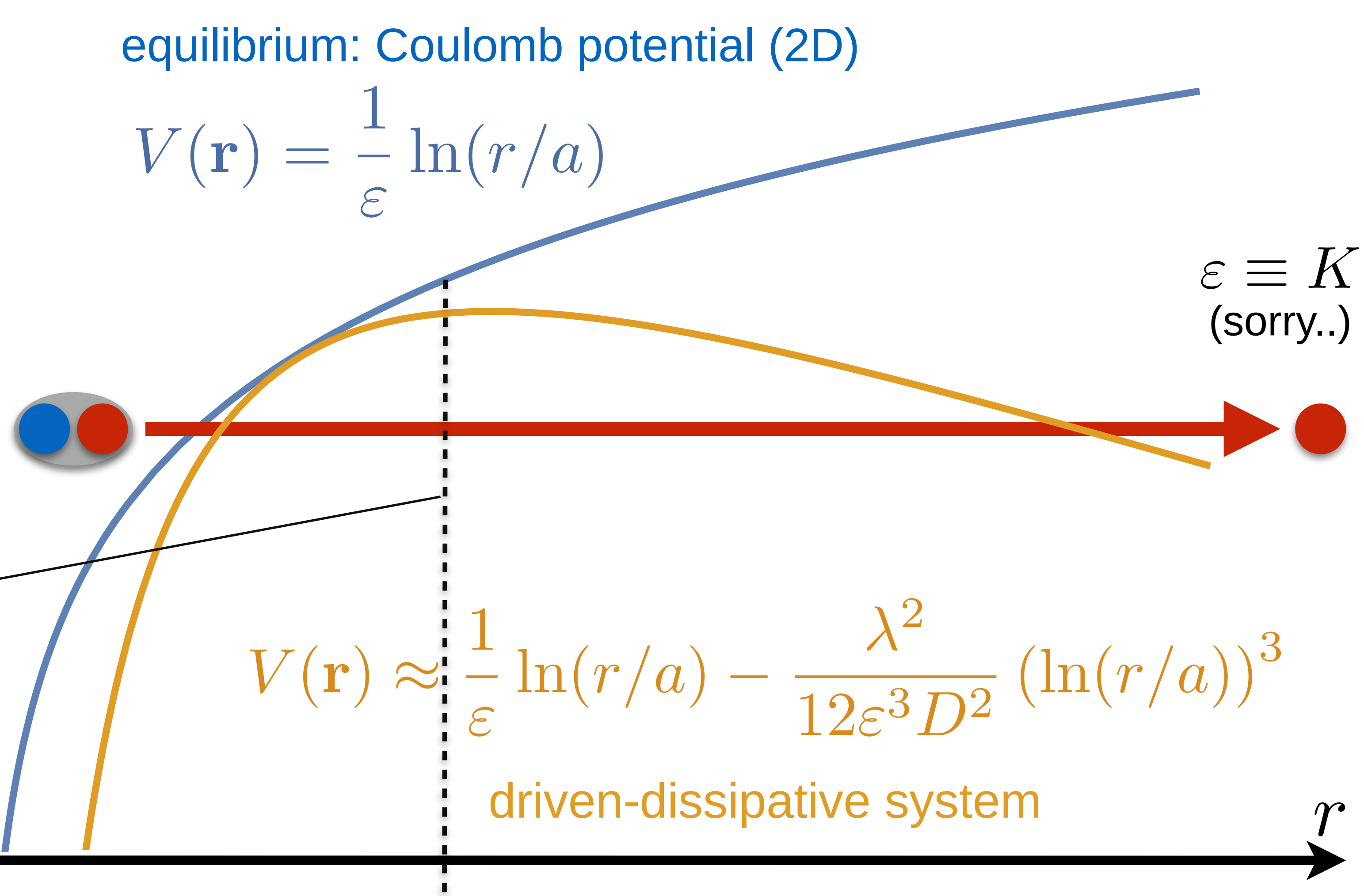




length scale:

$$L_v = a_0 e^{\frac{2D}{\lambda}}$$

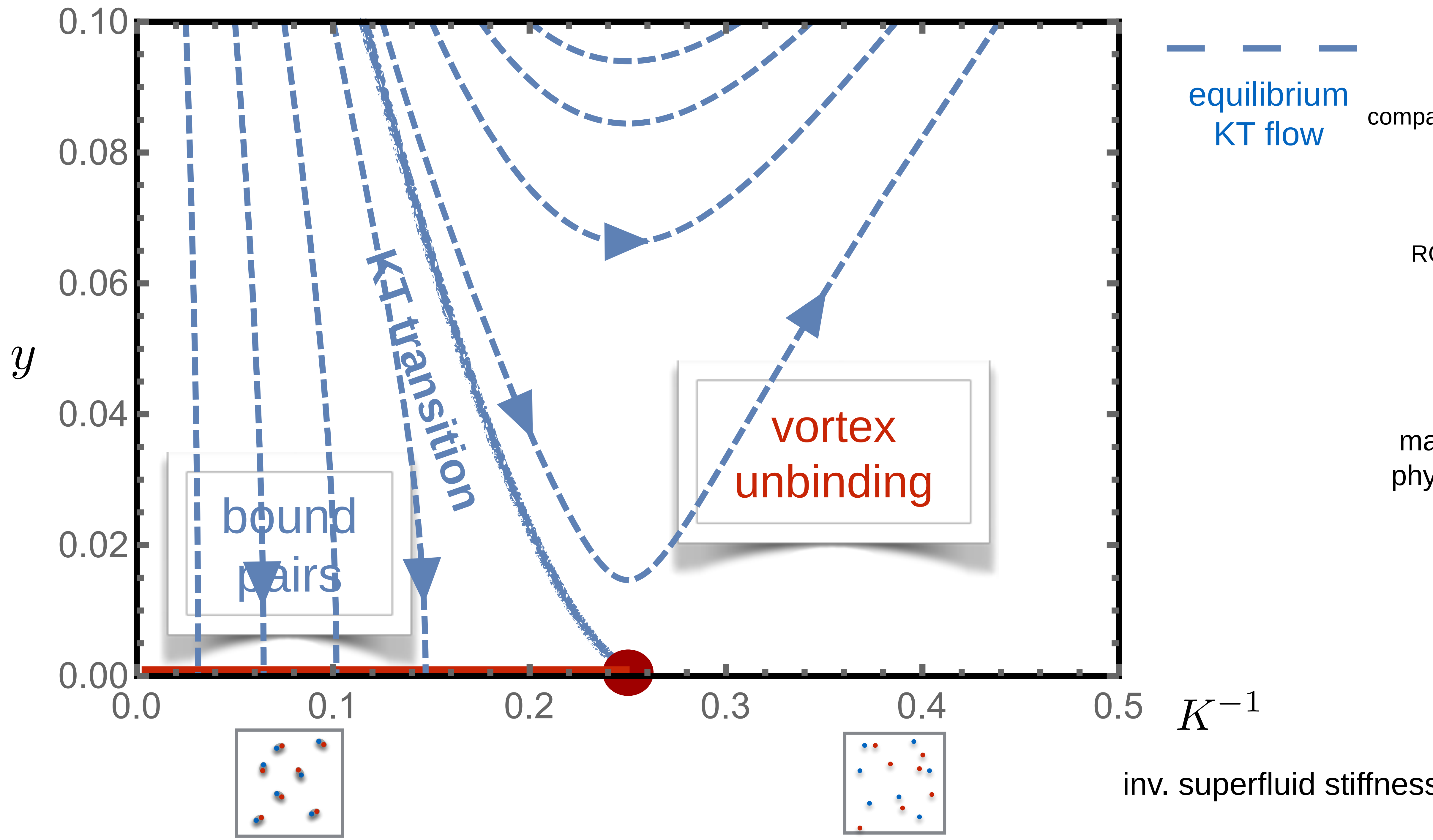
see also: I Aranson et al., PRB (1998) two-vortex problem

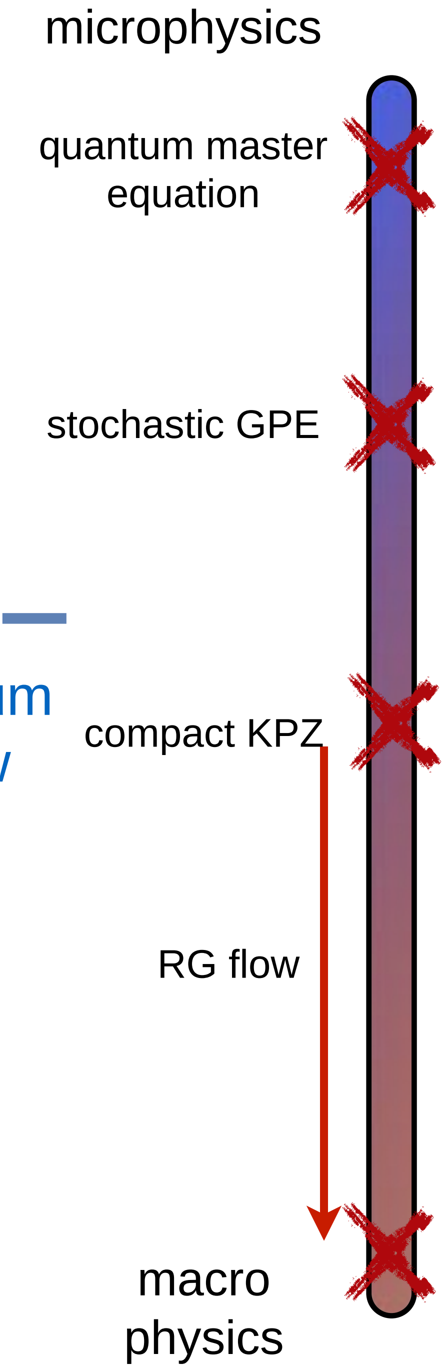
➡ noise-activated unbinding for a single pair (at exp small rate)

Many pairs: Corrections to Kosterlitz-Thouless flow

$$\frac{dK}{d\ell} = \frac{2\pi^2 y^2}{T} \qquad \frac{dy}{d\ell} = \left[2 - \frac{1}{2KT} + \cancel{\frac{\lambda^2}{4K^2D^2}}\left(\frac{1}{4}+\ell\right)\right] y \qquad \frac{dT}{d\ell} = \cancel{\frac{\lambda^2 T}{2K^2D^2}}\left(\frac{1}{4}+\ell\right)$$

- parameter y: ~ probability to create single vortex at distance l (flow parameter)





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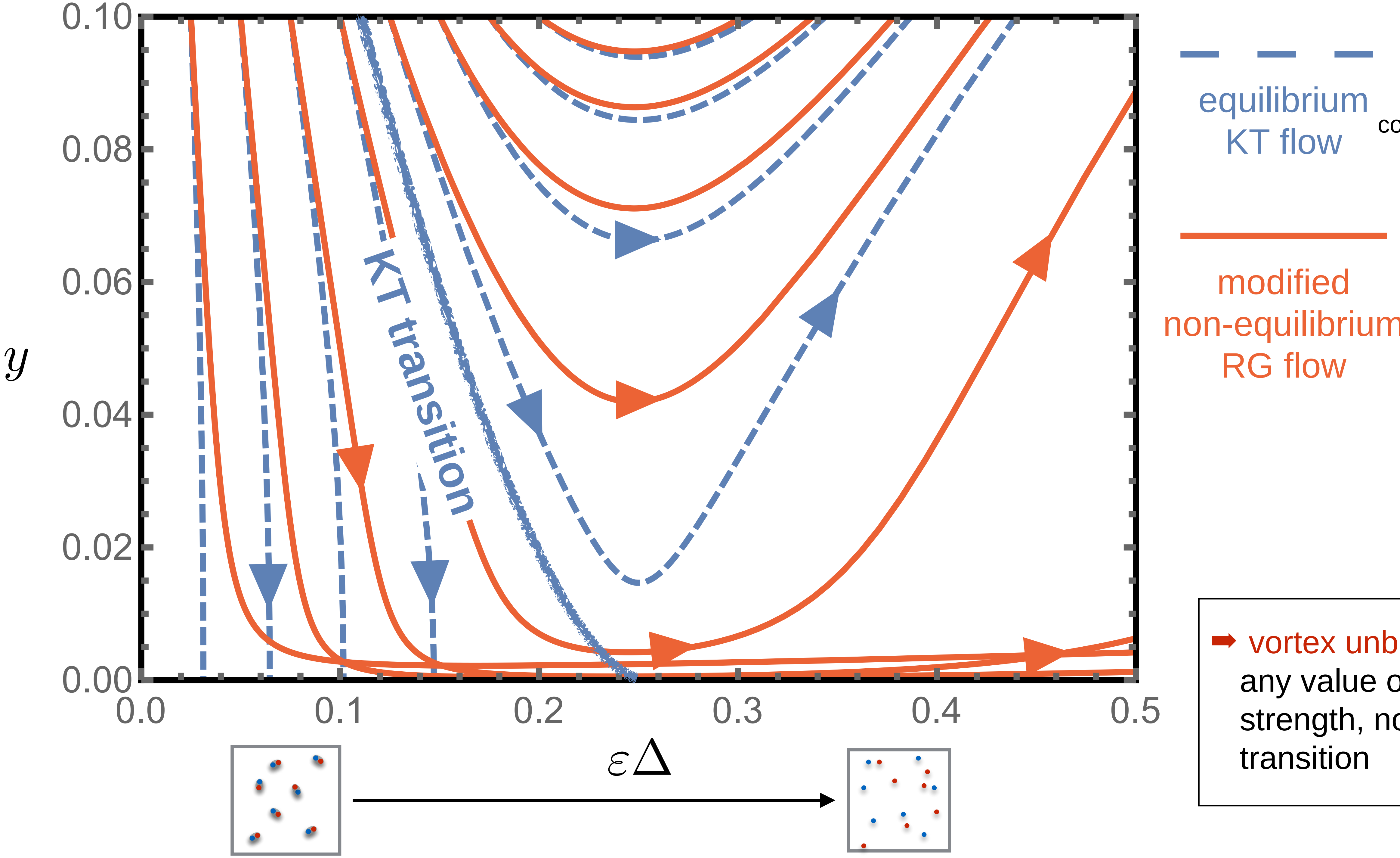
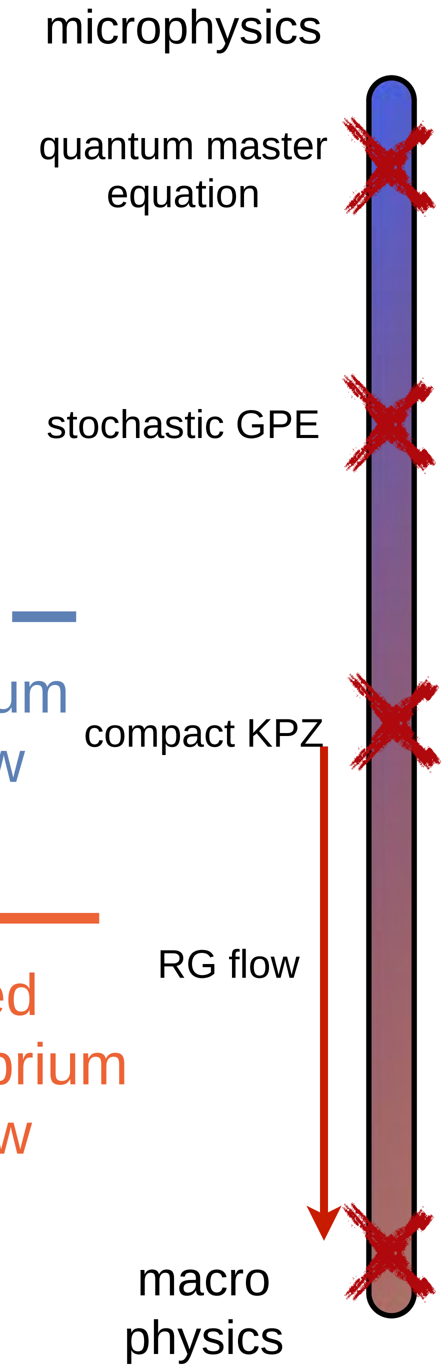




➡ vortex unbinding for any value of the noise strength, no phase transition

L. Sieberer, G. Wachtel, E. Altman, SD, PRB (2016)
G. Wachtel, L. Sieberer, SD, E. Altman, PRB (2016)

Competing length scales and suppression of KT

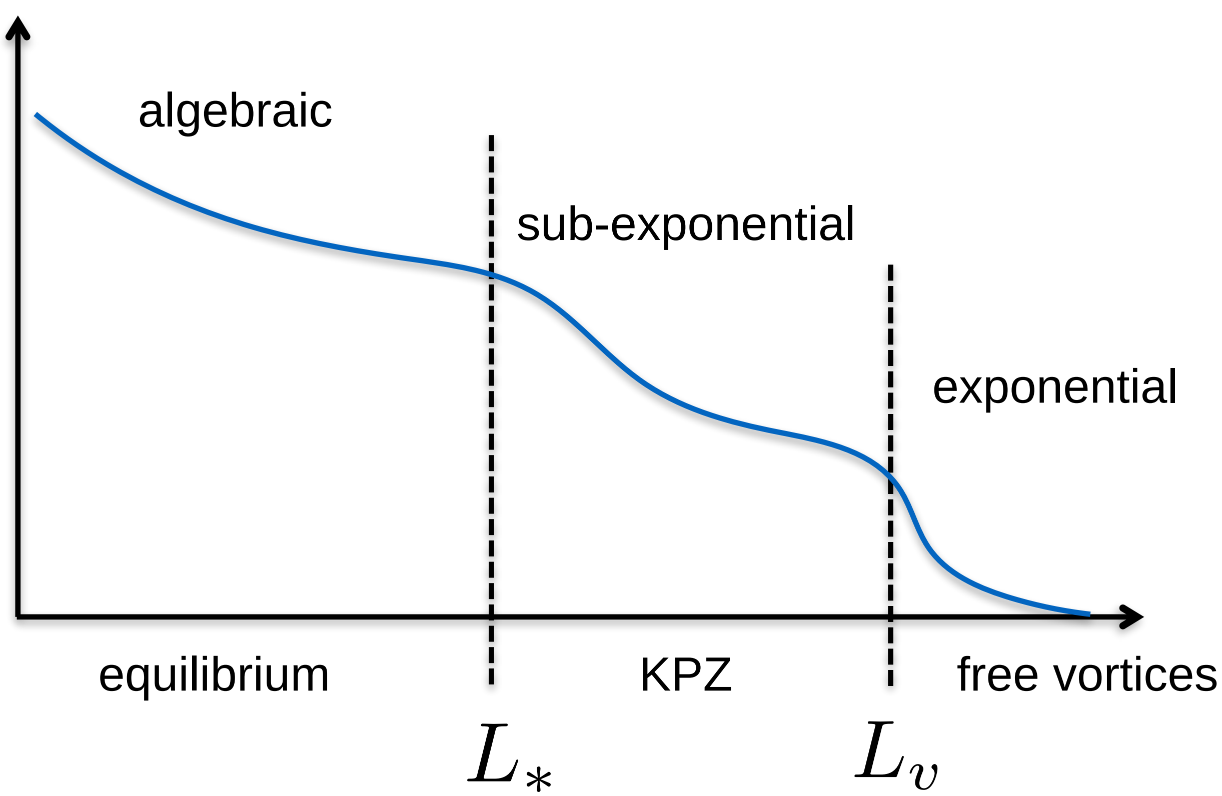
- two emergent length scales in complementary approaches:

$$L_* = a_0 e^{\frac{16\pi D^3}{\lambda^2 \Delta}}$$

KPZ length

$$L_v = a_0 e^{\frac{2D}{\lambda}}$$

vortex length



- full numerical confirmation of two-scale scenario in 1D (defects: vortices in (1+1)D space-time)

  L. He, L. Sieberer, SD PRL (2017)

- 2D simulations demanding

L. Sieberer, G. Wachtel, E. Altman, SD, PRB (2016)
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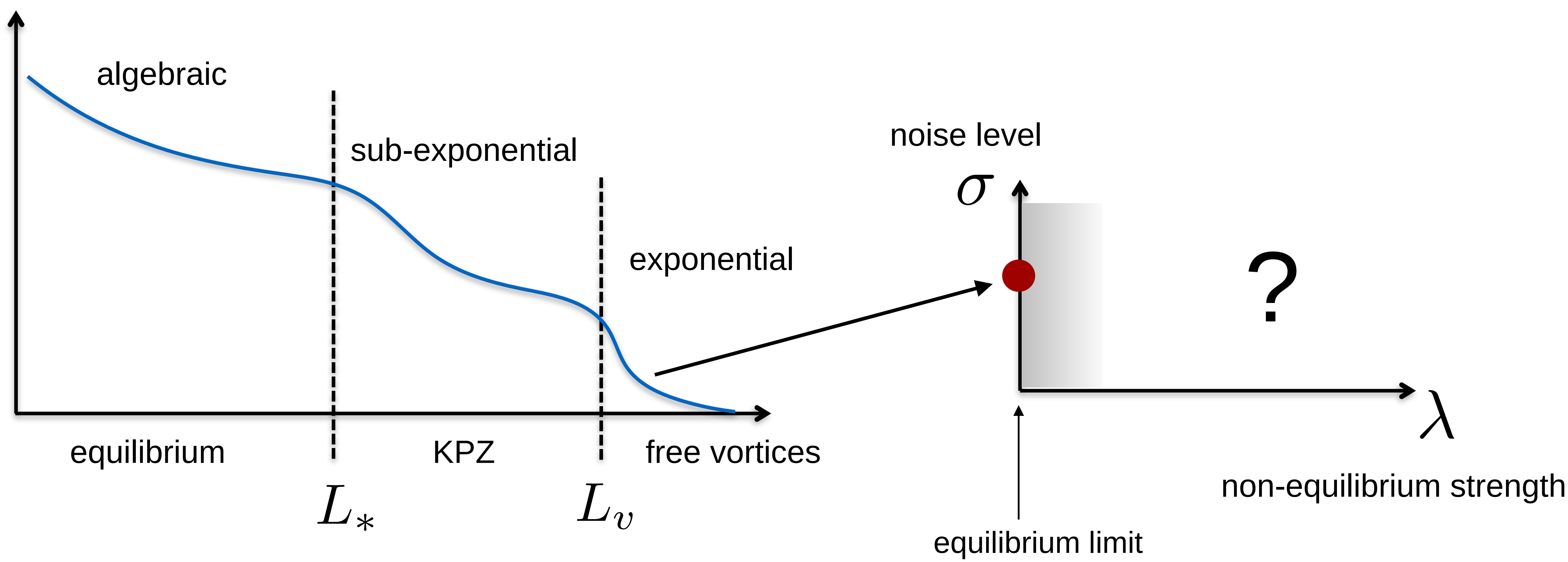
KPZ length   vortex length



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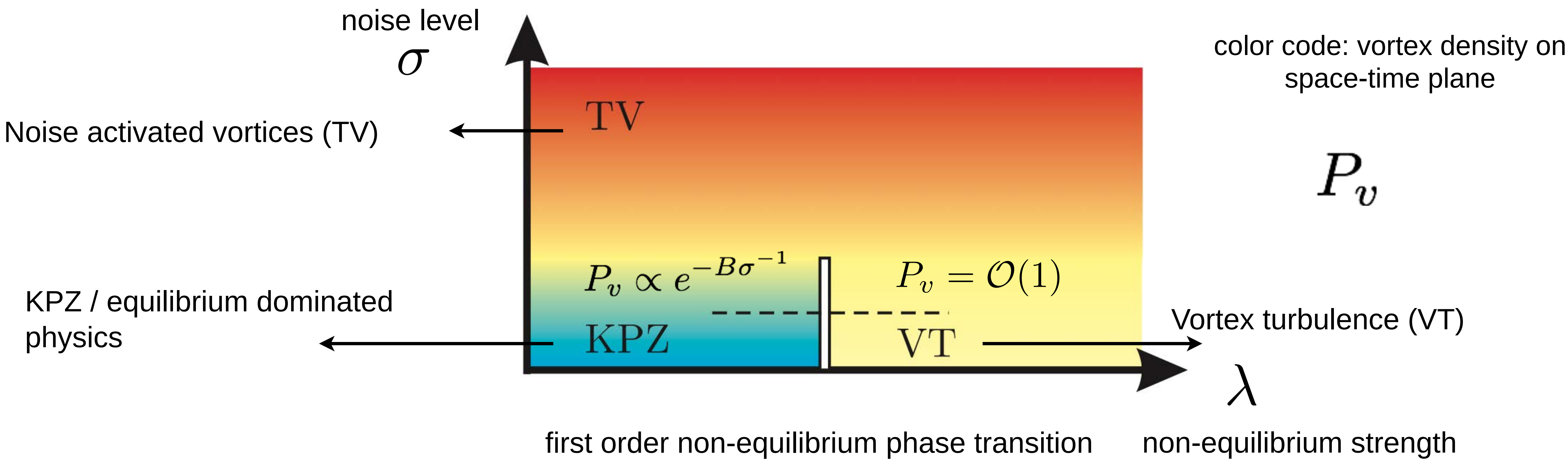
Kosterlitz-Thouless physics fragile to non-equilibrium perturbation

Strong non-equilibrium: Compact KPZ vortex turbulence

- In search of the phase diagram for XP condensates: 1+1 dimensions

L. He, L. Sieberer, SD PRL (2017)



Strong non-equilibrium: Compact KPZ vortex turbulence

- In search of the phase diagram for XP condensates: 1+1 dimensions

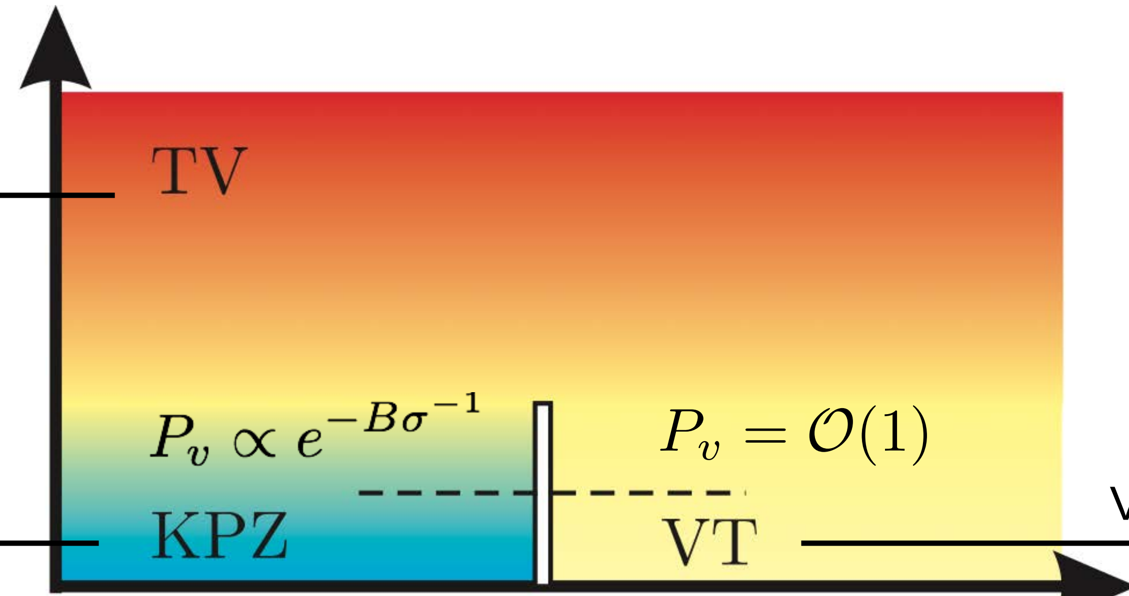
L. He, L. Sieberer, SD PRL (2017)

noise level

σ

Noise activated vortices (TV)

KPZ / equilibrium dominated physics



color code: vortex density on space-time plane

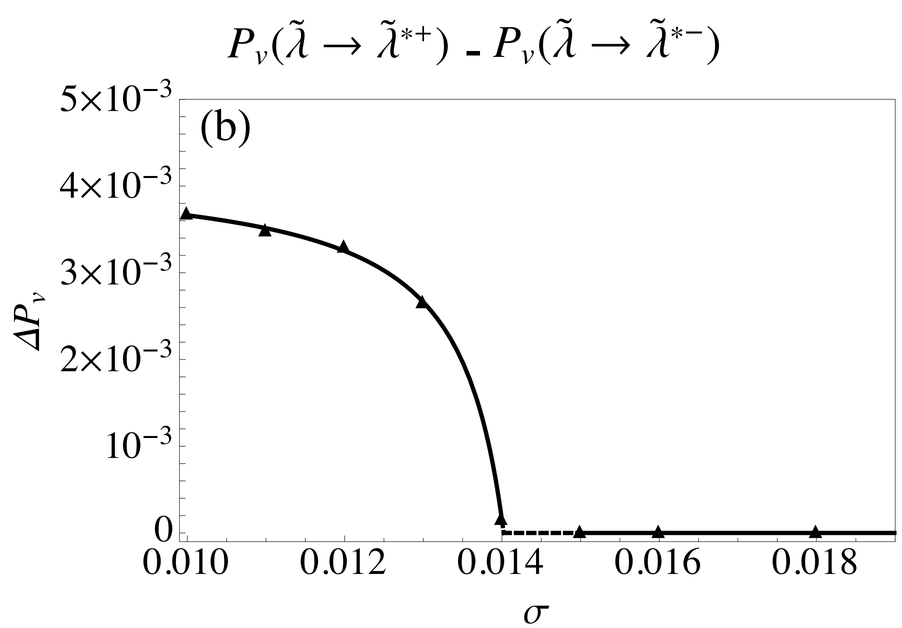
P_v

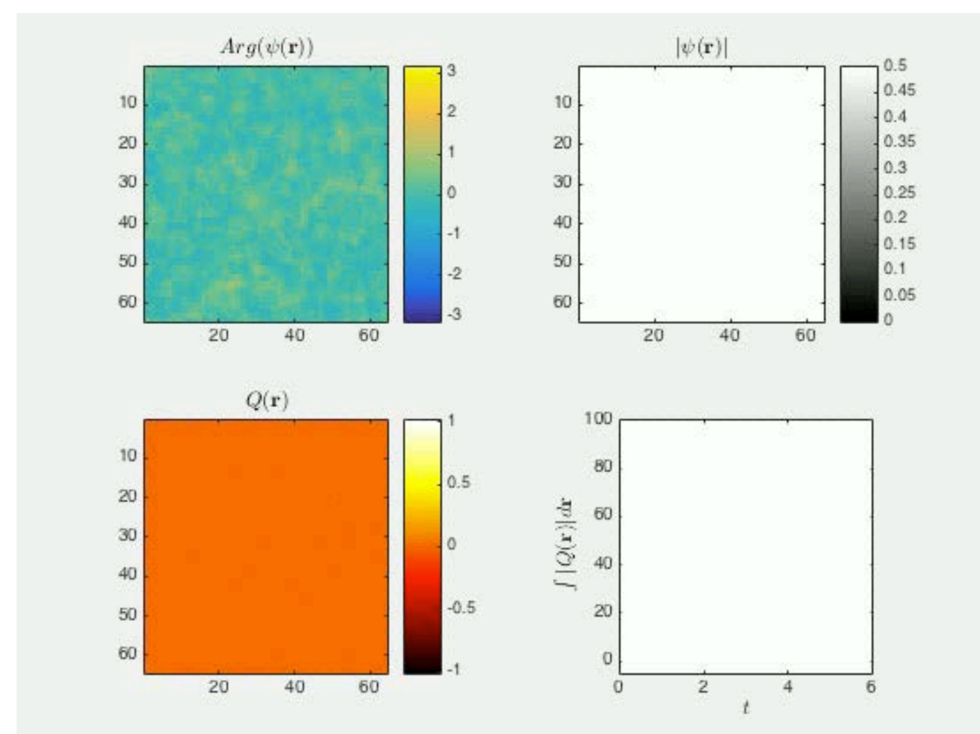
Vortex turbulence (VT)

λ

first order non-equilibrium phase transition

non-equilibrium strength





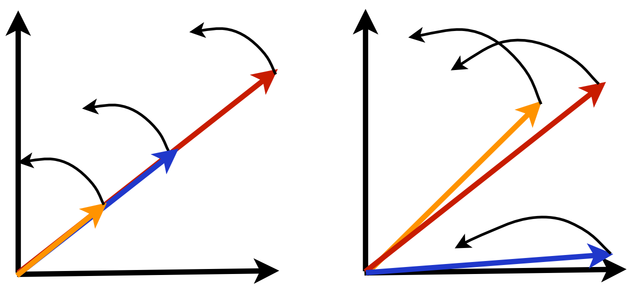
➡ deterministic limit: how does the system generate its own noise?

➡ KPZ physics announced in 1D XP condensates Bloch group, Paris

Further instances: Universality in driven open quantum systems

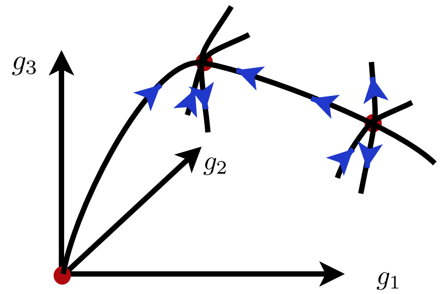
- criticality in non-equilibrium ϕ^4 model — Sieberer et al., PRL (2013)

  ➡ dynamical fine-structure distinguishing eq. from non-eq.



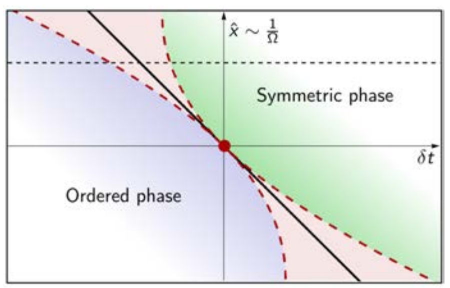
- Markovian quantum criticality — Marino, SD, PRL (2016)

  ➡ new fixed point in dark state models with quantum scaling



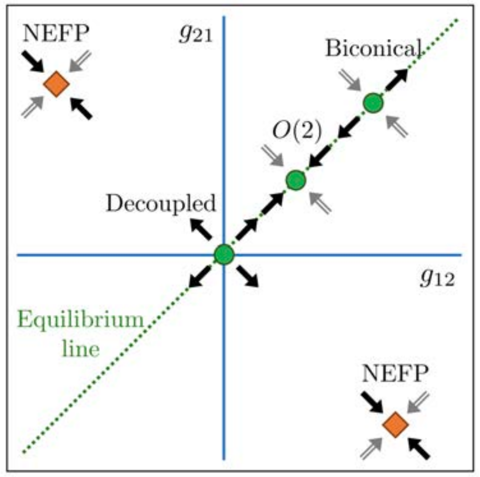
- phase transitions in open Floquet systems — Mathey, SD, PRL (2019)

  ➡ absence of criticality (dual to Kibble-Zurek mechanism)



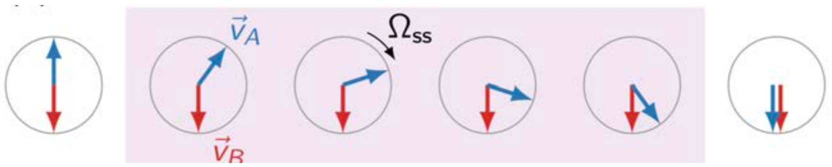
- coupled Ising models — Young et al., PRX (2020)

  ➡ new fixed point for strong non-equilibrium drive



- phase transitions with exceptional points — Fruchart et al., Nature (2021)

  ➡ universality yet to be discovered!



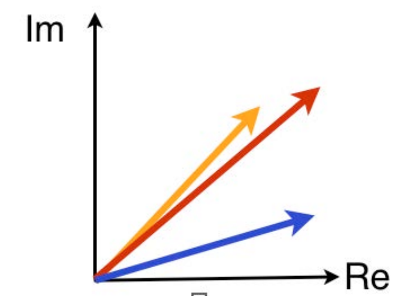
Summary lecture I

Review: L. Sieberer, M. Buchhold, SD, *Keldysh Field Theory for Driven Open Quantum Systems*, Reports on Progress in Physics (2016)

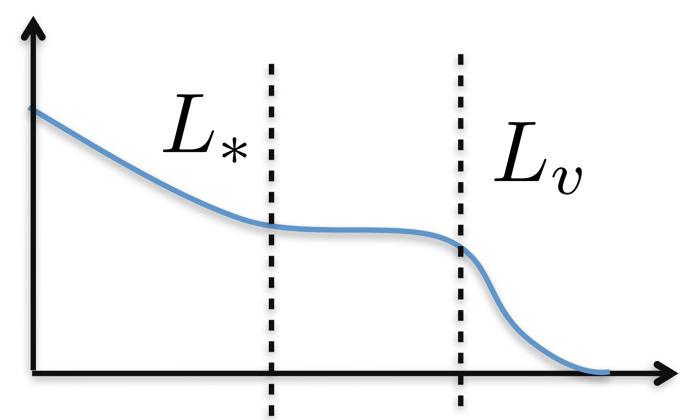
Driven open many-body systems: challenge to theory

microphysics → macrophysics

Many-Body Master Equation → 1-1 mapping → Keldysh functional integral



- mapping opens up QFT toolbox, today:
  - symmetries: eq. vs. non-eq.
  - control of IR fluctuations: understanding low dimensional gapless phases out of equilibrium
  - flexible choice of degrees of freedom: KPZ vs. vortices



2D: $L_v \ll L_*$ 1D: $L_v \gg L_*$
(time scales)

- mapping opens up QFT toolbox, next lectures:
  - symmetries: ‘weak’ and ‘strong’
  - responses out of equilibrium: topological gauge theory
  - replica field theory for measurements in many-body systems

Appendix:
Fermionic Lindblad-Keldysh functional integral

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+,\Phi_-]}$$

Fermionic Lindblad-Keldysh functional integral

- We start from the Lindblad equation

$$\partial_t \hat{\rho} = -i\big[\hat{H},\hat{\rho}\big] + \sum_\alpha \gamma_\alpha [2\hat{L}_\alpha \hat{\rho} \hat{L}_\alpha^\dagger - \hat{L}_\alpha^\dagger \hat{L}_\alpha \hat{\rho} - \hat{\rho} \hat{L}_\alpha^\dagger \hat{L}_\alpha] \tag{1}$$

  $\hat{H}, \hat{L}_\alpha$ are functions of fermionic creation and annihilation operators.

- Let us define monomial of creation and anhihilation operators as parity **even (odd)**, if they feature an even (odd) number of creation and annihilation operators. A Hamiltonian is always a sum of even operators (fermion superselection rule). However, the Lindblad operators can be either sums of even or sums of odd operators. In the second case, parity is exhanged with the bath, which is integrated out (the underlying system+bath Hamiltonian must be even). An important example of odd Lindblad operators are those linear in creation and annihilation operators, leading to Gaussian Lindblad dynamics.

- We focus on a single time step, denoting the density matrix after the n-th step by $\hat{\rho}_n = \hat{\rho}(t_n)$ ($t_n = t_i + n\delta_t$):

$$\hat{\rho}_{2,n+1} = (\mathbb{1} + \delta_t \mathcal{L})\,[\hat{\rho}_n] + O(\delta_t^2). \tag{2}$$

- We represent the density matrix in the basis of fermionic coherent states. They are defined as eigenstates of the annihilation operator $\hat{a}_i|\psi\rangle = \psi_i|\psi\rangle$, but the eigenvalues ψ_i are not complex number but Grassmann 'numbers', which anticommute with each other (see textbooks, e.g. Altland/Simons). At the time t_n, $\hat{\rho}_n$ can be written as

$$\hat{\rho}_n = \int d\psi^\dagger_{+,n} d\psi_{+,n} d\psi^\dagger_{-,n} d\psi_{-,n} e^{-\psi^\dagger_{+,n}\psi_{+,n} - \psi^\dagger_{-,n}\psi_{-,n}} \left\langle \psi_{+,n}|\rho_n| - \psi_{-,n}\right\rangle \left|\psi_{+,n}\right\rangle \left\langle -\psi_{-,n}\right|. \tag{3}$$

- The sign in the coherent states on the $-$ contour is motivated by the goal to construct a Keldysh partition function in terms of a functional integral. To represent this trace, one notes for any operator $\hat{O}$ acting in a fermionic Fock space spanned by states $|n\rangle$ the need to commute left and right fermion coherent states in the last step, according to

$$\mathrm{Tr}\,\hat{O} = \sum_n \langle n|\,\hat{O}\,|n\rangle = \sum_n \int d\psi^\dagger d\psi e^{-\psi^\dagger\psi} \left|\psi\right\rangle \left\langle\psi\right| \langle n|\,\hat{O}\,|n\rangle = \int d\psi^\dagger d\psi e^{-\psi^\dagger\psi} \left\langle\psi\right|\hat{O}\left|-\psi\right\rangle. \tag{4}$$

Fermionic Lindblad-Keldysh functional integral

- Next, the matrix element $\langle\psi_{+,n+1}|\hat\rho_{2,n+1}|\psi_{-,n+1}\rangle$, which appears in the coherent state representation of $\hat\rho_{2,n+1}$, is expressed in terms of the corresponding matrix element t_n. This requires evaluating

$$\begin{aligned}
&\langle\psi_{+,n+1}|\hat{\mathcal{L}}(|\psi_{+,n}\rangle\langle-\psi_{-,n}|)|-\psi_{-,n+1}\rangle \qquad (5)\\
=&-i\Big(\langle\psi_{+,n+1}|\hat H|\psi_{+,n}\rangle\langle-\psi_{-,n}|-\psi_{-,n+1}\rangle-\langle\psi_{+,n+1}|\psi_{+,n}\rangle\langle-\psi_{-,n}|\hat H|-\psi_{-,n+1}\rangle\Big)\\
&+\sum_\alpha\gamma_\alpha\Big[2\langle\psi_{+,n+1}|\hat L_\alpha|\psi_{+,n}\rangle\langle-\psi_{-,n}|\hat L_\alpha^\dagger|-\psi_{-,n+1}\rangle\\
&\quad-\Big(\langle\psi_{+,n+1}|\hat L_\alpha^\dagger\hat L_\alpha|\psi_{+,n}\rangle\langle-\psi_{-,n}|-\psi_{-,n+1}\rangle+\langle\psi_{+,n+1}|\psi_{+,n}\rangle\langle-\psi_{-,n}|\hat L_\alpha^\dagger\hat L_\alpha|-\psi_{-,n+1}\rangle\Big)\Big].
\end{aligned}$$

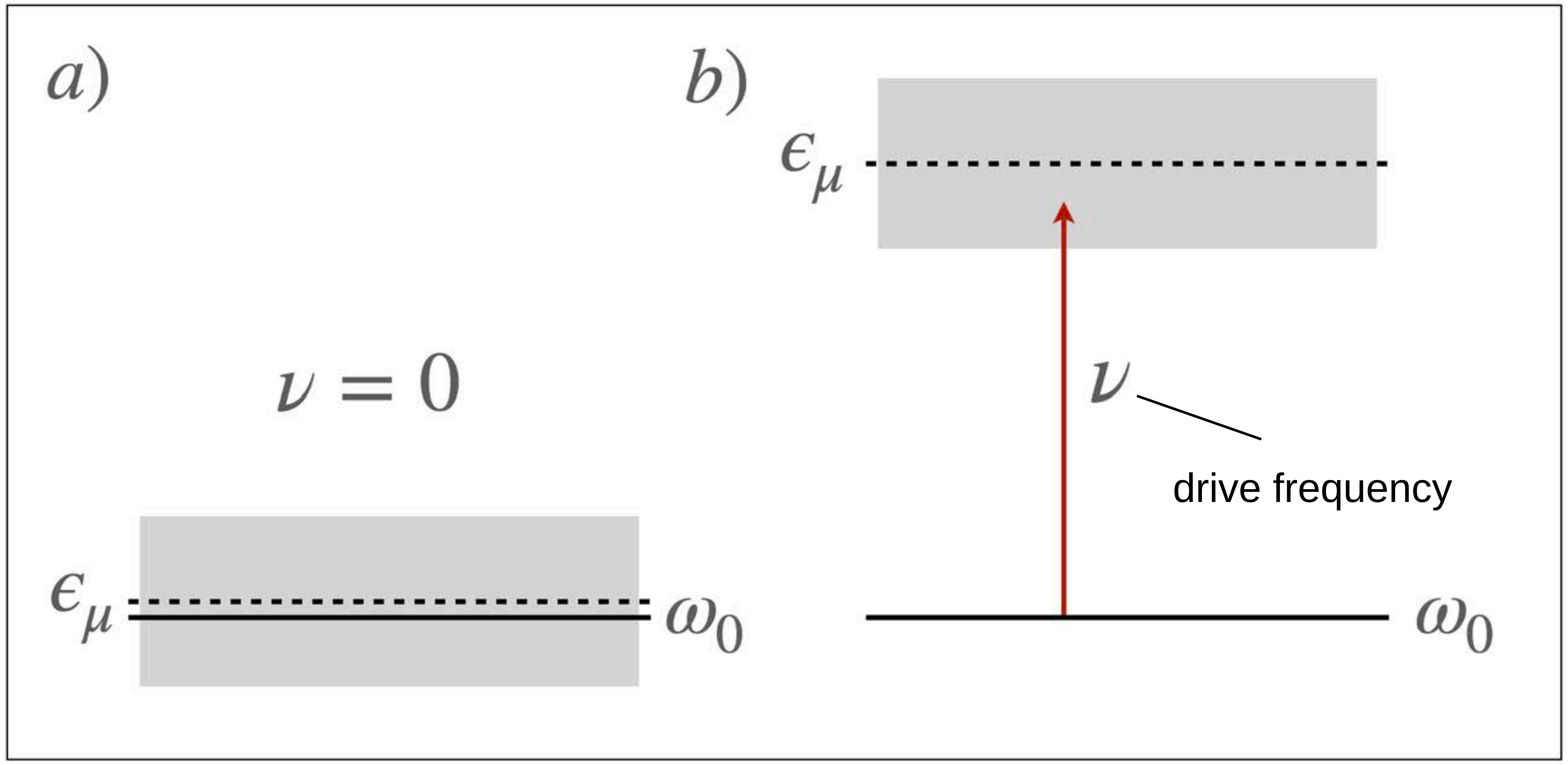
- Discarding sublteties of operator ordering, taking the continuum limit $N\to\infty,\delta t\to 0$, $t_f-t_i=\text{const.}$, and subsequently $t_f-t_i\to\infty$, by re-exponentiation, this leads to the Keldysh action

$$Z=\int D[\psi_\pm^\dagger,\psi_\pm]\,e^{iS[\psi_\pm]}, \qquad (6)$$

$$S[\psi_\pm]=\int_{t,\mathbf{x}}\Big[\psi_+^\dagger i\partial_t\psi_+-\psi_-^\dagger i\partial_t\psi_--i\gamma\big(2L_{-,\alpha}^\dagger L_{+,\alpha}-L_{+,\alpha}^\dagger L_{+,\alpha}-L_{-,\alpha}^\dagger L_{-,\alpha}\big)\Big]. \qquad (7)$$

- Note the order $2L_{-,\alpha}^\dagger L_{+,\alpha}$ for the contour coupling 'jump' term in the Lindblad Keldysh action, reversed compared to the Lindblad equation (1). This is chosen such that the formula is valid both for parity even and odd Lindblad operators. In the odd case, writing the action in the original ordering of the Lindblad equation, there is a prefactor (-1) in front of the jump term due the minus sign in the coherent states on the $-$ contour. This is compensated by exchanging the order to the form above. As a mnemonic, this way probability conservation is ensured.

Appendix:
Equilibrium vs. non-equilibrium dynamics



equilibrium

non-equilibrium

Equilibrium vs. non-equilibrium dynamics

- Single oscillator mode coupled to a bath (e.g. fermion)

$$\hat{H}_{\rm t} = \hat{H}_0 + \hat{H}_{\rm int} + \hat{H}_{\rm b}$$

$$\hat{H}_0 = \omega_0 a^\dagger a$$

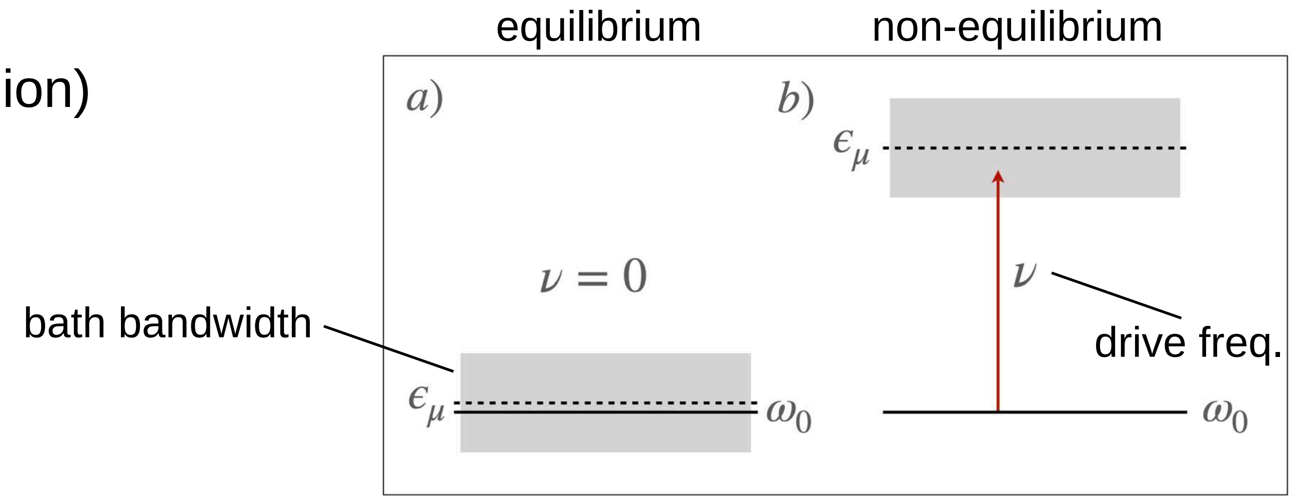
system

$$\hat{H}_{\rm b} = \sum_\mu \epsilon_\mu b^\dagger_\mu b_\mu$$

bath

$$\hat{H}_{\rm int} = \sum_\mu g_\mu e^{-i\nu t} a^\dagger b_\mu + \text{h. c.}$$

system-bath



$\nu = 0 \ll g_\mu \ll \omega_0 \approx \epsilon_\mu$ $\quad$ $g_\mu \ll \omega_0 \ll \nu \approx \epsilon_\mu$

- ω_0, ϵ_μ are the system and bath frequencies. The bath consists of discrete modes μ and we will take the continuum limit below. It is coupled to the system with real valued strength g_μ.
- ν is the driving frequency, which we will compare to the other energy scales below. For the simple drive considered here, we can choose a rotating frame where the system-bath Hamiltonian becomes time independent, and the system frequency is replaced $\omega_0 \to \delta = \omega_0 - \nu$, with detuning δ.
- However, once the state of the bath is fixed by a choice of bath correlation functions, the drive scale ν cannot be eliminated from the problem (cf. the last line in Eq. (4) below). The generator of time evolution $H_{\rm t}$ and the state of the (sub-)system then form an inseparable entity and must not be considered in autonomy.
- To directly compare equilibrium and Lindblad limits, we work in the Keldysh functional integral. The action $S_{\rm t} = S_0 + S_{\rm int} + S_{\rm b}$ corresponding to (1) is defined with

$$S_0 = \int dt a^\dagger(t)(i\partial_t - \omega_0)\tau_z a(t), \; S_{\rm int} = \int dt \sum_\mu g_\mu (e^{-i\nu t} a^\dagger(t)\tau_z b_\mu(t) + \text{h. c.}), \; S_{\rm b} = \int dt \sum_\mu b^\dagger_\mu(t) G^{-1}_\mu(t,t') b_\mu(t'). \quad (2)$$

Equilibrium vs. non-equilibrium dynamics

- Single oscillator mode coupled to a bath (e.g. fermion)

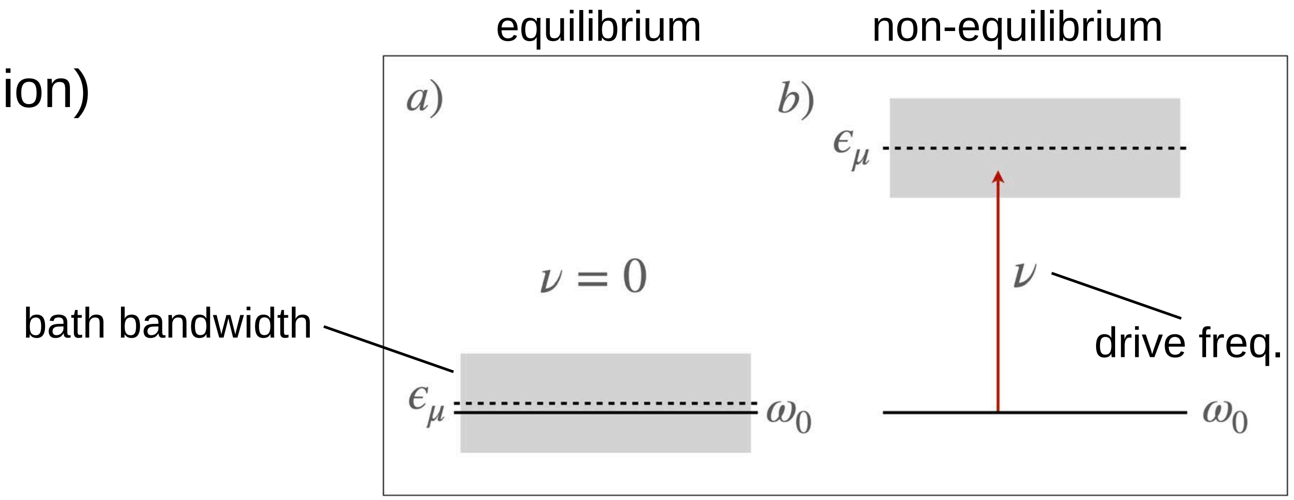
$$\hat{H}_{\mathrm{t}} = \hat{H}_0 + \hat{H}_{\mathrm{int}} + \hat{H}_{\mathrm{b}}$$

$$\hat{H}_0 = \omega_0 a^\dagger a \qquad \hat{H}_{\mathrm{b}} = \sum_\mu \epsilon_\mu b^\dagger_\mu b_\mu$$

system bath

$$\hat{H}_{\mathrm{int}} = \sum_\mu g_\mu e^{-i\nu t} a^\dagger b_\mu + \mathrm{h.\ c.}$$

system-bath



$$\nu = 0 \ll g_\mu \ll \omega_0 \approx \epsilon_\mu \qquad g_\mu \ll \omega_0 \ll \nu \approx \epsilon_\mu$$

- Each bath mode is here assumed to be in a state of thermodynamic equilibrium, described by the Fermi distribution function $n_F(\epsilon) = 1/(e^{-\beta\epsilon} + 1)$ for a mode at energy ϵ and temperature $T = 1/\beta$. The Green's function of the bath obtains by inversion of the G^{-1}_μ operator for all μ, which reads

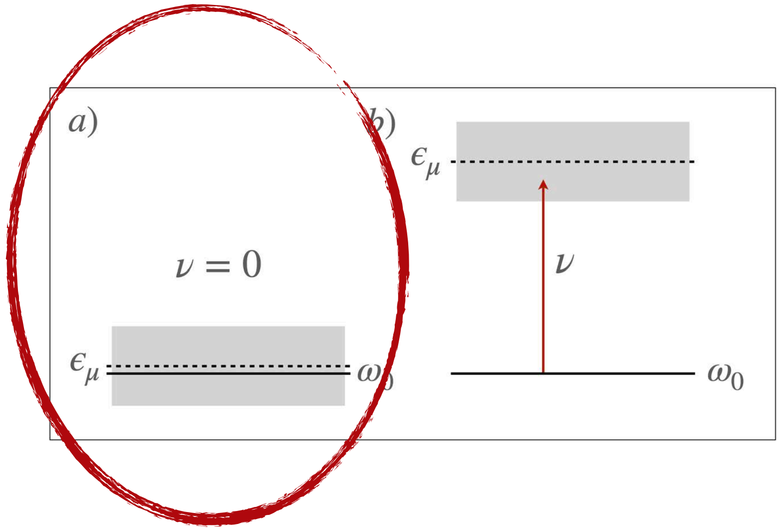
$$G_\mu(t-t') \;=\; -ie^{-i\epsilon_\mu(t-t')}\begin{pmatrix} (1-n_F(\epsilon_\mu))\theta(t-t') - n_F(\epsilon_\mu)\theta(t'-t) & -n_F(\epsilon_\mu) \\ (1-n_F(\epsilon_\mu)) & (1-n_F(\epsilon_\mu))\theta(t'-t) - n_F(\epsilon_\mu)\theta(t-t') \end{pmatrix}. \quad (3)$$

- Integrating out the bath, we obtain

$$\Delta S \;=\; \int dt\,dt'\,a^\dagger(t)e^{-i\nu(t-t')}B(t-t')a(t') = \int dt\,d\tau\,a^\dagger(t)e^{-i\nu\tau}B(\tau)a(t-\tau), \quad B(\tau) = -\sum_\mu g^2_\mu \tau_z G_\mu(\tau)\tau_z,$$

$$= \int \frac{d\omega}{2\pi} a^\dagger(\omega+\nu)B(\omega)a(\omega+\nu) = \int \frac{d\omega}{2\pi} a^\dagger(\omega)B(\omega-\nu)a(\omega), \qquad B(\omega) = \int d\tau e^{i\omega\tau}B(\tau). \quad (4)$$

Limit of thermodynamic equilibrium



- Single oscillator mode coupled to a bath (e.g. fermion)

$$\hat{H}_{\rm t} = \hat{H}_0 + \hat{H}_{\rm int} + \hat{H}_{\rm b}$$

- This is a system, where there is no drive,

$$\nu = 0 \ll g_\mu \ll \omega_0 \approx \epsilon_\mu. \qquad (5)$$

- We then find

$$B(\omega) \;=\; iJ(\omega)\omega \begin{pmatrix} 1-2n_F(\omega) & 2n_F(\omega) \\ -2(1-n_F(\omega)) & 1-2n_F(\omega) \end{pmatrix} - \Delta(\omega)\tau_z, \;\; J(\omega) = \pi \sum_\mu \frac{g_\mu^2}{\epsilon_\mu}\delta(\epsilon_\mu - \omega), \;\; \Delta(\omega) = \sum_\mu g_\mu^2 \frac{\mathcal{P}}{\epsilon_\mu - \omega}, \qquad (6)$$

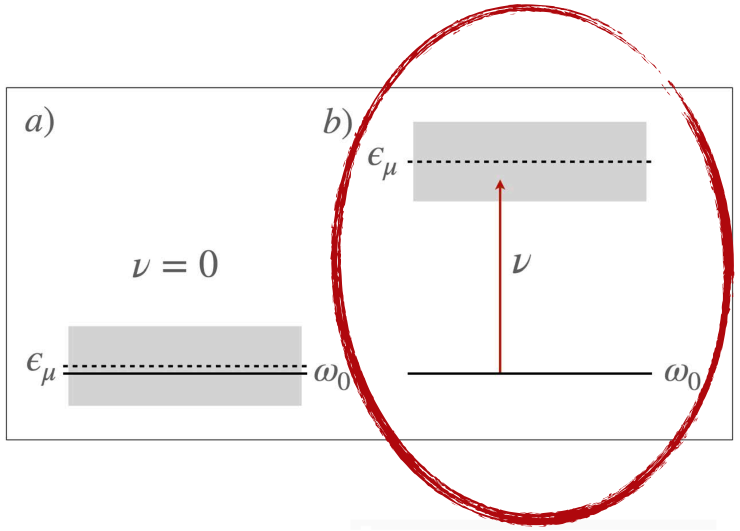
where we have introduced the bath spectral density $J(\omega)$ to parameterize the imaginary part. For many real baths, this function is constant to good approximation (ohmic bath). The bandwidth of the bath enters implicitly by the extent of the μ summation. There is also a frequency dependent Lamb shift $\Delta(\omega)$, which is a Hamiltonian term renormalizing the bare frequency ω_0.

- The system action then reads (neglecting the Lamb shift)

$$\begin{aligned} S_{\rm eq} \;=\; & \int dt[a_+^\dagger(i\partial_t - \omega_0)a_+ - a_-^\dagger(i\partial_t - \omega_0)a_-] \qquad (7) \\ & -i\int \frac{d\omega}{2\pi} J(\omega)\omega\{(1-n_F(\omega))[2a_-^\dagger a_+ - a_+^\dagger a_+ - a_-^\dagger a_-] + n_F(\omega)[2a_- a_+^\dagger - a_+ a_+^\dagger - a_- a_-^\dagger]\}. \end{aligned}$$

- This action obeys the thermal symmetry $\mathcal{T}_\beta$ introduced in the main part.

Non-equilibrium (Lindblad) limit



- Single oscillator mode coupled to a bath (e.g. fermion)

$$\hat{H}_{\rm t} = \hat{H}_0 + \hat{H}_{\rm int} + \hat{H}_{\rm b}$$

- Now we consider a driven system, where the drive is the largest scale:

$$g_\mu \ll \omega_0 \ll \nu \approx \epsilon_\mu. \tag{8}$$

- Again we work at weak coupling. The driving frequency is on the order of that of the bath oscillators, i.e. the bath has a sizeable density of states at the driving frequency, such that emission of excitations generated by the drive (carrying energy $\sim \nu$) into it can take place. However, the system itself evolves on much slower time scales ($\sim \omega_0$).

- This justifies the *Markov approximation*, where $a(t-\tau) \approx a(t)$ in the last expression in (4) – strictly speaking, we are working in the limit $\nu^{-1} = 0$. We then obtain

$$\Delta S \approx \int dt a^\dagger(t)[\int d\tau B(\tau) e^{+i\nu\tau}] a(t) = \int dt a^\dagger(t) B(\nu) a(t), \tag{9}$$

where, with the usual conventions for a decay rate κ and a Lamb shift Δ,

$$B(\nu) = i\kappa \begin{pmatrix} 1-2\bar{n} & 2\bar{n} \\ -2(1-\bar{n}) & 1-2\bar{n} \end{pmatrix} - \Delta\tau_z, \quad \kappa = \pi \sum_\mu g_\mu^2 \delta(\epsilon_\mu - \nu), \quad \Delta = \Delta(\nu), \quad \bar{n} = n_F(\nu), \tag{10}$$

- The Lindblad action is local in the time domain and reads, again dropping the Lamb shift

$$S_{\rm neq} = \int dt[a_+^\dagger(i\partial_t - \omega_0)a_+ - a_-^\dagger(i\partial_t - \omega_0)a_- - i\kappa\{(1-\bar{n})[2a_-^\dagger a_+ - a_+^\dagger a_+ - a_-^\dagger a_-] + \bar{n}[2a_- a_+^\dagger - a_+ a_+^\dagger - a_- a_-^\dagger]\}] \tag{11}$$

- This action does not obey the thermal symmetry $\mathcal{T}_\beta$.

Boulder Summer School 2021
July 5-30, 2021
Boulder, USA (via zoom)

thp Institute for Theoretical Physics University of Cologne

Lecture II:
Symmetry and topology
in driven open quantum systems

Sebastian Diehl

Institute for Theoretical Physics, University of Cologne

€ € € :

DFG Deutsche Forschungsgemeinschaft

erc European Research Council
Established by the European Commission

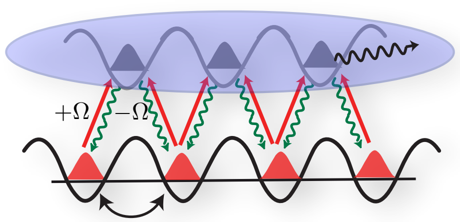
Outline

➡ Quantum Optics ➡ Many-Body Physics ➡ Statistical Mechanics

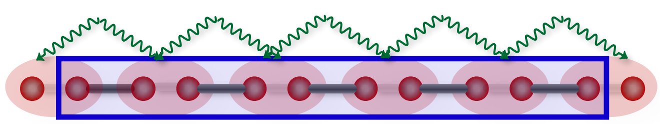
microphysics → macrophysics

States

- order by dissipation: quantum mechanical?
- phase coherence and entanglement for bosons and spins



- order by dissipation: topological ?
- fermionic dissipative ‘superconductors’



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microphysics → macrophysics

States

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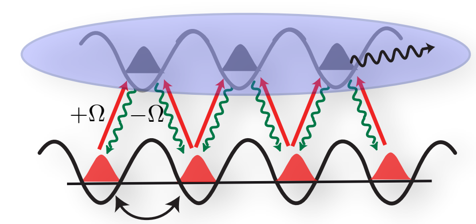

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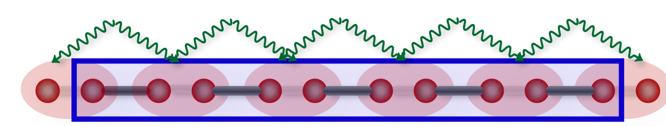
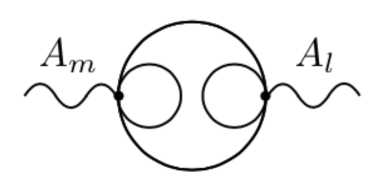
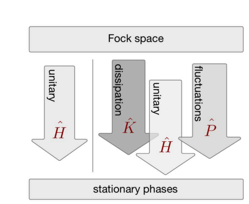


Dynamics

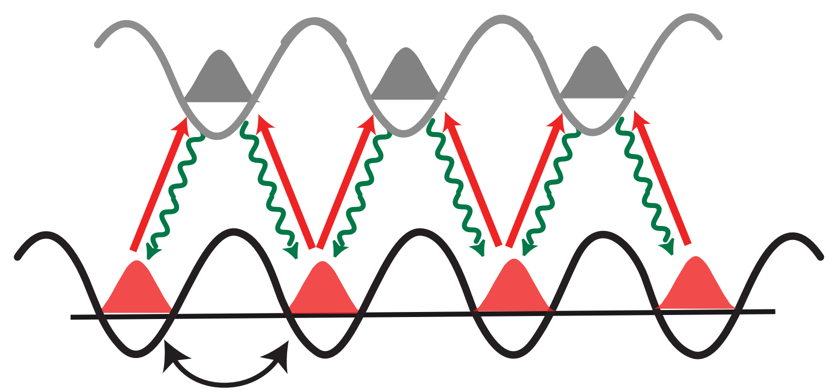
- Topology: Quantized, reversible response on top of a dissipative bulk?
- topological field theory for driven open quantum systems



- Symmetry: Classification principles out-of-equilibrium?
- Fock space bottom-up approach to non-hermitian system classification



Order by dissipation



SD, A. Micheli, A. Kantian, B. Kraus, H.P. Büchler, P. Zoller, Nat. Phys. (2008);
B. Kraus, H.P. Büchler, SD, A. Micheli, A. Kantian, P. Zoller, PRA (2008);
F. Verstraete, M. Wolf, J. I. Cirac, Nature Physics **5**, 633 (2009).

Microscopic
Quantum Optics

“Thermodynamic”
Many-body physics

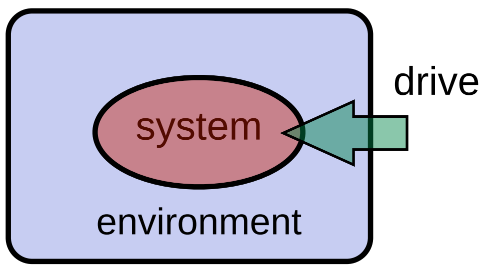
Long wavelength
Statistical mechanics

Dark states in Lindblad equations

- quantum master equation

Lindblad operators

$$\partial_t \rho = \underbrace{\cancel{-i[H,\rho]}}_{\text{coherent evolution}} + \underbrace{\kappa \sum_i \left(L_i \rho L_i^\dagger - \tfrac{1}{2}\{L_i^\dagger L_i, \rho\}\right)}_{\text{driven-dissipative evolution}}$$



- Key concept: Dark states

$$L_i |D\rangle = 0 \quad \forall i$$

➡ time evolution stops when $\rho = |D\rangle\langle D|$ *

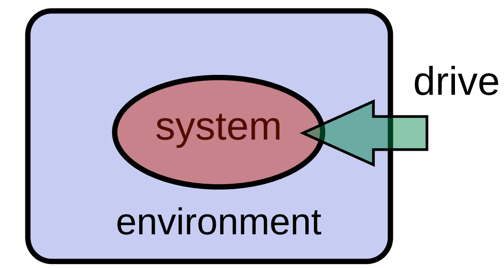
* for e.g. $H|D\rangle = E|D\rangle$

Dark states in Lindblad equations

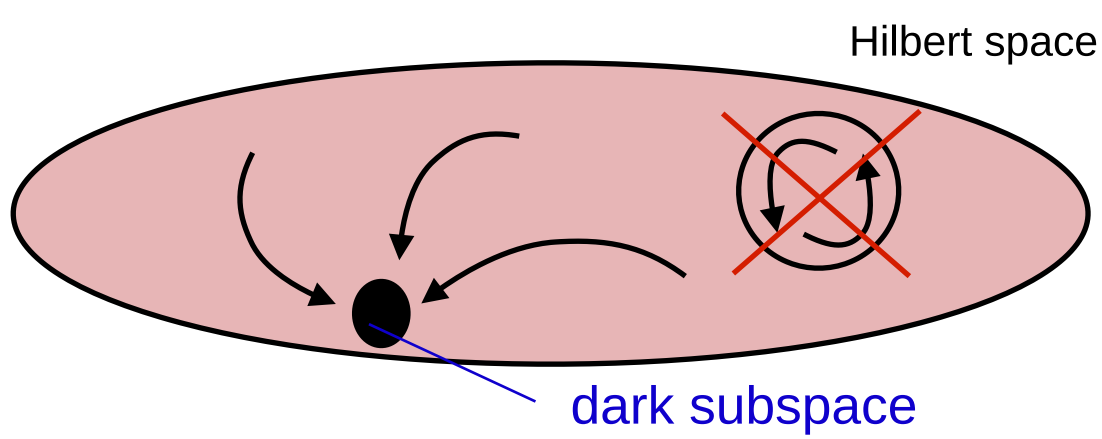
- quantum master equation

Lindblad operators

$$\partial_t \rho = \underbrace{-i[H,\rho]}_{\text{coherent evolution}} + \underbrace{\kappa \sum_i \left(L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\}\right)}_{\text{driven-dissipative evolution}}$$



- Interesting situation: unique dark state solution



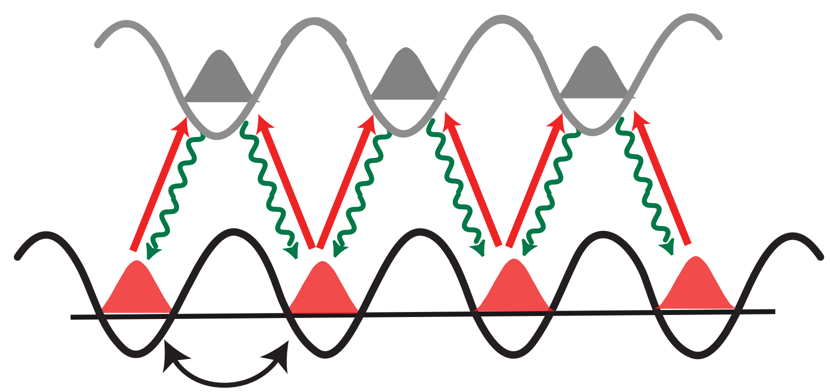
➡ you can enter, but never leave *

➡ directed motion in Hilbert space $\rho \xrightarrow{t\to\infty} |D\rangle\langle D|$

➡ dissipation removes entropy, increases purity

* for e.g. $H|D\rangle = E|D\rangle$

Order by dissipation:
Bosons



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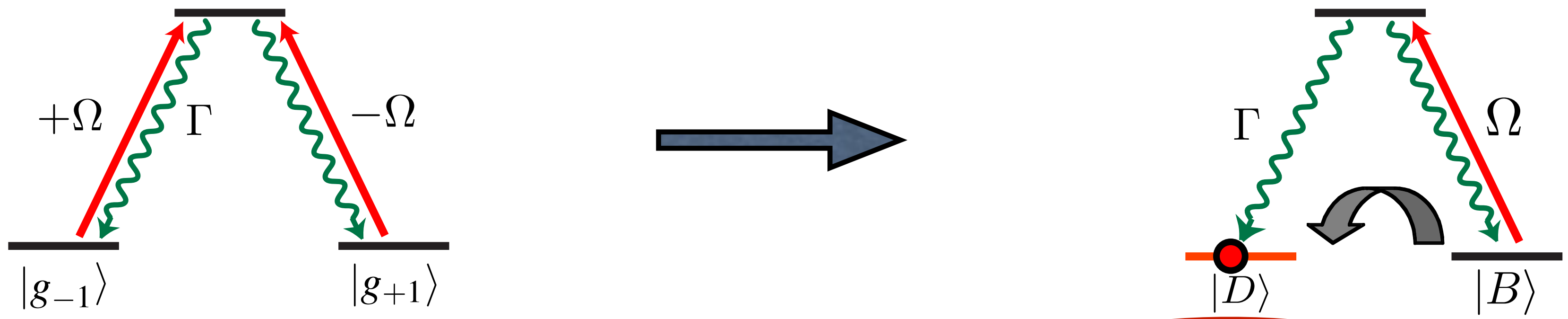
Dark states: From quantum optics to many particles

- optical pumping: three internal (electronic) levels   Aspect et al., PRL (1998); Kasevich, Chu, PRL (1992)



dark state

$$|D\rangle \sim |g_{+1}\rangle + |g_{-1}\rangle$$

bright state

$$|B\rangle \sim |g_{+1}\rangle - |g_{-1}\rangle$$

Dark states: From quantum optics to many particles

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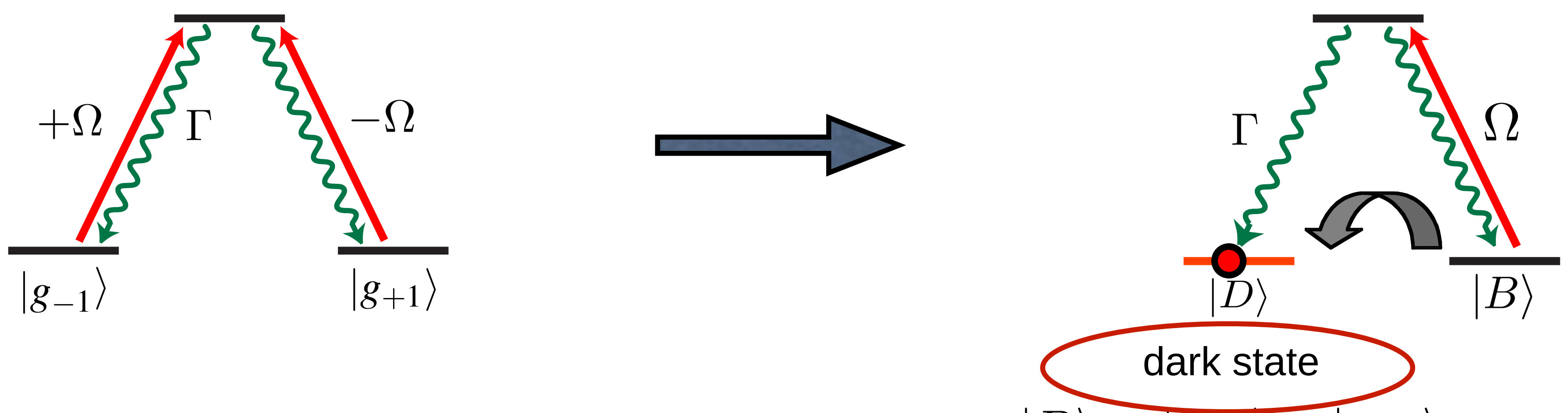
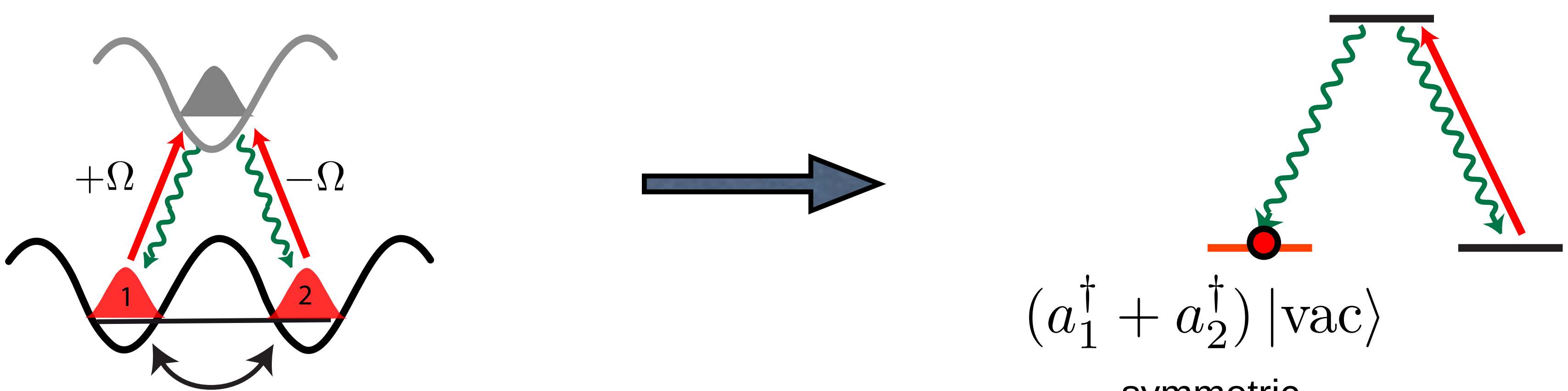


$$|D\rangle \sim |g_{+1}\rangle + |g_{-1}\rangle$$

- 1 atom on 2 sites: external (spatial) degrees of freedom (atoms on optical lattice)



$$(a_1^\dagger + a_2^\dagger)\,|\mathrm{vac}\rangle$$

symmetric
superposition

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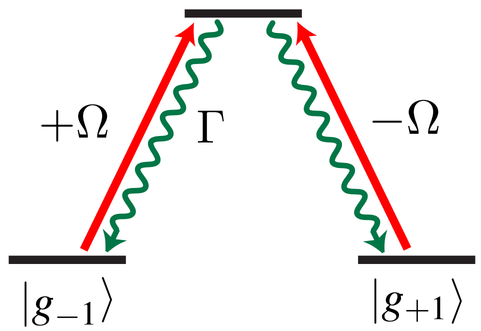

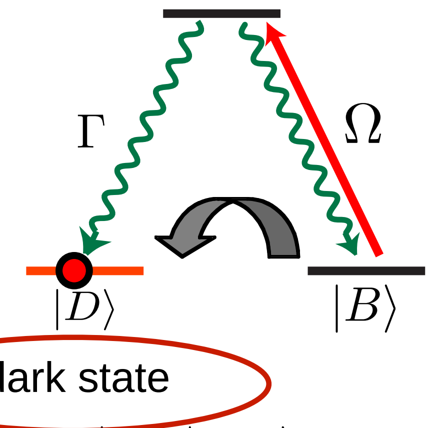

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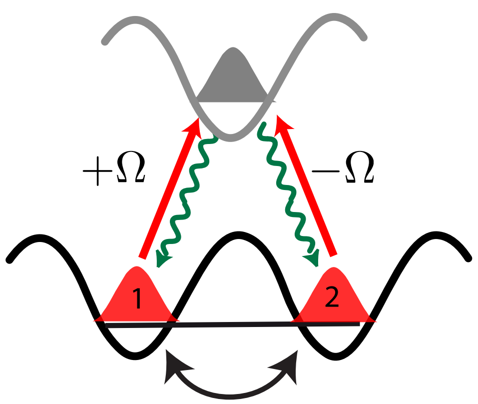
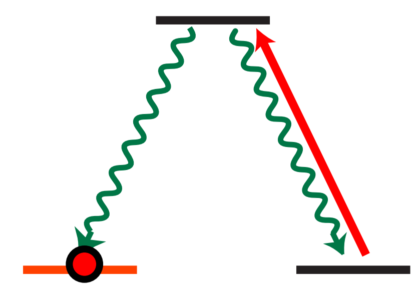
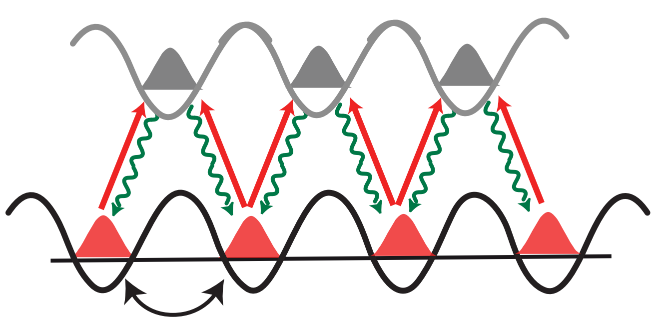




$$(a_1^\dagger + a_2^\dagger)\,|\mathrm{vac}\rangle$$

symmetric superposition

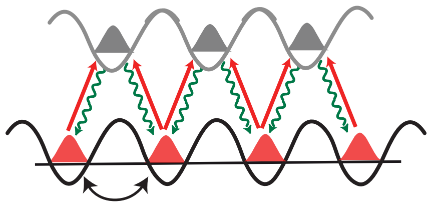
- N atoms on M sites (optical lattice)



$$|BEC\rangle = \frac{1}{N!}\Big(\sum_\ell a_\ell^\dagger\Big)^N |vac\rangle$$

➡ combination of drive and dissipation enables purification

Dissipative many-body state preparation

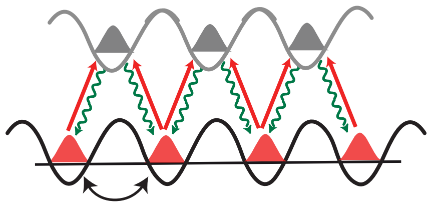


- Lindblad operators for BEC dark state:

$$L_i = (a_i^\dagger + a_{i+1}^\dagger)(a_i - a_{i+1}) \qquad\qquad L_i|\mathrm{BEC}\rangle = 0 \; \forall \, i$$

➡ Long range phase coherence/ boson condensation builds up from quasilocal dissipative operations

Dissipative many-body state preparation



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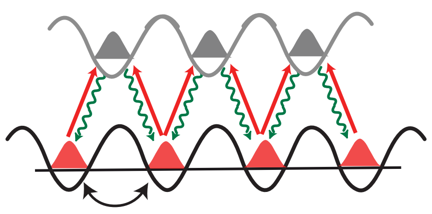
➡ Long range phase coherence/ boson condensation builds up from quasilocal dissipative operations

- uniqueness of stationary solution can be shown

➡ Ordered phase reached from arbitrary initial state

$$\Longrightarrow \rho(t) \longrightarrow |BEC\rangle\langle BEC| \text{ for } t \to \infty$$

Dissipative many-body state preparation



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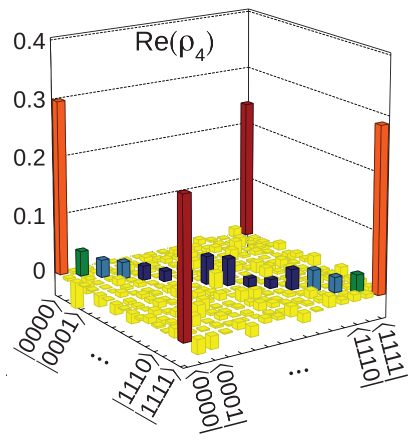
- experimental realizations: entanglement generation



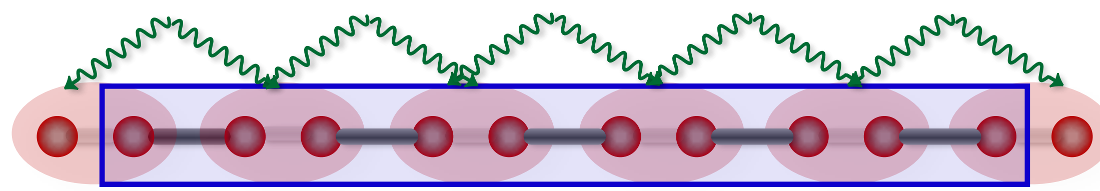
$$\rho = |D\rangle\langle D|$$

$$|D\rangle = \frac{1}{2}(|0000\rangle + |1111\rangle)$$

GHZ state of
four ions

Universal open-system quantum simulator,
Schindler et al., Nature (2011) (Blatt group)

Topological order by dissipation: dissipative Kitaev chain



SD, E. Rico, M. A. Baranov, P. Zoller, Nat. Phys. (2011)
Review: C.-E. Bardyn, C. Kraus, E. Rico, M. Baranov, A. Imamoglu, SD, NJP (2013)

Microscopic
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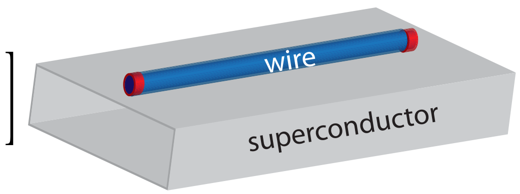
Kitaev‘s quantum wire (Hamiltonian scenario)

Kitaev, Physics Uspekhi (2001)

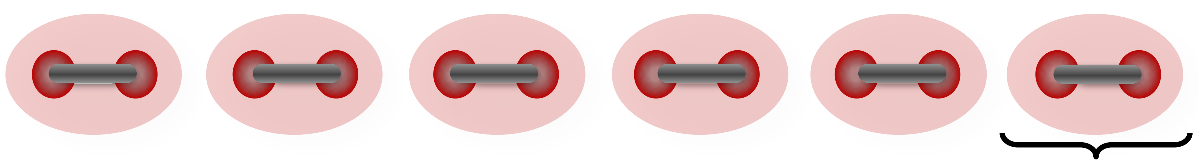
- spinless superconducting fermions on a lattice $H = \sum_i \left[-J a_i^\dagger a_{i+1} + \Delta a_i a_{i+1} + \text{h.c.} - \mu \left(a_i^\dagger a_i - \frac{1}{2} \right) \right]$



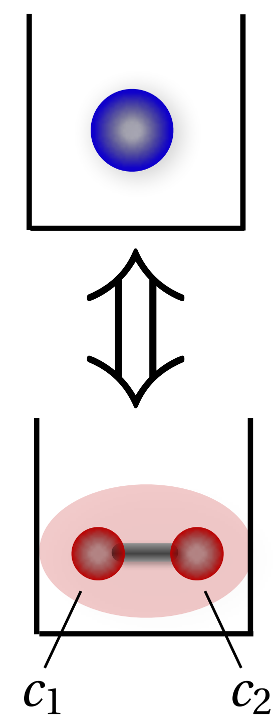
- Hamiltonian in Bogoliubov basis $H = \sum \epsilon_i \tilde{a}_i^\dagger \tilde{a}_i$ $\quad\quad \tilde{a}_i |G\rangle = 0 \, \forall \, i$

- two inequivalent representatives

$$\tilde{a}_i = a_i$$



trivial phase



complex basis

$$H = \epsilon \, a_1^\dagger a_1$$

$$a_1 \equiv \frac{1}{2}(c_1 + \mathrm{i} c_2)$$

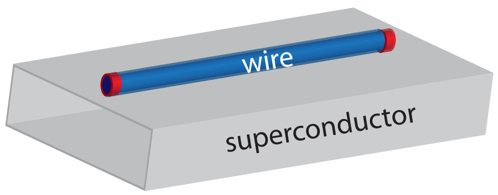
Majorana (real) basis

$$H = \frac{\mathrm{i}}{2} \epsilon \, c_1 c_2$$

fermion as onsite pairing of two Majoranas

Kitaev's quantum wire (Hamiltonian scenario)

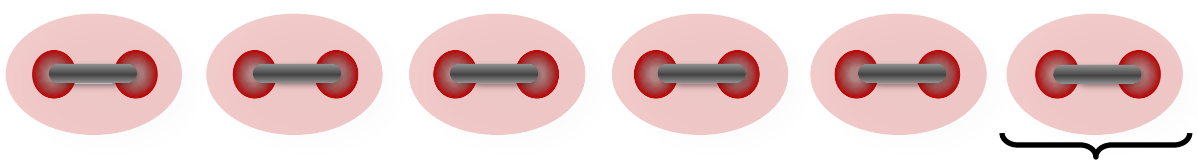
Kitaev, Physics Uspekhi (2001)



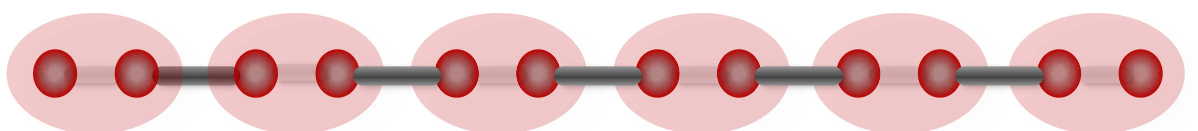
- spinless superconducting fermions on a lattice
- Hamiltonian in Bogoliubov basis $H = \sum \epsilon_i \tilde{a}_i^\dagger \tilde{a}_i$ $\qquad \tilde{a}_i|G\rangle = 0 \, \forall \, i$
- two inequivalent representatives

quasilocal!

$$\tilde{a}_i = a_i \qquad\qquad \tilde{a}_i = \frac{1}{2}(a_{i+1} + a_{i+1}^\dagger - a_i + a_i^\dagger)$$



physical site



trivial phase

nontrivial phase

bulk

- BCS p-wave superfluid in ground state

- gapped spectrum

edge

- unpaired zero energy Majorana edge modes, or

- non-local Bogoliubov fermion

Dissipative Majorana quantum wire

- reconsider simplemost Lindblad operators and dark state condition:

$$L_i = (a_i^\dagger + a_{i+1}^\dagger)(a_i - a_{i+1}) \equiv C_i^\dagger A_i \qquad\qquad L_i|D\rangle = 0 \;\forall\, i$$

- main insight:
  - $a_i^\dagger$ boson creation => $|D\rangle = |\mathrm{BEC}, N\rangle$ fixed number BEC dark state
  - $a_i^\dagger$ fermion creation => $|D\rangle = |\mathrm{BCS}, N\rangle$ fixed number BCS pair dark state

- this example: $|\mathrm{BCS}, N\rangle = |\mathrm{Kitaev}, N\rangle$

Dissipative Majorana quantum wire

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- this example: $|\mathrm{BCS}, N\rangle = |\mathrm{Kitaev}, N\rangle$

- connection to Kitaev model: emergent eigenoperators in thermodynamic limit

SD, E. Rico, M. A. Baranov, P. Zoller, Nat. Phys. (2011)

fixed number

fixed phase

$$L_i = \underbrace{(a_i^\dagger + a_{i+1}^\dagger)}\underbrace{(a_i - a_{i+1})} \xrightarrow[\hat{=}\text{“low energies”}]{\text{long times}} \ell_i = (\underbrace{a_i^\dagger + a_{i+1}^\dagger} + \underbrace{a_i - a_{i+1}})$$

Dissipative Majorana quantum wire

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  - $a_i^\dagger$ fermion creation => $|D\rangle = |\mathrm{BCS}, N\rangle$ fixed number BCS pair dark state

- this example: $|\mathrm{BCS}, N\rangle = |\mathrm{Kitaev}, N\rangle$

- connection to Kitaev model: emergent eigenoperators in thermodynamic limit

SD, E. Rico, M. A. Baranov, P. Zoller, Nat. Phys. (2011)

fixed number

fixed phase

$$L_i = \underbrace{(a_i^\dagger + a_{i+1}^\dagger)}\underbrace{(a_i - a_{i+1})} \xrightarrow{\text{long times}} \ell_i = \underbrace{(a_i^\dagger + a_{i+1}^\dagger} + \underbrace{a_i - a_{i+1})}$$

$\hat{=}$ "low energies"

$$\propto \tilde{a}_i$$

Kitaev's Majorana operators

Dissipative Majorana quantum wire

- reconsider simplemost Lindblad operators and dark state condition:

$$L_i = (a_i^\dagger + a_{i+1}^\dagger)(a_i - a_{i+1}) \equiv C_i^\dagger A_i \qquad\qquad L_i|D\rangle = 0 \;\forall\, i$$

- main insight:
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$$|D\rangle = |\mathrm{Kitaev}, \theta\rangle \qquad\qquad \propto \tilde{a}_i$$

standard Kitaev p-wave superfluid state

Kitaev's Majorana operators

Mechanism: Fixed number vs. fixed phase Lindblad operators

- spinless fermions for simplicity   SD, W. Yi. A. Daley, P. Zoller, PRL (2010)

- fixed number Lindblad operators

$$L_i = C_i^\dagger A_i$$

$$\Leftrightarrow$$

- fixed phase Lindblad operators

$$\ell_i = C_i^\dagger + r e^{\mathrm{i}\theta} A_i$$

- resulting dark state

$$|BCS, N\rangle = G^{\dagger\, N}|\mathrm{vac}\rangle$$

- resulting dark state (with $\Delta N \sim 1/\sqrt{N}$)

$$|BCS, \theta\rangle = \exp(r e^{\mathrm{i}\theta} G^\dagger)|\mathrm{vac}\rangle$$

- requirements

translation invariant creation and annihilation part

$$C_i^\dagger = \sum_j v_{i-j} a_j^\dagger \qquad C_k^\dagger = v_k a_k^\dagger$$

$$A_i = \sum_j u_{i-j} a_j \qquad A_k = u_k a_k$$

antisymmetry

$$\varphi_k = \frac{v_k}{u_k} = -\varphi_{-k}$$

$$G^\dagger = \sum_k \varphi_k c_{-k}^\dagger c_k^\dagger$$

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Spontaneous symmetry breaking and dissipative gap

- use equivalence of fixed number and fixed phase states in thermodynamic limit
- use exact knowledge of stationary state: linearized long time evolution

$$\mathcal{L}[\rho] = \kappa \sum_i [\ell_i \rho \ell_i^\dagger - \tfrac{1}{2}\{\ell_i^\dagger \ell_i, \rho\}] = \sum_{\mathbf{q}} \kappa_{\mathbf{q}} [\ell_{\mathbf{q}} \rho \ell_{\mathbf{q}}^\dagger - \tfrac{1}{2}\{\ell_{\mathbf{q}}^\dagger \ell_{\mathbf{q}}, \rho\}]$$

- properties
  - relation to microscopic operators

fixed by average particle number

fixed spontaneously

$$L_i \quad = \quad C_i^\dagger A_i \quad \xrightarrow[\text{“low energy limit”}]{t \to \infty} \qquad \ell_i = C_i^\dagger + r e^{\mathrm{i}\theta} A_i$$

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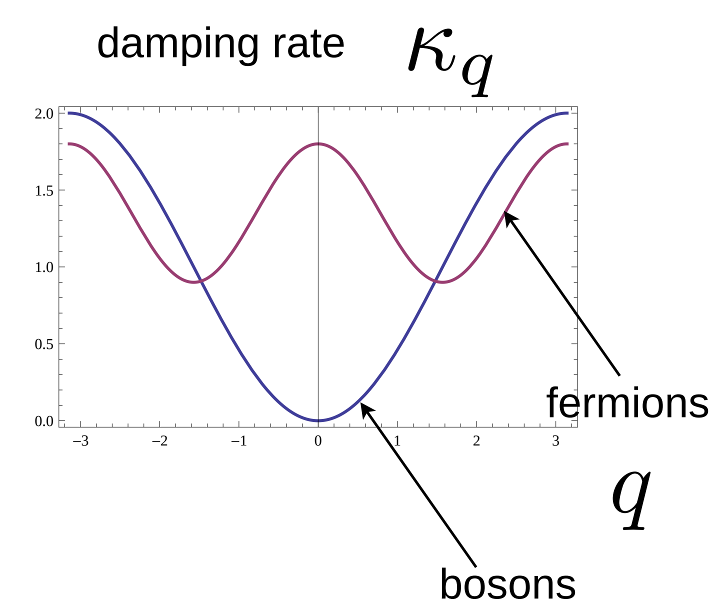
$$L_i = C_i^\dagger A_i \xrightarrow[\text{“low energy limit”}]{t \to \infty} \ell_i = C_i^\dagger + r e^{i\theta} A_i$$

  - effective fermionic quasiparticle operators

$$\ell_{\mathbf{q}} |BCS, \theta\rangle = 0$$; fulfill Dirac algebra -> uniqueness

  - dissipative gap in the damping rate

$$\kappa_{\mathbf{q}} = \kappa_0 \int_{\mathrm{BZ}} \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{|u_{\mathbf{k}} v_{\mathbf{k}}|^2}{|u_{\mathbf{k}}|^2 + |\alpha v_{\mathbf{k}}|^2} (|u_{\mathbf{q}}^2| + |v_{\mathbf{q}}^2|) \geq \kappa_0 n$$



➡ scale generated in long time evolution (single particle sector)

➡ robustness of prepared state against perturbations

Properties: Topological states induced by dissipation

✓ generic features of topological states

➡ insensitivity of edge modes against microscopic details in the bulk:

- ➡disorder
- ➡non-pure bulk states $\{j_i, j_j\} \neq 0$

✓braiding of dissipative Majoranas by adiabatically changing $\mathcal{L}$

cf. work by Avron, Fraas, Graf, J. Stat. Phys. (2012);
Avron, Fraas, Graf, Kenneth, New J. Phys. (2010)

✓ topological origin

➡ topological invariant of the bulk (for mixed, dissipative systems)

- reason:

$$\frac{d}{dt}\rho = -i[A,\rho] + \sum_{a,b} |a\rangle \dot{\rho}_{ab} \langle b|,$$

adiabatic connection $A = \mathrm{i}\dot{U}^{\dagger} U$

universal

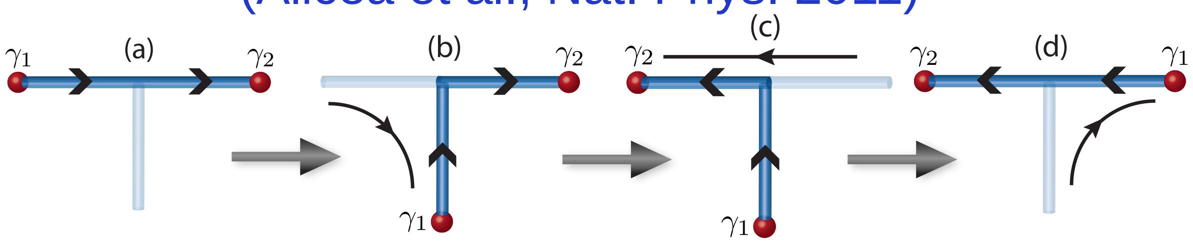
$\dot{\rho}_{ab} \equiv \langle a(t)|\partial_t \rho|b(t)\rangle$
phys. evolution

- implication:

➡ dissipative braiding in networks

➡ non-abelian statistics

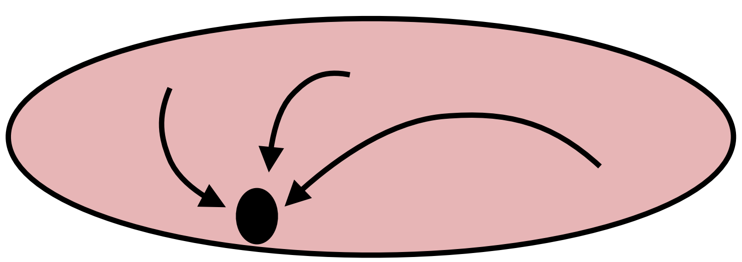
(Alicea et al., Nat. Phys. 2011)



Topological field theory far from equilibrium

F. Tonielli, J. Budich, A. Altland, SD
PRL (2020)

microphysics → macrophysics



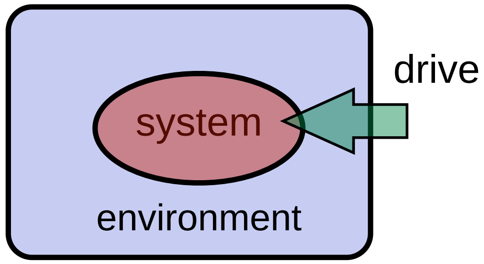
$$\frac{\theta}{4\pi}\int A_+\,dA_+ - A_-\,dA_-$$

Motivation: Quantum states vs. quantum dynamics

- quite general quantum evolution:

Lindblad operators

$$\partial_t \hat{\rho} = \underbrace{-i[\hat{H}, \hat{\rho}]}_{\text{coherent evolution}} + \underbrace{\kappa \sum_\alpha (\hat{L}_\alpha \hat{\rho} \hat{L}_\alpha^\dagger - \frac{1}{2}\{\hat{L}_\alpha^\dagger \hat{L}_\alpha, \hat{\rho}\})}_{\text{driven-dissipative evolution}}$$



- simple example: damped harmonic oscillator (quantum cavity)

$$H = \omega_0\, a^\dagger a \qquad\qquad L = a$$

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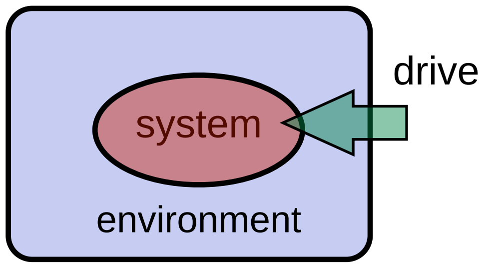

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$$\kappa = 0 \qquad\qquad \omega_0 = 0$$

evolution $\quad \partial_t \rho = -i[H, \rho] \qquad\qquad \partial_t \rho = \kappa\left(a\rho a^\dagger - \tfrac{1}{2}\{a^\dagger a, \rho\}\right)$

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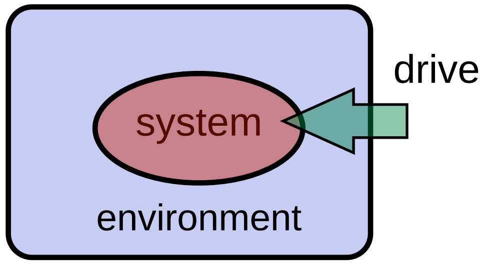

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stationary state

$$\rho_{\text{eq}} = e^{-\beta H} \qquad\qquad \rho_{\text{neq}}$$

$$\lim_{\beta\to\infty} e^{-\beta H} = |0\rangle\langle 0| \quad \xleftrightarrow{\text{coincide!}} \quad \rho_{\text{neq}} = |0\rangle\langle 0|$$

ground state

dark state $\quad L_i|D\rangle = 0\,\forall\, i$

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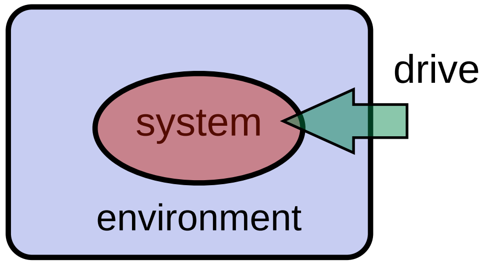

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ground state

dark state $L_i|D\rangle = 0\,\forall\, i$

dynamical response to $j^\dagger a + a^\dagger j$

$$\theta(t) e^{-i\omega_0 t} \qquad \xleftrightarrow{\text{qualitatively different!}} \qquad \theta(t) e^{-\gamma t}$$

reversible, equilibrium

irreversible, non-equilibrium

Motivation: Quantum states vs. quantum dynamics

- quite general quantum evolution: eliminate environment

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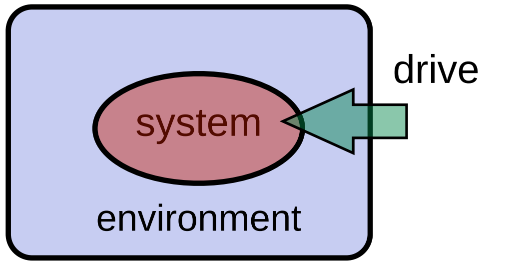
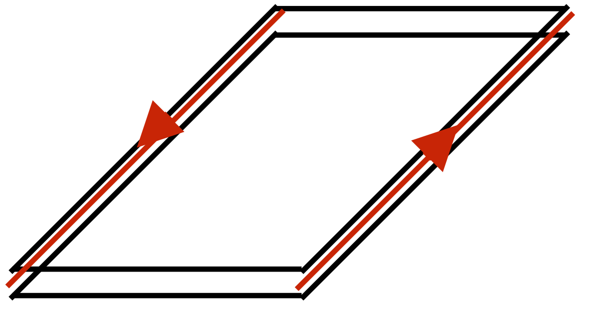


- enter topology:
  - topology is encoded in the state (wave function)
  - observables are oftentimes dynamical responses (to gauge fields)
- example: Chern insulator (e.g. quantum Hall)



cf., for single particle problems: Albert, Bradlyn, Fraas, Jiang, PRX (2016)
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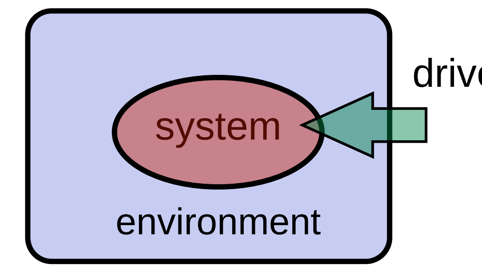

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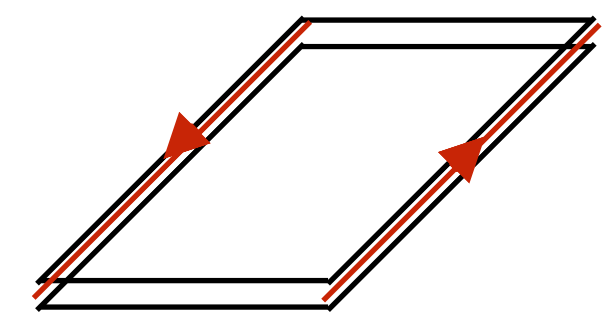

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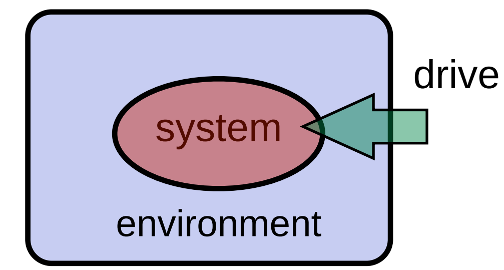

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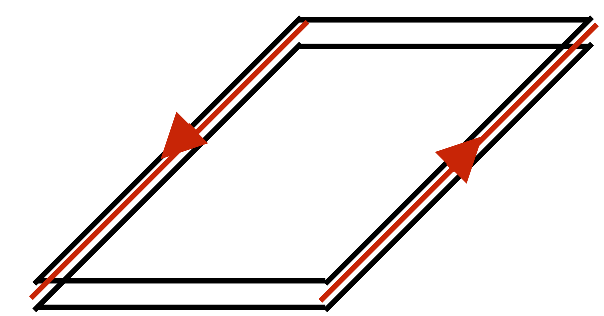

- example: Chern insulator (e.g. quantum Hall)

Questions:
Is there a topological response in irreversible out of equilibrium dynamics?
Are there chiral edge modes (reversible) on top of a dissipative bulk (irreversible)?

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dissipative topological superconductors: SD, Rico, Baranov, Zoller, Nat. Phys. (2011),
dissipative TIs: Goldstein, SciPost (2019); Shavit, Goldstein, PRB (2020)

Different dynamics: overview

State

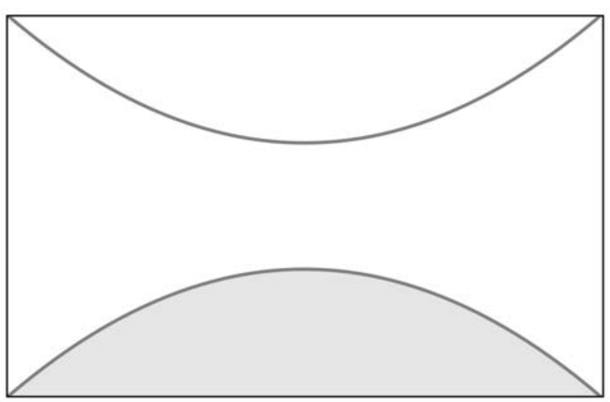
Hamiltonian scenario

- unique ground state

$$H|D\rangle = 0$$

Ex.: Chern insulator

band 1



band 2

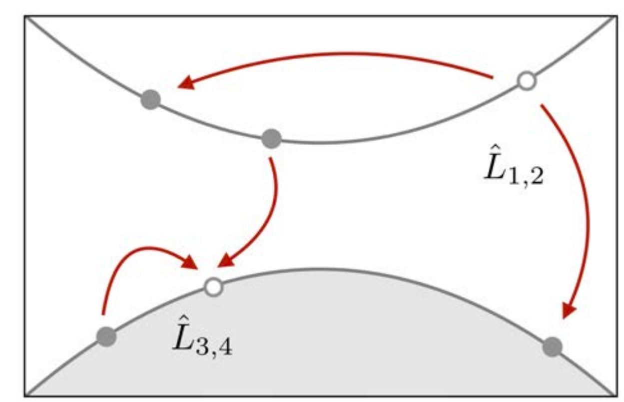
Lindblad scenario

- unique dark state

$$L_\alpha|D\rangle = 0 \quad \forall\alpha$$

Ex.: dissipative Chern insulator

coincide!



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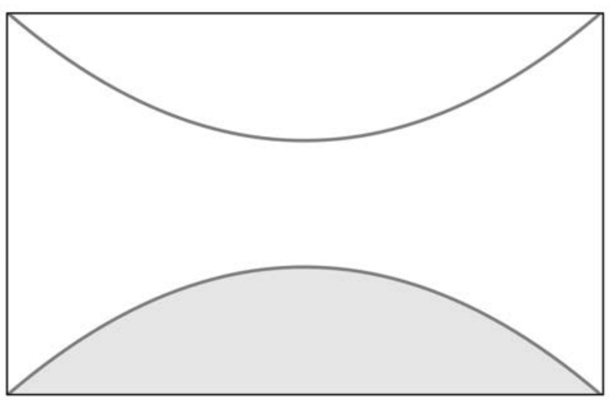
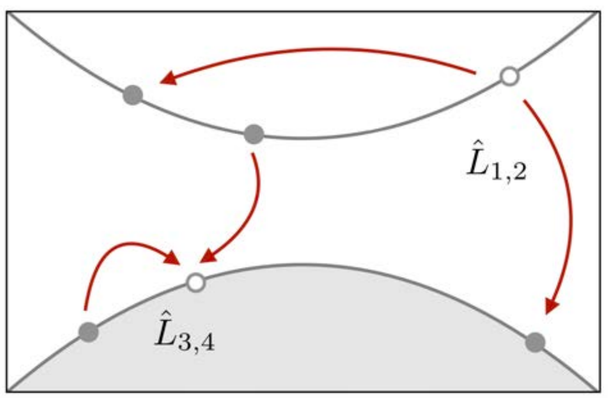


band 2



Dynamics

- (micro) reversible / unitary
- thermal equilibrium (detailed balance)

very different!

- irreversible / dissipative
- far from equilibrium: detailed balance violated see Lecture I

Sieberer, Chiocchetta, Taeuber, Gambassi, SD, PRB (2015)

$$\partial_t \hat{\rho} = -i[\hat{H}, \hat{\rho}]$$

$$\partial_t \hat{\rho} = \kappa \sum_\alpha (\hat{L}_\alpha \hat{\rho} \hat{L}_\alpha^\dagger - \tfrac{1}{2}\{\hat{L}_\alpha^\dagger \hat{L}_\alpha, \hat{\rho}\})$$

Model: Driven open Chern insulator

- starting point: class A insulator, no discrete symmetry to protect topological state

$$\hat{H} = \int_{\mathbf{q}} \hat{\psi}^{\dagger}_{\mathbf{q}} \left(\mathbf{d}_{\mathbf{q}} \cdot \boldsymbol{\sigma}\right) \hat{\psi}_{\mathbf{q}} = \int_{\mathbf{q}} \underbrace{\hat{\psi}^{\dagger}_{\mathbf{q}} V^{-1}_{\mathbf{q}}} \sigma_z \underbrace{V_{\mathbf{q}} \hat{\psi}_{\mathbf{q}}} \equiv \int_{\mathbf{q}} \hat{l}^{\dagger}_{\mathbf{q}} \sigma_z \hat{l}_{\mathbf{q}}$$

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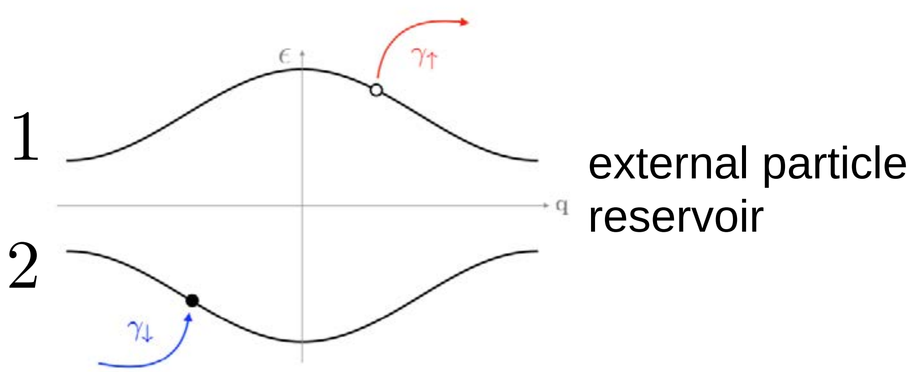
$$\mathbf{d}_{\mathbf{q}} \equiv (2mq_1, 2mq_2, -m^2 + \mathbf{q}^2)^T$$

- ground state condition $\quad \hat{l}_{1,\mathbf{x}}|D\rangle = \hat{l}^{\dagger}_{2,\mathbf{x}}|D\rangle = 0 \quad \forall \mathbf{x}$

model M1: no number conservation

$$\hat{L}_{\alpha} = \hat{l}_{1,\mathbf{x}},\, \hat{l}^{\dagger}_{2,\mathbf{x}}, \quad \gamma_1 = \gamma_2 \equiv \gamma_{M1}$$



quadratic, solvable

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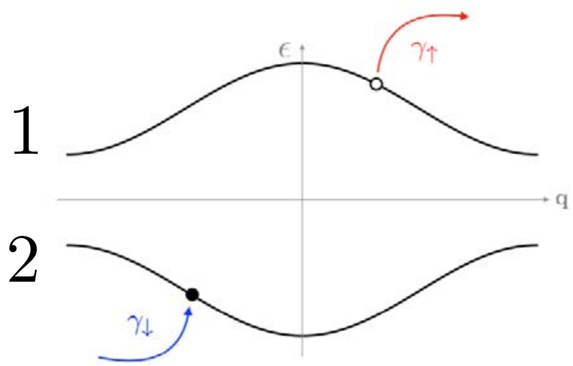
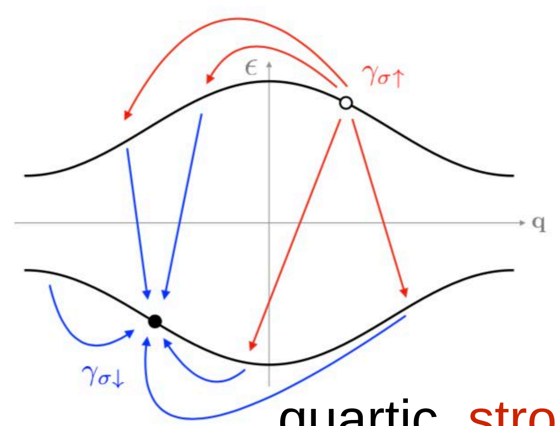


external particle reservoir

quadratic, solvable

model M2: number conservation $[\hat{L}_\alpha, \hat{N}] = 0 \,\forall\, \alpha$

$$\hat{L}_{1,2} = \hat{\psi}^\dagger_{1,2}\hat{l}_1,\, \hat{L}_{3,4} = \hat{\psi}_{1,2}\hat{l}^\dagger_2, \quad \gamma_{1...4} \equiv \gamma_{M2}$$



system acts as its own particle reservoir

quartic, strongly interacting

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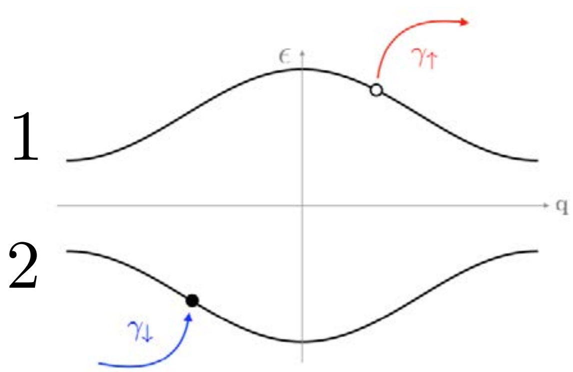
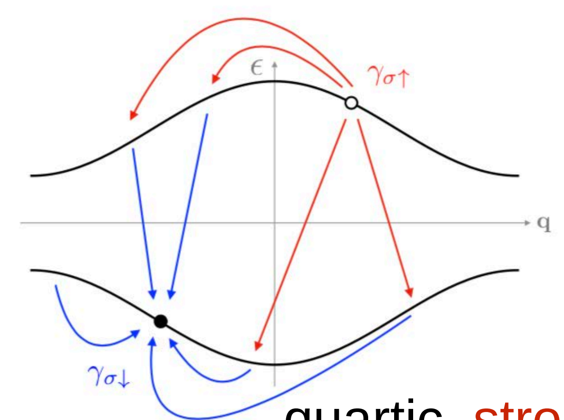


external particle reservoir

quadratic, solvable

model M2: number conservation $\quad [\hat{L}_\alpha, \hat{N}] = 0 \,\forall\, \alpha$

$$\hat{L}_{1,2} = \hat{\psi}^\dagger_{1,2}\hat{l}_1,\, \hat{L}_{3,4} = \hat{\psi}_{1,2}\hat{l}^\dagger_2, \quad \gamma_{1...4} \equiv \gamma_{M2}$$



system acts as its own particle reservoir

quartic, strongly interacting

gapped `continuity' equation $\quad \partial_t \hat{n}(\mathbf{x}) = \nabla \vec{J}(\mathbf{x}) + \mathcal{O}(\delta)$

continuity equation $\quad \partial_t \hat{n}(\mathbf{x}) = \nabla \vec{J}(\mathbf{x})$

➡ same dark state, different dynamics (non-equilibrium and conservation laws)

Model: Driven open Chern insulator

- starting point: class A insulator, no discrete symmetry to protect topological state

$$\hat H = \int_{\mathbf q} \hat\psi^\dagger_{\mathbf q}\,(\mathbf d_{\mathbf q}\cdot\boldsymbol\sigma)\,\hat\psi_{\mathbf q} = \int_{\mathbf q} \underbrace{\hat\psi^\dagger_{\mathbf q}\,V^{-1}_{\mathbf q}}_{}\sigma_z \underbrace{V_{\mathbf q}\,\hat\psi_{\mathbf q}}_{} \equiv \int_{\mathbf q} \hat l^\dagger_{\mathbf q}\,\sigma_z\,\hat l_{\mathbf q}$$

$$\hat\psi_{\mathbf q} \equiv (\hat\psi_{1,\mathbf q}, \hat\psi_{2,\mathbf q})^T$$

$$\hat l_{\mathbf q} \equiv (\hat l_{1,\mathbf q}, \hat l_{2,\mathbf q})^T$$

$$\mathbf d_{\mathbf q} \equiv (2mq_1, 2mq_2, -m^2 + \mathbf q^2)^T$$

- ground state condition $\quad \hat l_{1,\mathbf x}|D\rangle = \hat l^\dagger_{2,\mathbf x}|D\rangle = 0 \quad \forall \mathbf x$

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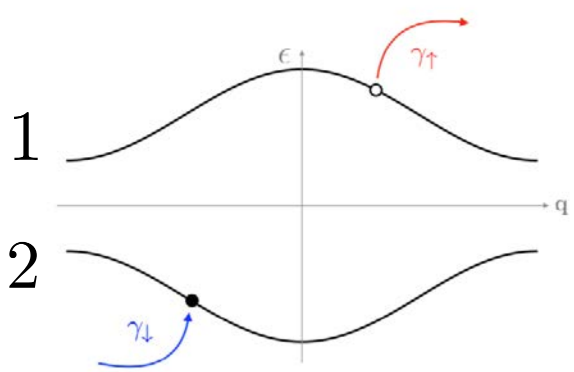

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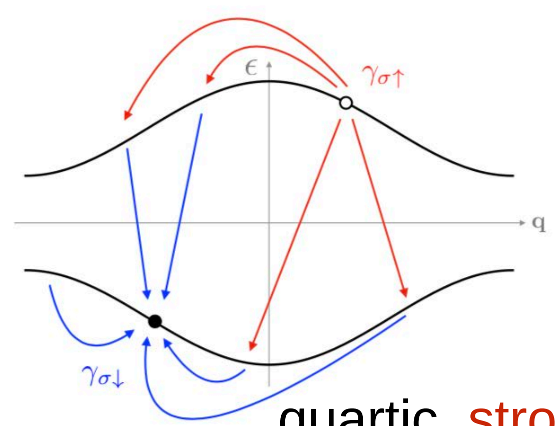

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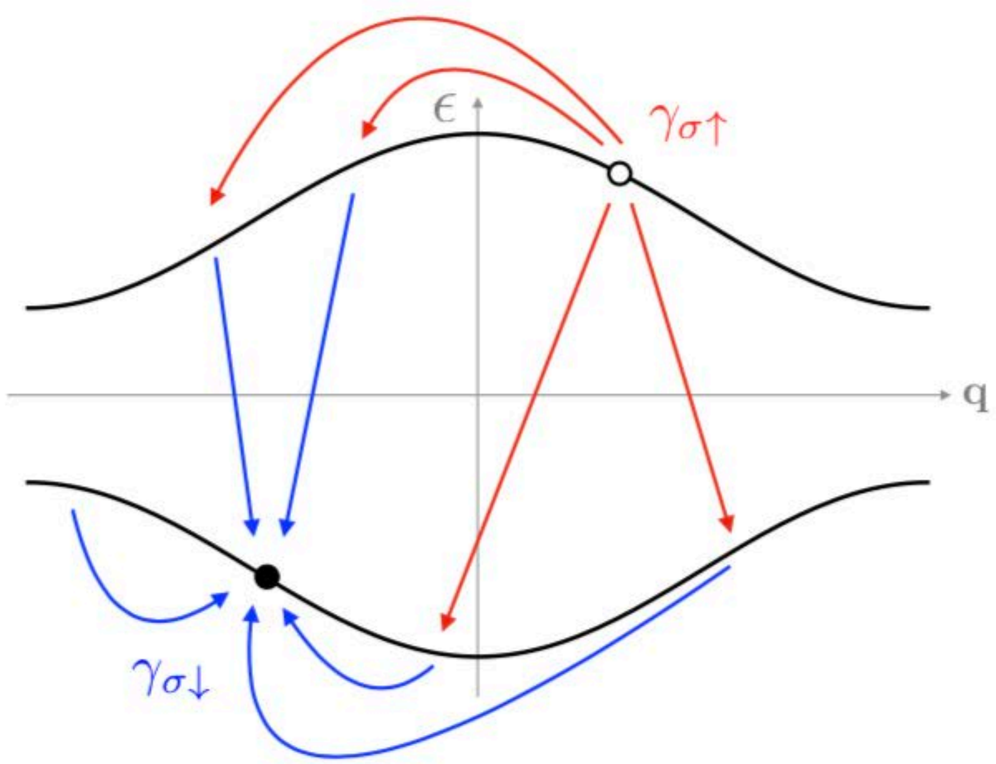
Program: Extracting the topological field theory

- map into Keldysh functional integral (better for concrete calculations)
- intermezzo: weak and strong symmetries in the Keldysh formalism
- minimal coupling to U(1) gauge field
- compute the effective gauge field action, encoding the long wavelength response

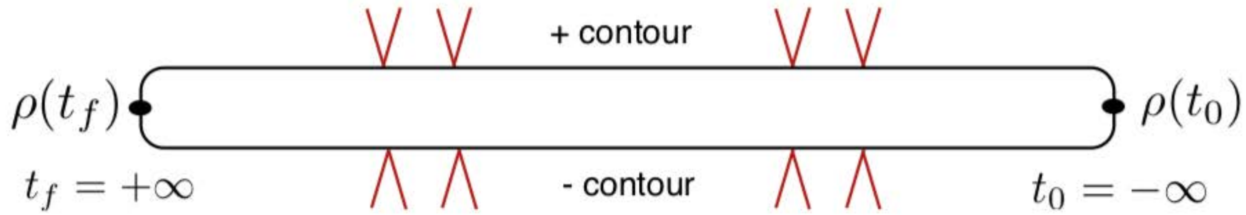


more details: Sieberer, Buchhold, SD, ROPP (2016)

Intermezzo: Symmetries in the Keldysh formalism

- closed time path (fermions, bosons) for recap & fermions: see appendix Lecture I

$$Z = \mathrm{tr}\hat{\rho}(t) =$$



$$= \int \mathcal{D}\psi_\pm e^{iS[\psi_\pm]}$$

more details: Sieberer, Buchhold, SD, ROPP (2016)

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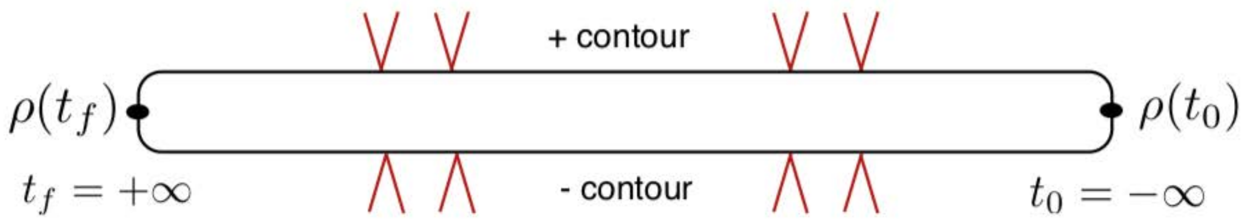

$$= \int \mathcal{D}\psi_\pm e^{iS[\psi_\pm]}$$

- Lindblad-Keldysh action $H_\pm = H[\psi_\pm], L_\pm = L[\psi_\pm]$

left action right action

$$S[\psi_\pm] = \int_{t,\mathbf{x}} \Big[\psi_+^\dagger i\partial_t \psi_+ - H_+ - (+ \to -) - i \sum_\alpha \gamma_\alpha \Big(2L^\dagger_{\alpha,-} L_{\alpha,+} - L^\dagger_{\alpha,+} L_{\alpha,+} - L^\dagger_{\alpha,-} L_{\alpha,-}\Big)\Big]$$

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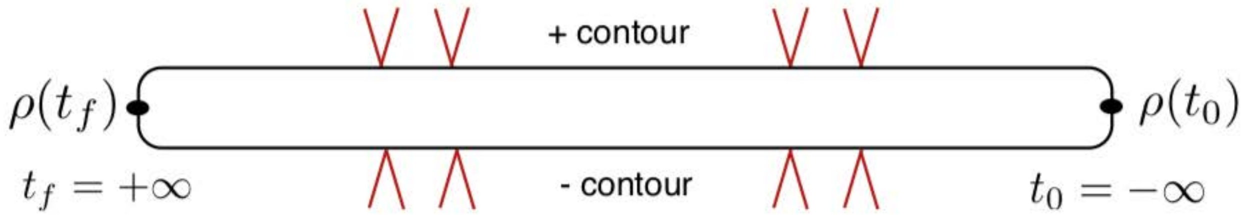

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- basis separating responses and correlations: $\psi_c = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-), \quad \psi_q = \frac{1}{\sqrt{2}}(\psi_+ - \psi_-)$

`classical field' `quantum field'

more details: Sieberer, Buchhold, SD, ROPP (2016)

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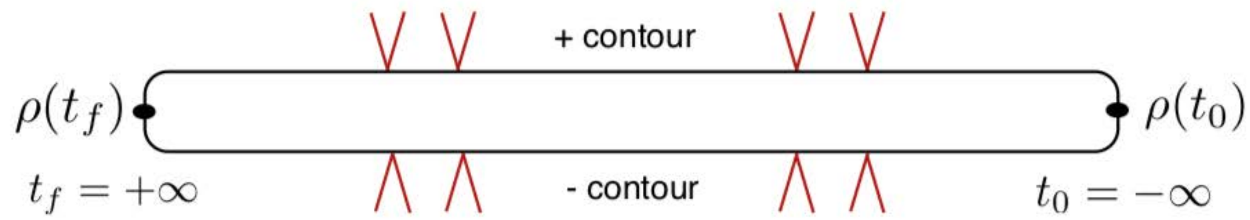

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`classical field' `quantum field'

- symmetries have contour structure, e.g. global U(1):

$$\psi_\pm(t,\mathbf{x}) \to e^{i\chi_\pm}\psi_\pm(t,\mathbf{x})$$

- symmetry generators $\chi_+, \quad \chi_- \quad \leftrightarrow \quad \chi_c = \chi_+ + \chi_-, \quad \chi_q = (\chi_+ - \chi_-)/2$

$U_+(1) \quad U_-(1) \qquad U_c(1) \qquad U_q(1)$

more details: Sieberer, Buchhold, SD, ROPP (2016)

Intermezzo: Classification of symmetries

Buca, Prosen, NJP (2013)
Lieu et al., PRL (2020)

- overview: strong and weak symmetries
- focus: continuous global symmetry U(1)

- strong symmetries: independent transformations on both contours

$$U_c(1) \times U_q(1)$$

➡ conservation laws, gapless hydrodynamic modes (Noether theorem)

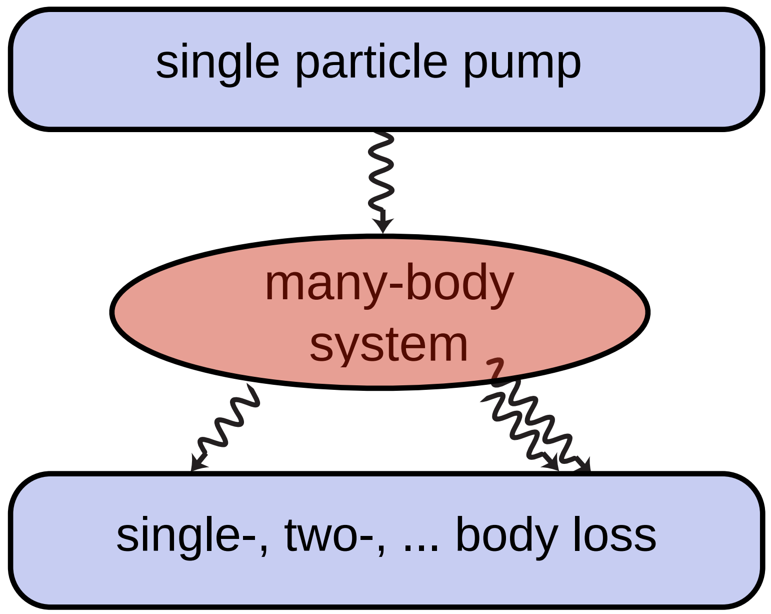
- weak symmetries: both contours ‘in phase’, $U_q(1)$ explicitly broken, $\chi_q = 0$

$$U_c(1)$$

➡ spontaneous breaking of ‘classical’ symmetry: gapless Goldstone modes (Goldstone theorem)

Intermezzo: Classical/weak symmetry

- recall generic Lindblad model



$$\partial_t \rho = -i[H,\rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho]$$

$$H = \int_{\mathbf{x}} \hat\phi^\dagger_{\mathbf{x}} \left(\tfrac{\triangle}{2M} - \mu\right) \hat\phi_{\mathbf{x}} + \tfrac{\lambda}{2}(\hat\phi^\dagger_{\mathbf{x}}\hat\phi_{\mathbf{x}})^2$$

$$\mathcal{D}[\rho] = \gamma_p \int_{\mathbf{x}} [\hat\phi^\dagger_{\mathbf{x}}\,\rho\,\hat\phi_{\mathbf{x}} - \tfrac{1}{2}\{\hat\phi_{\mathbf{x}}\hat\phi^\dagger_{\mathbf{x}},\rho\}] \quad + \quad \gamma_l \int_{\mathbf{x}} [\hat\phi_{\mathbf{x}}\,\rho\,\hat\phi^\dagger_{\mathbf{x}} - \tfrac{1}{2}\{\hat\phi^\dagger_{\mathbf{x}}\hat\phi_{\mathbf{x}},\rho\}] \quad +$$

single particle pump

single particle loss

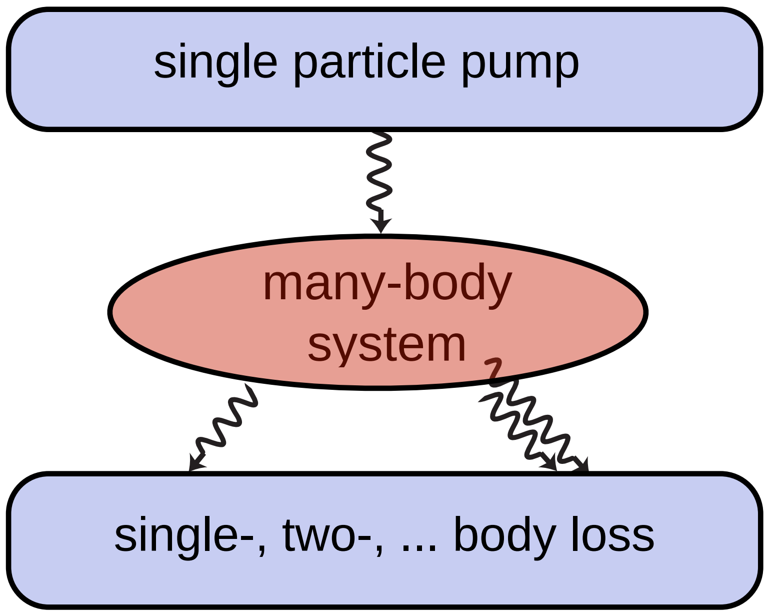
$$\kappa \int_{\mathbf{x}} [\hat\phi^2_{\mathbf{x}}\,\rho\,\hat\phi^{\dagger\,2}_{\mathbf{x}} - \tfrac{1}{2}\{\hat\phi^{\dagger\,2}_{\mathbf{x}}\hat\phi^2_{\mathbf{x}},\rho\}]$$

two particle loss

$U_c(1)$

Intermezzo: Classical/weak symmetry

- recall generic Lindblad model



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$$\underbrace{\kappa \int_{\mathbf{x}} [\hat{\phi}^2_{\mathbf{x}} \rho \hat{\phi}^{\dagger\,2}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^{\dagger\,2}_{\mathbf{x}} \hat{\phi}^2_{\mathbf{x}}, \rho\}]}_{\text{two particle loss}}$$

- symmetries:
  - $U_c(1)$ invariance: both contours transform 'in phase'

$$\hat{\phi} \to e^{i\chi} \hat{\phi}$$

  - but no $U_q(1)$ invariance

  - interesting physics is associated to the spontaneous breaking of $U_c(1)$

Intermezzo: Classical/weak symmetry, semiclassical limit

see also Lecture I: semiclassical similar $\hbar \to 0$

here: semiclassical as $N \to \infty$

- extracting the physics: associated Keldysh action

$$\mathcal{S} = \int_{t,\mathbf{x}} \left\{ (\phi_c^*, \phi_q^*) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} + 2i\kappa \phi_c^* \phi_c \phi_q^* \phi_q \; - \tfrac{1}{2} \left[(\lambda + i\kappa) \left(\phi_c^{*2} \phi_c \phi_q + \phi_q^{*2} \phi_c \phi_q \right) + c.c. \right] \right\}$$

$$P^R = i\partial_t + \nabla^2 + \mu + i\,\tfrac{1}{2}(\gamma_l - \gamma_p) \qquad P^A = (P^R)^\dagger \qquad P^K = i(\gamma_l + \gamma_p)$$

(more precisely: only q=0 mode scales)

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- assume a condensation is taking place:
  - classical / occupation field: macroscopic occupation $N \to \infty$ (more precisely: only q=0 mode scales)

$$\phi_c(q) \sim N^{1/2} \implies \phi_c(x) \sim \frac{N^{1/2}}{V^{1/2}} \sim N^0 \qquad \phi_c(x) = V^{-1/2} \sum_q e^{-iqx} \phi_c(q)$$

  - quantum field:

$$\phi_q(q) \sim N^0 \implies \phi_q(x) \sim \frac{1}{V^{1/2}} \sim N^{-1/2}$$

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➡ expand action up to second order in $\phi_q(q)$

➡ semiclassical (large occupation N) limit governed by Martin-Siggia-Rose (MSR) action

$$S = \int_{t,\mathbf{x}} \left\{ \phi_q^* \frac{\delta \bar{S}[\phi_c]}{\delta \phi_c^*} + c.c. + P^K \phi_q^* \phi_q \right\}$$

Martin, Siggia, Rose, PRA (1973); Janssen, Z. Phys. B (1976); DeDominicis, J. Phys. (1976)

$$\bar{S} = \int_{t,\mathbf{x}} \{ \phi_c^* i\partial_t \phi_c - \mathcal{H}_c + i\mathcal{H}_d \} \qquad \mathcal{H}_\alpha = r_\alpha |\phi_c|^2 + K_\alpha |\nabla\phi_c|^2 + \lambda_\alpha |\phi_c^*\phi_c|^4, \quad \alpha = c, d$$

➡ analyze like ordinary condensation problem (ϕ^4 potential)

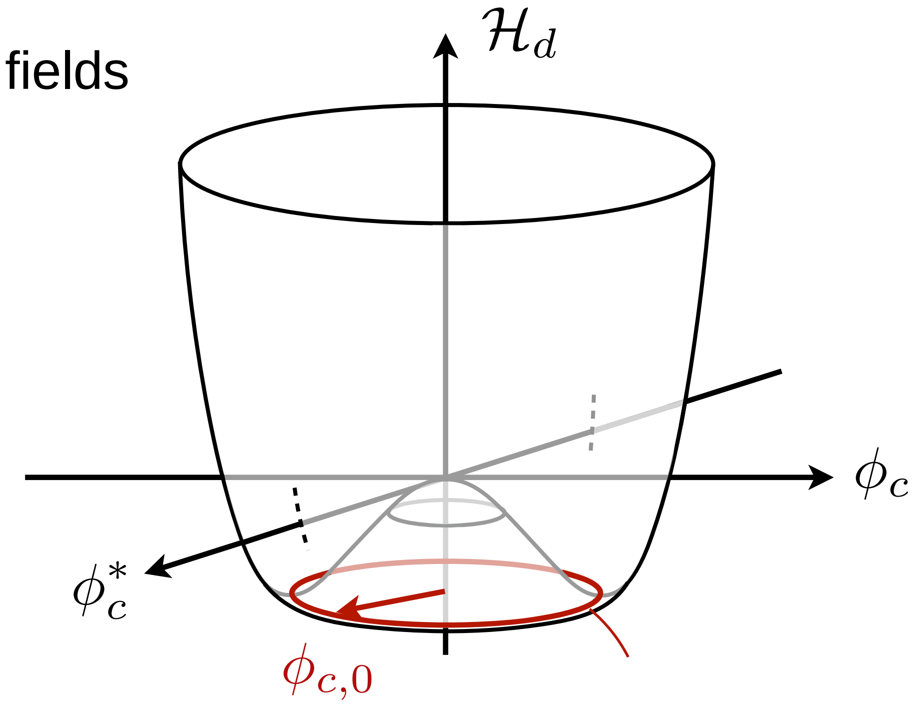
Intermezzo: Classical/weak symmetry, spontaneous breaking

- equation of motion:

$$i\partial_t \phi_c = \frac{\delta \mathcal{H}_c}{\delta \phi_c^*} - i\frac{\delta \mathcal{H}_d}{\delta \phi_c^*} + P^K \phi_q \qquad \mathcal{H}_\alpha = \int d^d x [r_\alpha |\phi_c|^2 + K_\alpha |\nabla \phi_c|^2 + g_\alpha |\phi_c|^4]$$

- $U_c(1)$ invariance: $\phi_c \to e^{i\chi_c}\phi_c, \quad \phi_q \to e^{i\chi_c}\phi_q$
- stat. state on mean field level: neglect quantum field, homogeneous fields
- stat. state determined by $\mathcal{H}_d$ alone (r_c adjusts in rotating frame)

➡ condensation for $r_d = \gamma_l - \gamma_p < 0$



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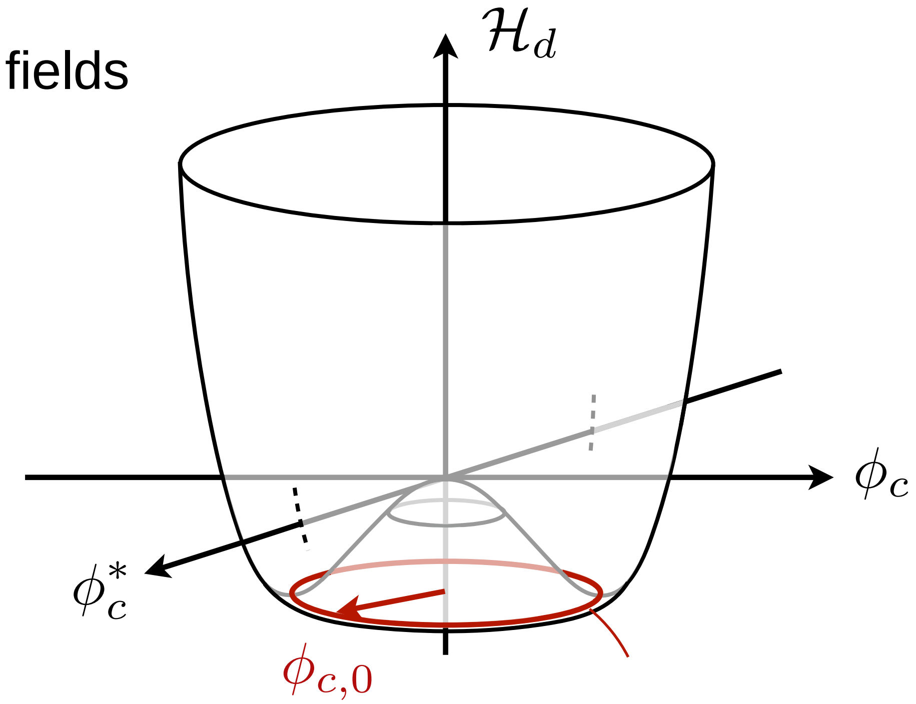
- visually clear:
  - ➡ Sombrero potential with degenerate manifold of minima
  - ➡ system chooses one of the minima spontaneously
  - ➡ motion along this manifold costs no action



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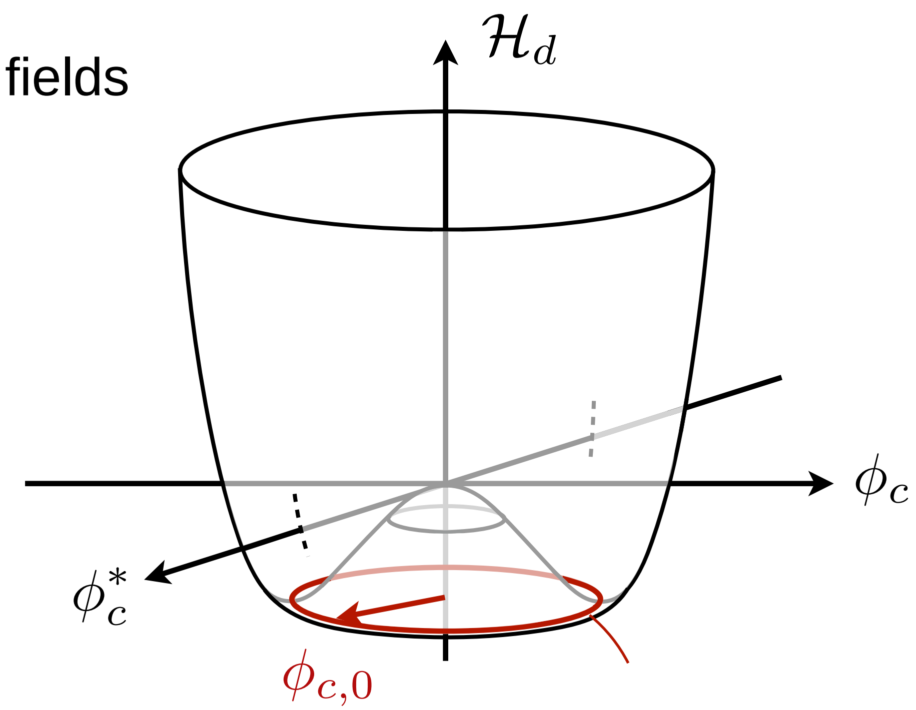


- more generally valid: Keldysh Goldstone theorem

If a global continuous weak symmetry is broken, there is an exact zero mode.

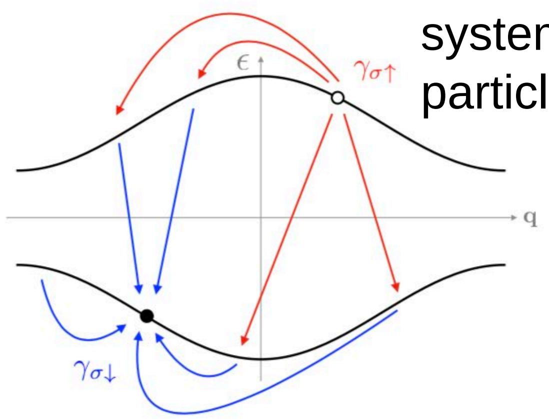
- holds including fluctuations and beyond semiclassical limit

more details: Sieberer, Buchhold, SD, ROPP (2016)

Intermezzo: Conservation laws and quantum/strong symmetry

- example: model M2, with number conservation $[\hat{L}_\alpha, \hat{N}] = 0 \,\forall\, \alpha$

$$\hat{L}_{1,2} = \hat{\psi}^\dagger_{1,2}\hat{l}_1,\; \hat{L}_{3,4} = \hat{\psi}_{1,2}\hat{l}^\dagger_2, \quad \gamma_{1...4} \equiv \gamma_{M2}$$



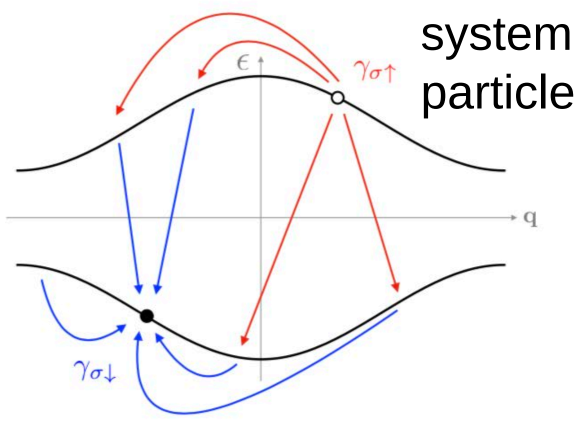
system acts as its own particle reservoir

- intuition: no number exchange with the bath
- on Keldysh contour:

$$\hat{L}_{1,2} \to (L_{1,2})_\pm$$

—> independent phase rotations, symmetry $U_+(1) \times U_-(1) = U_c(1) \times U_q(1)$

Intermezzo: Conservation laws and quantum/strong symmetry



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- Keldysh Noether construction: more details: Sieberer, Buchhold, SD, ROPP (2016)
  - promote $\theta_{c,q} \to \theta_{c,q}(t, \vec{x})$
  - change of action: $S \to S + \int dt d^2x \, [\, \partial_\mu \theta_c J^\mu_q + \partial_\mu \theta_q J^\mu_c \,]$
  - vanishing variation: $\frac{\delta S}{\delta \theta_{q,c}} \stackrel{!}{=} 0 \implies \partial_\mu J^\mu_{c,q} = 0$ or $\partial_t J^0_{c,q} = -\nabla \vec{J}_{c,q}$

where universally for time-local non-relativistic dynamics $J^0_c = \psi^*_c\psi_c + \psi^*_q\psi_q = \psi^*_+\psi_+ + \psi^*_-\psi_-$

and $\vec{J}_{c,q}$ is model specific $J^0_q = \psi^*_c\psi_q + \psi^*_q\psi_c = \psi^*_+\psi_+ - \psi^*_-\psi_-$

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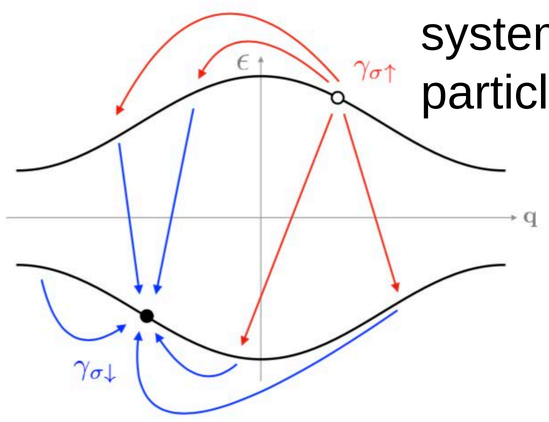

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  - promote $\theta_{c,q} \to \theta_{c,q}(t, \vec{x})$
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  - vanishing variation: $\frac{\delta S}{\delta \theta_{q,c}} \overset{!}{=} 0 \implies \partial_\mu J^\mu_{c,q} = 0$ or $\partial_t J^0_{c,q} = -\nabla \vec{J}_{c,q}$

  where universally for time-local non-relativistic dynamics $J^0_c = \psi^*_c\psi_c + \psi^*_q\psi_q = \psi^*_+\psi_+ + \psi^*_-\psi_-$

  and $\vec{J}_{c,q}$ is model specific $J^0_q = \psi^*_c\psi_q + \psi^*_q\psi_c = \psi^*_+\psi_+ - \psi^*_-\psi_-$

- Noether charges:

$$U_c(1): \quad N_q = \int d^2x \langle J^0_q \rangle = 0$$

$$\boxed{U_q(1): \quad N_c = \int d^2x \langle J^0_c \rangle = N}$$

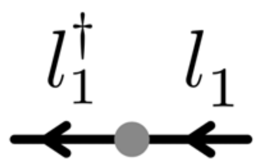
➡ formalising the obvious: particle number conserved in M2

➡ useful to build minimal coupling prescription

Spontaneous breaking of quantum/strong symmetry: M2 —> M1

- strongly interacting problem, but dark state exactly known: Self-consistent Born approximation (analogous to mean field theory for dissipative superconductor)

$$(L^\dagger_\alpha L_\alpha)_{\mathrm{mf}} =$$



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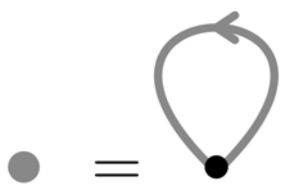

$$L^\dagger_{\alpha,1} L_{\alpha,1} = l_1^\dagger \psi_\alpha \psi_\alpha^\dagger l_1$$

$$\to l_1^\dagger \langle \psi_\alpha \psi_\alpha^\dagger \rangle l_1 = n l_1^\dagger l_1$$

- effective action in single particle sector: M2 reduces to M1 under replacements

$$L_{1,2} \to l_1, \quad L_{3,4} \to l_2^\dagger \qquad\qquad \gamma \to n\gamma \equiv \bar{\gamma}$$

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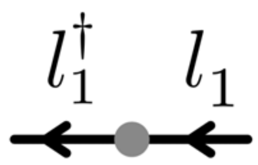
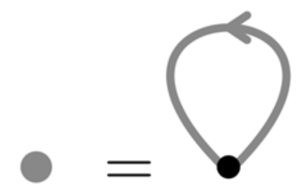
$(L^\dagger_\alpha L_\alpha)_{\mathrm{mf}} =$ 

with 

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- single particle Green's function $G^K_{\omega,\mathbf{q}} = -2i\bar\gamma \frac{\mathbf{d}_\mathbf{q}\cdot\boldsymbol{\sigma}}{\omega^2+\gamma_\mathbf{q}^2}, \quad G^R_{\omega,\mathbf{q}} = \frac{\mathbf{1}}{\omega + i\gamma_\mathbf{q}} \qquad \gamma_\mathbf{q} = \bar\gamma |\mathbf{d}_\mathbf{q}|$
    - emergent dissipative single-particle spectral gap $\gamma_{\mathbf{q}=0} = \bar\gamma m^2$
    - spectral Green's function topologically trivial, topology encoded in the Keldysh 'noise' component (would go unseen in non-Hermitian topology) Kawabata et al. PRX (2018); Zhou, Lee PRB (2018)
    - non-Hermitian single-particle Green's function, coincides with the one for model M1: particle number conservation masked / quantum symmetry spontaneously broken

➡ Quantization of the response? How does the system remember its number conserving origin?

Many-body gap: Dissipative insulator

- single particle gap emerges in mean field theory
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  - local particle-hole excitation on top of dark state in momentum space

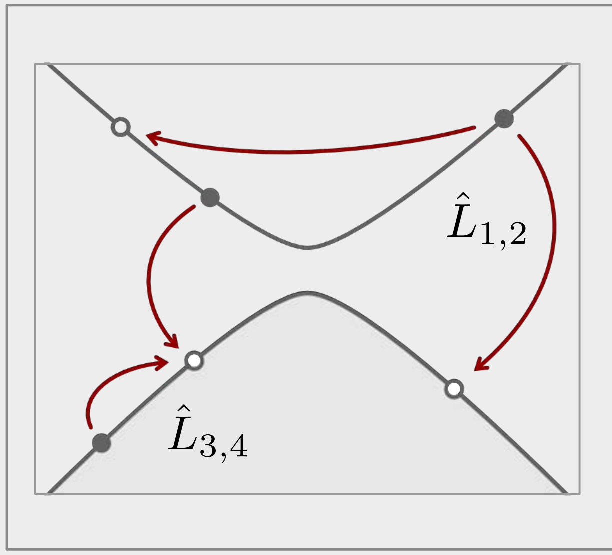
$$|\mathbf{k}\rangle = n^{-1/2} \int_{\mathbf{q}} \hat{c}^{\dagger}_{1,\mathbf{q}-\mathbf{k}} \hat{c}_{2,\mathbf{q}} |D\rangle$$

  - density matrix with up to one particle, one hole excitation

$$\hat{\rho} = \rho_0 |D\rangle\langle D| + \rho_{\mathbf{k}} |\mathbf{k}\rangle\langle\mathbf{k}|$$

  - closed set of evolution equations with leading behaviour

$$\partial_t \rho_0 = 2\bar{\gamma} m^2 \rho_{\mathbf{k}}, \qquad \partial_t \rho_{\mathbf{k}} = -2\bar{\gamma} m^2 \rho_{\mathbf{k}}$$



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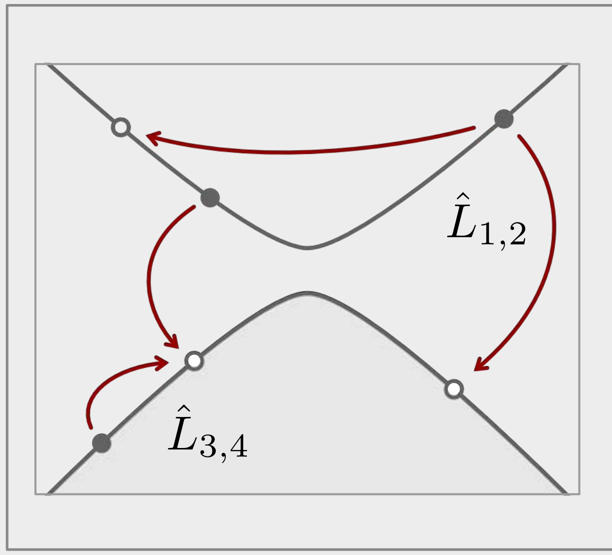

➡ particle-hole excitation damps exponentially fast with twice the gap of a single particle

➡ strong analogy to a Hamiltonian insulator despite different dynamics (e.g. no energy conservation)

Microscopic gauge-matter action: minimal coupling and response

- resolution: the full functional integral knows about number conservation, lost only by approximation
  - ➡ first couple to the gauge field, then do the approximations

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- Lindblad operators carry gauge field $L_\alpha =$ 

$l_{\mathbf{q}} = V_{\mathbf{q}} \psi_{\mathbf{q}}$

$$L_1 \to L_1 + \psi_1^\dagger (\partial_{q_i} V V^{-1} l)_1 \, A_i \approx L_1 - i\psi_1^\dagger (a_i l)_1 A_i$$

$U_{\mathbf{q}} = d_{\mathbf{q}}^{\frac{1}{2}} V_{\mathbf{q}}$ inessential diagonal matrix

Berry connection

$$a_i = i\partial_{q_i} U U^{-1}$$

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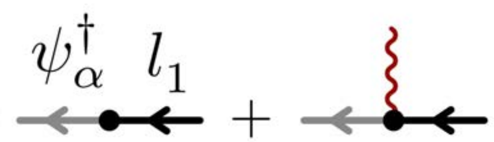
$L_\alpha =$ 

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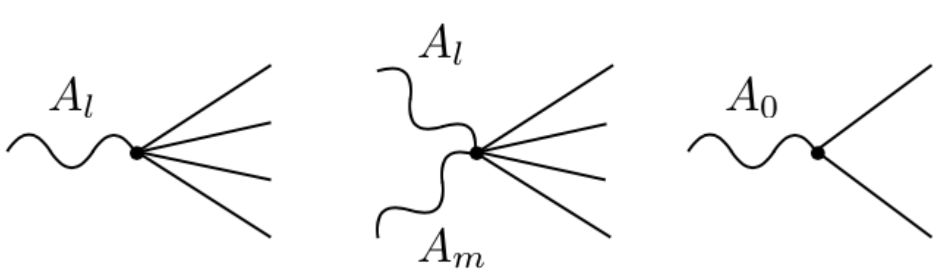
- gauge-matter (current) coupling in quartic. $L^\dagger_\alpha L_\alpha$:



$L^\dagger_\alpha L_\alpha$ spatial components (Lindblad operators)

temporal component

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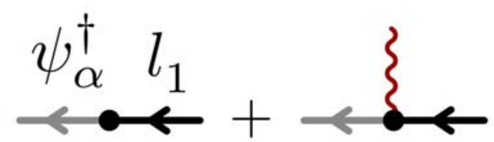
- Lindblad operators carry gauge field $\qquad L_\alpha =$ 

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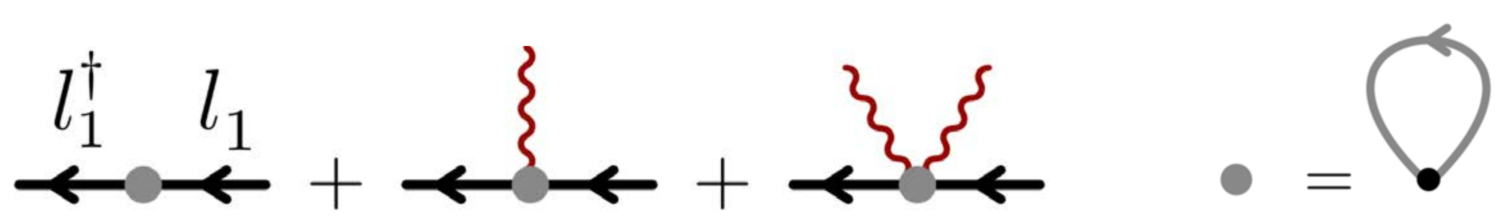
- gauge-matter (current) coupling in mean field decoupling:

$(L^\dagger_\alpha L_\alpha)_{\mathrm{mf}} =$ 

Microscopic gauge-matter action: minimal coupling and response

- Response: cumulant expansion to second order (= one-loop approximation)

$$Z[A] = e^{i(\langle X^{(2)}\rangle + \frac{i}{2}\langle X^{(1)} X^{(1)}\rangle_{c.})} \underset{A_0 = 0}{=} e^{i\int_{\omega,\mathbf{q}} A_{I,i}\Pi^{IJ}_{ij} A_{J,j}}$$

$$\Pi_{ij} = \begin{pmatrix} 0 & \Pi^{cq}_{ij} \\ \Pi^{qc}_{ij} & \Pi^{qq}_{ij} \end{pmatrix}$$

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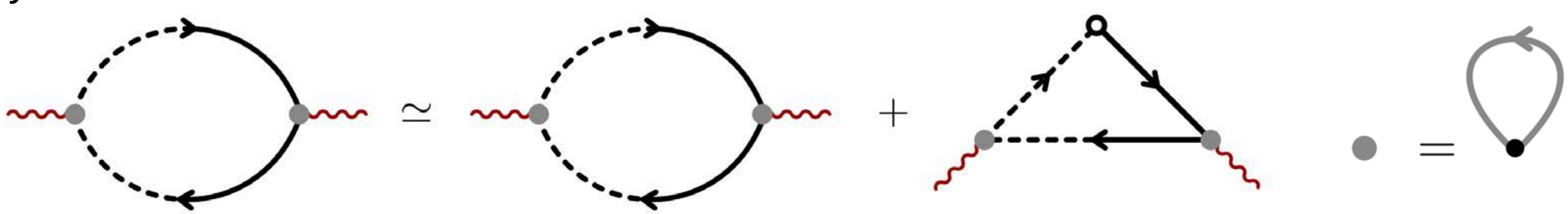

$\Pi_{cq}(\omega, \mathbf{0})$ $\qquad$ $\Pi_{cq}(0, \mathbf{0})$ $\qquad$ $\partial_\omega \Pi_{cq}(0, \mathbf{0})\omega$

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microphysics  macrophysics

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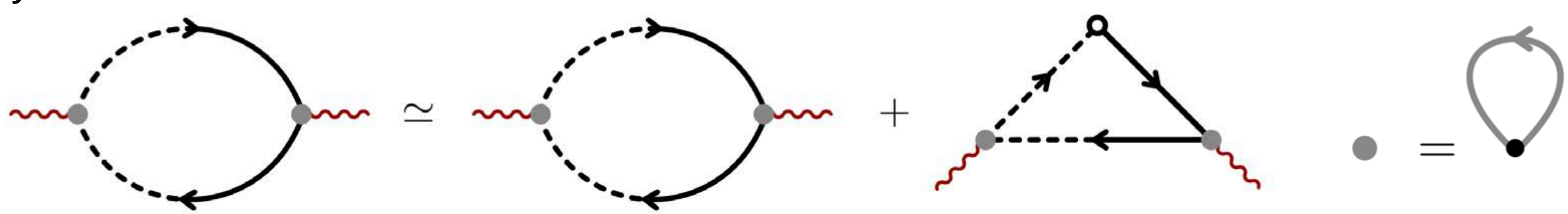

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- leading frequency term

$$\boxed{\omega\,\partial_\omega \Pi^{cq}_{ij}(0, \mathbf{0}) = \frac{i\omega}{2}\epsilon_{ij}\int_{\mathbf{p}} \mathrm{Tr}[\sigma^z F] = -i\omega\frac{\theta}{2\pi}\epsilon_{ij} \qquad \theta = -1}$$

with Berry curvature $\quad F = \partial_{q_1} a_2 - \partial_{q_2} a_1 \quad$ for above two-band model

➡ quantised response of the purely dissipative quantum system (within effective one-loop theory)

➡ gauge invariance demonstrated to first order in derivative expansion

Intermediate summary: symmetries and responses

- Effective single particle Green's function for number conserving M2 indistinguishable from number-violating M1

$$G^K_{\omega,\mathbf{q}} = -2i\bar{\gamma}\frac{\mathbf{d_q}\cdot\boldsymbol{\sigma}}{\omega^2+\gamma^2_{\mathbf{q}}}, \quad G^R_{\omega,\mathbf{q}} = \frac{\mathbf{1}}{\omega+i\gamma_{\mathbf{q}}}$$

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- Electromagnetic response: all possible contributions in 1-loop approximation

$$S[A_c,A_q] = \int\left[\frac{\theta}{4\pi}\,\epsilon^{\mu\nu\rho}\left(A_{\mu,c}\partial_\nu A_{\rho,q} \quad + A_{\mu,q}\partial_\nu A_{\rho,c}\right) + \lambda_1 B_c B_q + i\lambda_2 B_q^2\right] \qquad B_I = \epsilon_{ij}\partial_i A_{j,I}$$

  - ➡ $\theta = -1$ quantised
  - ➡ subleading Maxwell-like terms

Macroscopic gauge action: Keldysh topological field theory

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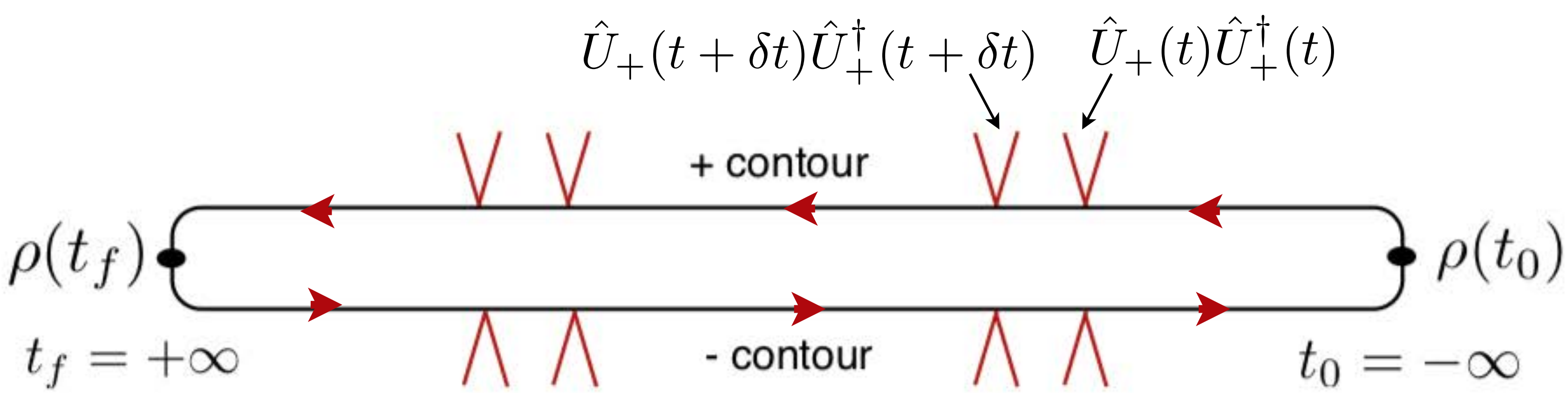

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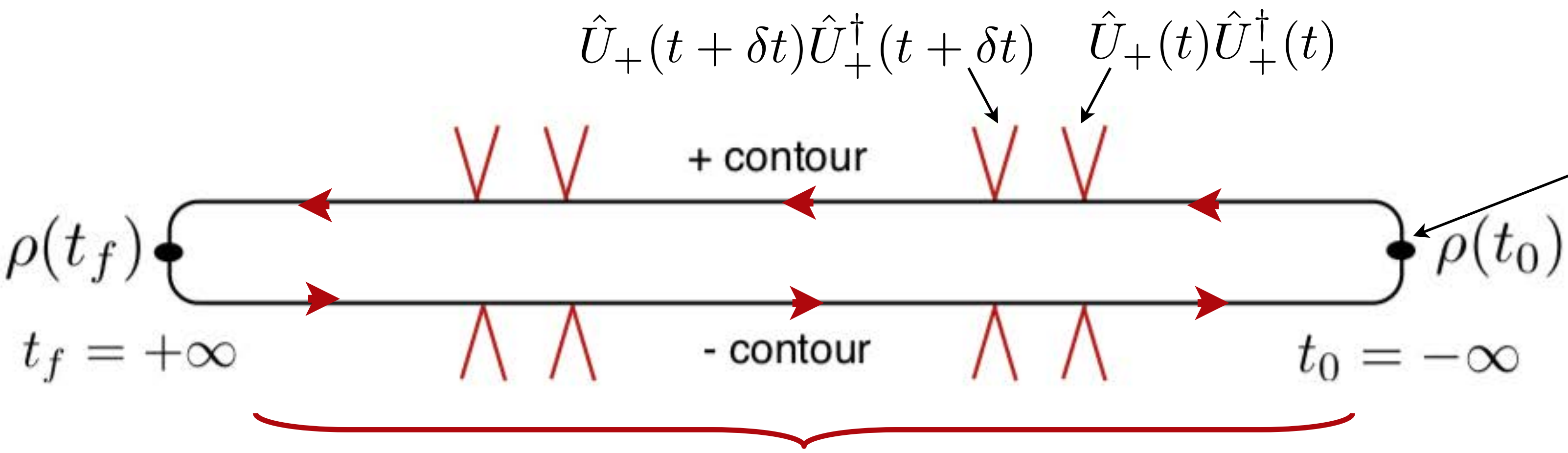

temporal bulk: invariance under $U_+(1) \times U_-(1)$

temporal boundary condition:

$$\hat{U}_+^\dagger(t_0)\hat{\rho}(t_0)\hat{U}_-^\dagger(t_0) \stackrel{!}{=} \hat{\rho}(t_0)$$

$\hat{\rho}(t_0)$ number eigenstate

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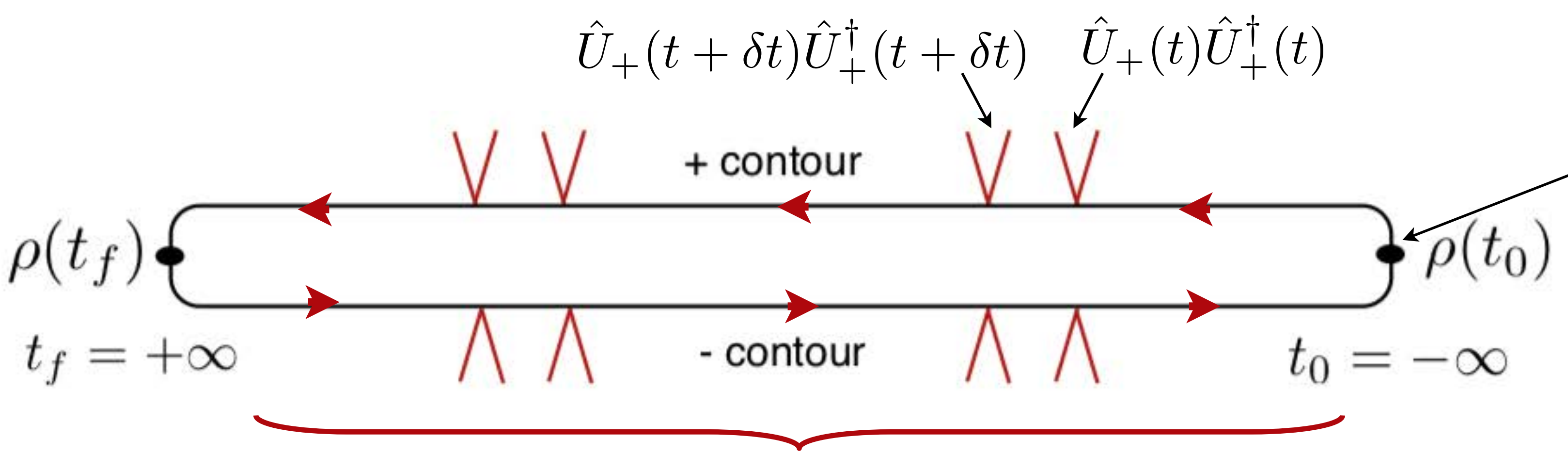

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- large gauge invariance: $\boxed{\chi_+(t_0) = \chi_-(t_0) + 2\pi n\,, \quad n \in \mathbb{Z}}$ phase accumulated along closed time path
- implication for effective theory: $e^{iS[A]} \to e^{iS[A]+2\pi i n} = e^{iS[A]}$ => constraint on action coefficient
- remark: more physical interpretation than at thermal equilibrium (periodicity due to Matsubara torus)

Macroscopic gauge action: Keldysh topological field theory (bulk)

- most general form of the Keldysh Chern-Simons action for pure states

$$S_{\mathrm{CS}}[A] = \int dt\, d^2x M_{IJ} \epsilon_{\mu\nu\rho}\, A_I^\mu \partial_\nu A_J^\rho$$

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$$S[A_c, 0] = 0$$

real: hermeticity of time dependent density matrix

$$S^*[A_c, A_q] = -S[A_c, -A_q]$$

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- standard Chern-Simons action emerges:

$$\boxed{S_{\mathrm{CS}}[A] = \frac{\theta}{4\pi} \int A_+\, dA_+ - A_-\, dA_-}$$

➡ contour decoupled Hamiltonian structure on top of purely dissipative bulk

Bulk-boundary correspondence

Chan, Hughes, Ryu, Fradkin, PRB (2013)

- adjustment of functional bosonization approach to Keldysh: lack of local $U_c(1) \times U_q(1)$ gauge invariance on boundary gives rise to new boundary degree of freedom

- add all possible leading terms allowed by global $U_c(1) \times U_q(1)$ invariance

$$\boxed{S|_{y=0} = \int_{t,x} \Big[\underbrace{-\,\theta\{\,\partial_t\varphi_q\partial_x\varphi_c + v\partial_x\varphi_q\partial_x\varphi_c + (c \leftrightarrow q)\}}_{\text{chiral propagation}} + \underbrace{D\{\partial_x\varphi_q\partial_x^2\varphi_c - (c \leftrightarrow q)\}}_{\text{diffusion}} + \underbrace{i\eta(\partial_x\varphi_q)^2}_{\text{conserved noise}} \Big]}$$

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➡ all terms consistent with particle number conservation are subleading: sharply defined chiral modes

$$\omega = vq + iDq^2$$

➡ qualitatively unmodified for mixed states

➡ potentially interesting non-linear fluctuating hydrodynamics different from equilibrium

Chen-Lin, Delcatraz, Harnoll, PRL (2019)

➡ unseen if mislead by `hidden' U_q(1) symmetry of effective single-particle action
—> implications for non-Hermitian bulk-boundary correspondence?

Kawabata, Shiozaki, Ueda, Sato, PRX (2018); Zhou, Lee PRB (2018)

Summary: Symmetry analysis

Key ingredients:

- global $U_c(1) \times U_q(1)$ invariance (number conservation)
- large gauge invariance (number quantization)
- pure states (but see outlook)

Implications:

- standard Chern-Simons action emerges (dissipative corrections subleading in derivative expansion)

$$\boxed{S_{\mathrm{CS}}[A] = \frac{\theta}{4\pi} \int A_+ \, dA_+ - A_- \, dA_-}$$

➡ contour decoupled Hamiltonian structure on top of purely dissipative bulk

- bulk-boundary correspondence: sharp chiral edge models with subleading width

$$\omega = vq + iDq^2$$ v determined by Lindblad parameters

Outlook: Topological field theory for mixed states

- basic riddle/paradox:

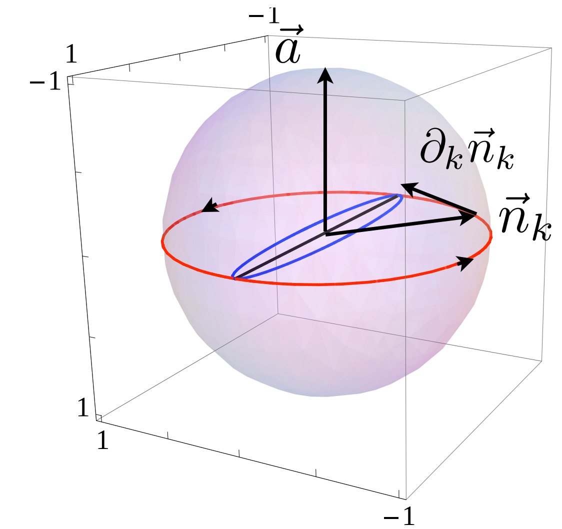
  e.g. Wang, Troyer, Dai, PRL (2013)

  - observables tend not to be topologically quantized for mixed states (eq/neq)
  - topology of density matrices well defined

  e.g. Viyuela, Rivas, Martin-Delgado, PRL (2014); Huang, Arovas, PRL 2014); Budich, SD, PRB (2015)

winding number

$$w = \oint \frac{dk}{2\pi} \vec{a} \cdot (\vec{n}_k \times \partial_k \vec{n}_k)$$



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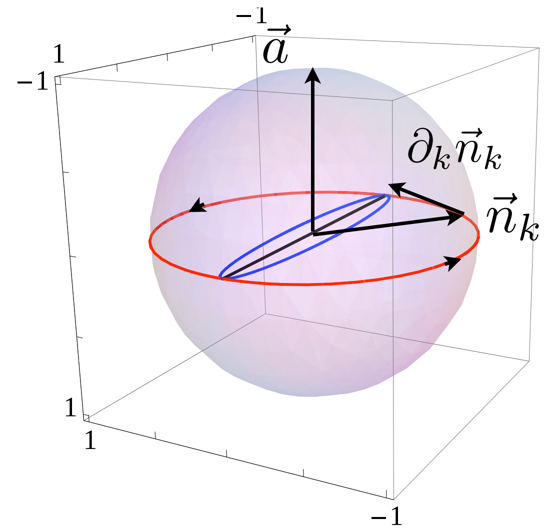

pure states: $|\vec{n}_k| = 1$

- resolution (for Dirac stationary theory in arbitrary dimension)

  Z. Huang, X. Sun, SD, in preparation

  - Effective real time gauge response action is topological but non-perturbative in the temporal gauge field, with structure (e.g. odd spacetime dim.)

$$S = c \int d^{2n}x \; \mathcal{P}[a_0^q, \beta|m|] \; C_{2n}[A_i^c]$$

Chern class density

$$C_{2n}[A_i^c] = \frac{1}{(2\pi)^n n!} \epsilon^{0\mu_1\mu_2\cdots} \partial_{\mu_1} A^c_{\mu_2} \ldots \partial_{\mu_{2n-1}} A^x_{\mu_{2n}}$$

Chern number (quantized)

only dim.less ratio: purity gap

$$a_0^q = \int dt A_0^q \qquad \mathcal{P}[a_0^q + 2\pi n, \beta|m|] = \mathcal{P}[a_0^q, \beta|m|] + 2\pi$$

under large $U_q(1)$

$$\lim_{\beta\to\infty} \mathcal{P}[a_0^q + 2\pi n, \beta|m|] = a_0^q$$

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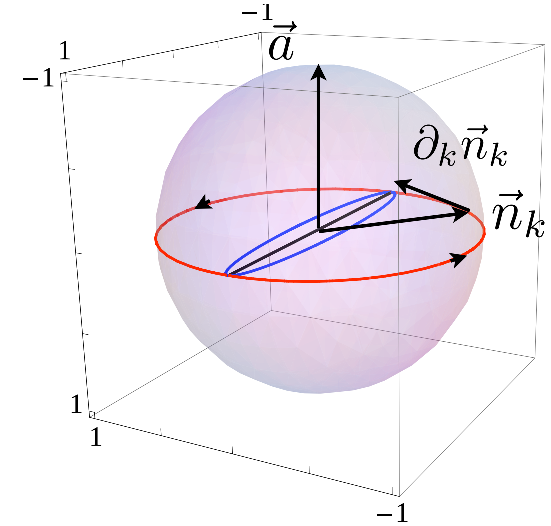

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- topological properties related to
  - $U_q(1)$ large gauge invariance i.e. number quantization
  - global $U_q(1)$ invariance, i.e. number conservation

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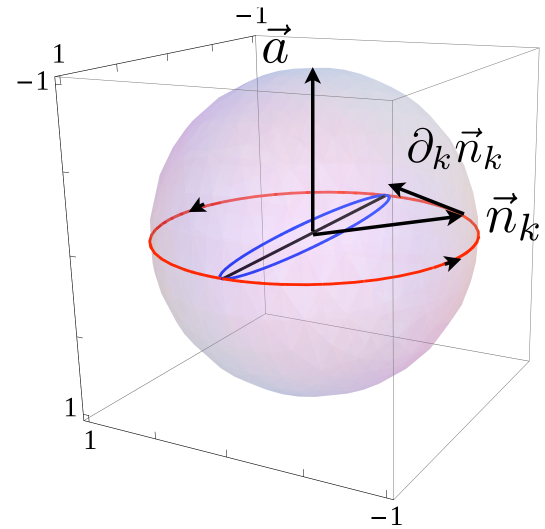

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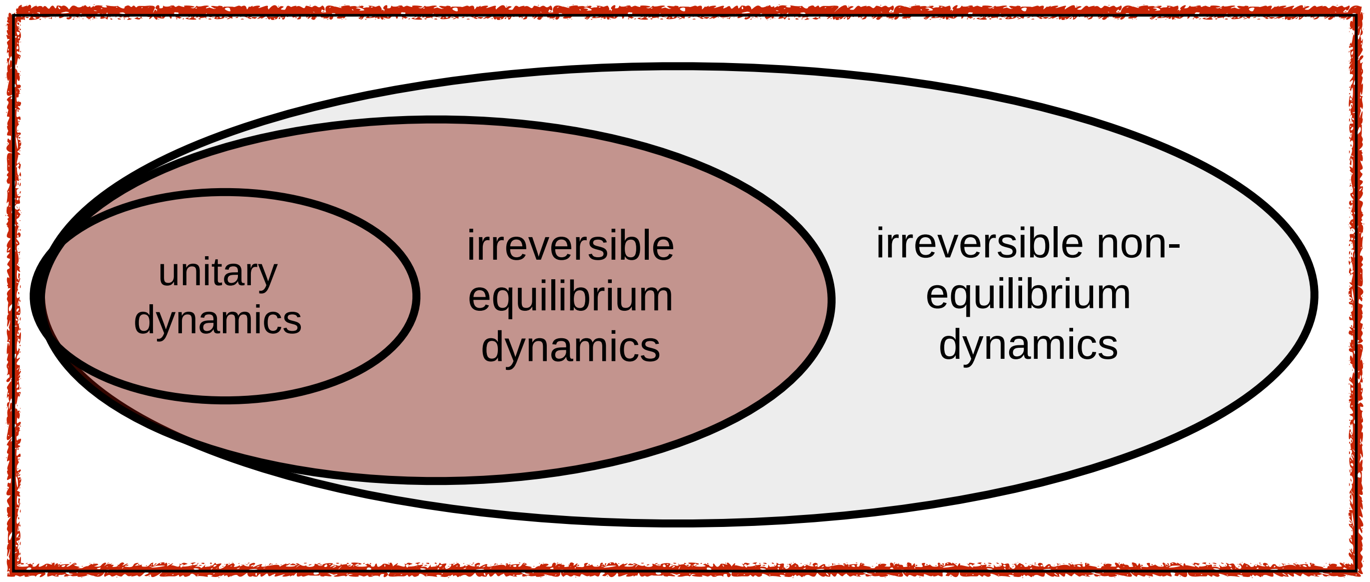
- linear response observables non-quantized
- non-linear response observables can be quantized

generalizes 'ensemble geometric phase'
Bardyn, Wawer, Altland, Fleischauer, SD PRX (2017)

Symmetry classes of open Fermionic quantum matter



A. Altland, M. Fleischhauer, SD, PRX (2021)

Motivation: Classification of open quantum systems

- physical setting: free systems (matrix representation)

closed system

- classified object: Hermitian matrices

$$H \qquad H^\dagger = H$$

- symmetry operations:
  - time reversal T
  - charge conjugation C
  - chiral symmetry S

- result: 10 classes

Altland, Zirnbauer PRB (1996)

- applications: classification of fermion topological states (topological insulators, superconductors)

Kitaev AIP (2009); Ryu, Furusaki, Schnyder, Ludwig NJP (2010)

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- physical setting: free systems (matrix representation)

closed system	open system
• classified object: Hermitian matrices	• classified object: non-Hermitian matrices
$H \qquad H^\dagger = H$	$K = H - iD \qquad H^\dagger = H,\ D^\dagger = D$
• symmetry operations:	• symmetry operations:
• time reversal T	• time reversal $\mathrm{T} \xrightarrow[\eta]{\text{Hermitian adjoint}} \mathrm{T}^\dagger$
• charge conjugation C	• charge conjugation $\mathrm{C} \xrightarrow[\eta]{\text{Hermitian adjoint}} \mathrm{C}^\dagger$
• chiral symmetry S	• chiral Symmetry $\mathrm{S} \xrightarrow[\eta]{\text{Hermitian adjoint}} \mathrm{S}^\dagger$
• result: 10 classes	• result: 38 classes
Altland, Zirnbauer PRB (1996)	Kawabata, Shiozaki, Ueda, Sato, PRX (2018); Zhou, Lee PRB (2018)

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  - time reversal $\quad \mathrm{T}$
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$$\begin{matrix} \mathrm{T} \\ \mathrm{C} \\ \mathrm{S} \end{matrix} \xrightarrow[\eta]{\text{Hermitian adjoint}} \begin{matrix} \mathrm{T}^\dagger \\ \mathrm{C}^\dagger \\ \mathrm{S}^\dagger \end{matrix}$$

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- open issues:
  - physical interpretation of non-Hermitian matrix?
  - particle statistics?
  - causality?
  - generalization to interacting systems?

➡ goal: first principles classification of general generators of fermion quantum dynamics

Classifying general fermionic quantum dynamics

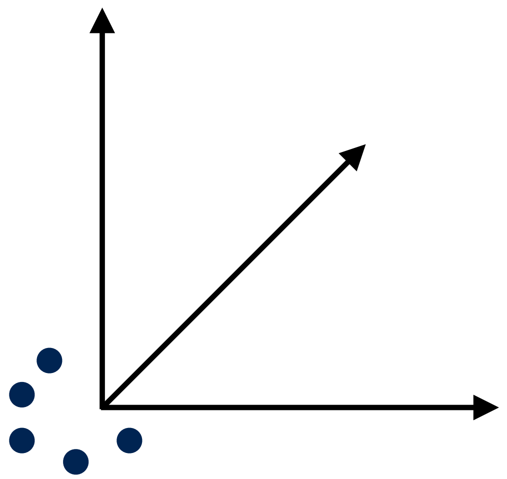
state $|\psi\rangle \in \mathcal{F}$

- starting point: symmetry operations in fermionic Fock space (state)

$$\{a_i, a_i^\dagger\}$$

- time reversal
- charge conjugation
- chiral symmetry

$$\begin{array}{lll}
\mathrm{T}: & \mathrm{T}a_i\mathrm{T}^{-1} \equiv u_{\mathrm{T}ij}\, a_j, & \mathrm{T}i\mathrm{T}^{-1} = -i, \\
\mathrm{C}: & \mathrm{C}a_i\mathrm{C}^{-1} \equiv u_{\mathrm{C}ij}\, a_j^\dagger, & \mathrm{C}i\mathrm{C}^{-1} = +i, \\
\mathrm{S}: & \mathrm{S}a_i\mathrm{S}^{-1} \equiv u_{\mathrm{S}ij}\, a_j^\dagger, & \mathrm{S}i\mathrm{S}^{-1} = -i.
\end{array}$$



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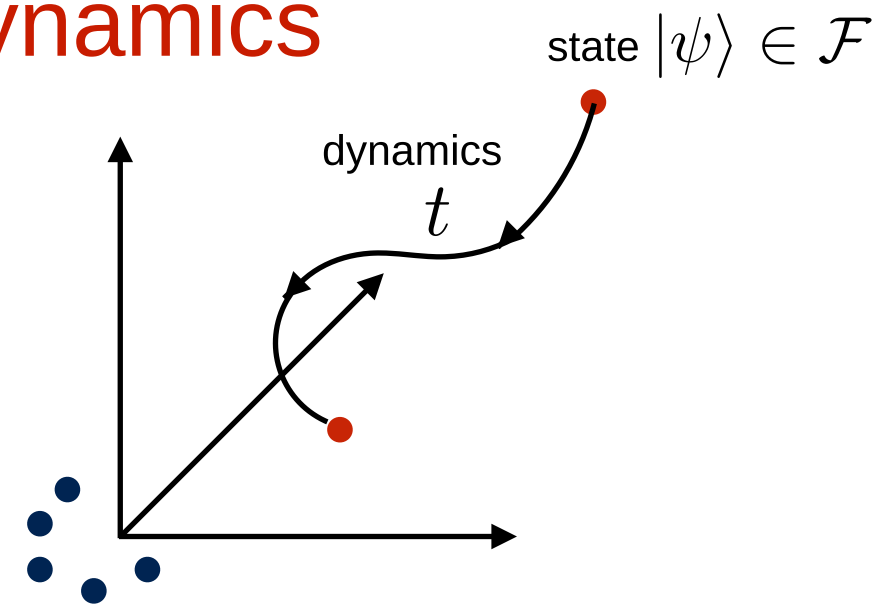

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\end{aligned}$$

- physical principle: invariance of the equation of motion (dynamics)

equilibrium unitary

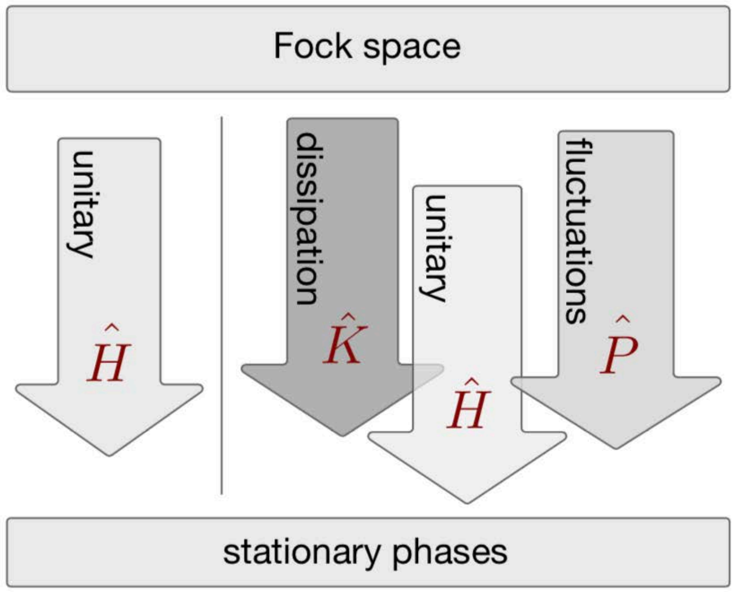
$$\partial_t \hat\rho = -i[\hat H, \hat\rho]$$

- reproduces ground state classification
- extends to irreversible eq. dyn.

non-equilibrium irreversible

$$\partial_t \hat\rho = -i[\hat H, \hat\rho] + \kappa \sum_\alpha (\hat L_\alpha \hat\rho \hat L_\alpha^\dagger - \tfrac{1}{2}\{\hat L_\alpha^\dagger \hat L_\alpha, \hat\rho\})$$

➡ object of classification: generator of dynamics (including interacting)



- classification of the state (incl. stationary) follows

Classifying general fermionic quantum dynamics

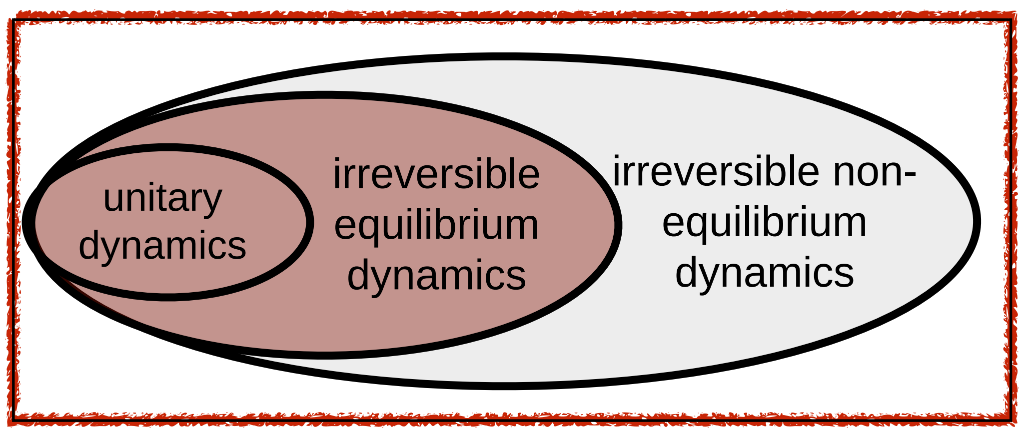
- results: A. Altland, M. Fleischhauer, SD, PRX (2021)

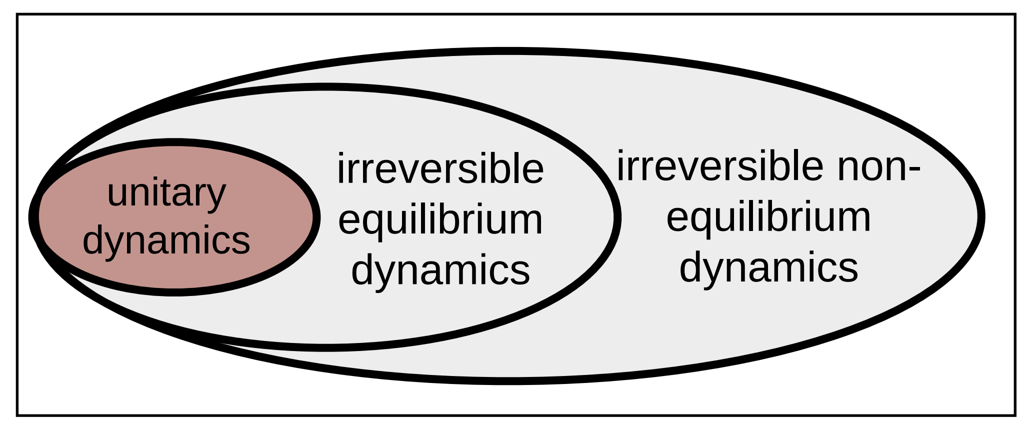
transformations		non-equilibrium			equilibrium	steady state
X	O_{X}	H	D	P	H	Θ/Γ
T	$U_{\mathrm{T}}\bar{O}U_T^\dagger$	$-$	$+$	$-$	$+$	$+$
C	$-U_{\mathrm{C}}^T O^T \bar{U}_{\mathrm{C}}$	$+$	$-$	$+$	$+$	$+$
S	$-U_{\mathrm{S}}^T O^\dagger \bar{U}_{\mathrm{S}}$	$-$	$-$	$-$	$+$	$+$

- states: 10 classes, irrespective to the dynamics stabilizing it
- dynamics: fundamental distinction of eq. and non-eq. dynamics

  ➡ 17 = 3 + (7 + 7) dynamical symmetry classes

  ➡ the ‘watershed’ is equilibrium vs. non-equilibrium, not reversible vs. irreversible





- implications (for out of equilibrium dynamics):
  - different representation / transformation rules of antiunitary symmetries: crucial for practical applications
  - non-Hermitian matrix classification not robust in presence of interactions

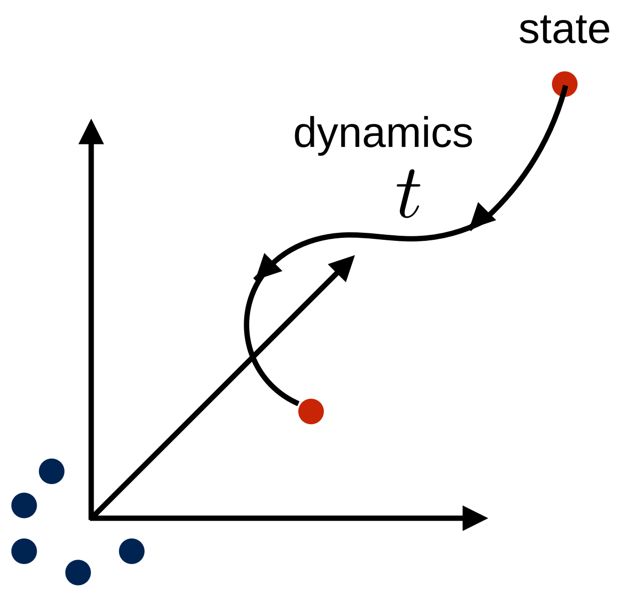
Origin of dynamical fine structure

- time reversal in quantum mechanics is a combined transformation
  - static: action in Fock space

$$\mathrm{T}: \qquad \mathrm{T} a_i \mathrm{T}^{-1} \equiv u_{\mathrm{T}ij}\, a_j, \qquad \mathrm{T} i \mathrm{T}^{-1} = -i,$$

  - dynamical: action in the time domain

$$t \to -t \qquad \text{in} \qquad i\partial_t |\psi\rangle = \hat{H}|\psi\rangle$$



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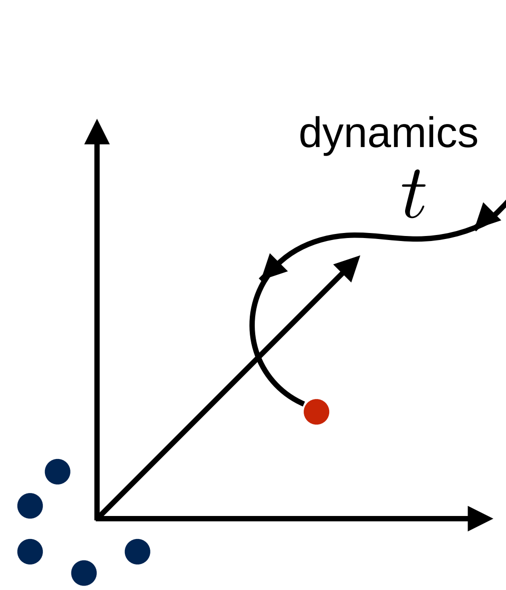
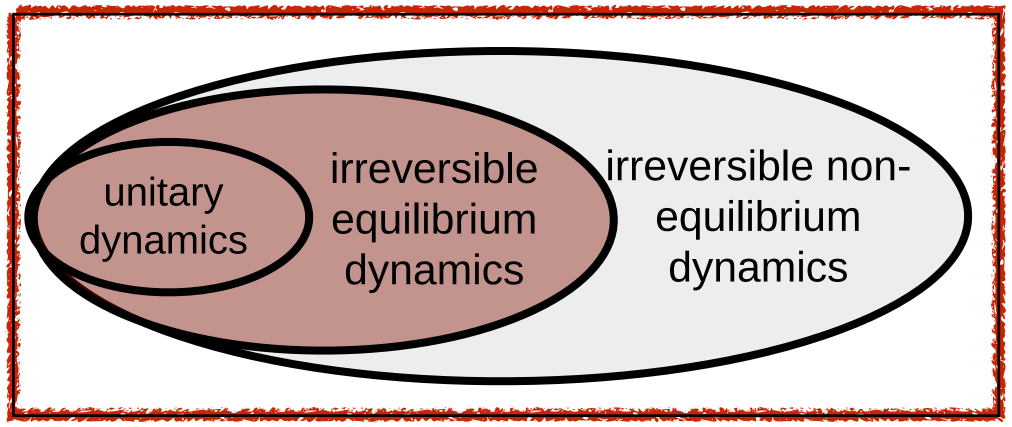
  - full quantum mechanical time reversal:

$$\mathrm{T} \circ \mathrm{E} \qquad \mathrm{E} f(t) = f(-t)$$

➡ ‘play movie backwards’
➡ does not make sense in general irreversible dynamics
➡ but can be extended to irreversible equilibrium dynamics





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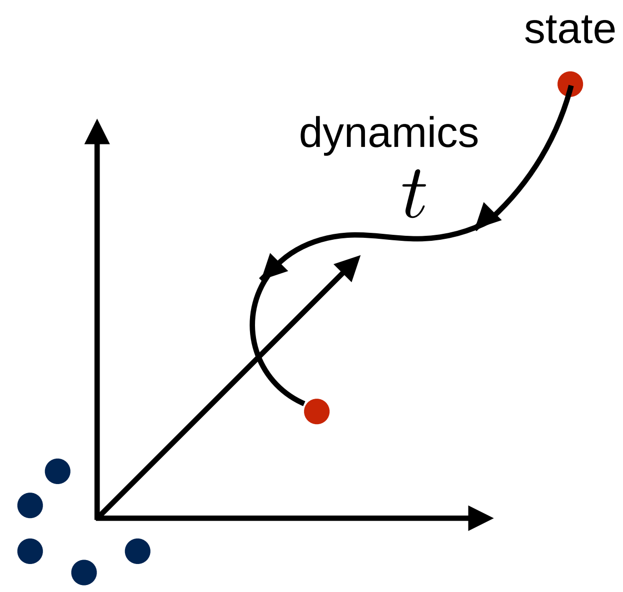

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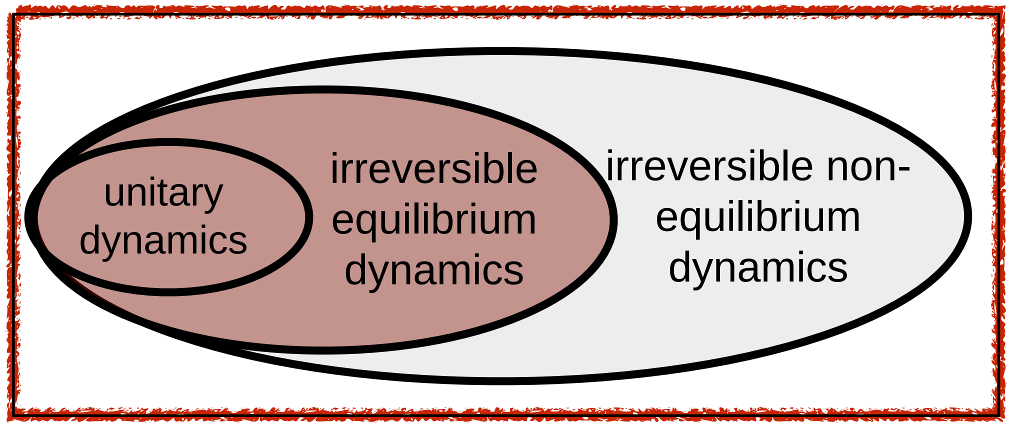

- thermal time reversal (Keldysh formulation): see lecture I

inverse temperature

$$\boxed{\mathrm{T} \circ \mathrm{E}_\beta \qquad \mathrm{E} f(t) = f(-t + i\beta)}$$

related to Kubo-Martin-Schwinger boundary conditions / FDT

➡ transformation laws of quantum mechanics smoothly extend to irreversible eq. dynamics
➡ out of equilibrium, $t \to +t$ and full time reversal gets represented differently (different trafo laws)

Implications for Gaussian dynamics

- classification encompasses interacting systems, but now make contact to matrix classifications
- Lindblad generator $$\partial_t \hat\rho = -i\,[\hat H, \hat\rho] + \sum_\alpha (2\hat L_\alpha \hat\rho \hat L_\alpha^\dagger - \{\hat L_\alpha^\dagger \hat L_\alpha, \hat\rho\})$$
- H quadratic, Lindblad operators linear: 3 structure matrices

$$\hat H = \tfrac{1}{2}\hat A^\dagger H \hat A \qquad \sum_\alpha \hat L_\alpha^\dagger \hat L_\alpha = \tfrac{1}{2}\hat A^\dagger (D - iP)\hat A \qquad \hat A_a = \begin{pmatrix} \hat a_a \\ \hat a_a^\dagger \end{pmatrix}$$

Nambu vector

acts in Nambu space

exercise: verify these relations!

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- with constraints:

acts in Nambu space

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  - Fermi statistics: $$H = -\sigma_x H^T \sigma_x, \quad D = \sigma_x D^T \sigma_x, \quad P = -\sigma_x P^T \sigma_x$$
  - Hermiticity of Lindblad map: $$H^\dagger = H, \quad D^\dagger = D, \quad P^\dagger = -P$$
  - complete positivity of Lindblad map: $$D \geq 0$$

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noise / fluctuation

dissipation

- Lindblad case $$D_\omega = D, \quad P_\omega = P$$ Markovian
- equilibrium case $$P_\omega = i\tanh\left(\frac{\beta\omega}{2}\right) D_\omega$$ non-Markovian, fixed by fluctuation-dissipation relation

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- steady state encoded in covariance matrix

$$\Gamma_{ab} = \mathrm{tr}(\hat\rho[\hat A_a, \hat A_b^\dagger]) = -2i\int \frac{d\omega}{2\pi}\frac{1}{\omega - K_\omega} P_\omega \frac{1}{\omega - K_\omega^\dagger}$$

equal time Keldysh Green’s function

Gaussian dynamics: two gaps

- classification requires robustness, encoded in action

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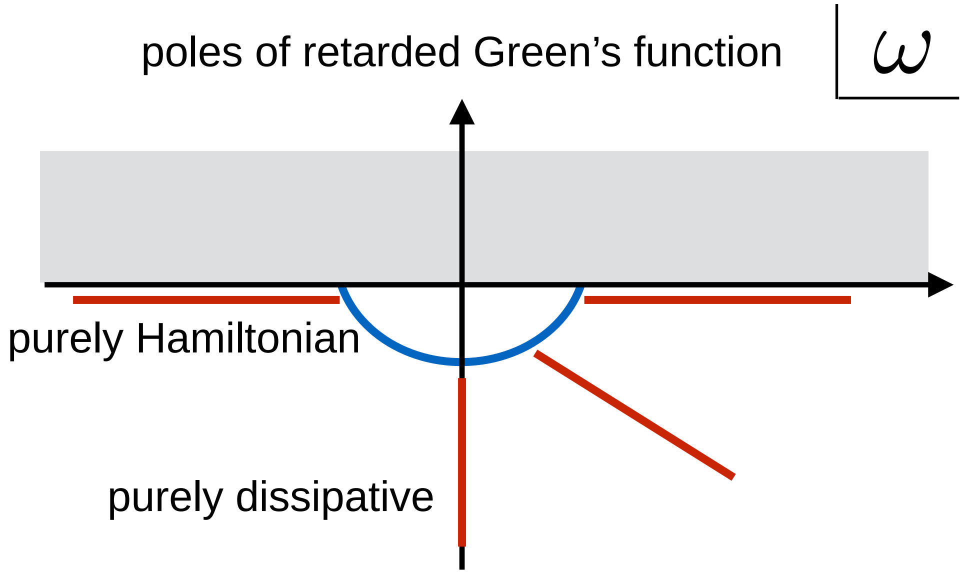
inv. ret. Green's function



- spectral gap: no zero modes of spectral matrix K_ω

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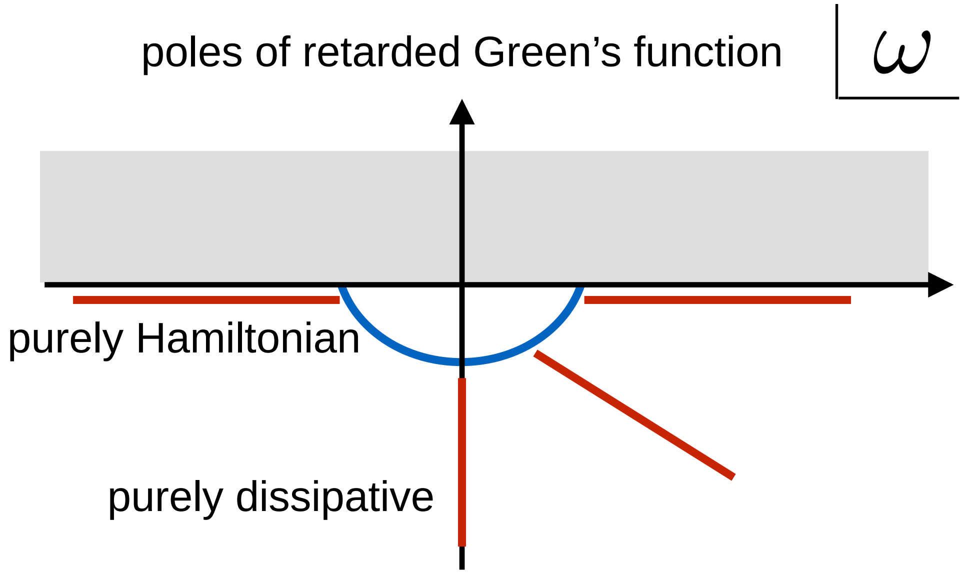
inv. ret. Green's function

poles of retarded Green's function



- spectral gap: no zero modes of spectral matrix K_ω

- purity gap: no zero modes of (Hermitian) covariance matrix

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$$= \mathrm{tr}(\hat\rho[\delta_{ab} - \hat A_b^\dagger \hat A_a]) = (\tanh\tfrac{\Theta}{2})_{ab}$$

for parameterization

$$\hat\rho = \frac{1}{\mathcal{N}}\, e^{-\frac{1}{2}\hat A^\dagger \Theta \hat A} \qquad \mathcal{N} = \mathrm{tr}(e^{-\frac{1}{2}\hat A^\dagger \Theta \hat A})$$

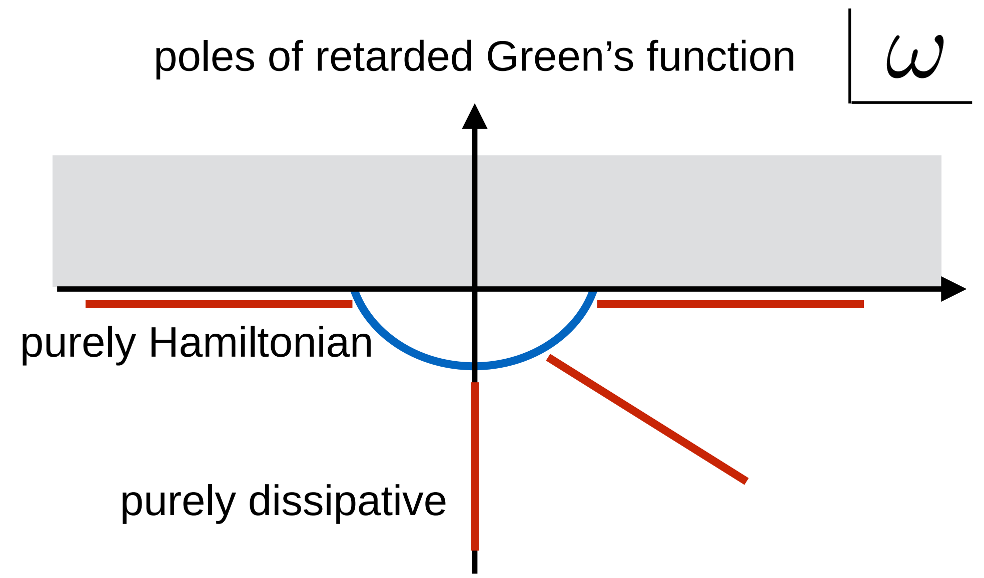
tanh structure seen in eigenbasis

Gaussian dynamics: two gaps

- classification requires robustness, encoded in action

$$S = \frac{1}{2}\int \frac{d\omega}{2\pi}(\psi_\omega^{c\,\dagger}, \psi_\omega^{q\,\dagger}) \begin{pmatrix} 0 & \omega - K_\omega^\dagger \\ \omega - K_\omega & 2P_\omega \end{pmatrix} \begin{pmatrix} \psi_\omega^c \\ \psi_\omega^q \end{pmatrix} \qquad K_\omega = H - iD_\omega$$

inv. ret. Green's function



- spectral gap: no zero modes of spectral matrix K_ω

- purity gap: no zero modes of (Hermitian) covariance matrix

$$\Gamma_{ab} = \mathrm{tr}(\hat\rho[\hat A_a, \hat A_b^\dagger]) = -2i\int \frac{d\omega}{2\pi}\left(\frac{1}{\omega - K_\omega} P_\omega \frac{1}{\omega - K_\omega^\dagger}\right)_{ab}$$

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tanh structure seen in eigenbasis

- example: equilibrium case $\Theta = \beta H$

  ➡ purity gap closing for zero modes of H (i.e. spectral gap closing

  ➡ or infinite temperature $\beta = 0$

- criterion in the presence of a spectral gap: no zero modes of fluctuation matrix P_ω

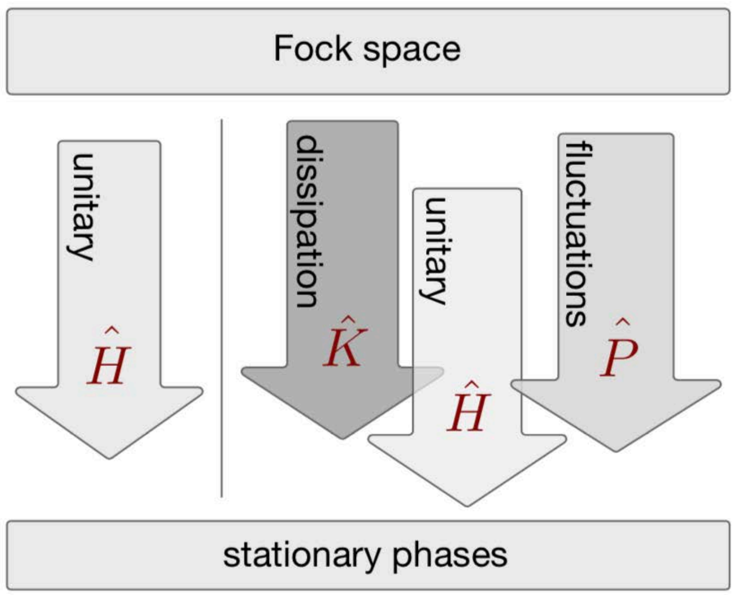
  ➡ purity gap = ‘fluctuation gap’

Different representation of anti-unitary symmetries

- table of transformation rules for Gaussian dynamics characterized by generators H, P, D

transformations		non-equilibrium			equilibrium	steady state
X	O_X	H	D	P	H	Θ/Γ
T	$U_T \bar{O} U_T^\dagger$	$-$	$+$	$-$	$+$	$+$
C	$-U_C^T O^T \bar{U}_C$	$+$	$-$	$+$	$+$	$+$
S	$-U_S^T O^\dagger \bar{U}_S$	$-$	$-$	$-$	$+$	$+$

Class	T	C	S	1	2	3	4
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	$+$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	$+$	0	0	0	0	0	$\mathbb{Z}$
BDI	$+$	$+$	$+$	$\mathbb{Z}$	0	0	0
D	0	$+$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
DIII	$-$	$+$	$+$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
AII	$-$	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CII	$-$	$-$	$+$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C	0	$-$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
CI	$+$	$-$	$+$	0	0	$\mathbb{Z}$	0



➡ anti-unitary transformation with i —> - i (T,S) rules different from Altland-Zirnbauer classification for dynamics out of equilibrium: 17 = 3 + 7 + 7 dynamical classes

- ➡ in 3 classes, no anti-unitary symmetry present
- ➡ in 7 classes (6 with T, 1 with S), anti-unitary transformations laws of equilibrium $t \to -t + i\beta$ present
- ➡ in 7 classes (6 with T, 1 with S), anti-unitary transformations laws of non-equilibrium $t \to +t$ present

➡ both equilibrium and non-equilibrium collapse onto Altland-Zirnbauer classification in stationary state

➡ study concrete example

Different representation of anti-unitary symmetries (qualit.)

transformations		non-equilibrium			equilibrium	steady state
X	O_X	H	D	P	H	Θ/Γ
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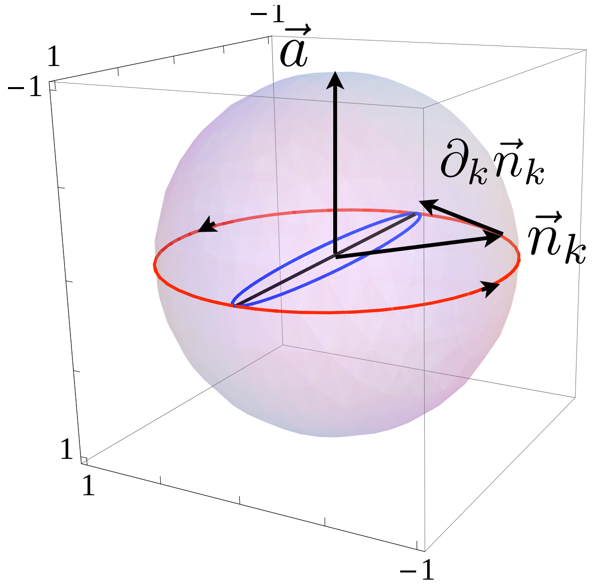
S	$-U_S^T O^\dagger \bar{U}_S$	$-$	$-$	$-$	$+$	$+$

- example: chiral 2-band insulator in 1D (SSH)
- covariance matrix for stationary steady state (trans. inv.)

$$\Gamma_k = \vec{n}_k \cdot \vec{\Sigma} \qquad |\vec{n}_k| \leq 1$$

- impose chiral symmetry with $U_S = \Sigma_z : \quad U_S \Gamma U_S^\dagger = -\Gamma$

➡ characterized by winding number $w = \oint \frac{dk}{2\pi} \vec{a} \cdot (\vec{n}_k \times \partial_k \vec{n}_k)$



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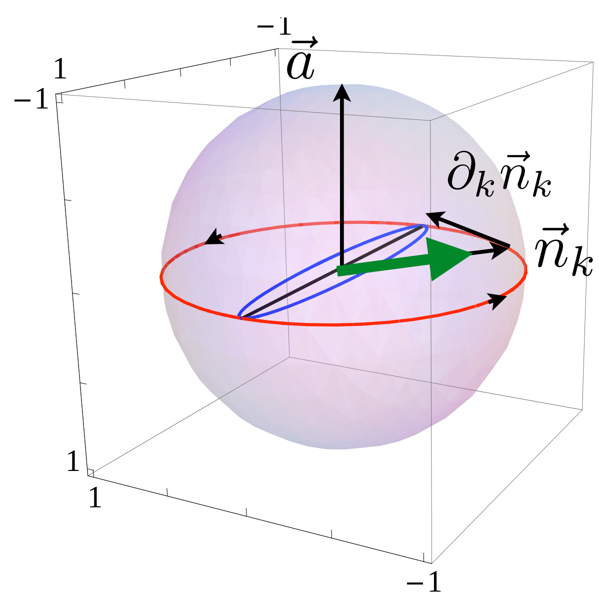

equilibrium

$$\vec{n}_k = \frac{\vec{h}_k}{|\vec{h}_k|} \tanh \frac{\beta |\vec{h}_k|}{2}$$

$$U_{\mathrm{S}} H U_{\mathrm{S}}^\dagger = -H$$ <-> in x-y-plane

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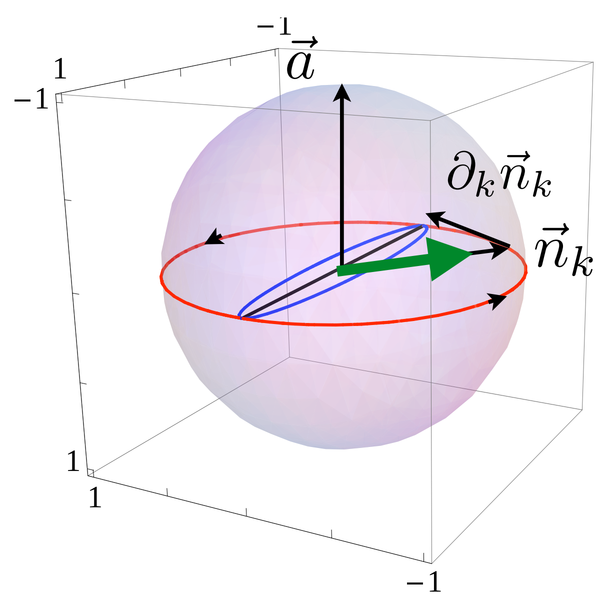

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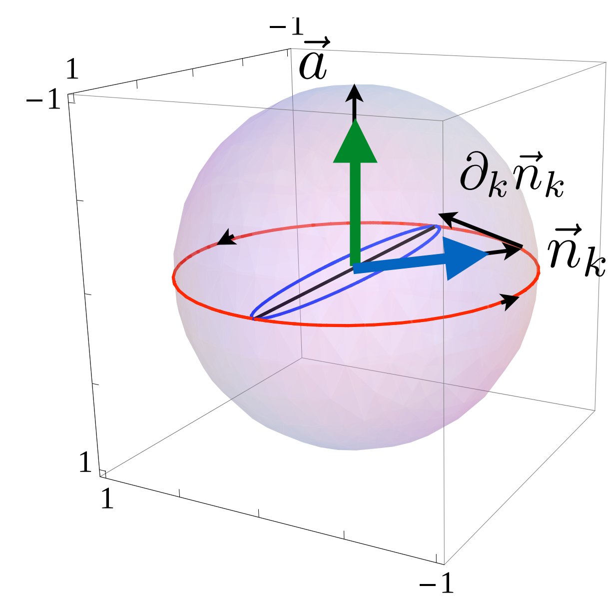


non-equilibrium

$$\vec{n}_k = \vec{\gamma}_k - \frac{1}{\kappa^2 + |\vec{h}_k|^2} \left(\kappa + \vec{h}_k \times\right) (\vec{\gamma}_k \times \vec{h}_k)$$

next pg. for details

$$U_{\mathrm{S}} H U_{\mathrm{S}}^\dagger = +H$$ <-> in z-direction

➡ to stabilize a chiral state, H must not be chiral in the usual sense out of equilibrium

Different representation of antiunitary symmetries (formulas)

transformations		non-equilibrium			equilibrium	steady state
X	O_X	H	D	P	H	Θ/Γ
T	$U_T \bar{O} U_T^\dagger$	$-$	$+$	$-$	$+$	$+$
C	$-U_C^T O^T \bar{U}_C$	$+$	$-$	$+$	$+$	$+$
S	$-U_S^T O^\dagger \bar{U}_S$	$-$	$-$	$-$	$+$	$+$

- example: chiral 2-band dissipative insulator in 1D
- covariance matrix for stationary steady state (trans. inv.)

$$\Gamma_k = \vec{n}_k \cdot \vec{\Sigma} \qquad |\vec{n}_k| \leq 1$$

- Lindbladian: cools into SSH model ground state (rate κ)

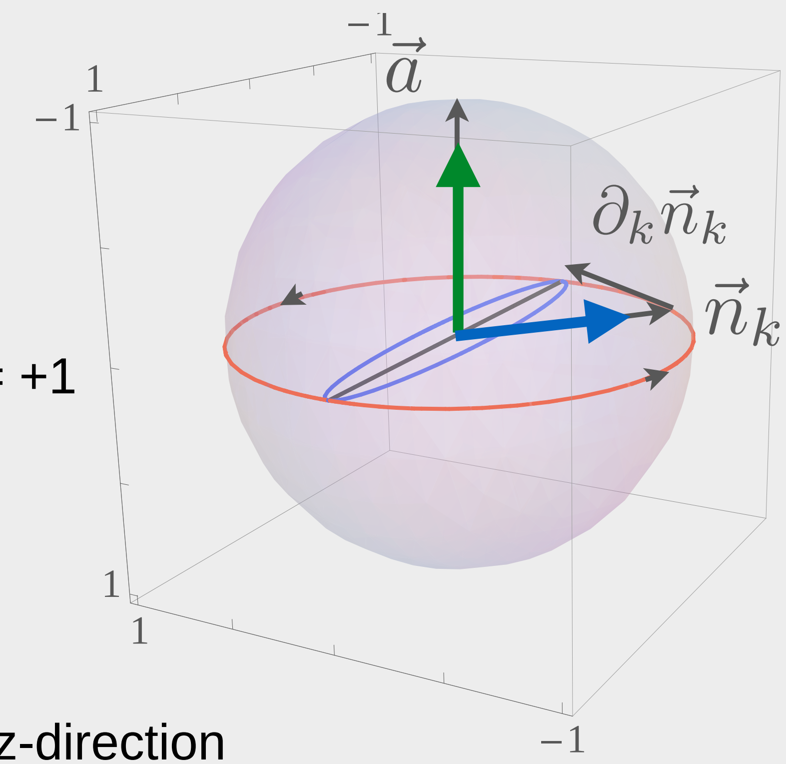
$$\hat{L}_{1,i} = \tfrac{1}{\sqrt{2}}(a_{1,i+1} + a_{2,i}), \quad \hat{L}_{2,i} = \tfrac{1}{\sqrt{2}}(-a_{1,i+1} + a_{2,i})$$

- symmetry class BDI with $U_\mathrm{T} = 1, U_\mathrm{S} = U_\mathrm{C} = \Sigma_z$
- $\vec{n}_k = \vec{\gamma}_k = (\cos k, \sin k, 0)^T$

winding number

$$w = \oint \frac{dk}{2\pi} \vec{a} \cdot (\vec{n}_k \times \partial_k \vec{n}_k)$$

w = +1



- add Hamiltonian $H_k = \vec{h}_k \cdot \vec{\Sigma}$

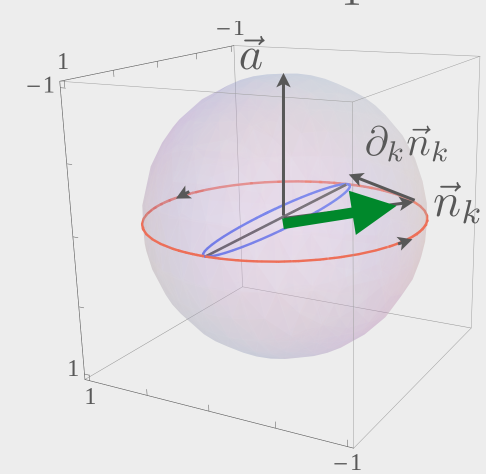
$$\vec{n}_k = \vec{\gamma}_k - \frac{1}{\kappa^2 + |\vec{h}_k|^2} \left(\kappa + \vec{h}_k \times\right) (\vec{\gamma}_k \times \vec{h}_k)$$

➡ chirally symmetric stationary state for $U_\mathrm{S} H U_\mathrm{S}^\dagger = +H$ in z-direction

- compare to equilibrium

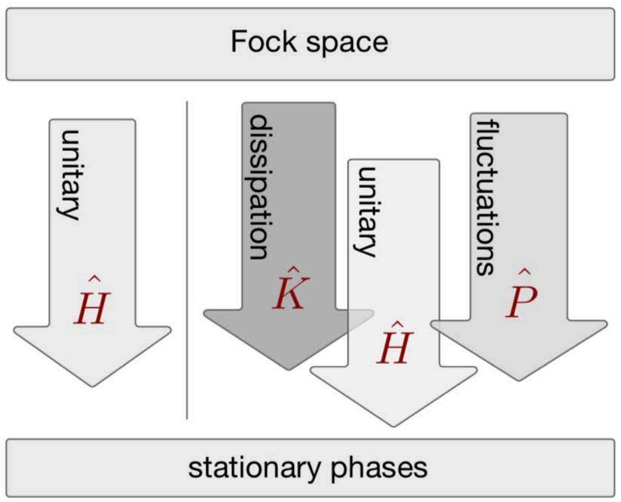
$$\vec{n}_k = \frac{\vec{h}_k}{|\vec{h}_k|} \tanh \frac{\beta |\vec{h}_k|}{2}$$

➡ chirally symmetric stationary state for $U_\mathrm{S} H U_\mathrm{S}^\dagger = -H$ in x-y-plane



All 3 generators needed for robust classification out of eq.

- the most fundamental description of quantum system is in terms of a system-bath Hamiltonian.
- Q: does a robust classification of symmetries exist in terms of that Hamiltonian?
- A: not out of equilibrium, since this ignores the state of the bath



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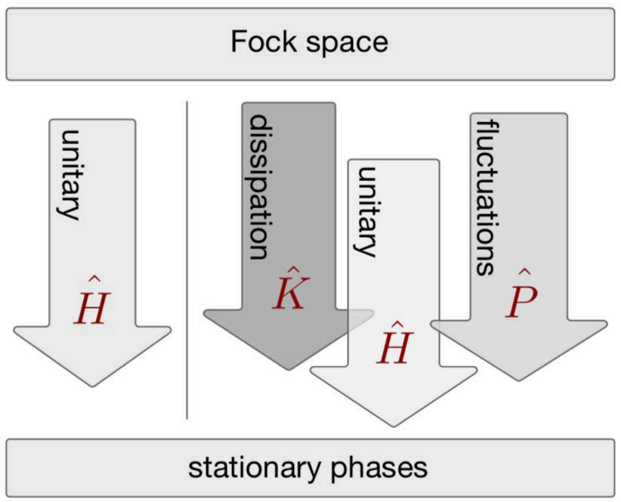
- equilibrium

reversible dissipation

$$K_\omega = H - iD_\omega$$

$$P_\omega = i \tanh\left(\frac{\beta\omega}{2}\right) D_\omega$$

fluctuation



- D, P from self-thermalization or equilibrium bath
- transformation rules of fixed by system-bath H
- D, P not independent

- non-Hermitian classification can work
- but only 10 classes compatible with causality

Lieu, McGinley, Cooper PRL (2020)

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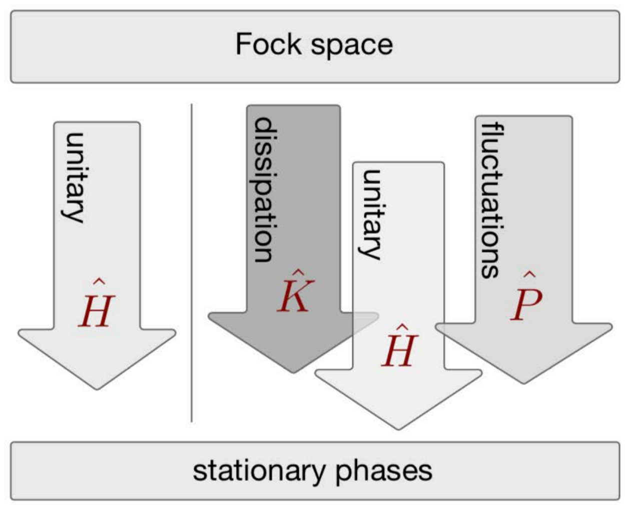
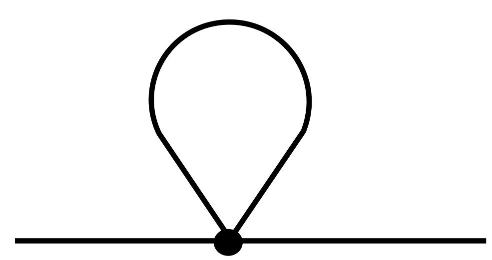


$$\sim (\omega - K)^{-1} P (\omega - K^\dagger)^{-1}$$



concrete ex: Altland, Fleischhauer, SD PRX (2021)

- (Markovian) non-equilibrium

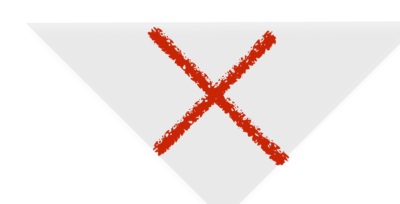
$$K = H - iD, \quad P$$

- D, P from non-equilibrium baths (drive, different temperatures, chem. pot. etc.)
- D, P independent
- single non-Hermitian matrix classification not perturbatively stable

$$\Delta K = \Delta K[H, D, P]$$

- non-Hermitian classification can work
- but only 10 classes compatible with causality

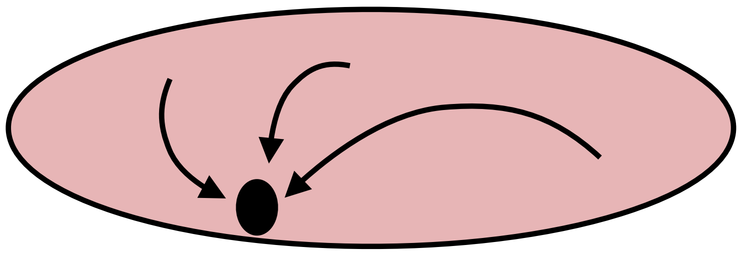
Lieu, McGinley, Cooper PRL (2020)



- non-Hermitian matrix classification not robust
- different ruleset

Summary lecture II

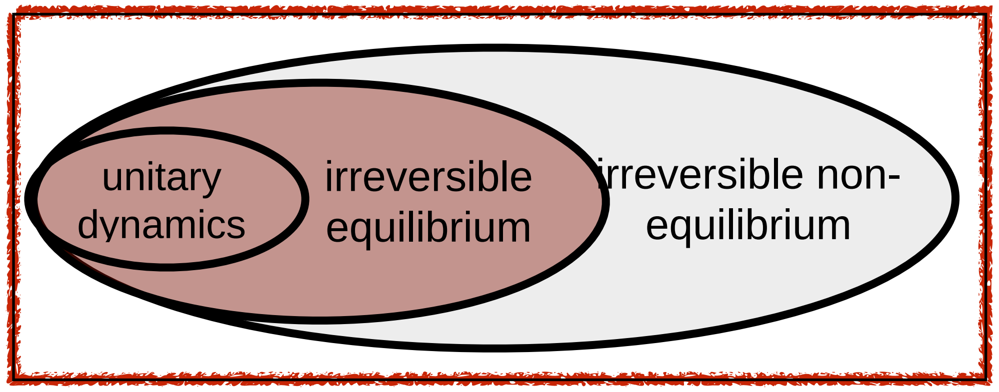
- states
  - ➡ topologically ordered states can be induced by Lindblad dynamics



- dynamical universality: topology beats dynamics
  - ➡ ‘basin of attraction’ for topological field theory encompasses equilibrium and non-equilibrium dynamics
  - ➡ topologically protected emergent reversible dynamics on top of irreversible bulk

$$\frac{\theta}{4\pi}\int A_+\,dA_+ - A_-\,dA_-$$

- dynamical fine structure: symmetry classes
  - ➡ states: 10 classes, irrespective to the dynamics stabilizing it
  - ➡ dynamics: fundamental distinction of eq. and non-eq. dynamics
    - ➡ 17 = 3 + 7 + 7 dynamical symmetry classes
    - ➡ The ‘watershed’ is equilibrium vs. non-equilibrium, not reversible vs. irreversible



Boulder Summer School 2021
July 5-30, 2021
Boulder, USA (via zoom)

Institute for Theoretical Physics University of Cologne

Lecture III

Lindblad-Keldysh 2.0: measurement induced phase transitions

- Background: weak continuous measurements
- Measurement induced phase transitions of fermions
- Replica Keldysh field theory approach

Sebastian Diehl

Institute for Theoretical Physics, University of Cologne

€ € € :

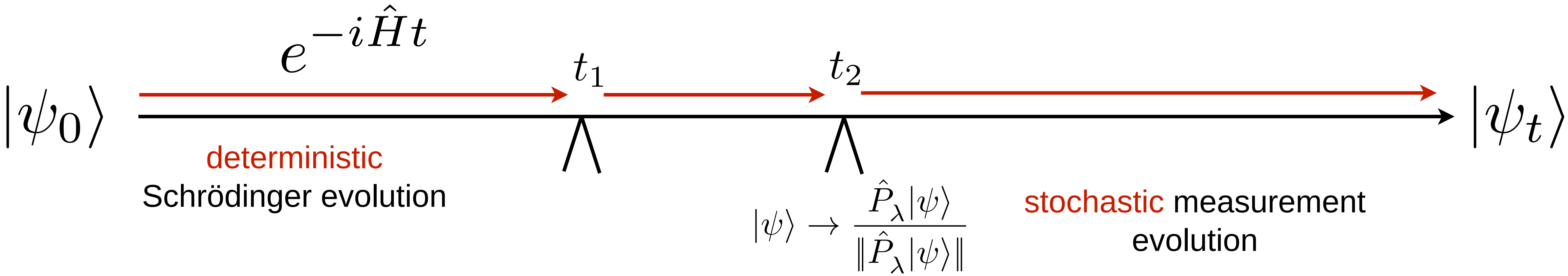
DFG Deutsche Forschungsgemeinschaft

erc European Research Council
Established by the European Commission

Introduction

Small quantum systems: measurements

- two types of quantum dynamics



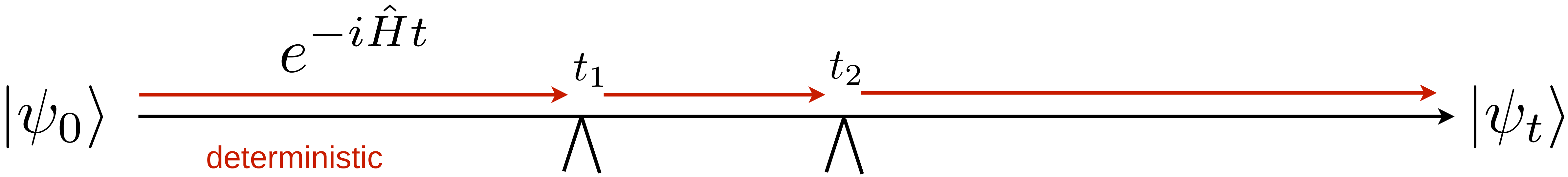
for measurement observable $\hat{M} = \sum_\lambda m_\lambda |\lambda\rangle\langle\lambda| \equiv \sum_\lambda m_\lambda \hat{P}_\lambda$

- dynamics non-trivial (eigenstates not shared) once $[\hat{H}, \hat{M}] \neq 0$

Introduction

Small quantum systems: measurements

- two types of quantum dynamics



deterministic
Schrödinger evolution

$$|\psi\rangle \to \frac{\hat{P}_\lambda|\psi\rangle}{\|\hat{P}_\lambda|\psi\rangle\|}$$

stochastic measurement
evolution

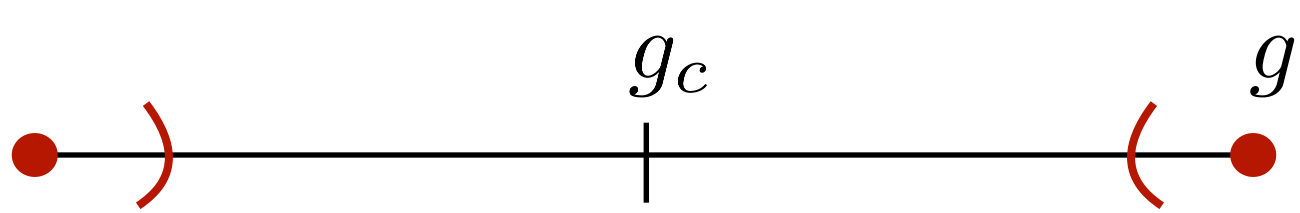
for measurement observable $\hat{M} = \sum_\lambda m_\lambda |\lambda\rangle\langle\lambda| \equiv \sum_\lambda m_\lambda \hat{P}_\lambda$

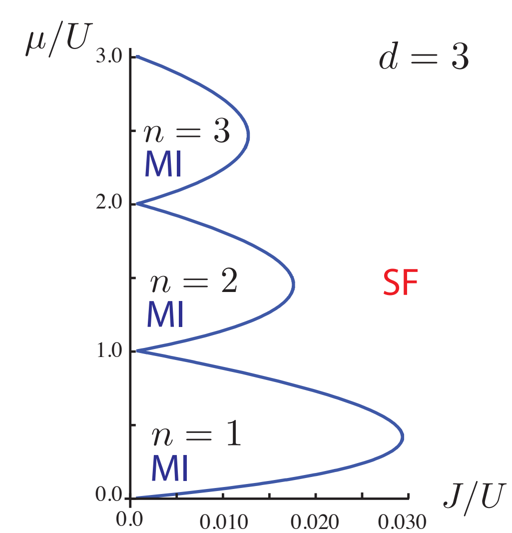
- dynamics non-trivial (eigenstates not shared) once $[\hat{H}, \hat{M}] \neq 0$

Many-body systems: Phase transitions

- non-commuting operators lead to (quantum) phase transitions

$$\hat{H} = \hat{H}_1 + g\hat{H}_2 \qquad [\hat{H}_1, \hat{H}_2] \neq 0$$





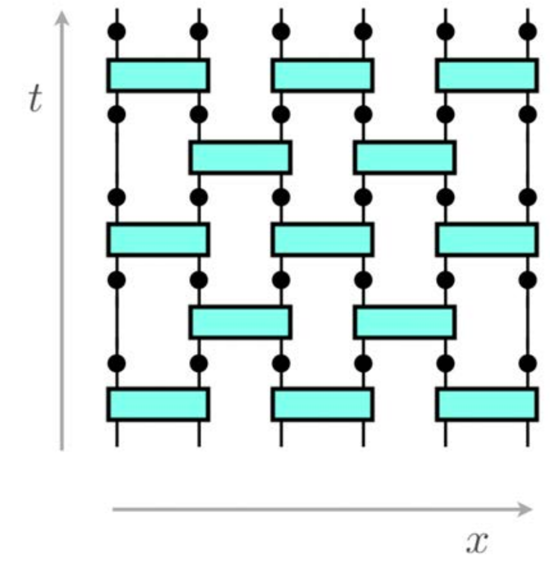
e.g. Mott-insulator to superfluid
transition in cold atoms

➡ combine measurement and many particles: similar scenario?

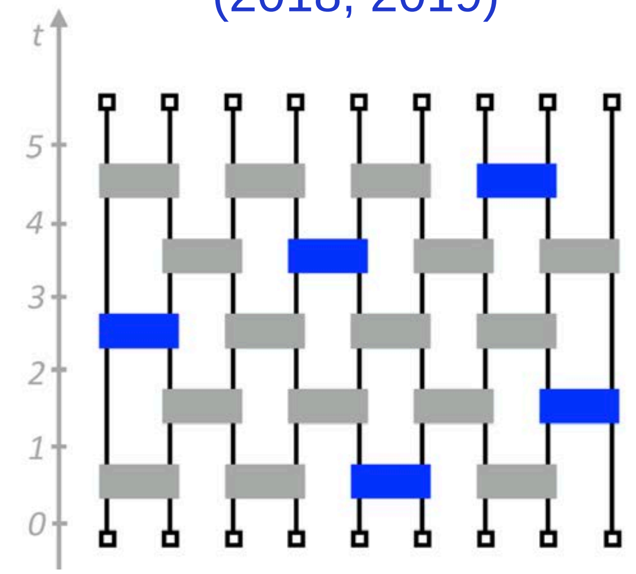
Entanglement phase transitions in random circuits

- model and key ingredients:
  - randomly chosen local entangling unitary gates
  - projective local measurement of non-commuting observables

Skinner, Ruhman, Nahum PRX (2019)



Li, Chen, Fisher, PRB (2018, 2019)



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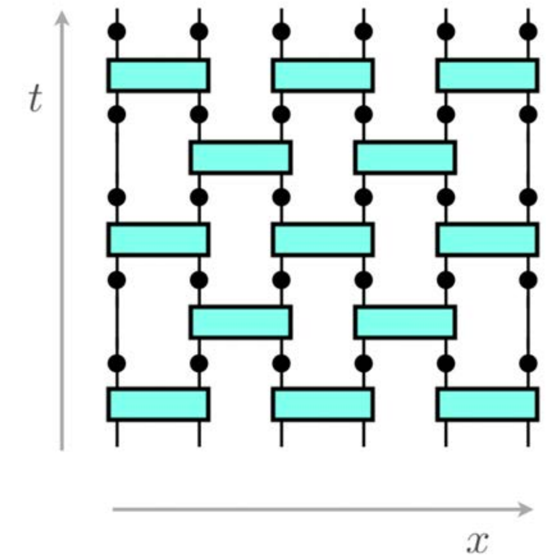

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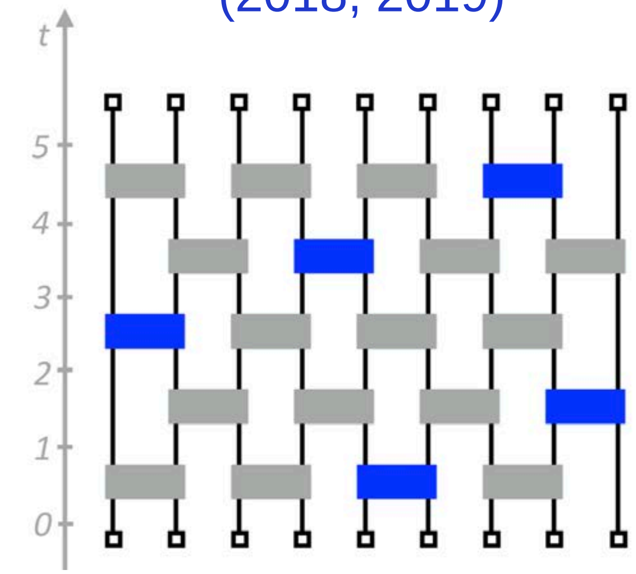

- basic picture: competition in many-body context (single trajectory, exactly local meas. of σ_i^z)

$g = 0$ $\qquad\qquad$ $g^{-1} = 0$ $\qquad$ $g = \dfrac{\text{\# measurements/time}}{\text{\# unitaries/time}}$

chaotic / non-integrable dynamics

➡ entanglement growth

convergence to product state $|\{\sigma_i\}\rangle = \prod_i |\sigma_i\rangle$ $\quad \sigma_i = \uparrow, \downarrow$

➡ entanglement saturation

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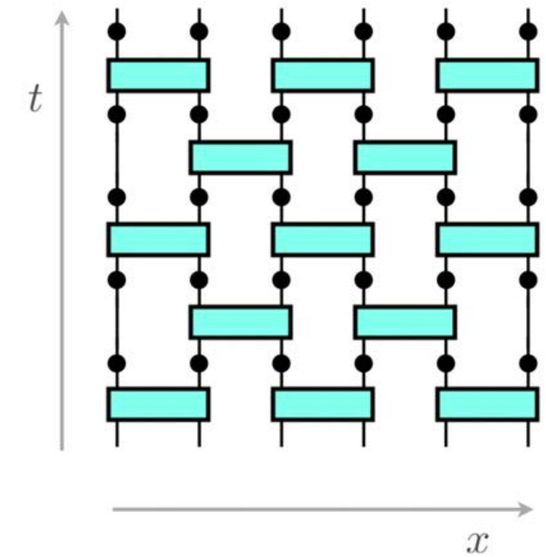

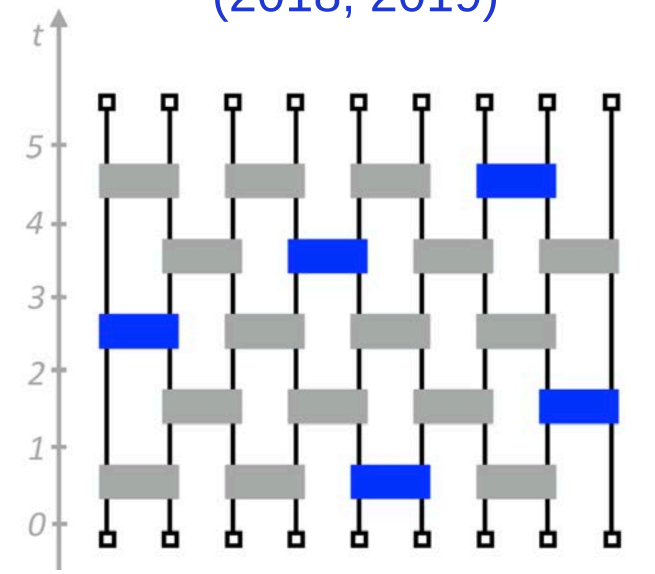

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convergence to product state $|\{\sigma_i\}\rangle = \prod_i |\sigma_i\rangle$ $\quad \sigma_i = \uparrow, \downarrow$

➡ entanglement saturation

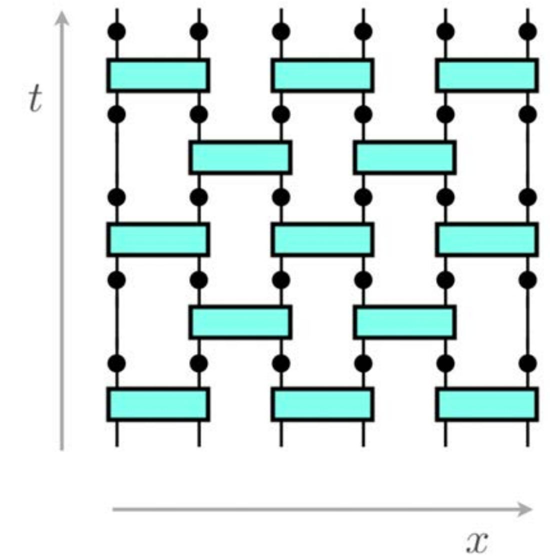
- Procedure
  - track single quantum trajectories (pure states)
  - compute the quantity of interest (e.g. entanglement entropy)
  - average over trajectory ensemble

non-commuting

$\overline{|\{\sigma_i\}\rangle\langle|\{\sigma_i\}|} \sim \mathbf{1}$

Entanglement phase transitions in random circuits

Skinner, Ruhman, Nahum PRX (2019)



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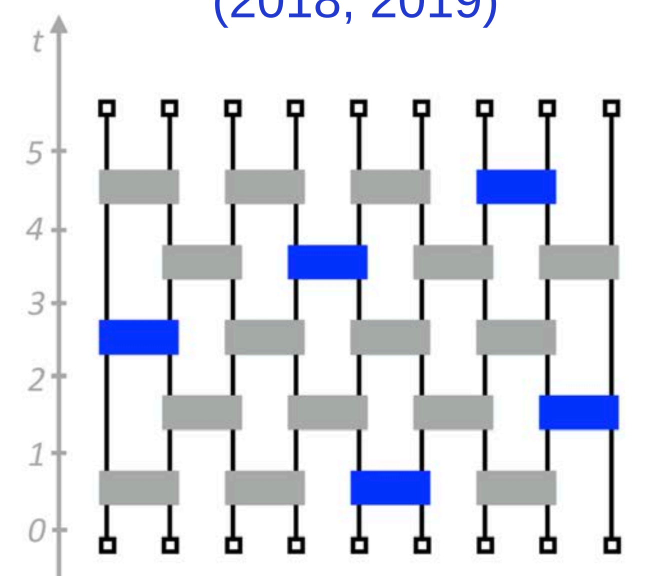


- model and key ingredients:
  - randomly chosen local entangling unitary gates
  - projective local measurement of non-commuting observables

- basic picture: competition in many-body context (single trajectory, exactly local meas. of σ_i^z)

$g = 0$ ——— g_c ——— $g^{-1} = 0$

$S \sim L$ $\qquad$ $S \sim \text{const.}$

$$g = \frac{\text{\# measurements/time}}{\text{\# unitaries/time}}$$

chaotic / non-integrable dynamics

➡ volume law

convergence to product state $|\{\sigma_i\}\rangle = \prod_i |\sigma_i\rangle$ $\quad \sigma_i = \uparrow, \downarrow$

➡ area law

- Procedure
  - track single quantum trajectories (pure states)
  - compute the quantity of interest (e.g. entanglement entropy)
  - average over trajectory ensemble

non-commuting

$$\overline{|\{\sigma_i\}\rangle\langle|\{\sigma_i\}|} \sim \mathbf{1}$$

➡ Phase transition in entanglement growth at finite competition ratio g

Entanglement phase transitions: Physical pictures

$g = 0$ $\qquad$ g_c $\qquad$ $g^{-1} = 0$

$$g = \frac{\text{\# measurements/time}}{\text{\# unitaries/time}}$$

chaotic / non-integrable dynamics

convergence to product state

$\boxed{\text{non-commuting unitary and measurement dynamics}}$

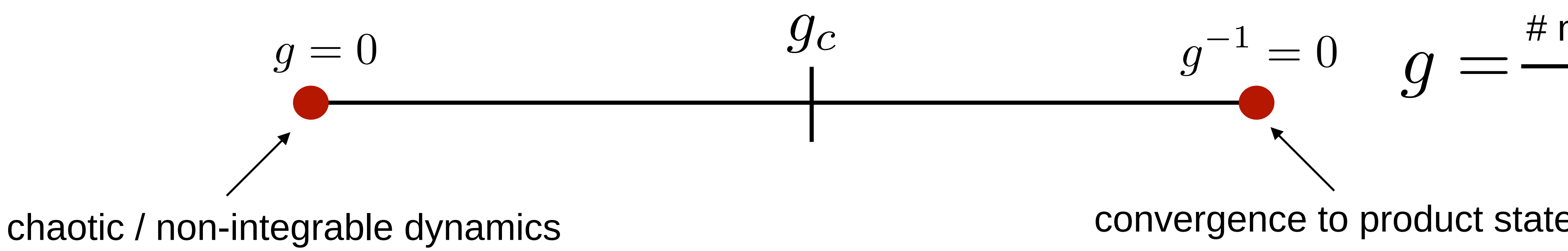
- entanglement picture

Skinner, Ruhman, Nahum PRX (2019) Li, Chen, Fisher, PRB (2018, 2019)

scrambling -> extensive
entanglement entropy

disentangling evolution

Entanglement phase transitions: Physical pictures



$$g = \frac{\text{\# measurements/time}}{\text{\# unitaries/time}}$$

non-commuting unitary and measurement dynamics

- entanglement picture — Skinner, Ruhman, Nahum PRX (2019) Li, Chen, Fisher, PRB (2018, 2019)

  scrambling -> extensive entanglement entropy

  disentangling evolution

- quantum error correction picture — Choi, Bao, Qi, Altman, PRL (2020); Fan et al. arxiv (2020); Li Fisher PRB (2021)

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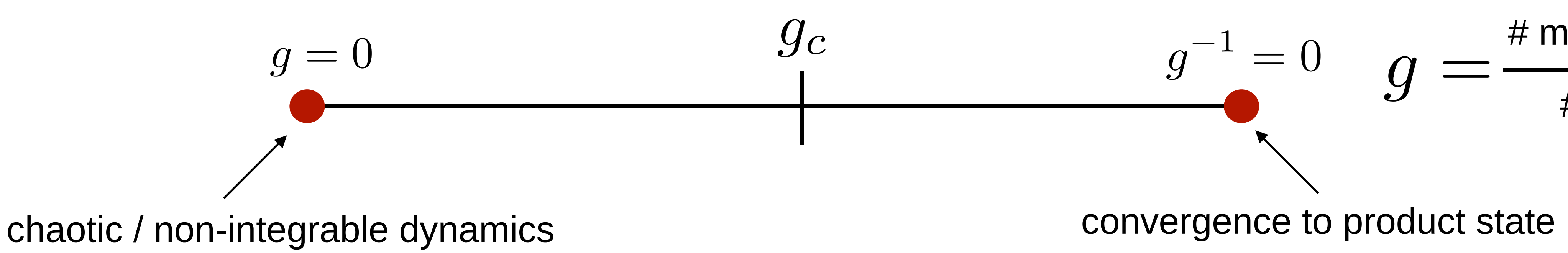

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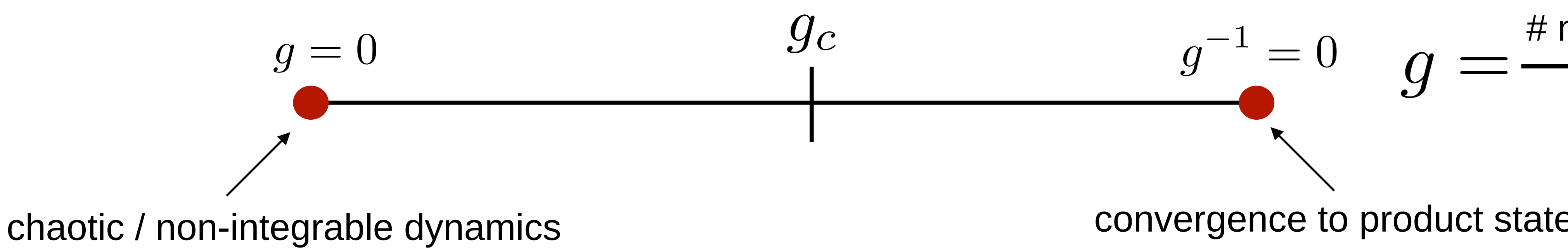

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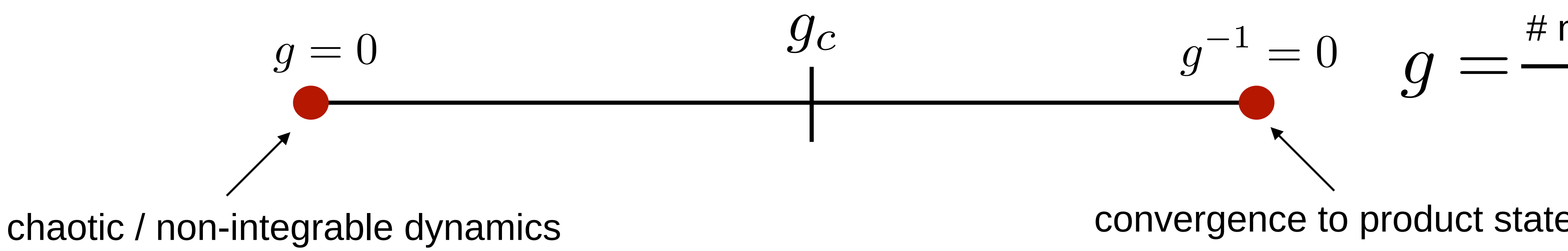
- statistical mechanics picture — e.g. mapping to spin model partition function: Jian, You, Vasseur Ludwig, PRB (2020)
  field theory based on replica symmetry: Nahum, Roy, Skinner, Ruhman, PRX Quantum (2021)

  long ranged correlation functions? which ones?

  short ranged correlation functions? which ones?

➡ Here: non-equilibrium statistical mechanics approach to a monitored fermion chain

Outline lecture III



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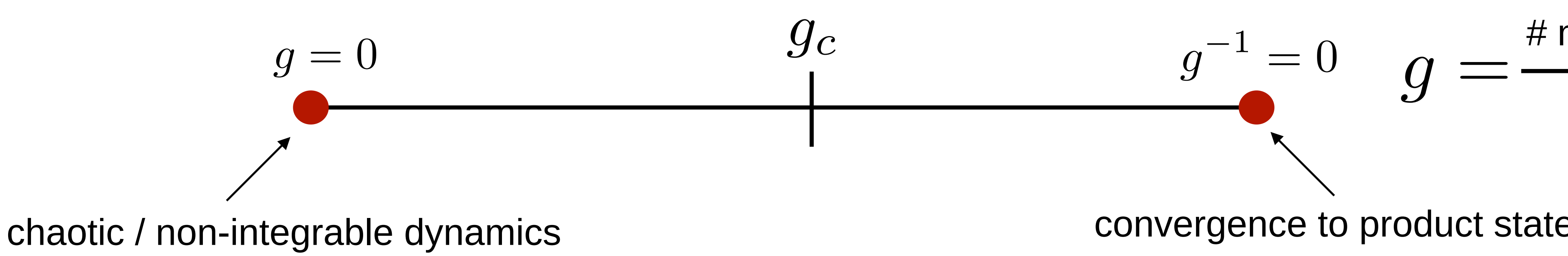

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- measurements: description?

- strong projective vs. weak continuous measurements
- ‘observables’

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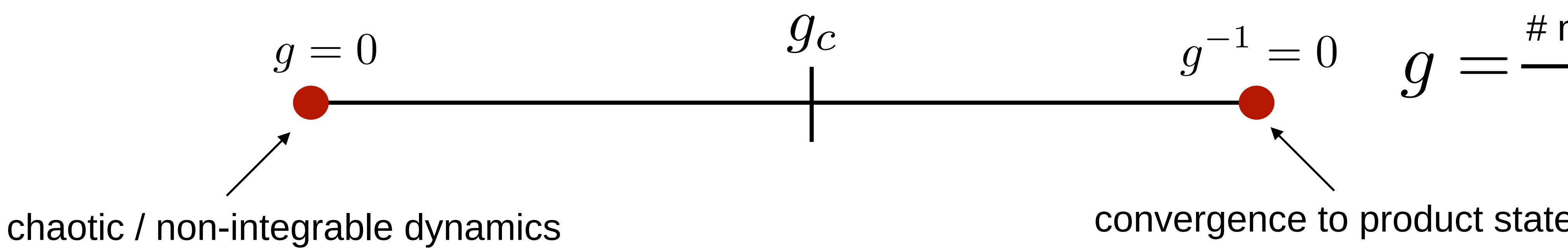

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  - continuously monitored lattice fermions
  - BKT type phase transition from critical to area law phase
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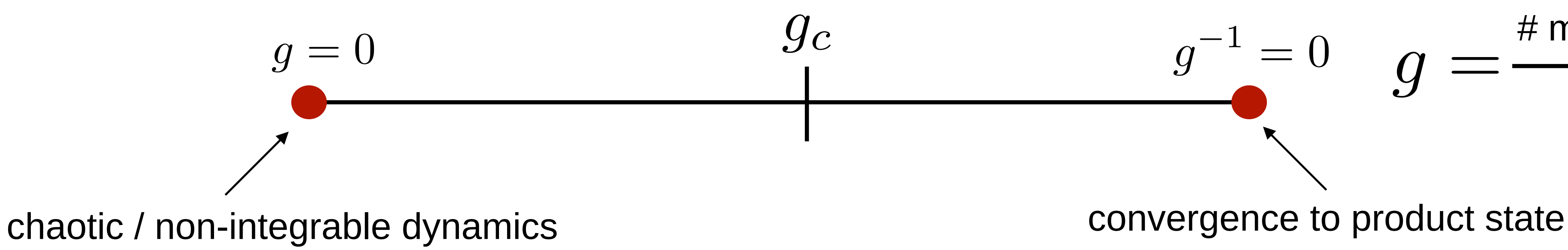

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  - strong projective vs. weak continuous measurements
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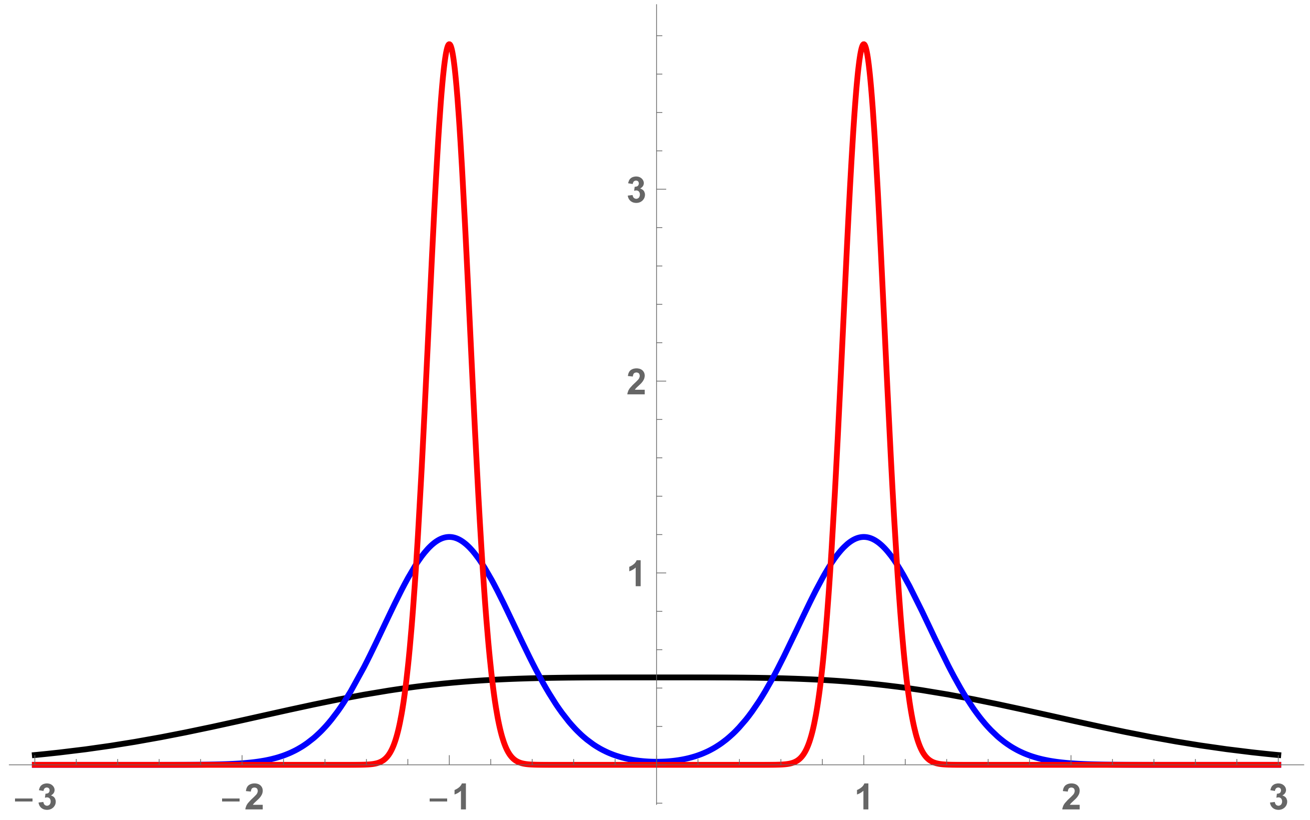
- many-body problem: phase transitions?
  - continuously monitored lattice fermions
  - BKT type phase transition from critical to area law phase
  - signatures beyond entanglement entropy
- how to understand?
  - Lindblad-Keldysh 2.0:
  - replicated Lindblad equation
  - replicated Keldysh field theory

➡ Quantum phase transition in trajectory wavefunction, revealed in non-linear-in-state observables

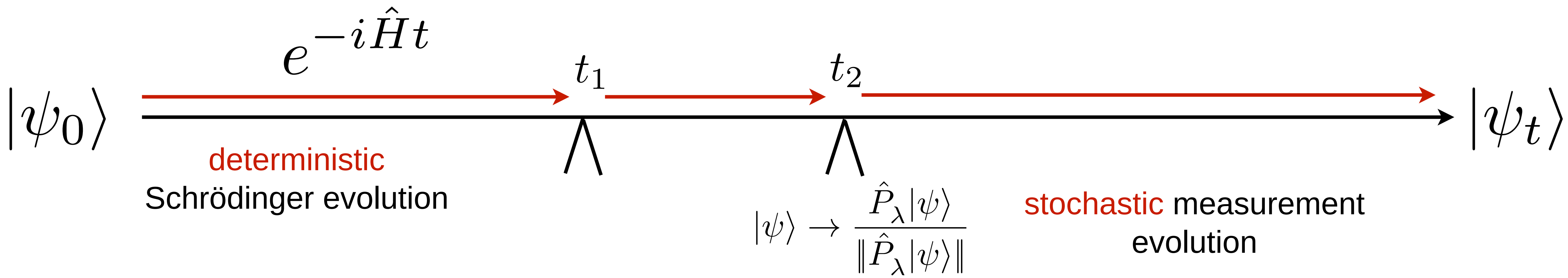
Theory background:
Strong projective vs. weak continuous measurements



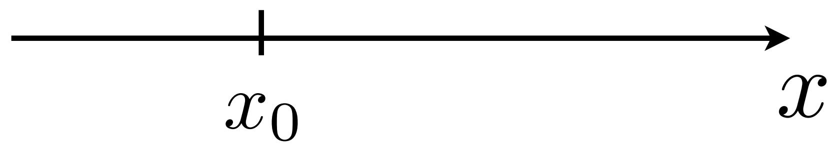
Projective vs. weak measurements

Review: Jacobs, Steck, Contemp. Phys. (2006)

- projective measurement: acquire full knowledge about observable



- example: position measurement, detector at x_0



$$\hat{A}(x_0) = |x_0\rangle\langle x_0| = \int dx\, \delta(x - x_0)\, |x\rangle\langle x|$$

➡ know post-measurement position x_0 with certainty

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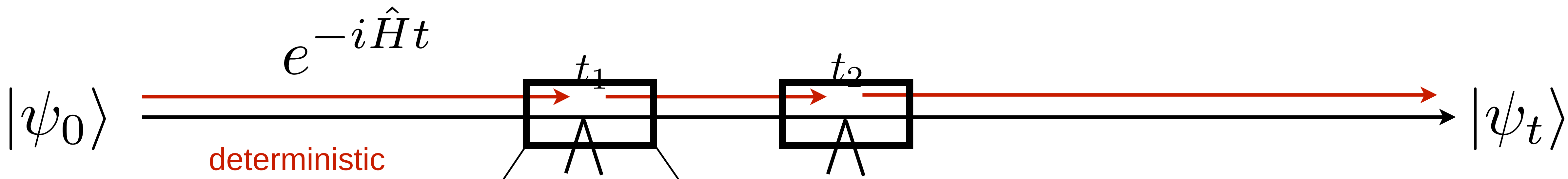

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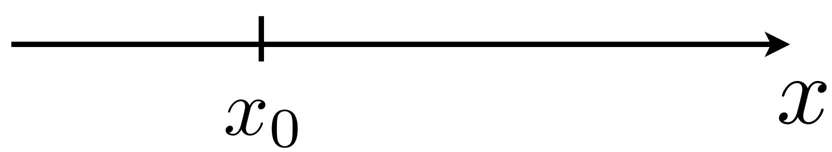

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- instantaneous collapse of wave function, but real measurements take time: continuous observation

$$\hat{A}(x_0) = (\tfrac{2\gamma\Delta t}{\pi})^{1/4} \int dx\, e^{-\gamma\Delta t(x-x_0)^2}\, |x\rangle\langle x|$$

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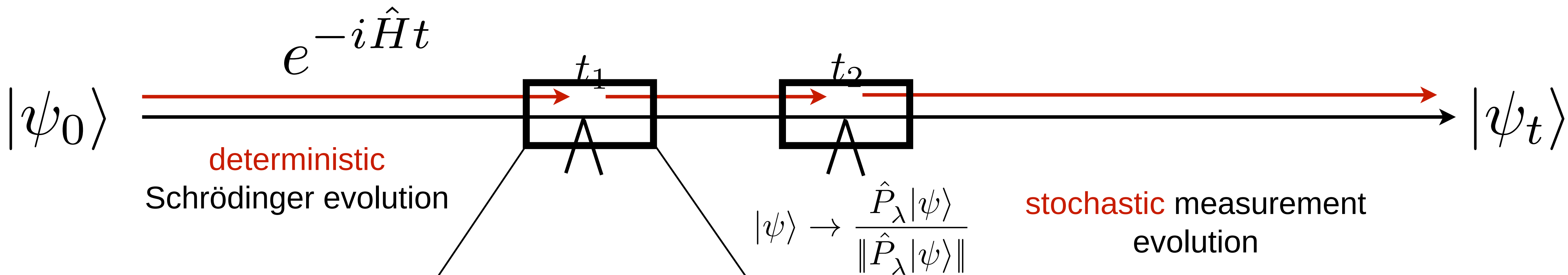

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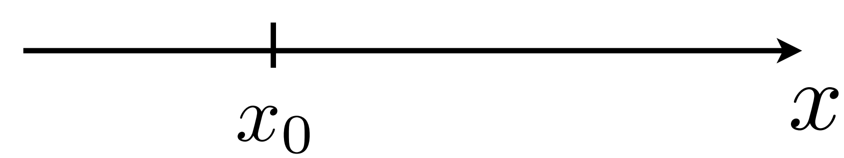

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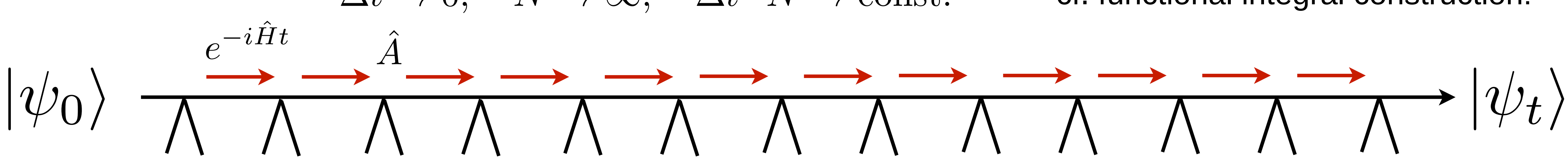
$$\hat{A}(x_0) = (\tfrac{2\gamma\Delta t}{\pi})^{1/4} \int dx\, e^{-\gamma\Delta t(x-x_0)^2}\, |x\rangle\langle x| \;\xrightarrow{\Delta t \to 0}\; (\tfrac{2\gamma\Delta t}{\pi})^{1/4} e^{-\gamma\Delta t(\hat{x}-x_0)^2}$$

➡ strong projective measurement for $\Delta t \to \infty$

➡ weak continuous measurement for small Δt, but large number of repetitions N:

$$\Delta t \to 0, \quad N \to \infty, \quad \Delta t \cdot N \to \text{const.}$$

cf. functional integral construction!



Probabilistic character of weak measurements

Review: Jacobs, Steck, Contemp. Phys. (2006)

- probability to measure x_0 on state $|\psi\rangle = \int dx \psi(x)|x\rangle$ in time $\Delta t \to 0$

$$p(x_0) = ||\hat{A}(x_0)|\psi\rangle|| = \mathrm{tr}[\hat{A}^\dagger(x_0)\hat{A}(x_0)|\psi\rangle\langle\psi|]$$

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exercise: verify this!

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- stochastic formulation (cf. Fokker-Planck vs. Langevin): parameterize

$$x_0 = \langle\hat{x}\rangle + \frac{\Delta W}{\Delta t}$$

Gaussian random variable

$$\overline{\Delta W} = 0$$

$$\overline{\Delta W \Delta W} = \frac{\Delta t}{4\gamma} \implies \Delta W \sim \sqrt{\Delta t}$$

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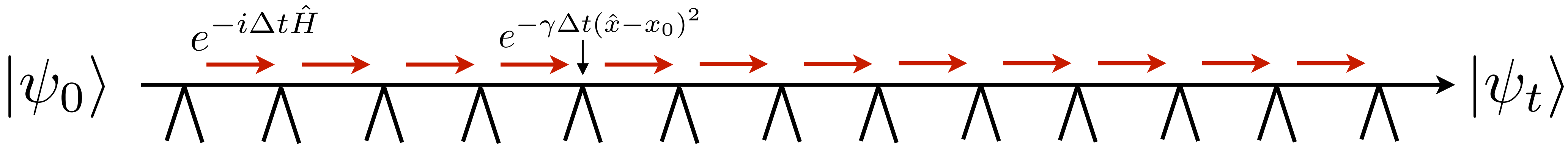
- for short observation times: expand $|\psi_{t+\Delta t}\rangle = \hat{A}(x_0)|\psi_t\rangle$ to linear order $\Delta t \equiv dt \to 0$ exercise: verify this!

$$d|\psi_t\rangle \equiv |\psi_{t+dt}\rangle - |\psi_t\rangle = \{[-\tfrac{1}{2}\gamma(\hat{x} - \langle\hat{x}\rangle)^2]dt + (\hat{x} - \langle\hat{x}\rangle)]dW\}|\psi\rangle$$

Weak measurements: stochastic Schrödinger equation (SSE)



- generalize: stochastic Schrödinger equation for quantum trajectory $|\psi_t\rangle$ Belavkin (1987); Gisin, Percival (1993)

$$d|\psi_t\rangle = dt(-i\hat{H} - \frac{\gamma}{2}\sum_l (\hat{n}_l - \langle\hat{n}_l\rangle_t)^2|\psi_t\rangle + \sum_l \underbrace{dW_l\,(\hat{n}_l - \langle\hat{n}_l\rangle_t)\,|\psi_t\rangle}$$

$$\langle\hat{n}_l\rangle_t = \langle\psi_t|\hat{n}_l|\psi_t\rangle$$

- Hamiltonian added, many degrees of freedom, $\hat{n}_l = \hat{n}_l^\dagger$
- works for measurement operators with discrete spectrum (measurement record continuous, e.g. Stern-Gerlach)

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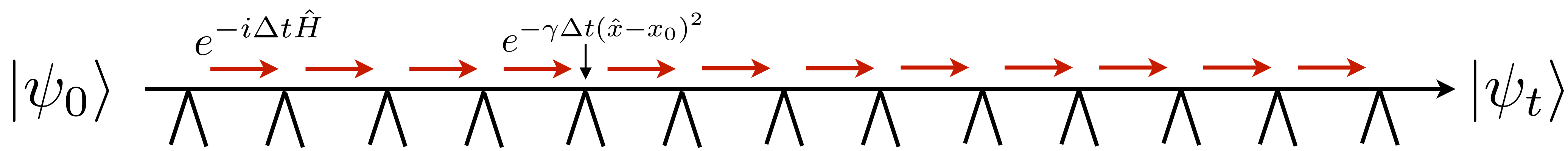

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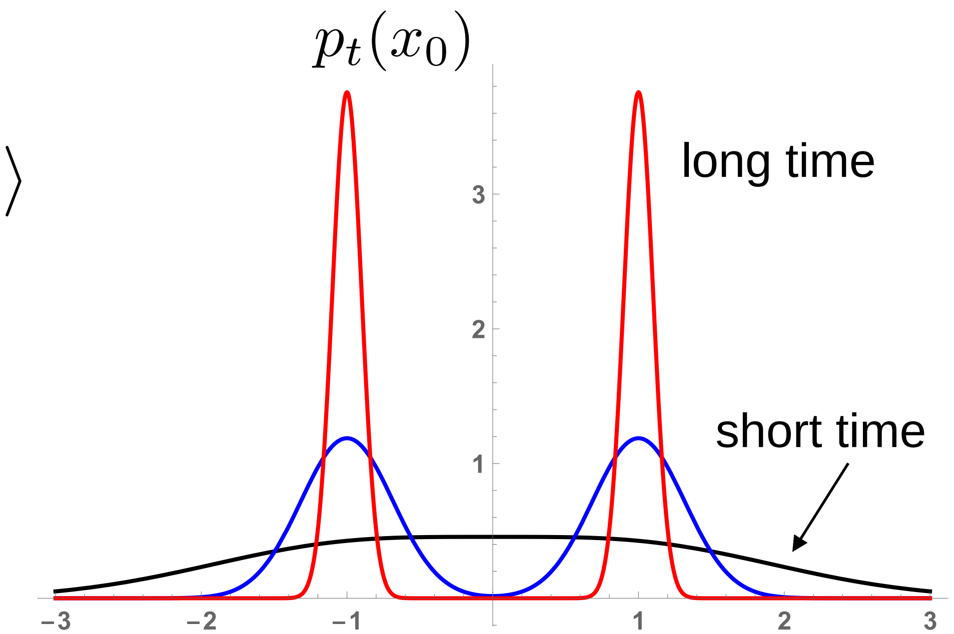
- measurement only dynamics H =0: measurement dark states
  - dW 'multiplicative noise': inactive when $\hat{n}_l|\psi_t\rangle = \langle\hat{n}_l\rangle_t|\psi_t\rangle$
  - e.g. in eigenstate $\hat{n}_l|\psi_t\rangle = n|\psi_t\rangle$

dark state of
measurement operator

➡ continuous collapse: convergence to measurement eigenstate for long times (more gen. for) $[\hat{H},\hat{n}_l] = 0$

measurement of a spin 1/2



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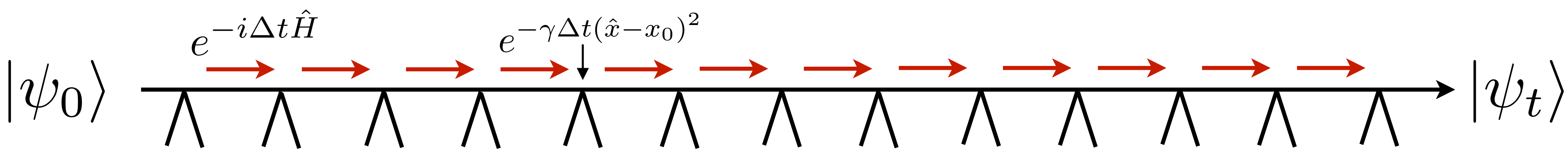

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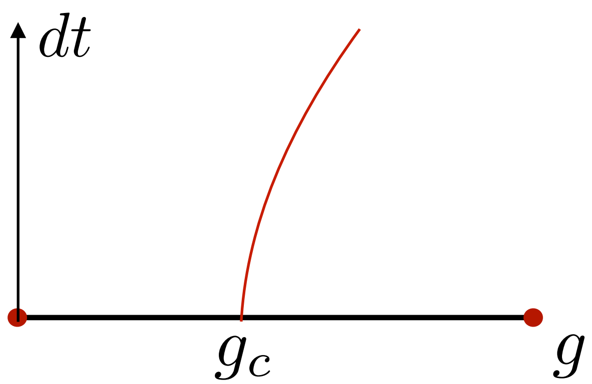
  - Hamiltonian added, many degrees of freedom, $\hat{n}_l = \hat{n}_l^\dagger$
  - works for measurement operators with discrete spectrum (measurement record continuous, e.g. Stern-Gerlach)

- Can we expect a measurement induced phase transition similar to projective measurements? Yes!
  - no competition => no phase transition upon interpolation $\hat{A}(x_0) = (\frac{2\gamma\Delta t}{\pi})^{1/4} e^{-\gamma\Delta t(\hat{x}-x_0)^2} \overset{\Delta t\to\infty}{=} |x_0\rangle\langle x_0|$
  - Demonstrated numerically Szyniszewski, Romito, Schomerus, PRB (2019)



➡ expect no extra phase transition upon taking temporal continuum limit

➡ continuum limit useful for analytical approach measurement induced phase transitions (e.g. Keldysh field theory approach)

Monitored dynamics: Extracting information

- stochastic Schrödinger equation for projector $\hat{\rho}_t = |\psi_t\rangle\langle\psi_t|$ ⟵ random variable

$$d\hat{\rho}_t = dt(-i[\hat{H}, \hat{\rho}_t] - \gamma \sum_l (2\hat{n}_l\hat{\rho}_t\hat{n}_l - \hat{n}_l^2\hat{\rho}_t - \hat{\rho}_t\hat{n}_l^2]) + \sum_l dW_l\{\hat{n}_l - \langle\hat{n}_l\rangle_t, \hat{\rho}_t\}$$

➡ statistical analysis: consider trajectory ensemble

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- first try: statistical average of SSE for projector
  - deterministic, linear Lindblad equation for $\rho_t = \overline{|\psi_t\rangle\langle\psi_t|}$
  - $\hat{n}_l$ Hermitian: heating to infinite temperature $\longrightarrow$ $\rho_t \sim \mathbf{1}$

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quantum average $\longrightarrow \langle\hat{O}\rangle = \langle\psi_t|\hat{O}|\psi_t\rangle$ $\xrightarrow{\text{statistical average}}$ $\overline{\langle\hat{O}\rangle} = \overline{\langle\psi_t|\hat{O}|\psi_t\rangle} = \mathrm{tr}[\hat{O}\overline{\hat{\rho}_t}]$

➡ evaluated on featureless infinite T state

➡ use state-dependent observables $\overline{\langle\hat{O}(|\psi\rangle)\rangle} = \mathrm{tr}\overline{\hat{O}(\hat{\rho})\hat{\rho}}$

more promising, because in general $F(\overline{\hat{\rho}}) \neq \overline{F[\hat{\rho}]}$

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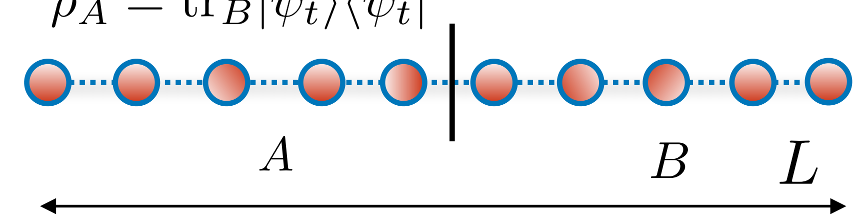
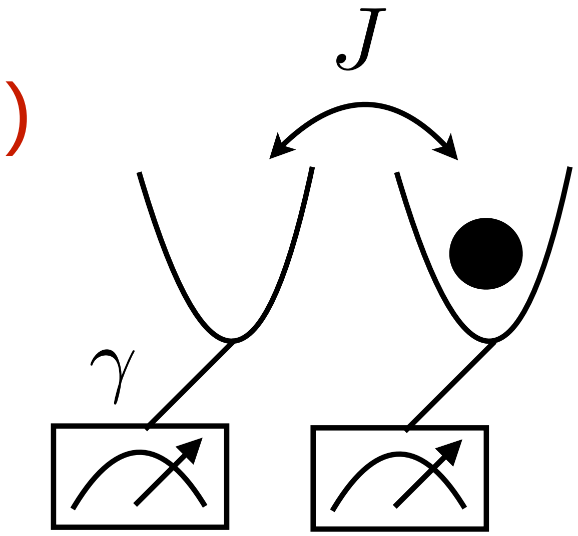
$\rho_A = \mathrm{tr}_B|\psi_t\rangle\langle\psi_t|$



- examples:
  - von Neumann entropy $\overline{S_{vN}(l,L)} = \overline{\langle\log(\rho_A)\rangle}$ arbitrarily high power of state projector
  - correlation function $\overline{\langle\hat{n}_i\rangle\langle\hat{n}_j\rangle}$ quadratic in state projector

An example: one fermion on two sites (two-level system)



- toy model: trajectory evolution of single fermion on two sites

$$|\psi_{t+dt}\rangle = |\psi_t\rangle - idt\hat{H}_{\text{eff}}|\psi_t\rangle + \sum_{l=1}^{2} dW_l \left(\hat{n}_l - \langle\hat{n}_l\rangle_t\right)|\psi_t\rangle$$

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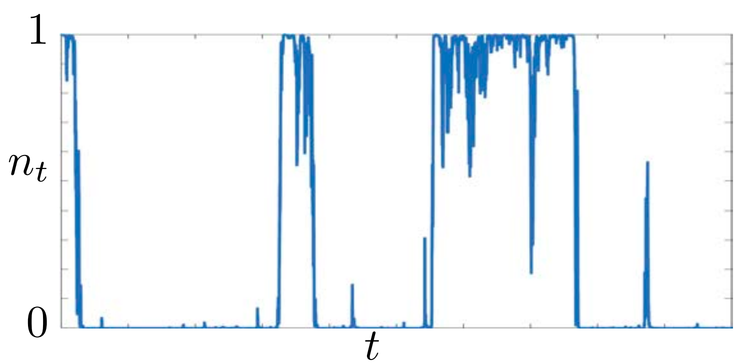
➡ H=0: collapse into dark state at long times $\hat{n}_l|\psi_t\rangle = \langle\hat{n}_l\rangle|\psi_t\rangle \Longrightarrow n_l = 0, 1$

- strong monitoring $J/\gamma \ll 1$



➡ pinning to measurement eigenstate

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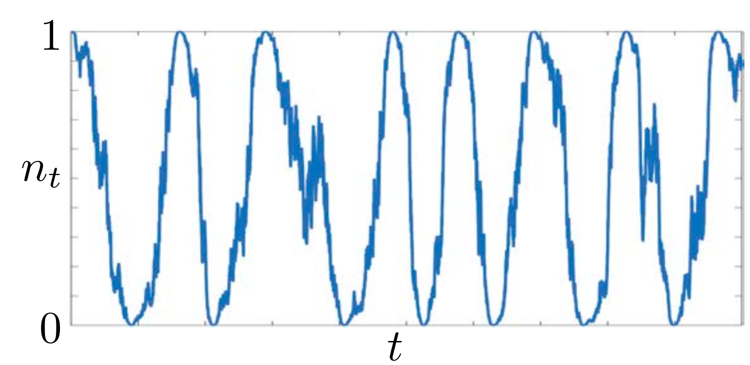
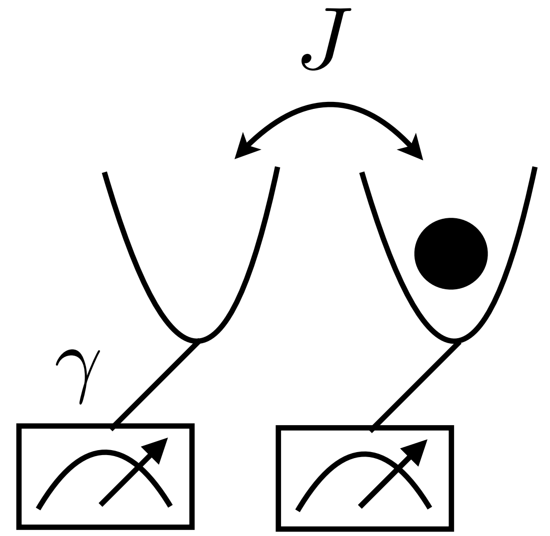


➡ vanishing time spent in eigenstate

An example: one fermion on two sites (two-level system)



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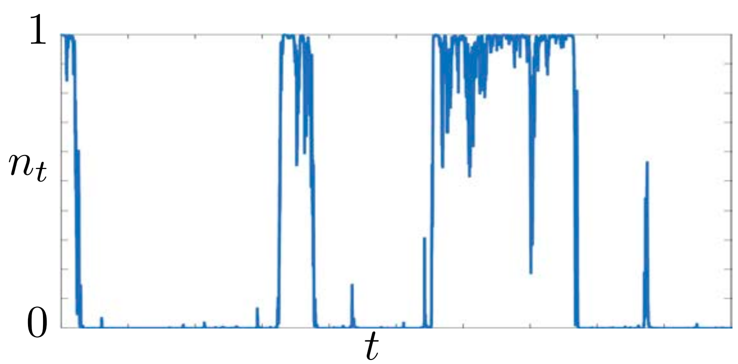

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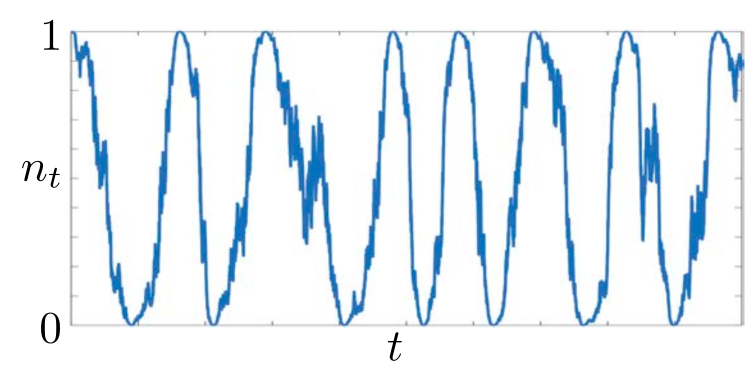
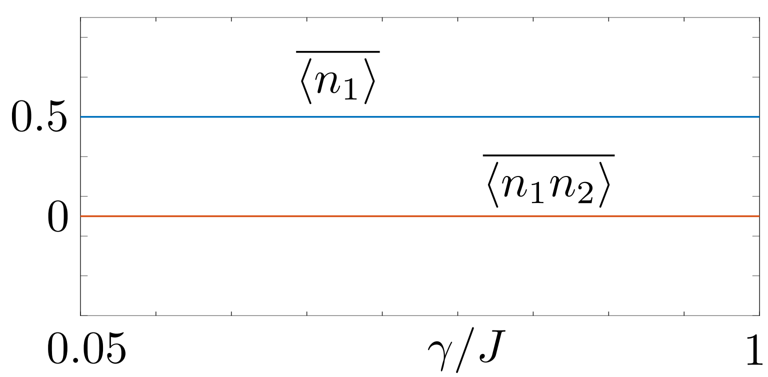
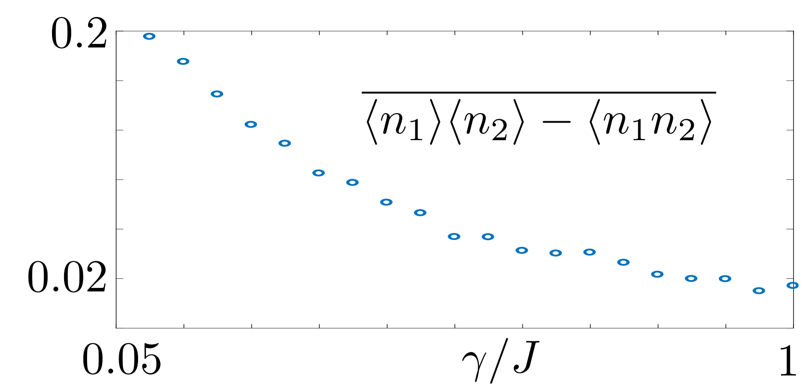


➡ vanishing time spent in eigenstate

- invisible in linear averages



- seen in averaged trajectory covariance matrix



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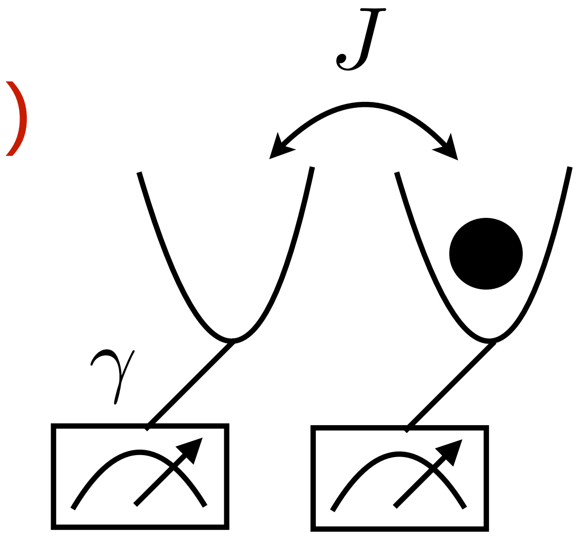

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guiding physical picture:

➡ thermodynamic limit: pinning quantum phase transition may happen at sharply defined point

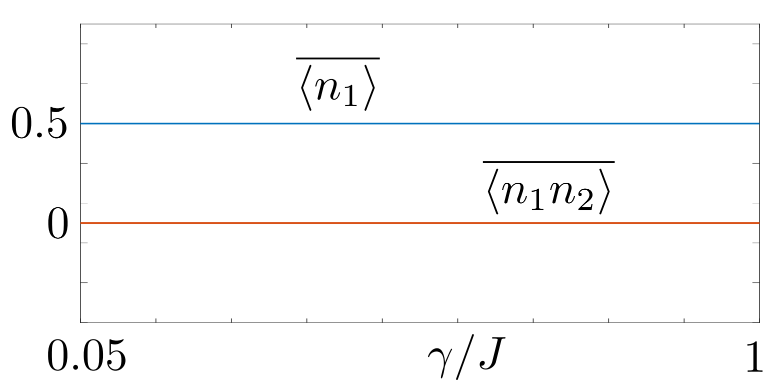
➡ signalled in nonlinear-in-state ‘observable’, like the covariance matrix

t t

➡ pinning to measurement eigenstate

➡ vanishing time spent in eigenstate

- invisible in linear averages



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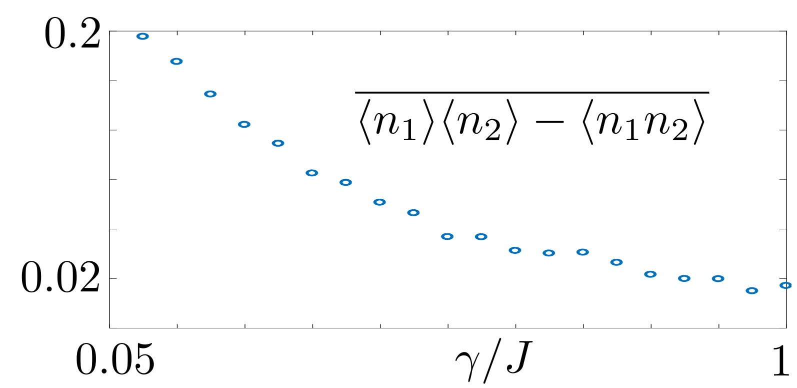
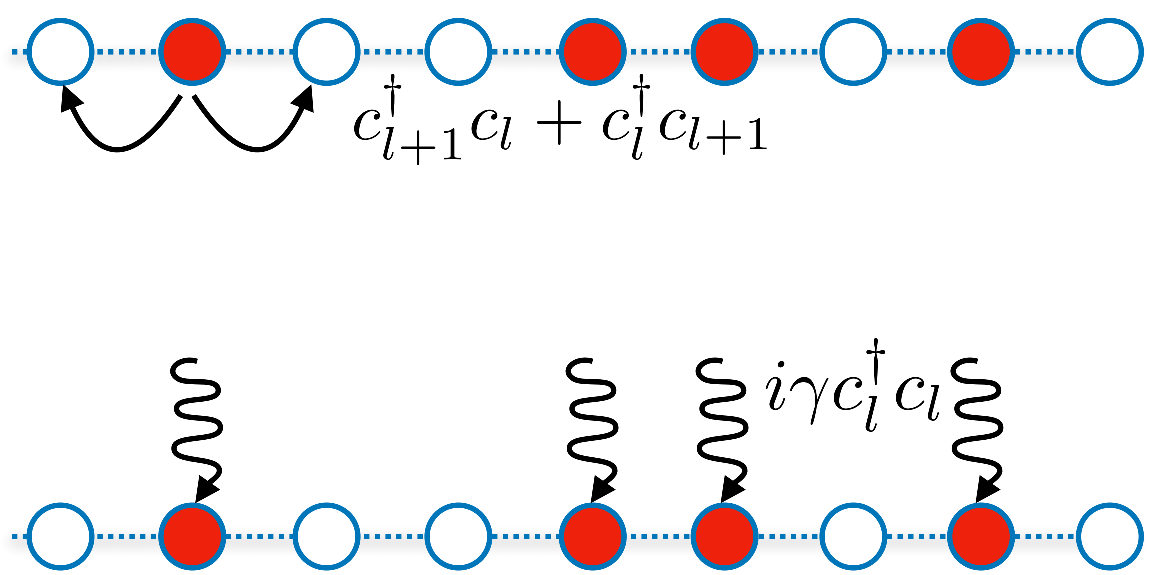


Entanglement phase transition in a monitored fermion chain

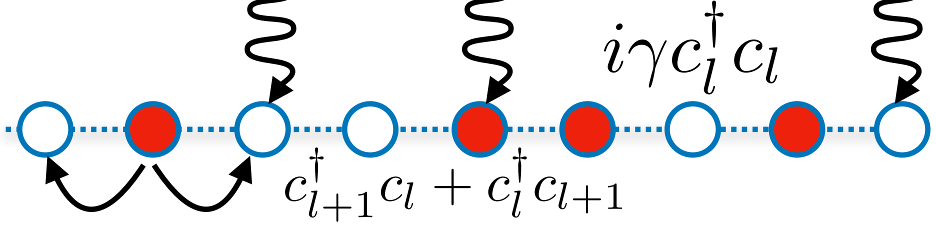
Hamiltonian: entanglement growth

Monitoring: entanglement saturation



O. Alberton, M. Buchhold, SD, PRL (2021)

Monitored fermion dynamics

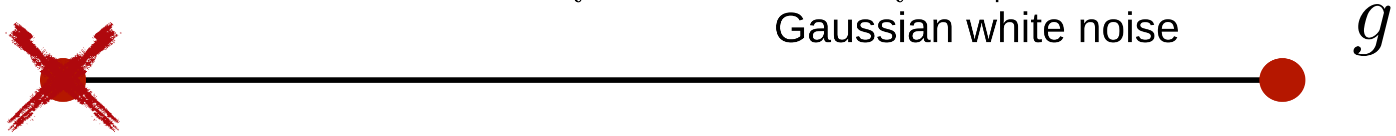


- Weak continuous measurements in many-body system

$$d|\psi_t\rangle = dt(-i\hat{H} - \tfrac{\gamma}{2}\sum_l \hat{M}_l^2|\psi_t\rangle + \sum_l dW_l \hat{M}_l|\psi_t\rangle$$

Gaussian white noise

- competition: $g = \dfrac{\gamma}{J}$



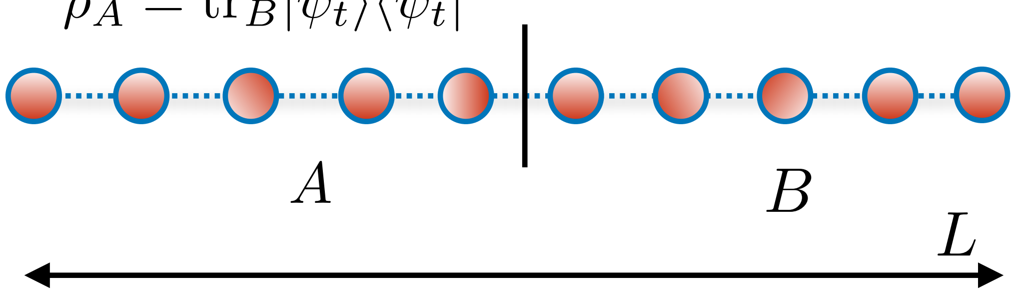
- unitary dynamics: hopping

$$H = -J\sum_l \left(c_l^\dagger c_{l+1} + c_{l+1}^\dagger c_l\right)$$

➡ volume law entanglement entropy

$$S_{vN}(L/2, L) = \mathrm{tr}\rho_A \log(\rho_A) \overset{t\to\infty}{\sim} L$$

$$\rho_A = \mathrm{tr}_B|\psi_t\rangle\langle\psi_t|$$



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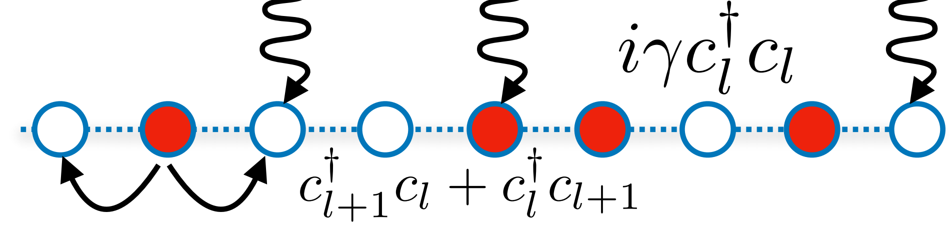

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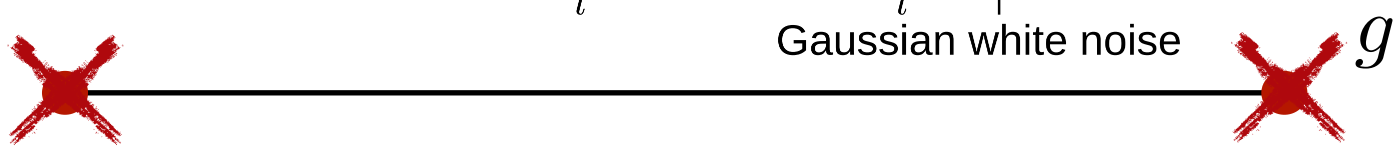

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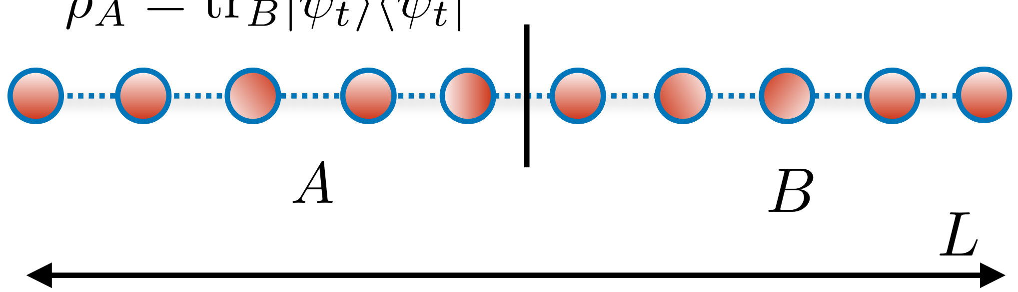


- measurement operators $\hat{M}_l = \hat{n}_l - \langle\hat{n}_l\rangle_t$

- H = 0: evolution stops after collapse into dark state

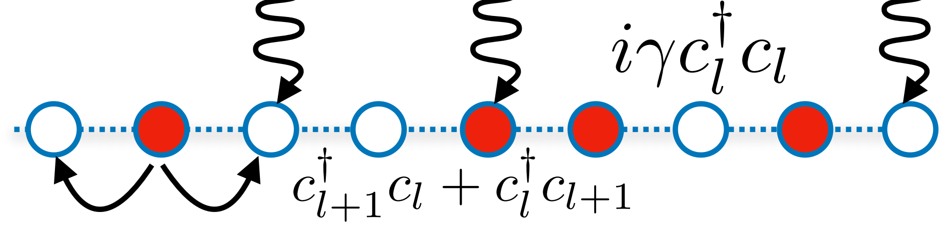
$$\hat{M}_l|\psi_t\rangle = 0 \quad \text{for} \quad \hat{n}_l|\psi_t\rangle = n_l|\psi_t\rangle$$

eigenstate of measurement operator

➡ area law entanglement entropy

$$S_{vN}(L/2, L) = s_0$$

Monitored fermion dynamics



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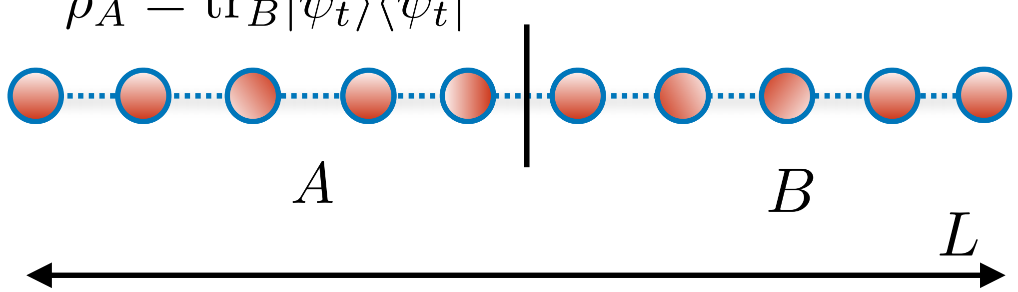


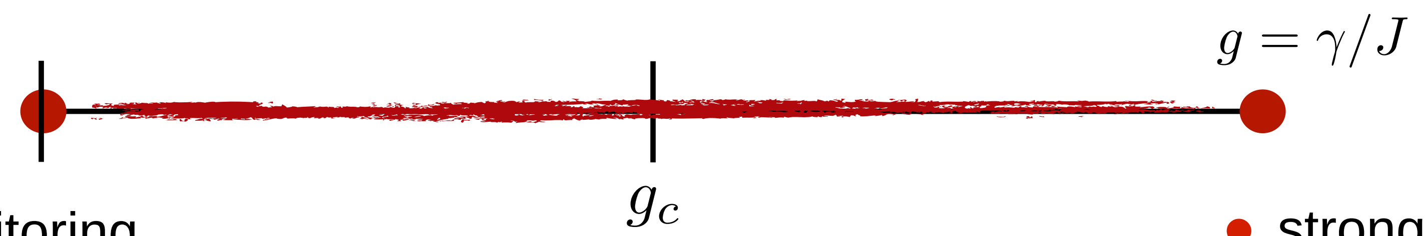
➡ area law entanglement entropy

$$S_{vN}(L/2, L) = s_0$$

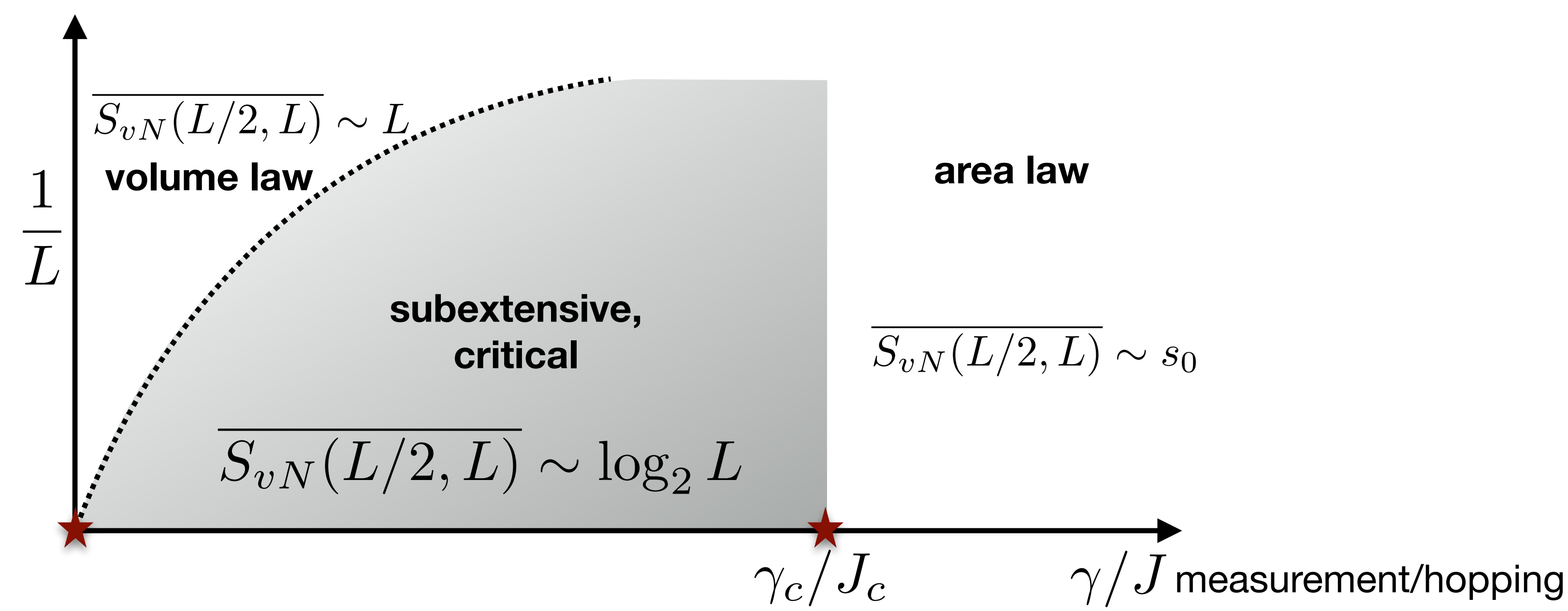
- caveat: $|\psi_t\rangle$ is a random variable
- binary measurement outcomes generate extensive configurational entropy

➡ 'observables': entanglement entropy, traj. averaged correlators

Trajectory ensemble phase diagram: Entanglement entropy



- weak monitoring
- strong monitoring



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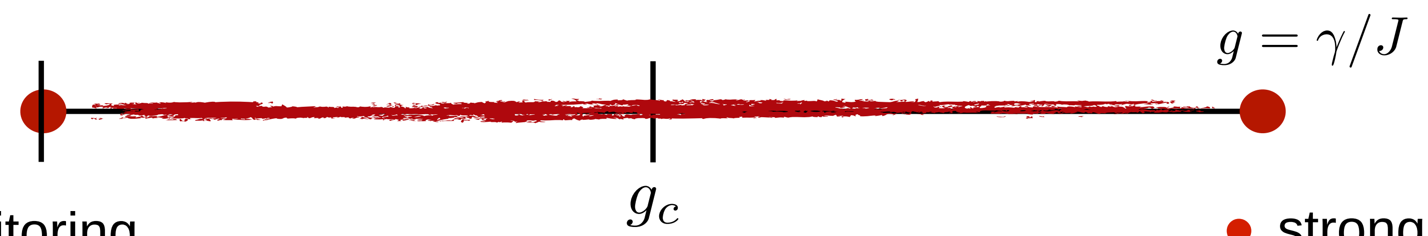

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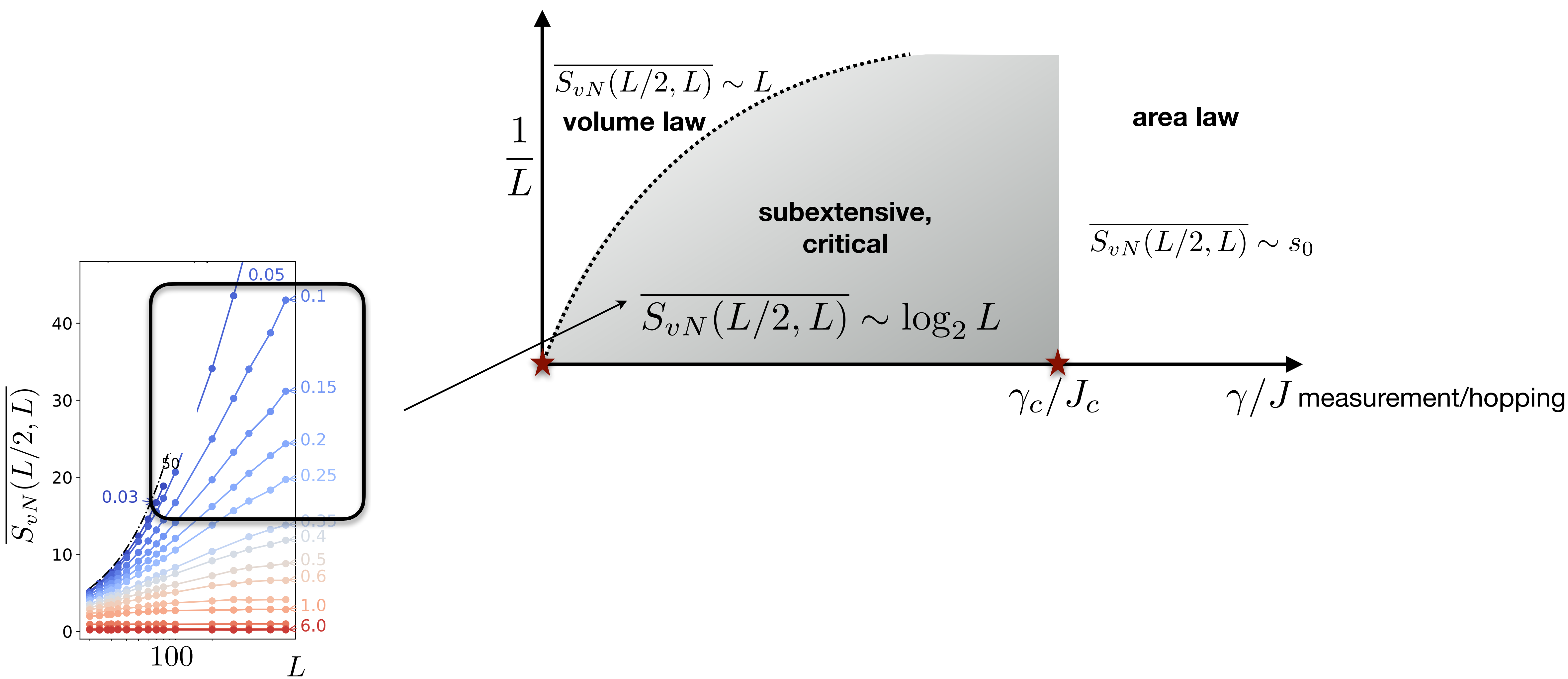

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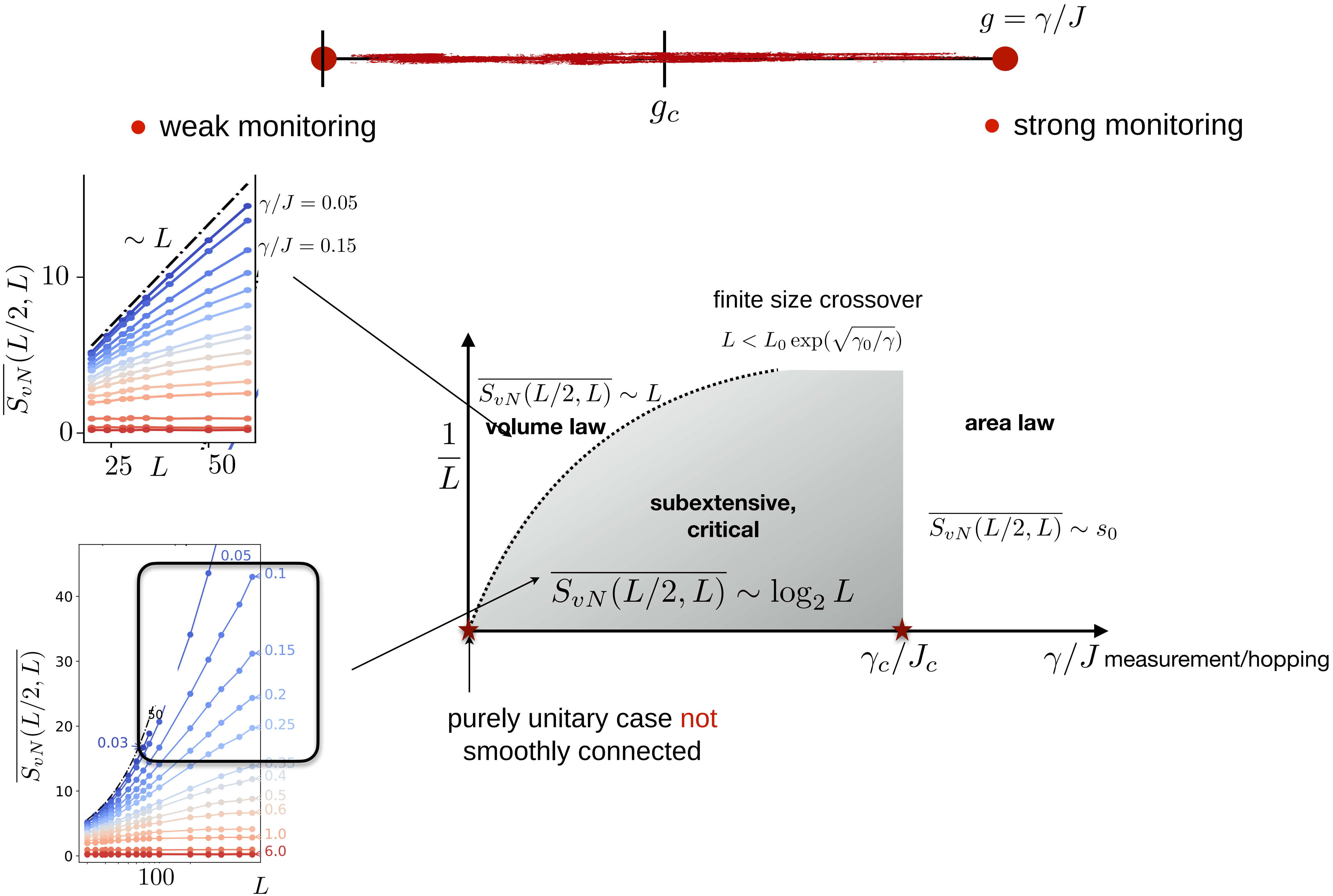

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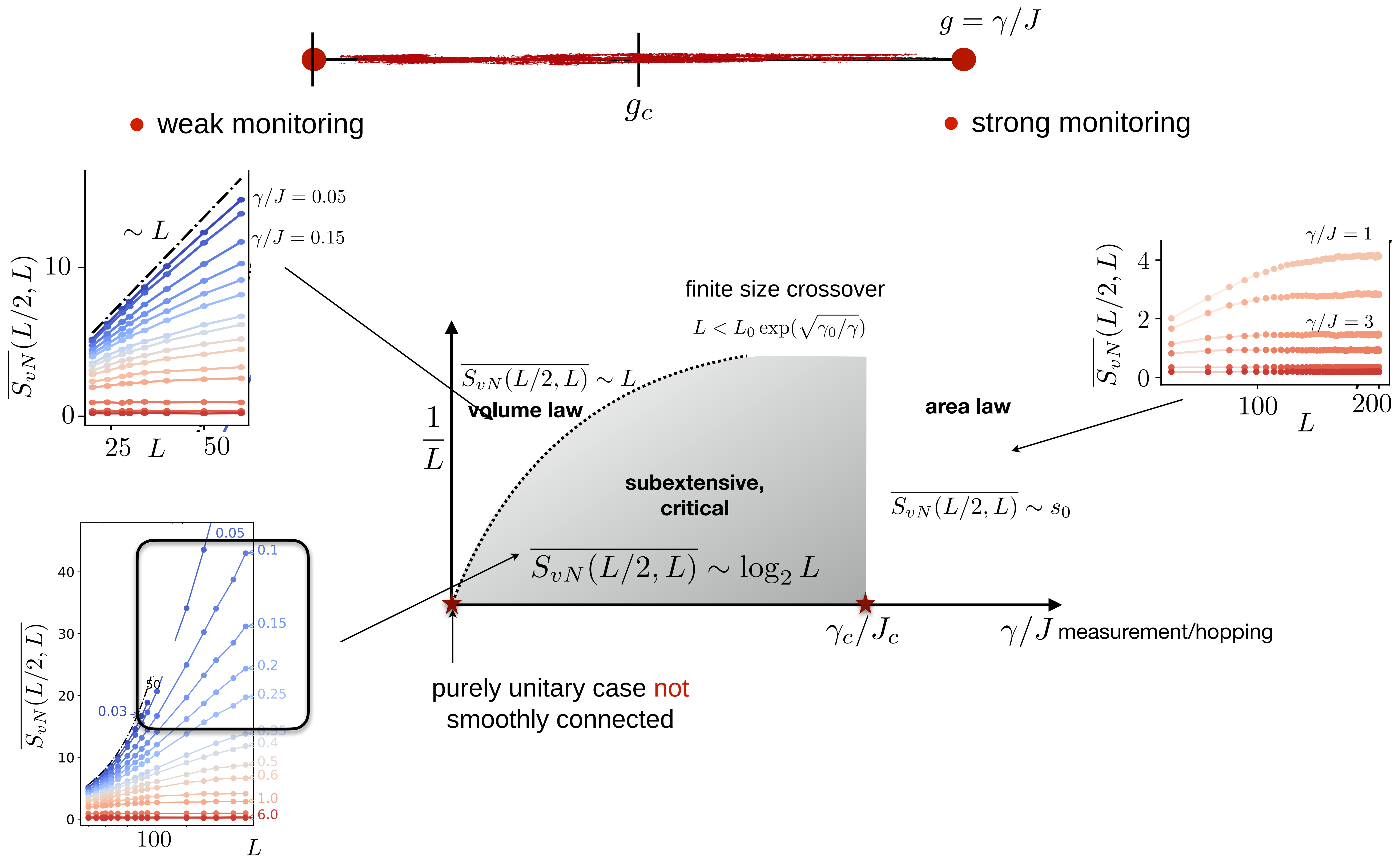

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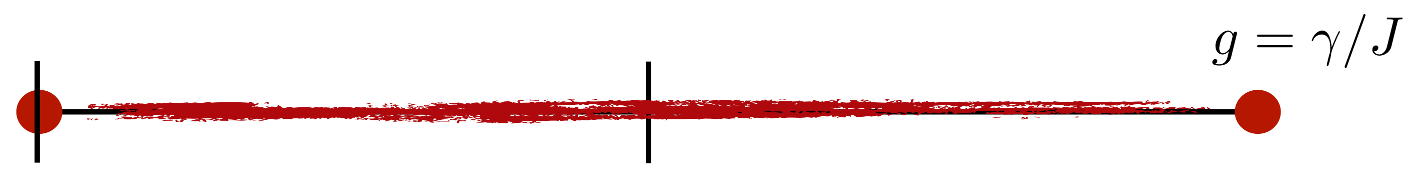

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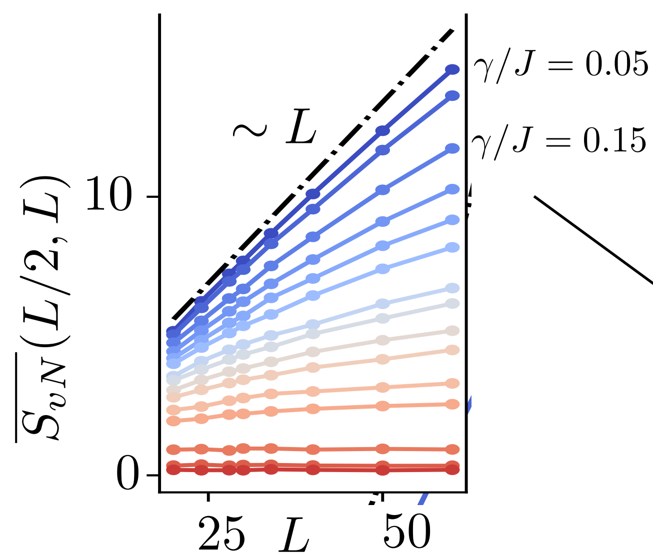
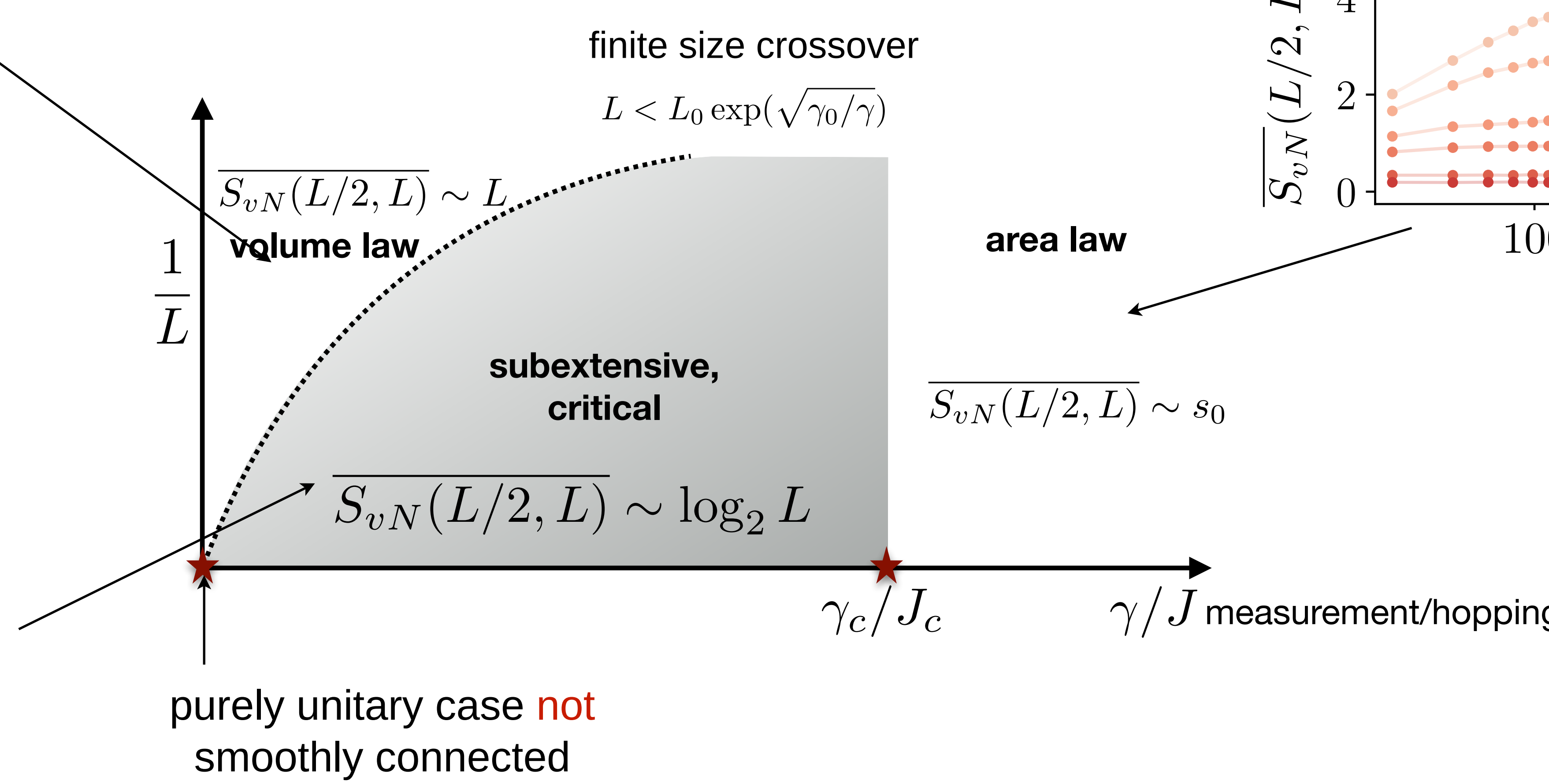
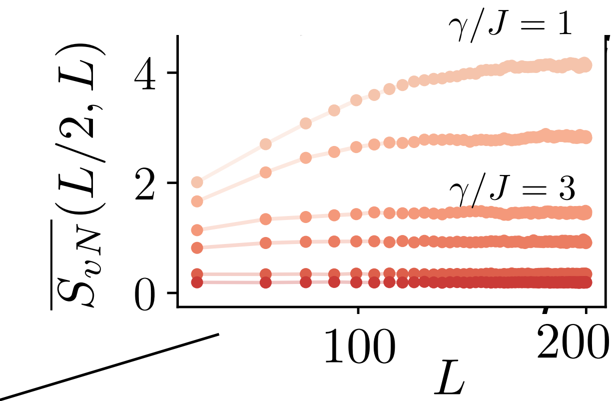
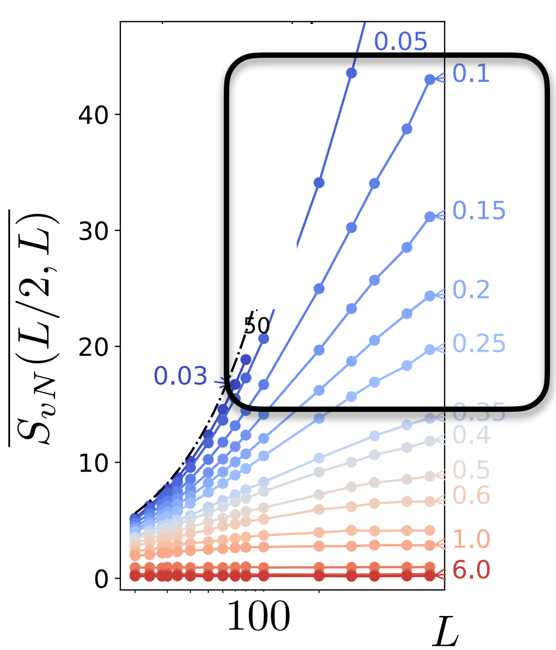








➡ extended subextensive, critical phase at intermediate monitoring

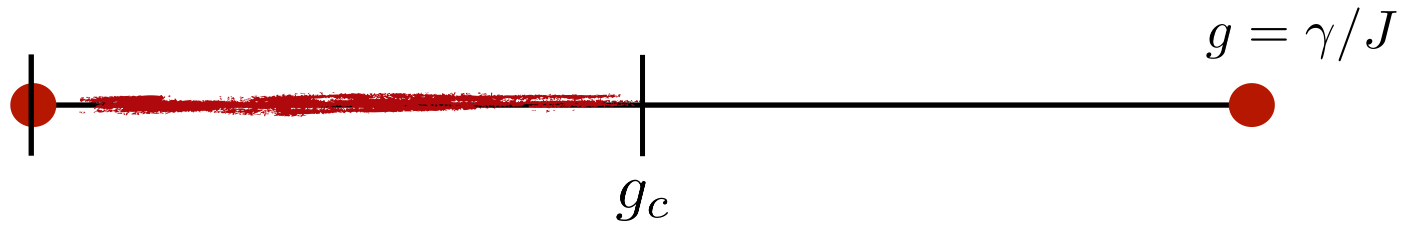
see also Chen, Li, Fisher, Lucas PRR (2020)

➡ new phase transition for physical measurement protocol

➡ consistent with Cao, Tilloy, De Luca SciPost (2019)

only excluding volume law in thermodynamic limit

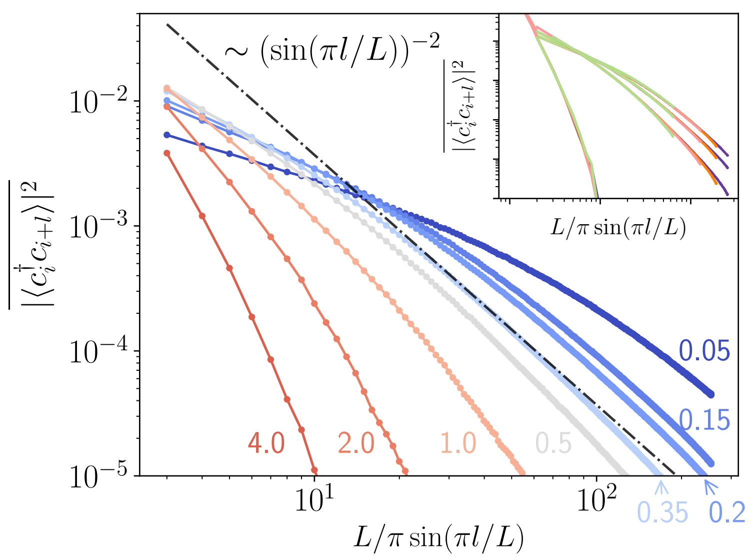
Characterizing the weak monitoring phase



- extended criticality: Connected correlation function

$$C_{i,i+l} = \overline{\langle \hat{n}_i \rangle \langle \hat{n}_{i+l} \rangle} - \overline{\langle \hat{n}_i \hat{n}_{i+l} \rangle}$$

$$\sim l^{-2}$$

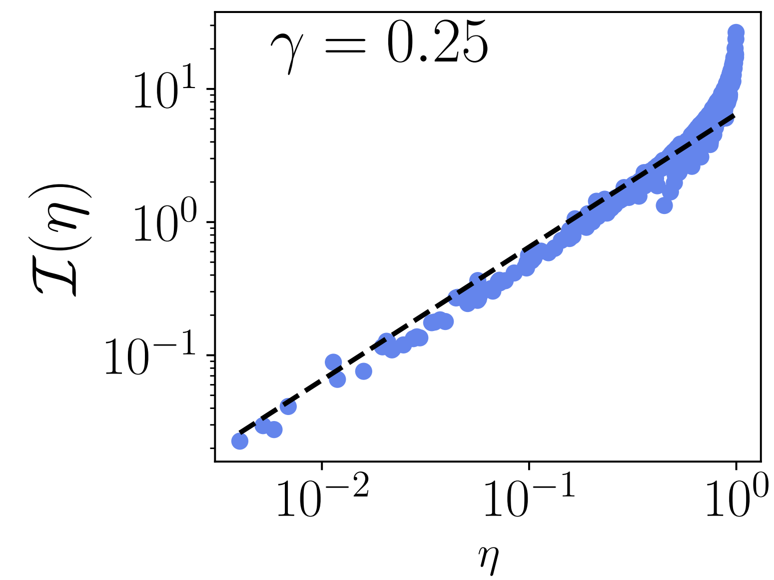


- captures all distinct phases:

$$C_{i,i+l} \sim \begin{cases} 0 & \text{for } H = 0 \\ \exp(-l/\xi) & \text{for } \gamma \gg J \\ l^{-2} & \text{for } \gamma \ll J \\ l^{-1} & \text{for } \gamma = 0 \end{cases}$$

- emergent conformality: Mutual information

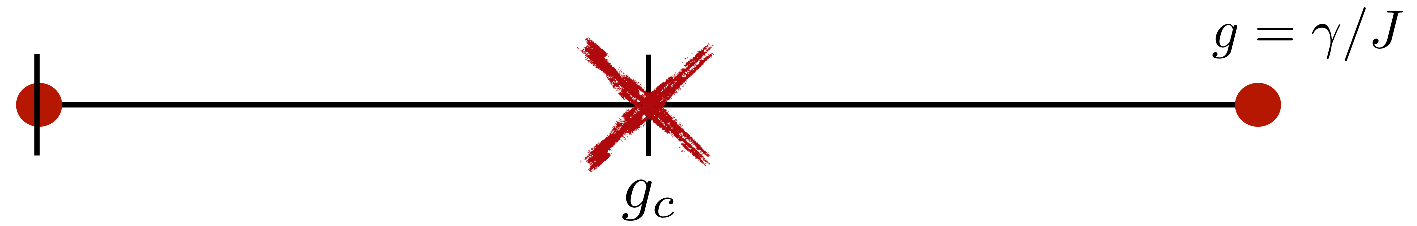
$$\mathcal{I}(A,B) = \overline{S_{vN}(A)} + \overline{S_{vN}(A)} + \overline{S_{vN}(A \cup B)}$$



conformally invariant critical point: Nahum et al. PRX (2019); Li Chen Fisher PRB (2019); Jian et al. PRB (2020);

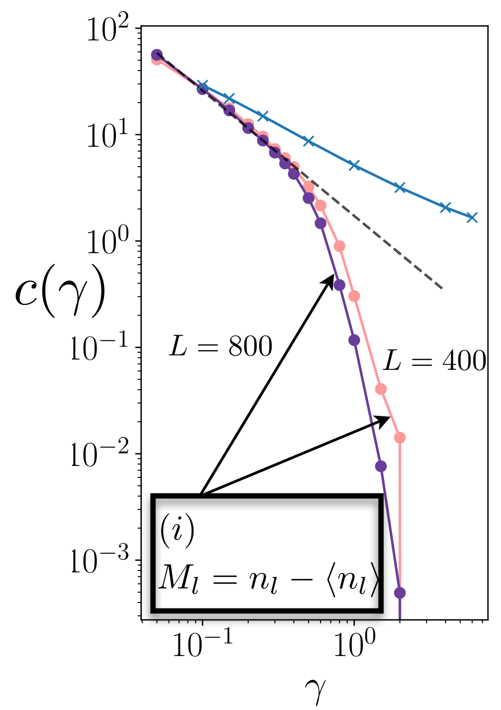
➡ emergent conformally invariant critical phase for weak monitoring

Characterizing the phase transition



- effective central charge $c(\gamma)$

$$\overline{S_{vN}(l,L)} = \frac{c(\gamma)}{3}\log_2\left[\frac{L}{\pi}\sin\left(\frac{\pi l}{L}\right)\right] + s(\gamma)$$

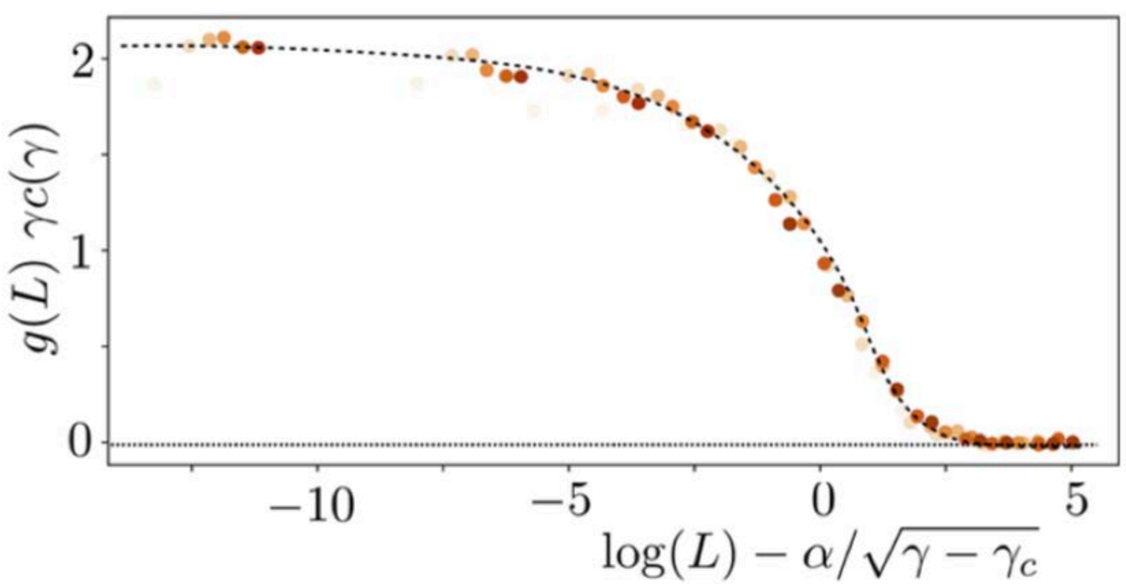


parameter dependent c

random systems: Cardy Jacobsen PRL (1997);
Refael, Moore PRL (2004)

➡ sudden jump reminiscent of BKT

- essential scaling of the central charge: scaling collapse above phase transition

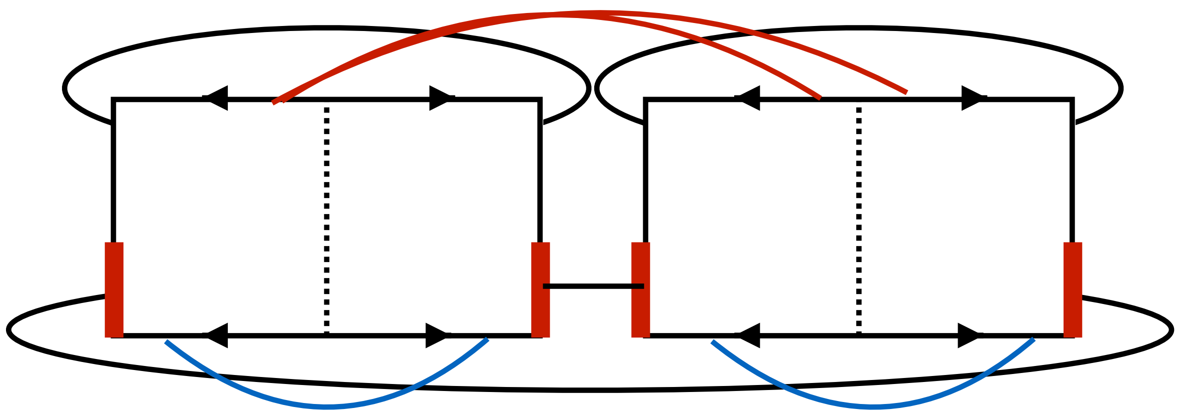


➡ BKT universal behavior

➡ establishes BKT type phase transition

➡ further: measurement protocol dependence, trajectory entanglement distribution as probe of transition…

Keldysh replica field theory approach
to
measurement induced phase transitions



M. Buchhold, Y. Minoguchi, A. Altland, SD, arxiv:2102.08381

microphysics → macrophysics

Continuum (1+1) dimensional model

- preface: model obtains from naive continuum limit and bosonization of lattice fermion model

fermionic variant $\longrightarrow$ bosonized variant

- Hamiltonian: massless Dirac fermions $\hat{\Psi}_x = (\hat{\psi}_{R,x}, \hat{\psi}_{L,x})^T$

Luttinger liquid

$$\hat{H} = iv \int_x \hat{\Psi}_x^\dagger \sigma_z \partial_x \hat{\Psi}_x \longrightarrow \hat{H} = \frac{v}{2\pi} \int_x [(\partial_x \hat{\theta}_x)^2 + (\partial_x \hat{\phi}_x)^2]$$

phase density

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- measurement operators: current and vertex operators

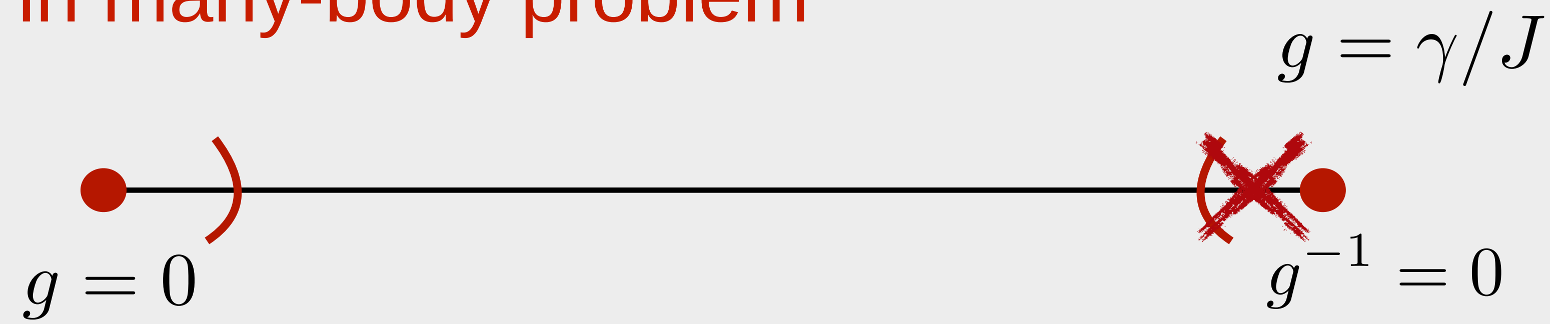
rate γ_1 : $\hat{O}_{1,x} = \Psi_x^\dagger \Psi_x = \hat{J}_x^{(0)} \longrightarrow \hat{O}_{1,x} = -\frac{1}{\pi} \partial_x \hat{\phi}_x$ linear gapless

rate γ_2 : $\hat{O}_{2,x} = \Psi_x^\dagger \sigma_x \Psi_x \longrightarrow \hat{O}_{2,x} = \underbrace{m}_{\mathcal{O}(1)} \cos(2\hat{\phi}_x)$ nonlinear

common eigenstates: $\hat{\phi}_x |\Psi_D\rangle = \phi_x |\Psi_D\rangle$

- stabilize product dark states: exactly local
- realize competition: do not commute with H (phase fluctuations)

Signatures of phase transition in many-body problem



- Analysis of limiting cases:
  evolution of covariance matrix

$$C_{ij} = \overline{\langle n_i n_j \rangle} - \overline{\langle n_i \rangle \langle n_j \rangle}$$

- strong monitoring: perturbation theory

  - hierarchy of equations of motions expanded to leading order 1/g

$$C_{ij} = \frac{(-1)^{|i-j+1|}(2|i-j|-3)!!}{2^{|i-j+1|}|i-j|!}\left(\frac{2J^2}{\gamma^2}\right)^{|i-j|}$$

$$\Longrightarrow \boxed{C_{ij} \sim e^{-|i-j|/\xi}} \qquad \xi = \frac{1}{2}\log^{-1}\left(\frac{\gamma}{J\sqrt{2}}\right)$$

➡ leading exponential decay, in line with numerics

Signatures of phase transition in many-body problem

$$g = \gamma / J$$

- Analysis of limiting cases:
  evolution of covariance matrix

$$C_{ij} = \overline{\langle n_i n_j \rangle} - \overline{\langle n_i \rangle \langle n_j \rangle}$$

- weak monitoring: Ricatti equations   V. P. Belavkin (1987); Kalman filtering in cavity optomechanics, Wieczorek et al. PRL (2015)

  - quadratic theory: $H = \frac{v}{2\pi} \int_x (\partial_x \theta_x)^2 + (\partial_x \phi_x)^2$ $M_x \approx -\frac{1}{\pi}(\partial_x(\phi_x - \langle \phi_x \rangle))$

  - evolution of covariance matrix governed by Ricatti equation, steady state solvable in momentum space

$$C_k = C(\gamma) \sin(|k|/2) \quad \Longrightarrow \quad \boxed{C_{ij} = \frac{C(\gamma)}{|i-j|^2}} \qquad C(\gamma) = \frac{v}{4\gamma}\sqrt{2\left(\sqrt{\frac{16\gamma^2}{\pi^2 v^2} + 1} - 1\right)}$$

➡ algebraic decay reproducing numerical result at weak monitoring

➡ ground state Luttinger liquid result reproduced as

$$\gamma \to 0: \quad C_k = \frac{|k|}{2\pi}$$

Signatures of phase transition in many-body problem

- Analysis of limiting cases:
  evolution of covariance matrix

$$C_{ij} = \overline{\langle n_i n_j \rangle} - \overline{\langle n_i \rangle \langle n_j \rangle}$$

- back to strong monitoring: Ricatti equations

  - absence of Hamiltonian: systems evolves into product of local number eigenstates

$$\phi_x |\psi\rangle = \varphi_x |\psi\rangle \qquad \text{for all x}$$

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$$\phi_x |\psi\rangle = \varphi_x |\psi\rangle \qquad \text{for all x}$$

  - expand about homogeneous configuration $\varphi_x = \varphi$

$$n_x - \langle n_x \rangle = -\left(m + \frac{\partial_x}{\pi} \right) (\delta\phi_x - \langle \delta\phi_x \rangle) \approx -m(\delta\phi_x - \langle \delta\phi_x \rangle) \qquad m = \sqrt{\frac{16}{\pi^2 \alpha^2} - \frac{1}{4}}$$

➡ pinning seen as a gap opening in the effective Hamiltonian

Signatures of phase transition in many-body problem

- Analysis of limiting cases:
  evolution of covariance matrix

$$C_{ij} = \overline{\langle n_i n_j \rangle} - \overline{\langle n_i \rangle \langle n_j \rangle}$$

- back to strong monitoring: Ricatti equations

  - absence of Hamiltonian: systems evolves into product of local number eigenstates

$$\phi_x |\psi\rangle = \varphi_x |\psi\rangle \qquad \text{for all x}$$

  - expand about homogeneous configuration $\varphi_x = \varphi$

$$n_x - \langle n_x \rangle = -\left(m + \frac{\partial_x}{\pi} \right) (\delta\phi_x - \langle \delta\phi_x \rangle) \approx -m(\delta\phi_x - \langle \delta\phi_x \rangle) \qquad m = \sqrt{\frac{16}{\pi^2 \alpha^2} - \frac{1}{4}}$$

➡ pinning seen as a gap opening in the effective Hamiltonian

  - solution of the Ricatti equation for gapped quadratic theory

$$C_{ij} \sim m \sqrt{\frac{\nu}{8\pi\gamma}} e^{-\frac{|i-j|}{\xi}} \qquad \xi = \frac{1}{\log(2\pi\gamma m^2/\nu)}$$

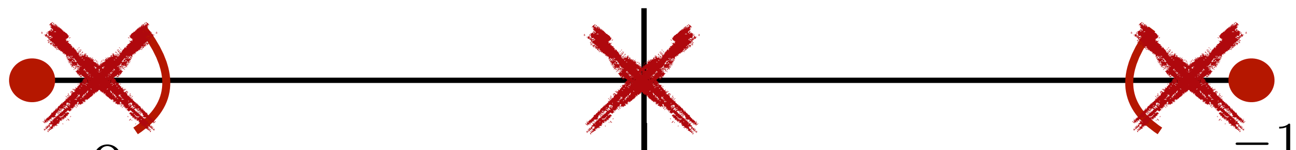
➡ reproduces the qualitative behavior of fermionic strong monitoring approach

Intermediate summary

- Analysis of limiting cases: evolution of covariance matrix

$$C_{ij} = \overline{\langle n_i n_j \rangle} - \overline{\langle n_i \rangle \langle n_j \rangle}$$

$$g = \gamma / J$$



$g = 0$ $\qquad$ $g^{-1} = 0$

- Luttinger + Ricatti approach

$$C_{ij} = \frac{C(\gamma)}{|i-j|^2}$$

➡ algebraic decay

➡ depinned phase

- fermionic and Luttinger + Ricatti approach

$$C_{ij} \sim e^{-\frac{|i-j|}{\xi}}$$

➡ exponential decay

➡ pinned phase

➡ there must be a phase transition at some finite g_c

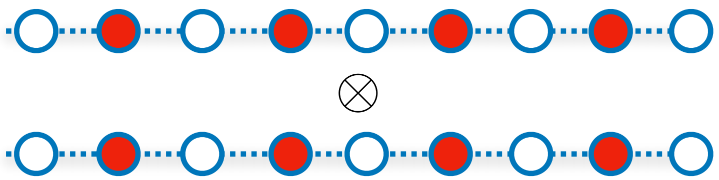
➡ strongly reminiscent of a deconfined to pinning transition in Sine-Gordon models

➡ develop field theory approach capture it

Towards the relevant degrees of freedom: Replica approach

- Q: What is the structure of $C_{xy} = \overline{\langle \hat{n}_x \hat{n}_y \rangle} - \overline{\langle \hat{n}_x \rangle \langle \hat{n}_y \rangle}$

- Introduce replicas in Hilbert space $|\Psi_t\rangle = |\psi_t^{(1)}\rangle \otimes |\psi_t^{(2)}\rangle =$



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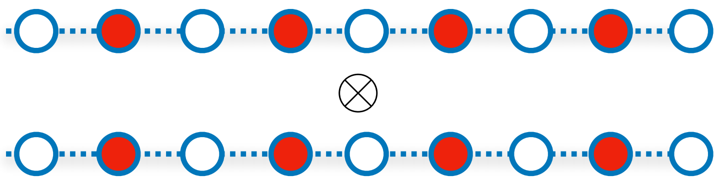

$$\hat{n}_x^{(1)} = \hat{n}_x \otimes \mathbf{1}$$

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- Then $C_{xy} = \frac{1}{2} \mathrm{Tr} \left[\left(\hat{n}_x^{(1)} - \hat{n}_x^{(2)} \right) \left(\hat{n}_y^{(1)} - \hat{n}_y^{(2)} \right) \rho^{2R} \right]$

$$\rho^{2\mathrm{R}} = \overline{|\Psi_t\rangle\langle\Psi_t|}$$

➡ linear statistical average of replica density matrix

➡ correlations of relative replica coordinate

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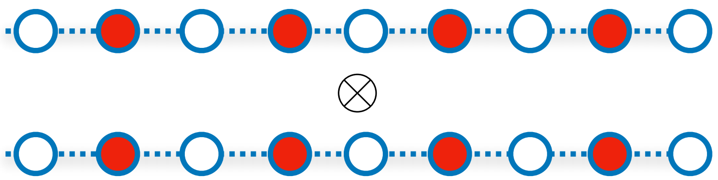

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$\rho^{2\mathrm{R}} = \overline{|\Psi_t\rangle\langle\Psi_t|}$

➡ linear statistical average of replica density matrix

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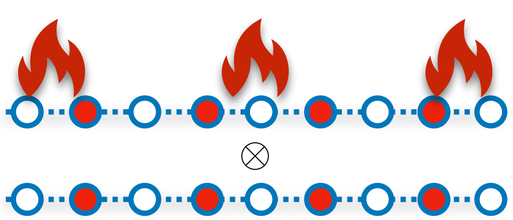
- Quantum master equation (truncate coupling to ρ^{3R})

$\partial_t \rho^{2\mathrm{R}} =$



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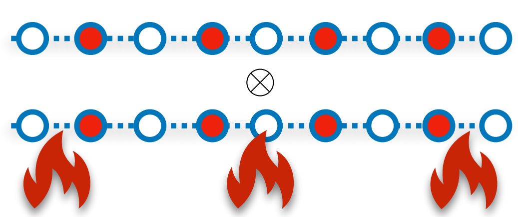

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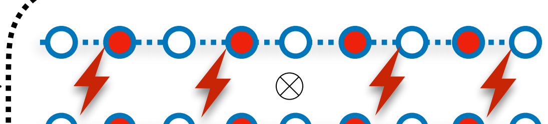


$i[\rho^{2R}, H^{(\alpha)}] - \frac{\gamma}{2} [\hat{M}_x^{(\alpha)}, [\hat{M}_x^{(\alpha)}, \rho^{2R}]]$

individual heating Lindbladians

$\gamma \{ \hat{M}_x^{(1)}, \{ \hat{M}_x^{(2)}, \rho^{2R} \} \}$

replica coupling

➡ study structure of 2-replica theory

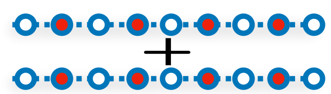
Boson replica quantum master equation

- Boson measurement $H^{(\alpha)} = \frac{\nu}{2\pi}\int_x (\partial_x \hat{\phi}^{(\alpha)})^2 + (\partial_x \hat{\theta}^{(\alpha)})^2$

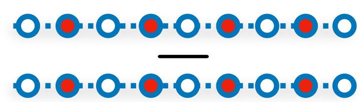
$$\hat{O}_{1,x}^{(\alpha)} = -\frac{1}{\pi}\partial_x \hat{\phi}_x^{(\alpha)} \qquad \hat{O}_{2,x}^{(\alpha)} = 0$$

linear case first

- New degrees of freedom



: $\hat{\phi}^{(a)} = \hat{\phi}^{(1)} + \hat{\phi}^{(2)}$ average coordinate



: $\hat{\phi}^{(r)} = \hat{\phi}^{(1)} - \hat{\phi}^{(2)}$ replica fluctuations

➡ Master equation becomes separable

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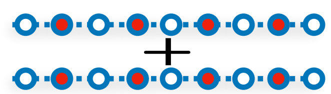
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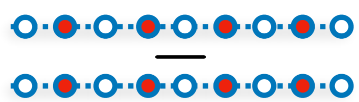
$: \hat{\phi}^{(r)} = \hat{\phi}^{(1)} - \hat{\phi}^{(2)}$ replica fluctuations

➡ Master equation becomes separable

- Average coordinate: heating to infinite temperature (<—> unbounded growth of mode occupation)

$$\partial_t \rho^{(a)} = i[\rho^{(a)}, H^{(a)}] + \frac{2\gamma}{\pi}\sum_l \left(\partial_x \hat{\phi}^{(a)} - \overline{\langle \partial_x \hat{\phi}^{(a)} \rangle}\right) \rho^{(a)} \left(\partial_x \hat{\phi}^{(a)} - \overline{\langle \partial_x \hat{\phi}^{(a)} \rangle}\right)$$

only jump term!

Boson replica quantum master equation

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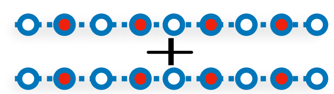

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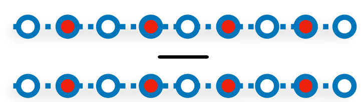

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only jump term!

- Relative coordinate: cooling/damping into dark state

$$\partial_t\rho^{(r)} = i[\rho^{(r)}, H^{(r)}] - \frac{\gamma}{\pi}\sum_l \left\{(\partial_x\hat{\phi}^{(r)})^2, \rho^{(r)}\right\} \longleftarrow$$ no jump term!

- further separable: $\partial_t|\psi_t^{(r)}\rangle = -iH_{\text{eff}}|\psi_t^{(r)}\rangle$ $\boxed{H_{\text{eff}} = \frac{\nu}{2\pi}\int_x (\partial_x\hat{\theta})^2 + (1-i\eta^2)(\partial_x\hat{\phi})^2}$ $\eta^2 = \frac{2\gamma}{\nu}$

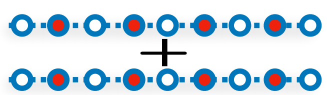
gapless non-Hermitean Hamiltonian

Boson replica quantum master equation

- Boson measurement $H^{(\alpha)} = \dfrac{\nu}{2\pi} \int_x (\partial_x \hat{\phi}^{(\alpha)})^2 + (\partial_x \hat{\theta}^{(\alpha)})^2$

$$\hat{O}^{(\alpha)}_{1,x} = -\frac{1}{\pi} \partial_x \hat{\phi}^{(\alpha)}_x \qquad \hat{O}^{(\alpha)}_{2,x} = m \cos(2\hat{\phi}_x)$$ general case

- $\hat{O}^{(\alpha)}_{2,x}$ couples relative and absolute degrees $\sim m \cos(\sqrt{2}(\hat{\phi}^{(a)}_x \pm \hat{\phi}^{(r)}_x))$

➡ how do  degrees enter? $\longrightarrow$ $\langle \hat{\phi}^{(a)}_x \hat{\phi}^{(a)}_x \rangle = \infty$ nonlinearity irrelevant for $\hat{\phi}^{(a)}_x$

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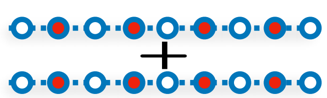
➡ how do  degrees enter? ⟶ $\langle\hat{\phi}^{(a)}_x\hat{\phi}^{(a)}_x\rangle = \infty$ nonlinearity irrelevant for $\hat{\phi}^{(a)}_x$

➡ integrate out $\hat{\phi}^{(a)}_x$ in Gaussian approx. for $\hat{\rho}^{(a)}$

- Non-hermitian Schrödinger equation for relative coordinate

$$\partial_t|\psi^{(r)}_t\rangle = -iH_{\mathrm{eff}}|\psi^{(r)}_t\rangle$$ ➡ cooling into dark state

$$\boxed{H_{\mathrm{eff}} = \frac{\nu}{2\pi}\int_x(\partial_x\hat{\theta})^2 + (1-i\eta^2)(\partial_x\hat{\phi})^2 \; -i\frac{\gamma m}{\pi}\underbrace{\int_x[1-\cos(\sqrt{8}\hat{\phi}_x)]}_{\text{effect of non-linearity}}}$$

➡ non-Hermitian Sine-Gordon: pinning via cos term, depinning via theta term

➡ extract physics in path integral approach

Effective non-Hermitian Hamiltonian and path integral

➡ Sine-Gordon action with complex coefficients

Fendley, Saleur, Zamolodchikov,
International Journal of Modern Physics (1993)

$$\boxed{S = \int_{t,x} \left\{ \frac{K}{16\pi} \left[\frac{1}{\eta} (\partial_t \phi)^2 - \eta (\partial_x \phi)^2 \right] - i\lambda \cos(\phi) \right\}}$$

- 'Wick rotation' brings free part to standard Euclidean (2+0) dimensional form $(x, t) \to (\eta^{\frac{1}{2}} x, i\eta^{-\frac{1}{2}} t)$

➡ RG flow: standard KT flow with complex K, λ

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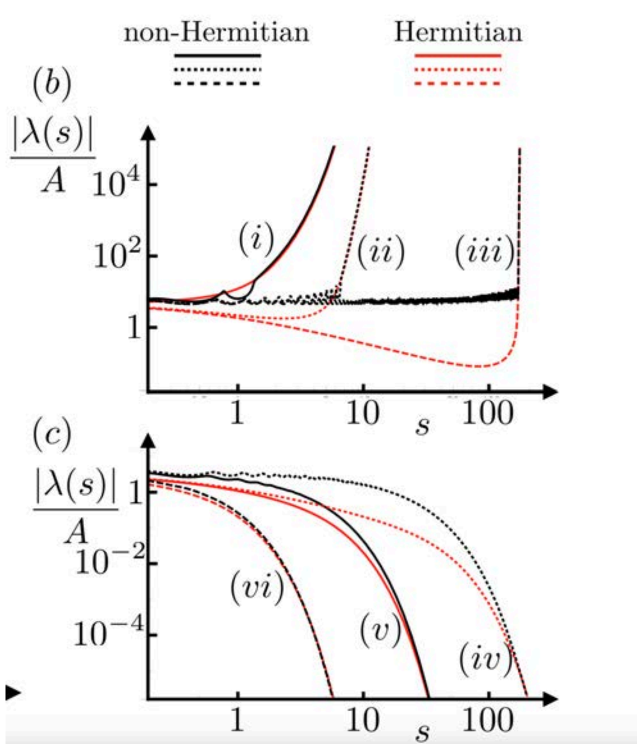


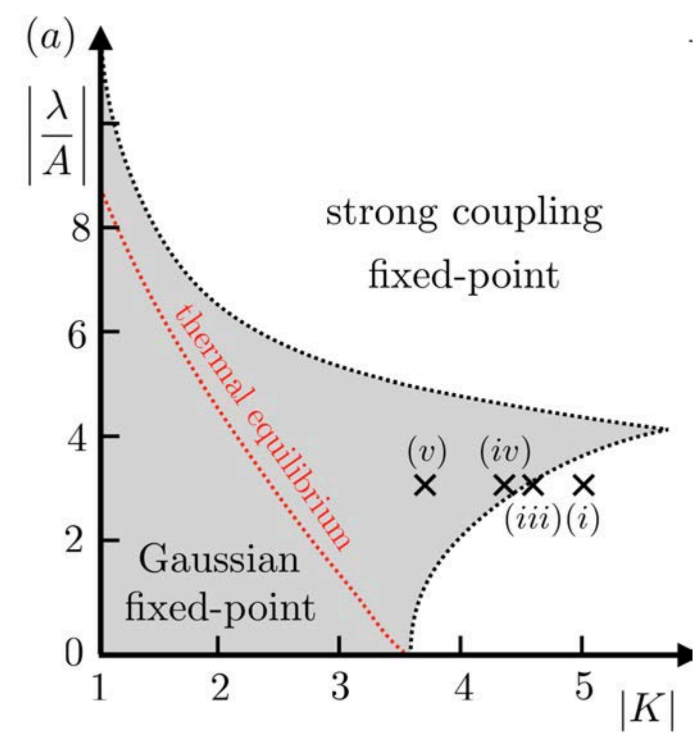
$$\partial_s \lambda = \left(2 - \frac{8\pi}{K}\right)\lambda,$$
$$\partial_s K = -\lambda^2$$

- UV flow modified

➡ shift of phase border

- IR flow reaches standard KT flow

➡ same long wavelength properties



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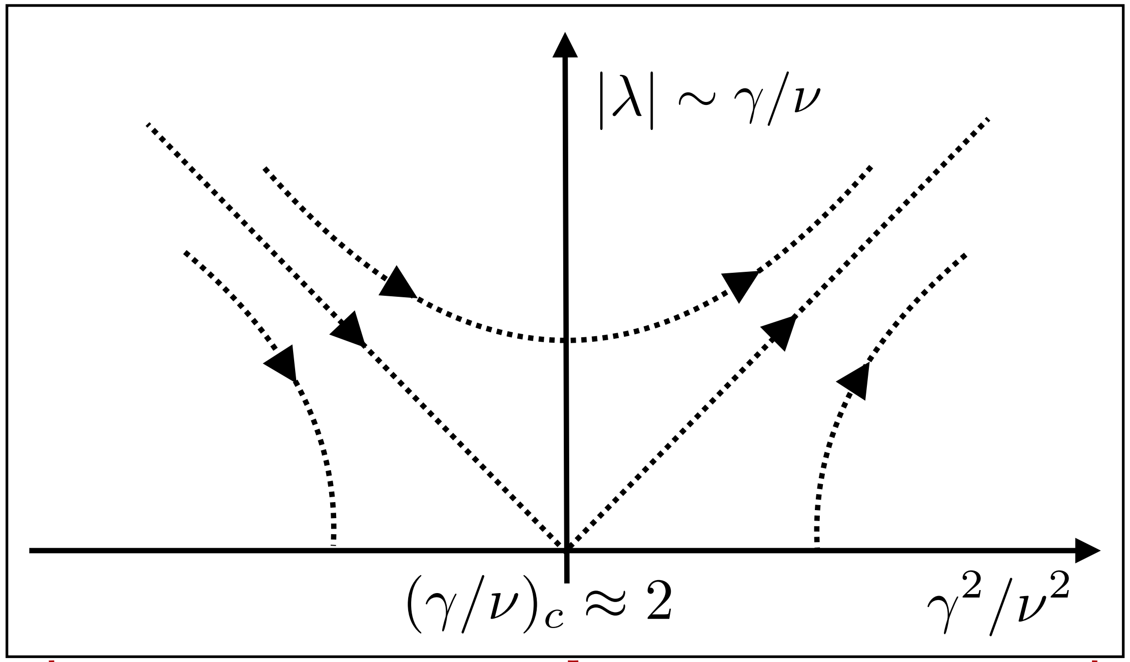
weak measurement
depinned phase



strong measurement
pinned phase

interaction irrelevant: flow to free gapless theory

interaction relevant: flow to free massive theory

➡ gapless generalized CFT phase with algebraic correlations and varying exponent

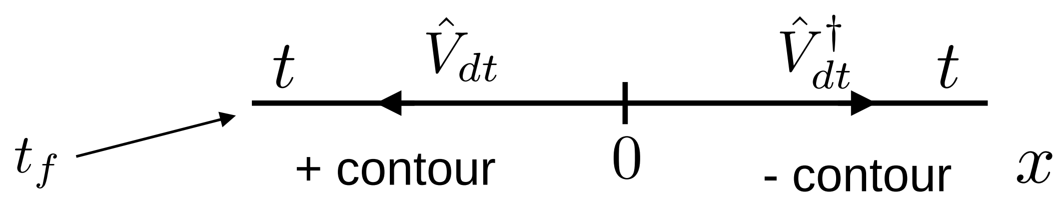
➡ phase transition in the BKT universality class in line with numerics

n replicas: Lindblad-Keldysh 2.0

n-replica Hamilton-Keldysh: Aleiner, Faoro, Ioffe, AoP (2016); Tsuji, Werner, Ueda, PRA (2017); Shenker, Stanford, JHEP (2015); Ansari, Nazarov, JETP (2016)

- Motivation:
  - Generalization of ‘hot’ and ‘cold’ modes?
  - Entanglement entropies?

- Lindblad-Keldysh construction for n replicas
  - evolution operator $\hat{V}_{dt} = \exp\left[-(i\hat{H} + \hat{M}_t^2)dt + \xi \hat{M}_t\right]$ expansion to second order: SSE
  - single replica



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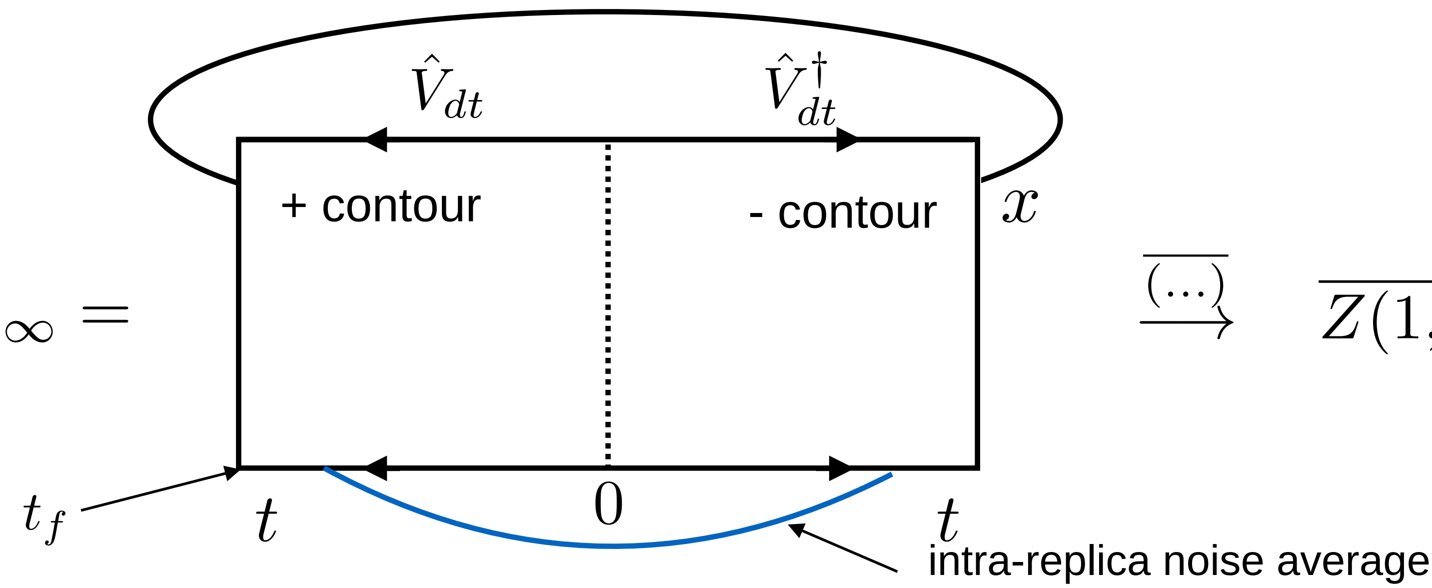
$$Z(1, \{\xi\}) = \mathrm{tr}\rho_{t_f\to\infty} =$$



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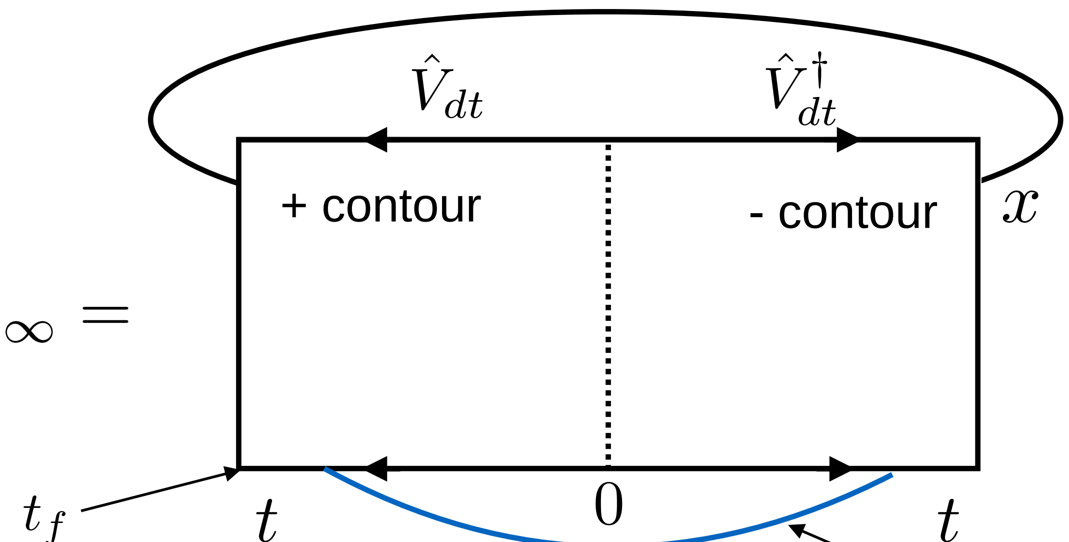

$$\xrightarrow{\overline{(\dots)}} \quad \overline{Z(1,\{\xi\})} \equiv Z(1)$$

$$Z(1,\{\xi\}) = \int \mathcal{D}[\Psi] \exp\left[i(S_{1,H}[\Psi] + S_{1,\xi}[\Psi])\right]$$

$$\Longrightarrow \quad Z(1) = \int \mathcal{D}\Psi \ \exp\left[i(S_{1,H}[\Psi] + S_{1,M}[\Psi])\right]$$

➡ measurement expectation values cancel

➡ Lindblad-Keldysh functional integral reproduced (Herm. Lindblads)

$$S_{1,H}[\Psi] = \sum_{\sigma=\pm} \sigma \int_t \left(\bar{\psi}_\sigma i\partial_t \psi_\sigma - H[\bar{\psi}_\sigma, \psi_\sigma]\right)$$

$$S_{1,\xi}[\Psi] = i \sum_{\sigma=\pm} \int_t \left[M_\sigma^2 - \xi M_\sigma\right]$$

$$S_{1,M}[\Psi] = i \int_t [M_+^2 + M_-^2 - \tfrac{1}{2}(M_+ + M_-)^2]$$

$$= -i \int_t [O_+ O_- - \tfrac{1}{2} O_-^2 + \tfrac{1}{2} O_-^2]$$

$$M = O - \bar{O}$$

n replicas: Lindblad-Keldysh 2.0

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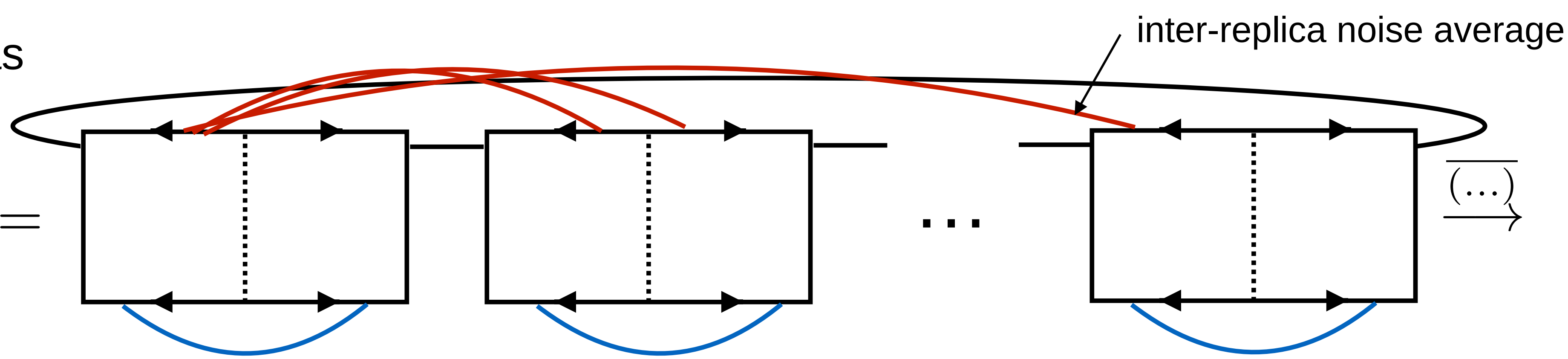
$\overline{Z(n,\{\xi\})} =$



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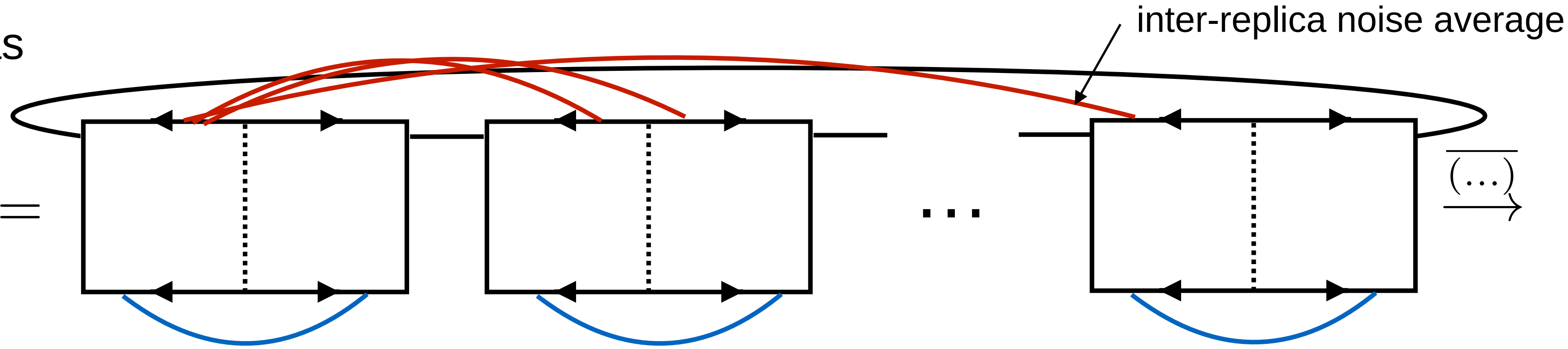

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$$\Longrightarrow \quad Z(n) = \int \mathcal{D}\Psi \ \exp\left[i(S_{n,H}[\Psi] + S_{n,M}[\Psi])\right]$$

➡ collective coupling to noise

➡ structural simplification for linear measurement dynamics

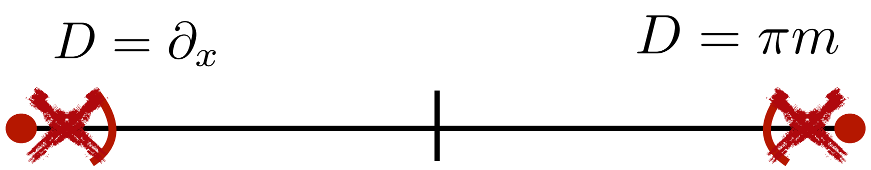
$$S_{n,H}[\Psi] = \sum_{\sigma=\pm}\sum_{l=1}^{n} \sigma \int_t \left(\bar{\psi}_\sigma^{(l)} i\partial_t \psi_\sigma^{(l)} - H[\bar{\psi}_\sigma^{(l)}, \psi_\sigma^{(l)}]\right)$$

$$S_{n,\xi}[\Psi] = i \sum_{\sigma=\pm}\sum_{l=1}^{n} \int_t \left[(M_\sigma^{(l)})^2 - \xi M_\sigma^{(l)}\right]$$

collective coupling to noise!

$$S_{n,M}[\Psi] = i \int_t \sum_{l=1}^{n} \left[\left([M_+^{(l)}]^2 + [M_+^{(l)}]^2\right) - \tfrac{1}{2}\left(\sum_{l=1}^{n} M_+^{(l)} + M_-^{(l)}\right)^2 \right]$$

n replicas: Decoupling of Gaussian theories



- practical importance: reduction to linear / Gaussian bosonic theory in limiting cases

quadratic

- bosonized action in the presence of noise:

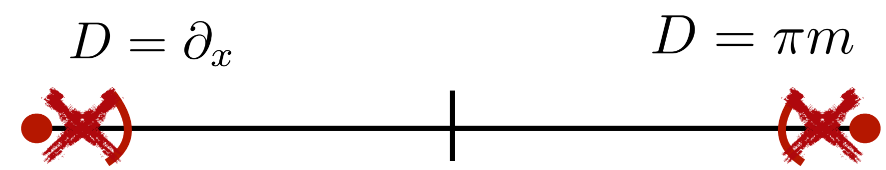
bosonic field

$$S[\phi] = S_{n,H}[\phi] + S_{n,\xi}[\phi]$$

$$S_{n,H}[\phi] = -\frac{1}{2\pi}\sum_{l=1}^{n}\sum_{\sigma=\pm}\sigma\int_{t,x}\phi_\sigma^{(l)}(\partial_t^2 - \partial_x^2)\phi_\sigma^{(l)} \qquad S_{n,\xi}[\phi] = i\sum_{\sigma=\pm}\sum_{l=1}^{n}\int_{t,x}\left[(O_\sigma^{(l)} - \bar{O})^2 - \xi(O_\sigma^{(l)} - \bar{O})\right]$$

$$O_\sigma^{(l)} = D\phi_\sigma^{(l)}, \quad D = \pi m, \partial_x$$

n replicas: Decoupling of Gaussian theories



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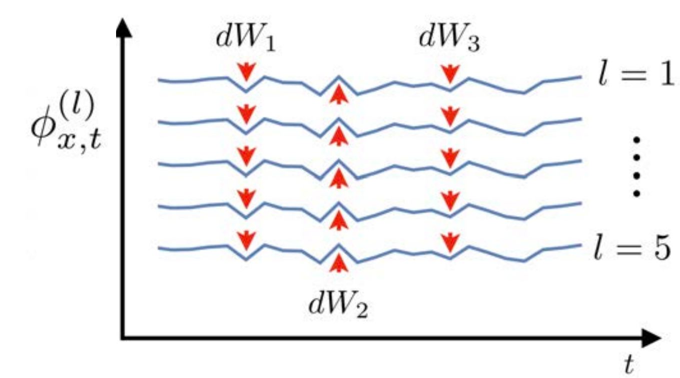
replica space picture



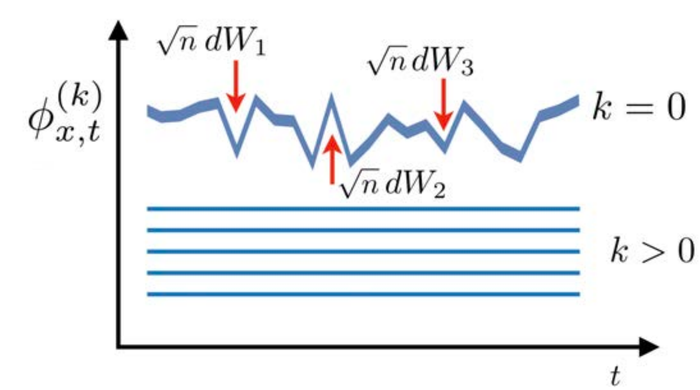
- decoupling of center-of-mass and relative modes
  - Fourier expansion in replica space

$$\phi^{(l)}_{\sigma,t,x} = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} e^{-i\frac{2\pi k l}{n}}\phi^{(k)}_{\sigma,t,x}$$

  - equation of motion $\frac{\delta S}{\delta \phi} = 0$

$$\boxed{\partial_t^2\phi_+^{(k)} = (\partial_x^2 - \frac{\gamma i}{\pi v}D^2)\phi_+^{(k)} + \sqrt{n}\xi\delta_{k,0}}$$

reciprocal replica space picture



➡ no reference to measurement expectation value

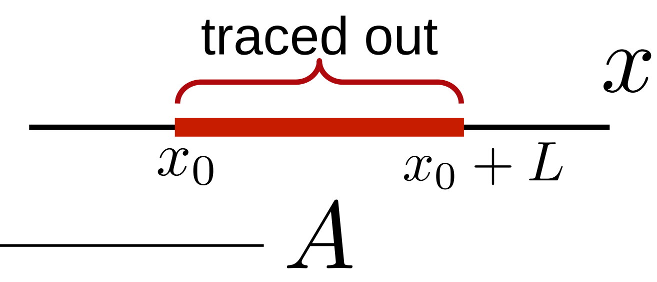
➡ exact decoupling into:

➡ 1 collective ‘hot’ mode, heating to infinite temperature

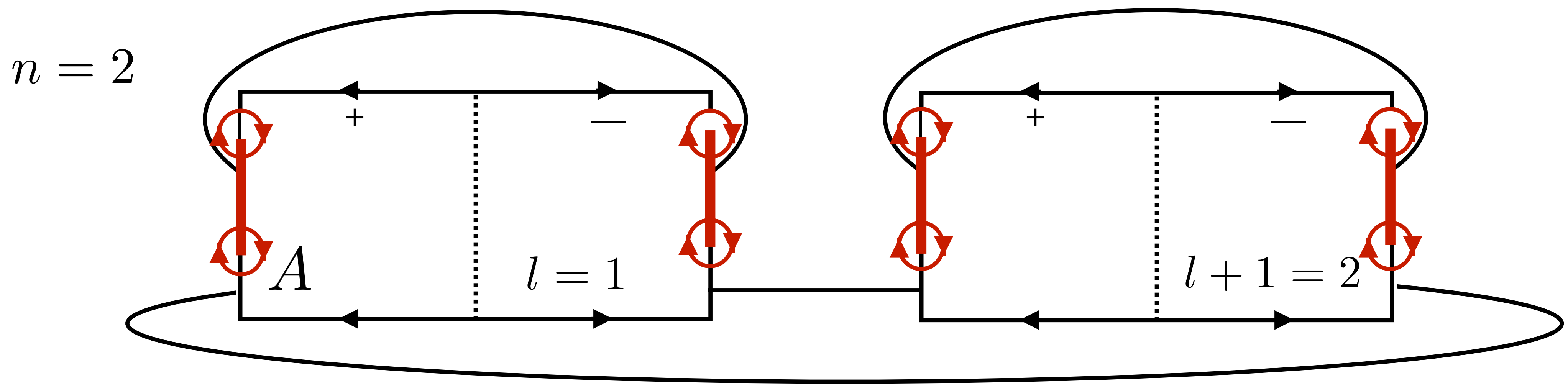
➡ (n-1) ‘cold’ modes, do not ‘see’ the noise, cool to ground state of non-Hermitian Hamiltonian

Calabrese, Cardy, JPA (2009)

Entanglement entropies from replica approach



- Keldysh path integral representation of $Z_A(n, \{\xi\}) = \mathrm{tr}\rho_A^n$



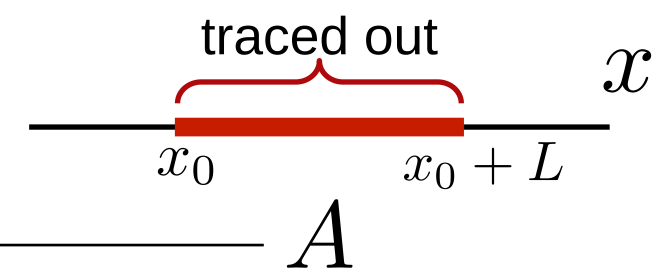
- formulate boundary condition via operator insertion (in the presence of noise):

$$Z_A(n), \{\xi\}) = \int \mathcal{D}[\Psi]\, T\, e^{iS_n[\Psi]} \qquad T\Psi^{(l)}_{+,x,t_f} = \begin{cases} -\Psi^{(l)}_{-,x,t_f} & \forall x \notin A \\ -\Psi^{(l+1)}_{-,x,t_f} & \forall x \in A \end{cases}$$

translation by one in replica space

Entanglement entropies from replica approach

Calabrese, Cardy, JPA (2009)



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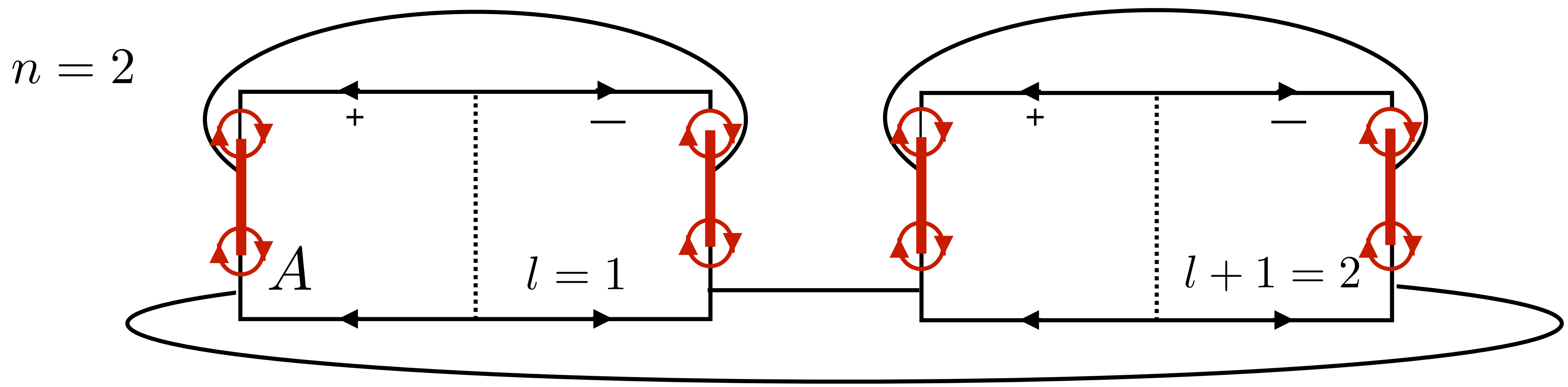

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translation by one in replica space

- e.g. free massless Dirac fermions (after bosonization into equivalent Luttinger Hamiltonian): factorization

$$Z_A(n,\{\xi\}) = \langle \prod_{k=1}^{n} \exp\left(-\sqrt{2}i\frac{k}{n}\int dx[\delta(x-x_0) - \delta(x-(x_0+L))]\phi^{(k)}_{c,x,t}\right)\rangle$$

ground states: Casini, Fosco, Huerta, J. Stat. Mech. (2005)

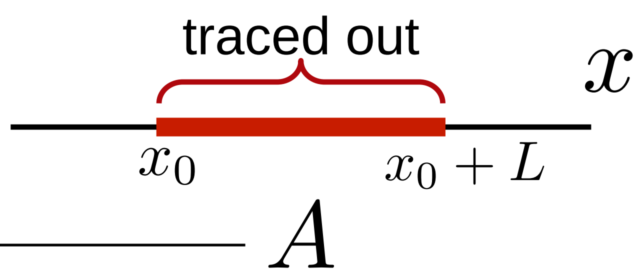
➡ boundary conditions appear as opposite charges

➡ $k = 0$ mode does not contribute!

➡ $k \neq 0$ independent of noise ξ !

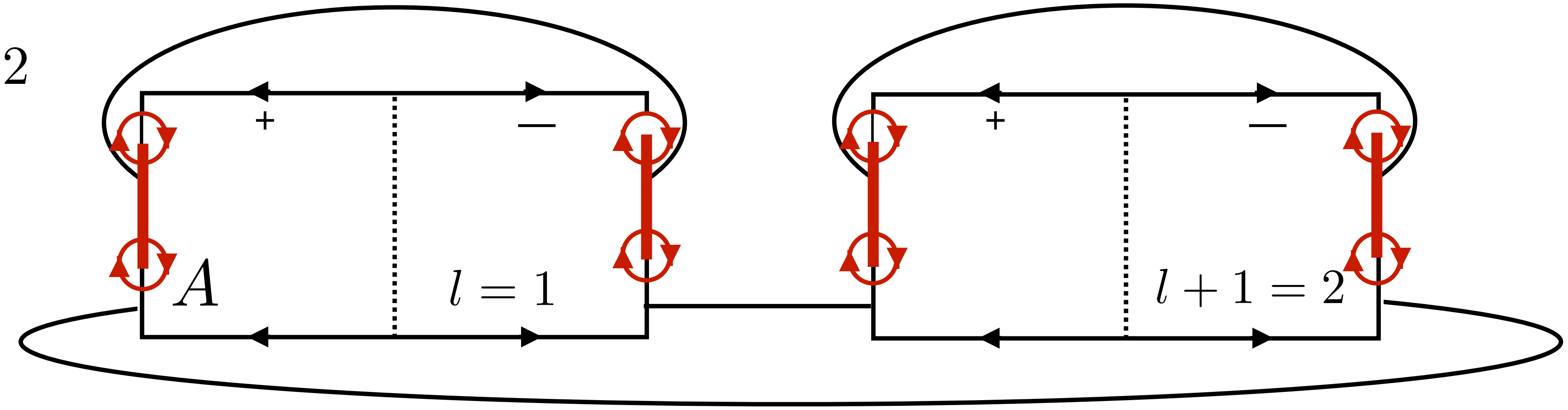
Calabrese, Cardy, JPA (2009)

Entanglement entropies from replica approach



- Keldysh path integral representation of $Z_A(n,\{\xi\}) = \mathrm{tr}\rho_A^n$

$n = 2$



- n-th Rényi entropy

$$S_n = \frac{1}{1-n} \overline{\log Z_A(n,\{\xi\})}$$

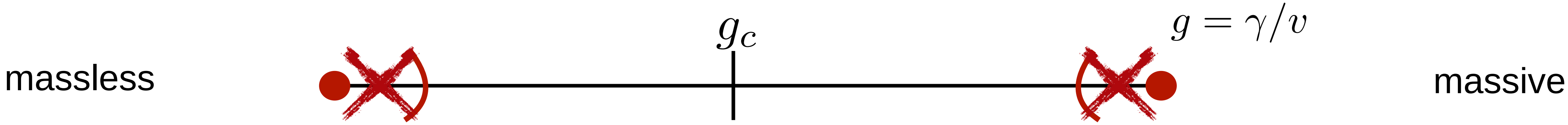
- von Neumann entropy

$$\boxed{S = \lim_{n\to 1} S_n = \frac{2}{3}\langle \phi_{c,x_0}^{(k>0)} \phi_{c,x_0+L}^{(k>0)} \rangle}$$

➡ correlator in the Gaussian dark state wave function

Entanglement transition from replica approach

- focus on von Neumann entropy S in Gaussian limiting regimes



$$S = \frac{1}{3}c(\gamma)\log(m^{-1}) \sim L^0$$

- saturation to area law

Entanglement transition from replica approach

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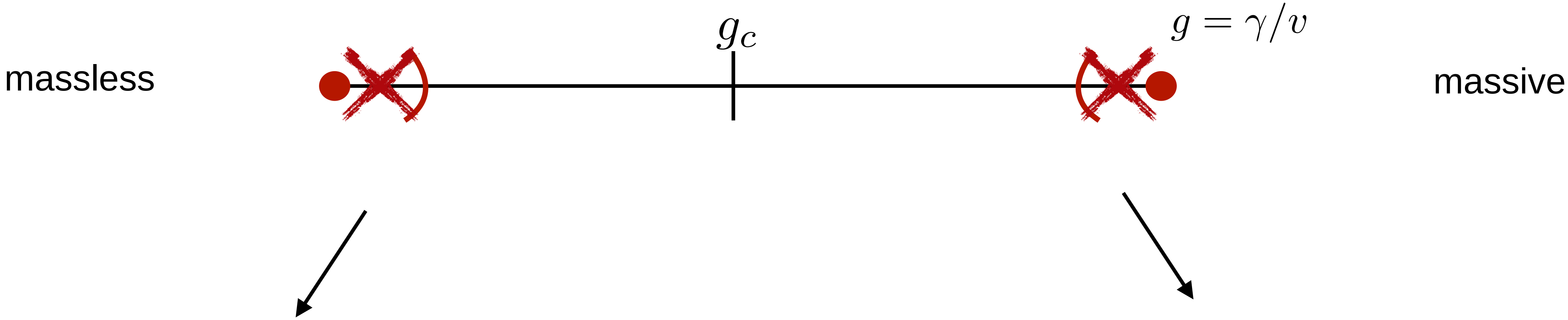

$$S = \frac{1}{3}c(\gamma)\log(L)$$

- sub-volume log-law
- $c(\gamma \to 0) \to 1$

➡ ground state entropy of massless Dirac

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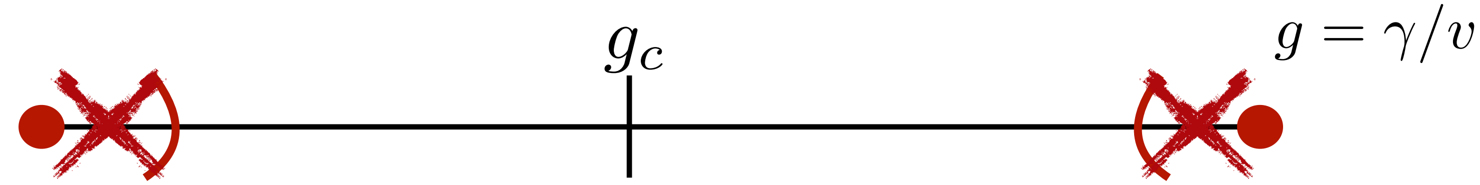
massless



massive

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➡ ground state entropy of massless Dirac

- in Gaussian state:

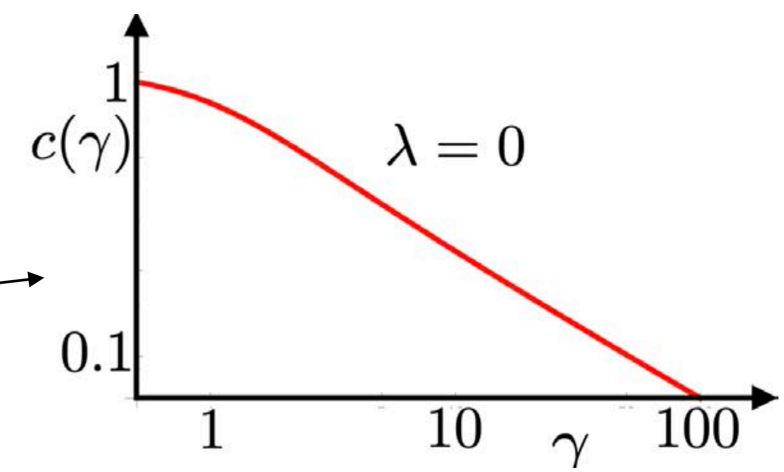
$$c(\gamma \to \infty) \sim \gamma^{-1/2}$$

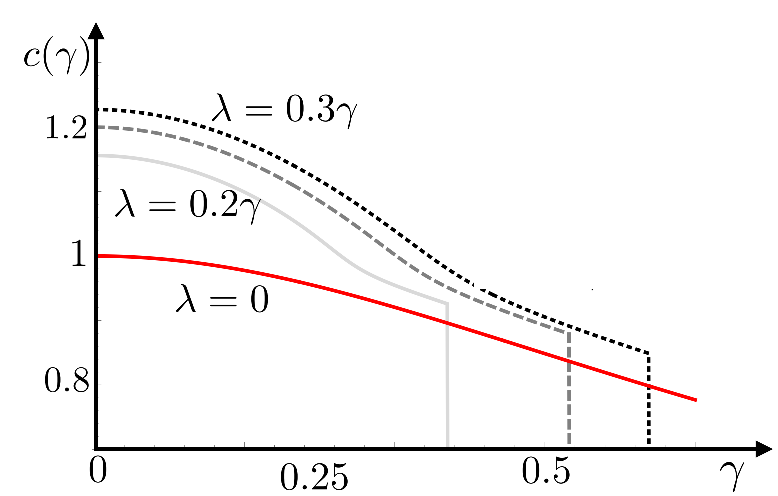
➡ compatible with numerics in critical phase

- with RG improvement, qualitatively similar to numerics

$$S = \tfrac{1}{3}c(\gamma)\log(m^{-1}) \sim L^0$$

- saturation to area law





Entanglement transition from replica approach

- focus on von Neumann entropy S in Gaussian limiting regimes

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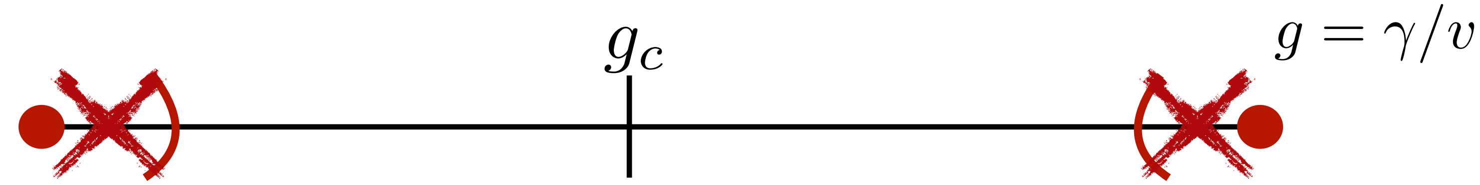

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➡ compatible with numerics in critical phase

- non-commuting limit: $\gamma = 0$
  finite temperature initial state

$$S \sim L$$

➡ volume law <—> finite temperature massless Dirac

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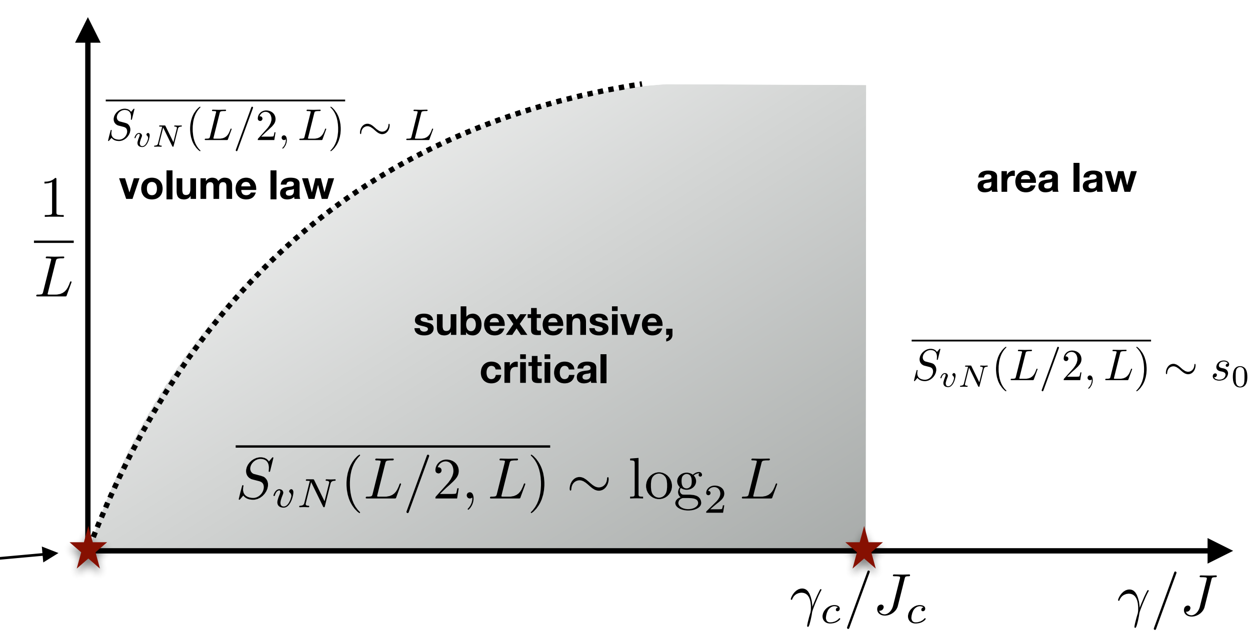
- saturation to area law



Entanglement transition from replica approach

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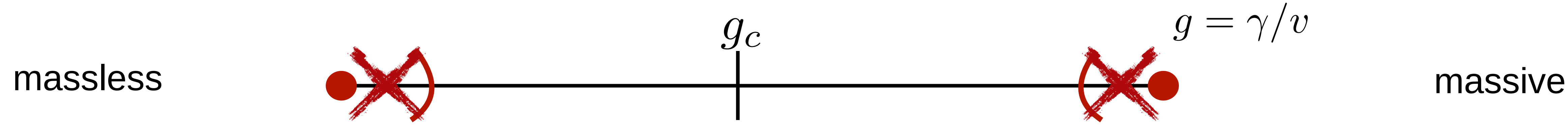

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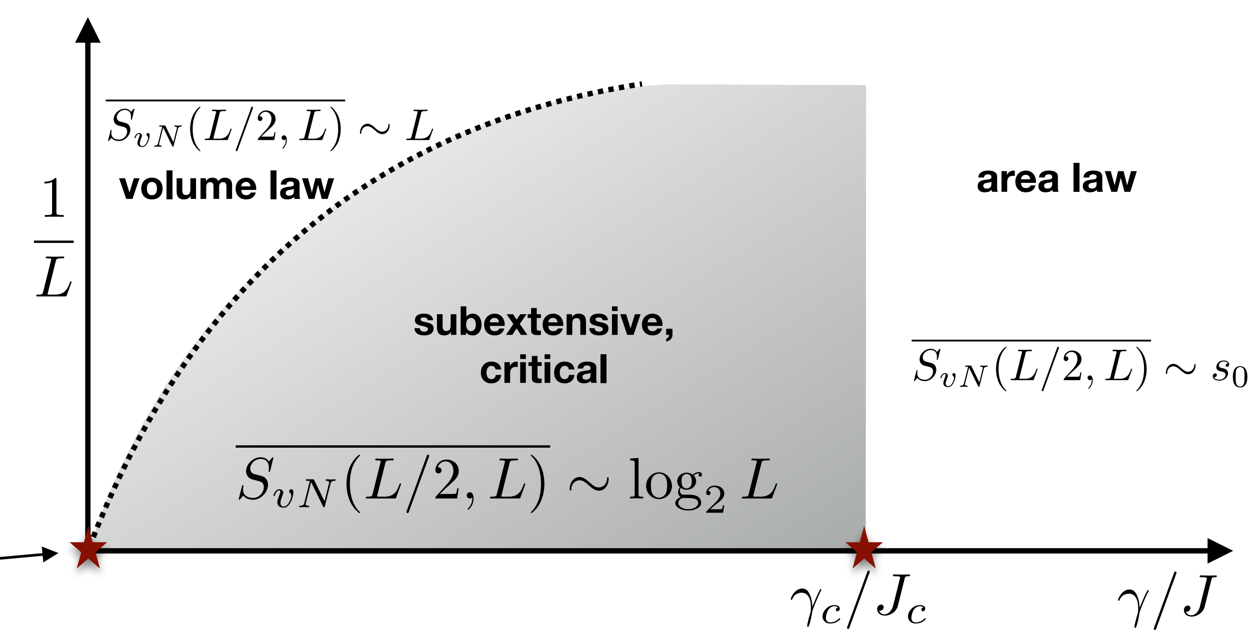


summary:

➡ underpins entanglement transition at finite critical g

➡ picture qualitatively in line with numerics

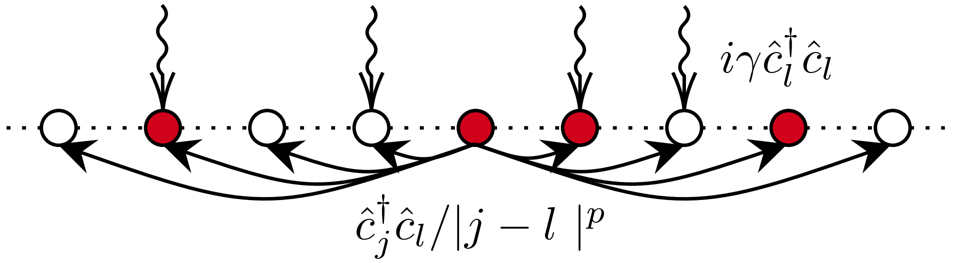
Other incarnations of measurement induced phase transitions

Th. Mueller, SD, M. Buchhold, arxiv:2105.08076

see also Minato et al, arXiv:2104.09118; Block et al. arxiv:2104.13372

- long ranged hopping model

$$\hat{H}_{LR} = \sum_{l \neq m} \frac{\hat{c}_l^\dagger \hat{c}_m}{|l-m|^p} \qquad 1 < p < \infty$$



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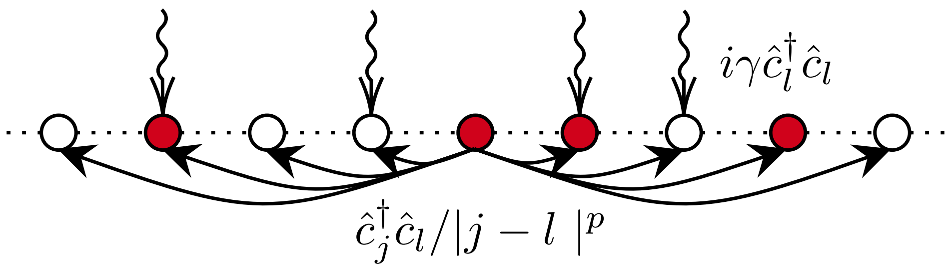
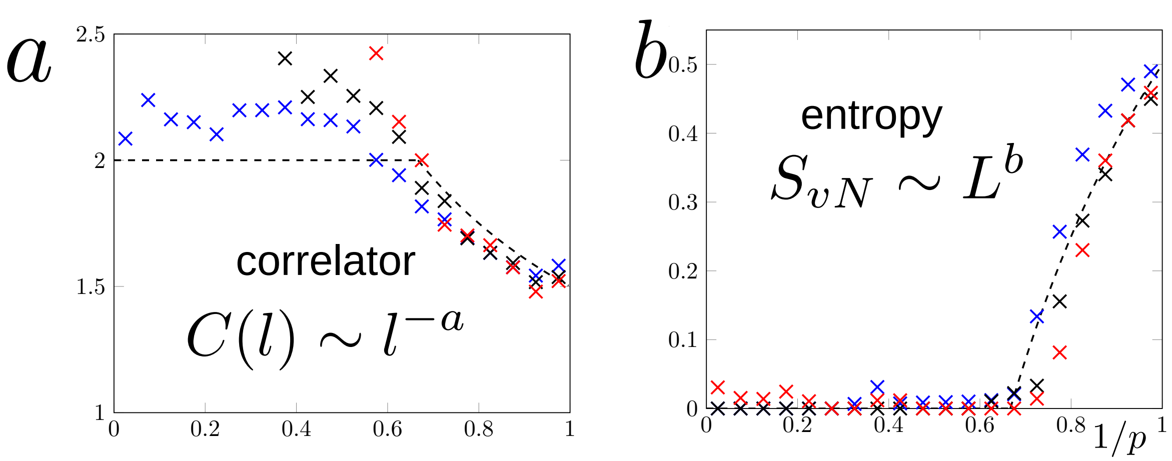
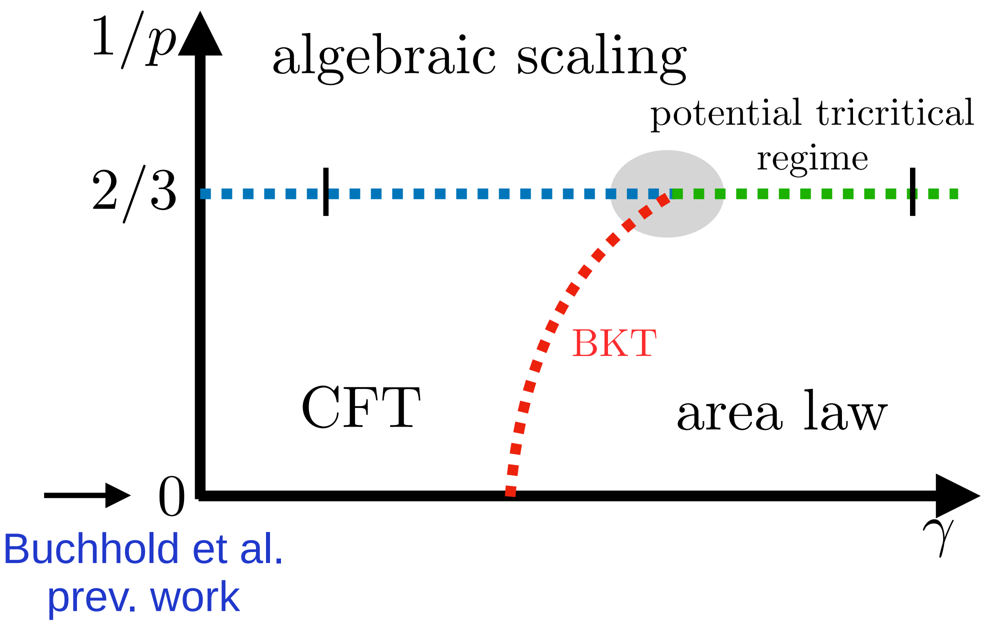


- new scaling behavior & new phase transition





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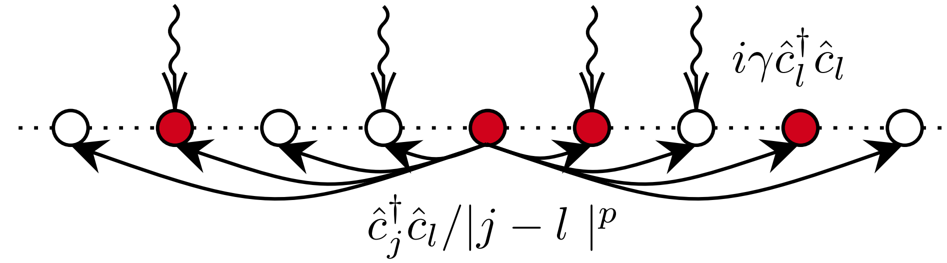

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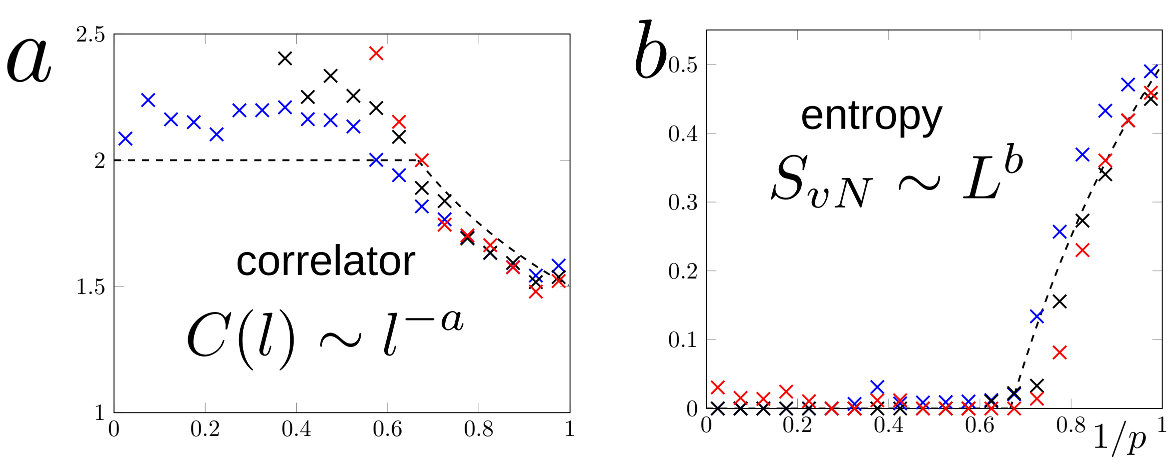
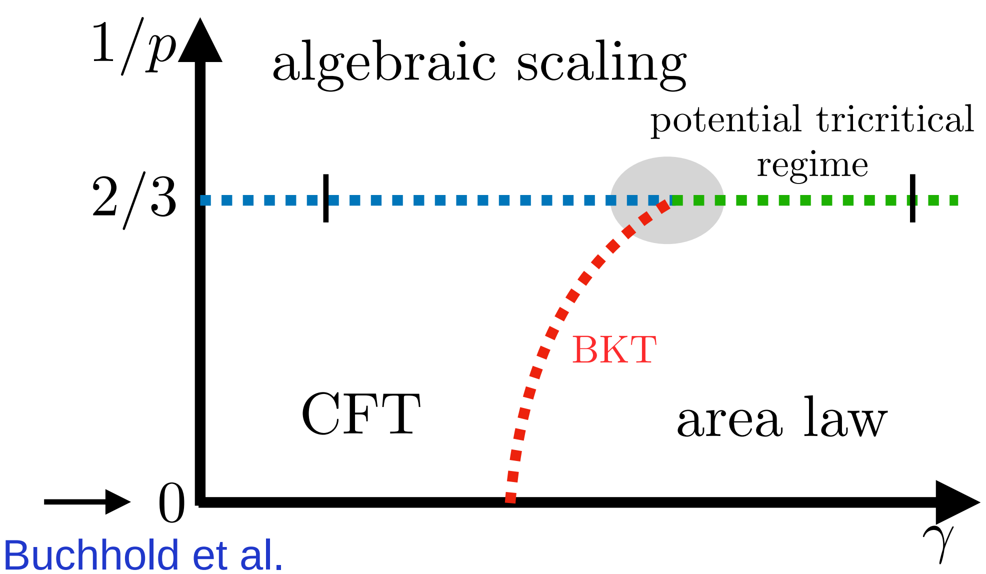




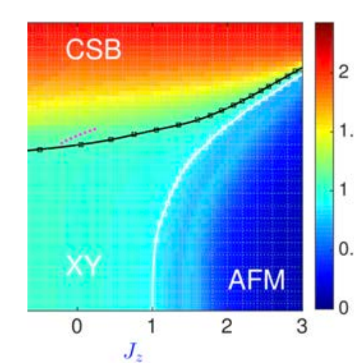
Buchhold et al. prev. work

- replica-bosonization approach explains numerical findings: new cos non-linearity from phase fluctuations
  - critical point of novel transition p = 3/2
  - scaling behavior

$$C(l) \sim \begin{cases} l^{-2} & p > 3/2 \\ l^{-(p+1/2)} & 1 < p < 3/2 \\ \text{ill-defined} & p < 1 \end{cases} \qquad S_{vN}(l) \sim \begin{cases} \log_2 l & p > 3/2 \\ l^{3/2-p} & 1 < p < 3/2 \\ \text{ill-defined} & p < 1 \end{cases}$$

- striking parallel to ground state phase diagram

Maghrebi et al., PRL (2017)



Other incarnations of measurement induced phase transitions

T. Botzung, SD, M. Mueller, arxiv:2106.10092

- robustness of log-area transitions: other competition patterns
  - critical phase stabilized by engineered dissipation

$$c_l = c_{\langle i,j \rangle} = (\sigma_i^+ + \sigma_j^+)(\sigma_i^- - \sigma_j^-)$$

  - area law phase stabilized by staggered Hamiltonian

$$H = V \sum_i^N (-1)^i \sigma_i^z$$

dissipation vs. dissipation: Regemortel et al., PRL (2021)

unique Dicke dark state:

$$|D_N^{(k)}\rangle \approx \Big(\sum_{i=1}^N \sigma_i^+ \Big)^k |{\downarrow}\rangle^{\otimes N}$$

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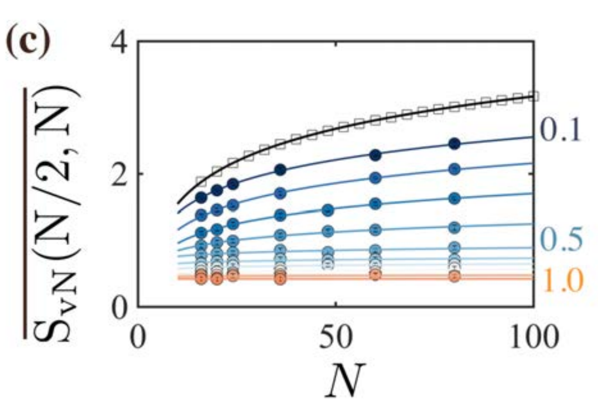
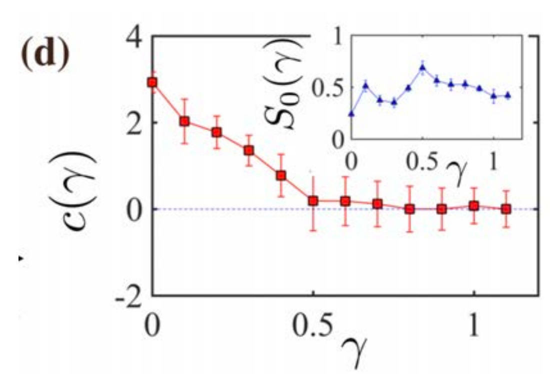
- interacting model -> trajectory MPS approach up to 80 sites

see also Fuji, Ashida, PRB (2020)

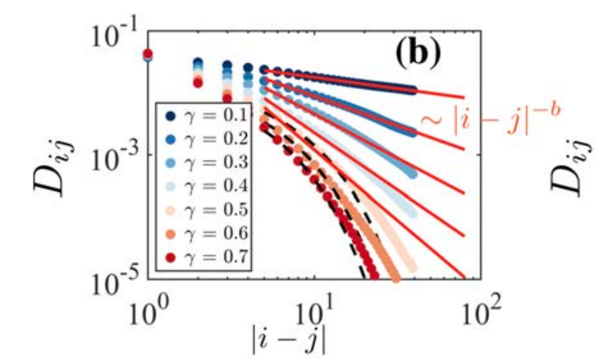
entropy scaling



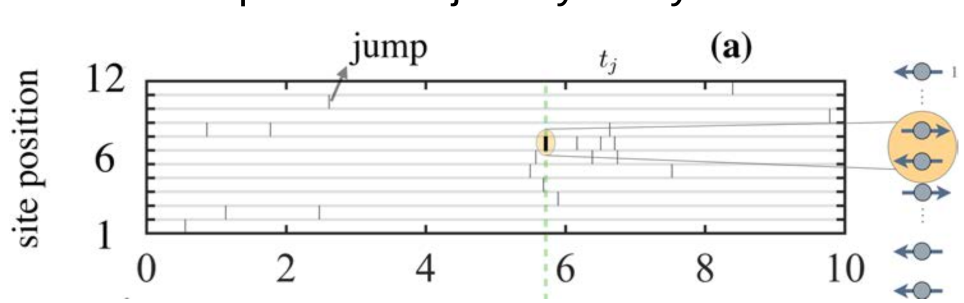
crit. point via eff. central charge



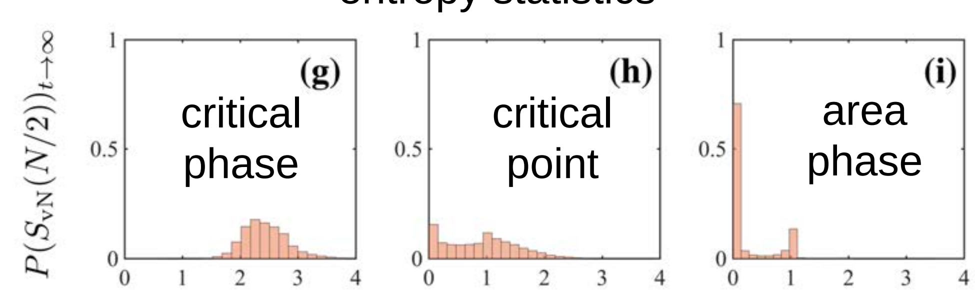
scaling of correlation function



quantum trajectory analysis



entropy statistics



➡ full phenomenology established in large scale MPS approach

➡ bosonization study?

Concept for experimental observation

with T. Bintener, M. Buchhold, J. Yago, P. Kirton, A. Daley (Strathclyde)

- recent protocols for measuring entanglement entropies work for deterministic dynamics only

  Elben et al., PRL (2018); Vermersch et al. PRA (2019)

- ideas for spin systems: entropy of auxiliary entangled system

  Gullans, Huse, PRL (2020);

  exp. in trapped ions (Monroe group): Noel et al. arxiv (2021)

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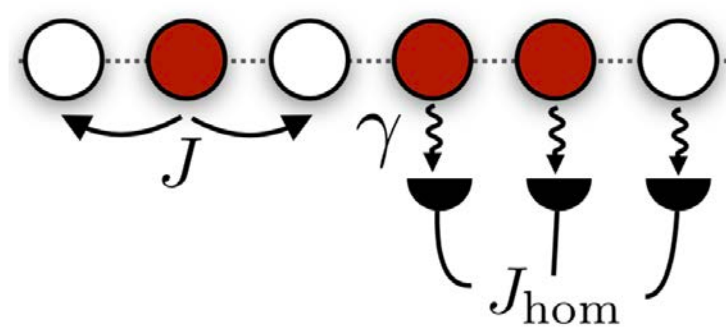


- here: witness phase transition directly via recorded trajectories:
  homodyne detection

- recall form of recorded signal

$$J_{i,t} = \langle \hat{n}_i \rangle_t + \frac{\Delta W_{i,t}}{\Delta t}$$

homodyne current — trajectory wavefunction expectation value — noise

$$\overline{\Delta W_{i,t}} = 0$$

$$\overline{\Delta W_{i,t} \Delta W_{j,t'}} = 4\gamma \Delta t \delta_{t,t'} \delta_{i,j}$$

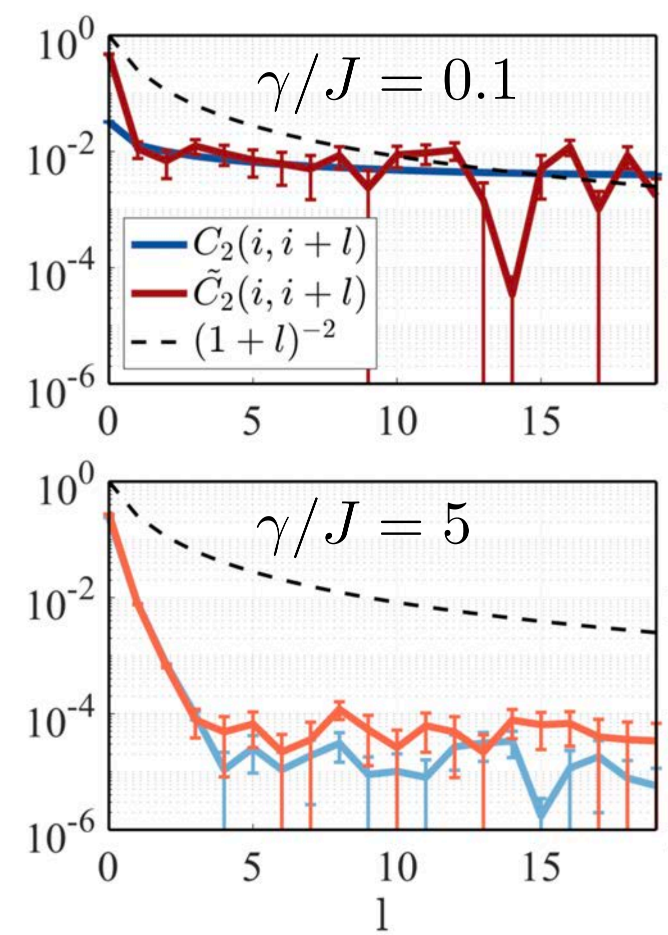
- correlation functions from averaging over exp. runs

$$\overline{\langle \hat{n}_i \rangle_c(t) \langle \hat{n}_j \rangle_c(t)}$$

- platforms: superconducting circuits/Rydberg tweezers?

Majer et al. Nature (2007); Mallet et al. Nat. Phys. (2009); Nagiloo et al. Nature Comm. (2016); Gu et al. Phys. Reports (2017)

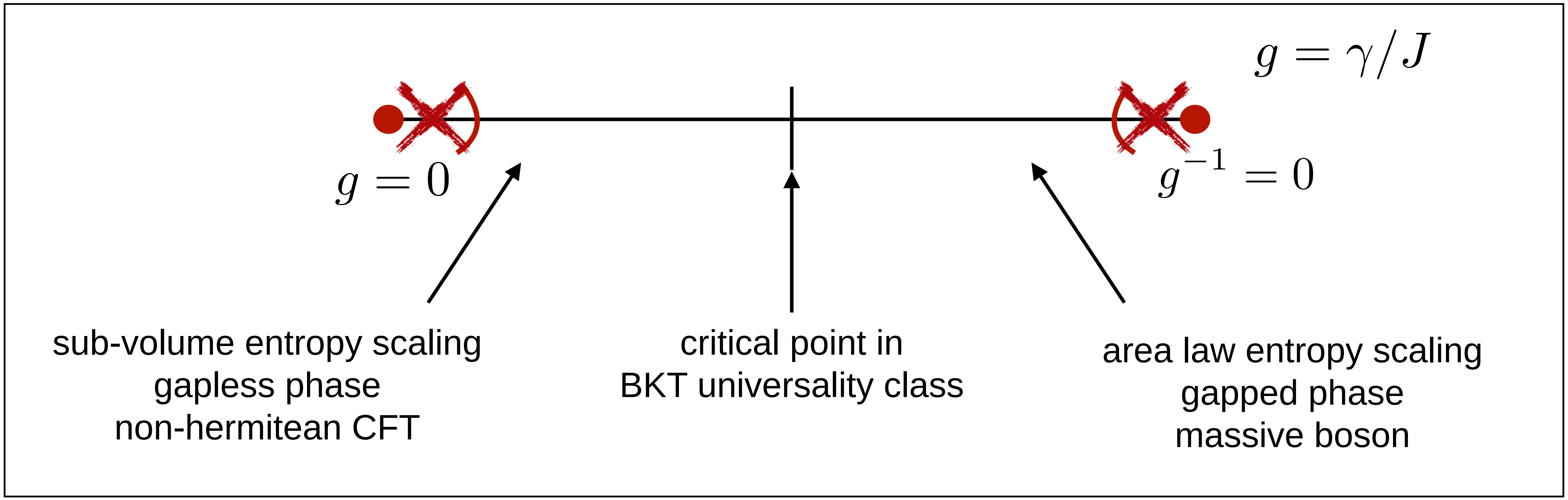
connected covariance (blue), signals (red)



Summary lecture III

O. Alberton, M.Buchhold, SD PRL 126. 170602 (2021)
M. Buchhold, Y. Minoguchi, A. Altland, SD, arXiv:2102.08381

- monitored fermions: new type of measurement induced phase transition



- nonlinear-in-state ‘observables’ beyond entanglement entropy
- ‘hot’ and ‘cold’ modes as relevant degrees of freedom for the transition via replica field theory
- physical picture: transition induced by pinning into measurement operator eigenstates

Directions:

- area-to-volume law transitions as incomplete decoupling of ‘hot’ and ‘cold’ modes?
- stabilization of pure state phases by continuous measurement?
- experimental observability in weakly monitored quantum simulation platforms?

integrability vs. non-integrability
O. Lunt, A. Pal, PRR (2020)

purification perspective:
Gullans, Huse, PRX (2020);
entropy measurement
Noel et al. arxiv (2021)