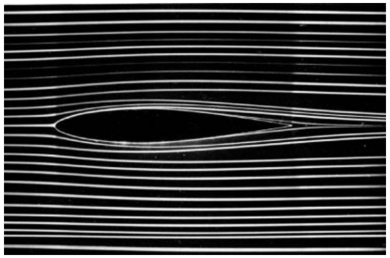
Lecture 2

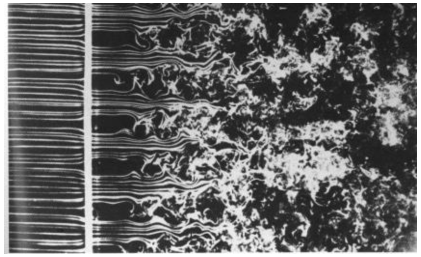
Blackboard interlude 2

- Laminar-turbulent transition in convection
  - Linear stability analysis
  - Supercritical transition

- Laminar-turbulent transition in pipe
  - Linear stability analysis
  - Subcritical transition

What is turbulence?





laminar flow

steady

predictable

laminar-turbulent transition

critical behavior

fully-developed turbulence

fluctuating

unpredictable

Re < 1600

Re ~ 2000

Re > 10^5

$$\nabla \cdot \mathbf{u} = 0$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + \mu \nabla^2 \mathbf{u}$$

$$\mathrm{Re} = \frac{UL}{\nu}$$

$$\upsilon = \mu/\rho$$

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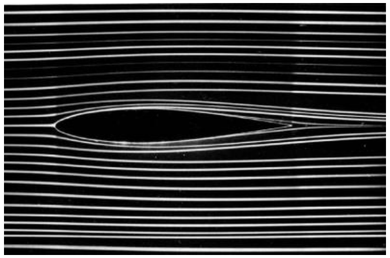

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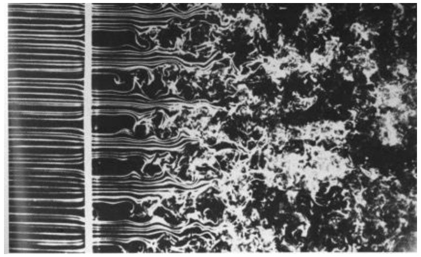

fully-developed turbulence

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unpredictable

$Re < 1600$ | **$Re \sim 2000$** | $Re > 10^5$

- Boundary conditions <u>not</u> periodic
- Turbulence generated by instabilities
  - Linear and long-wavelength
  - Nonlinear and spatially-localized
  - Not artificial noise
- Turbulence interacts with mean flow
  - Turbulence can generate emergent mean flows

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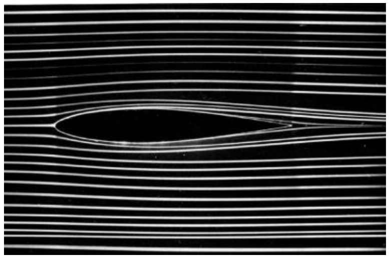

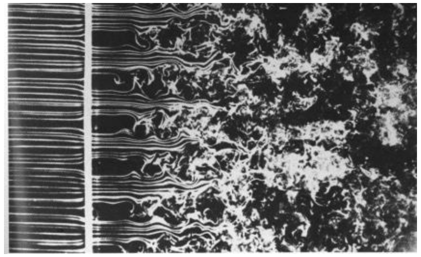

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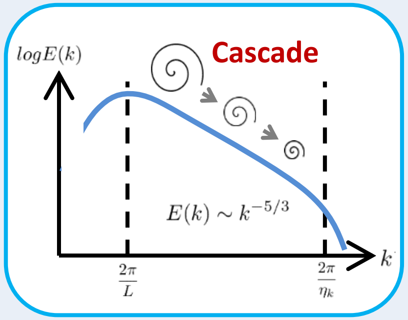
Fully-developed turbulence

- **Energy cascade: Non-equilibrium steady state**
  Energy transfers step by step → power law
- **Dissipative anomaly:**
  energy dissipation rate = 0 when viscosity = 0
  energy dissipation rate ≠ 0 when viscosity → 0

$$\epsilon = \nu |\nabla u|^2 \neq 0 \text{ as } \nu \to 0$$



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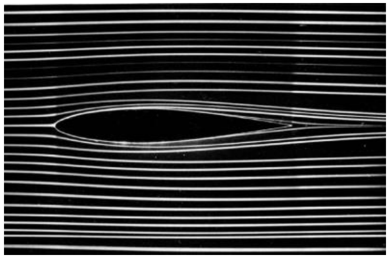

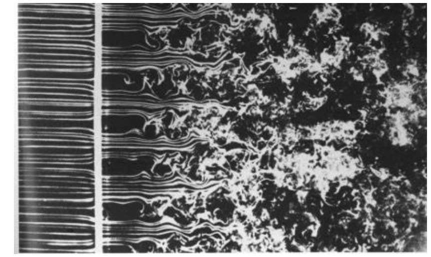

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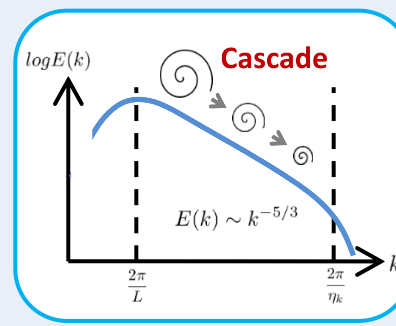
Transitional pipe flow

Puff decays

Splitting puffs

laminar | metastable puffs | spatiotemporal intermittency | expanding slugs

Re 1775 2100 2500



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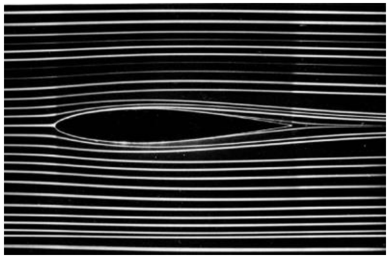

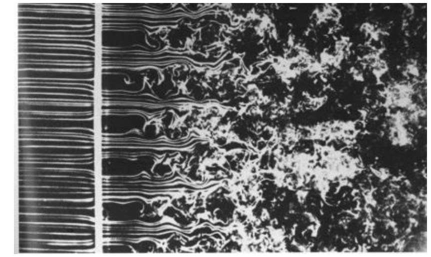

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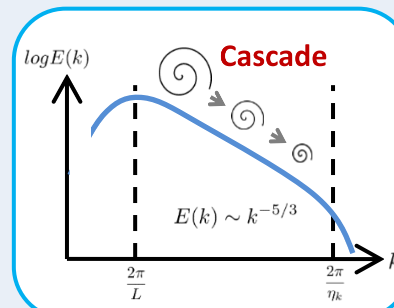
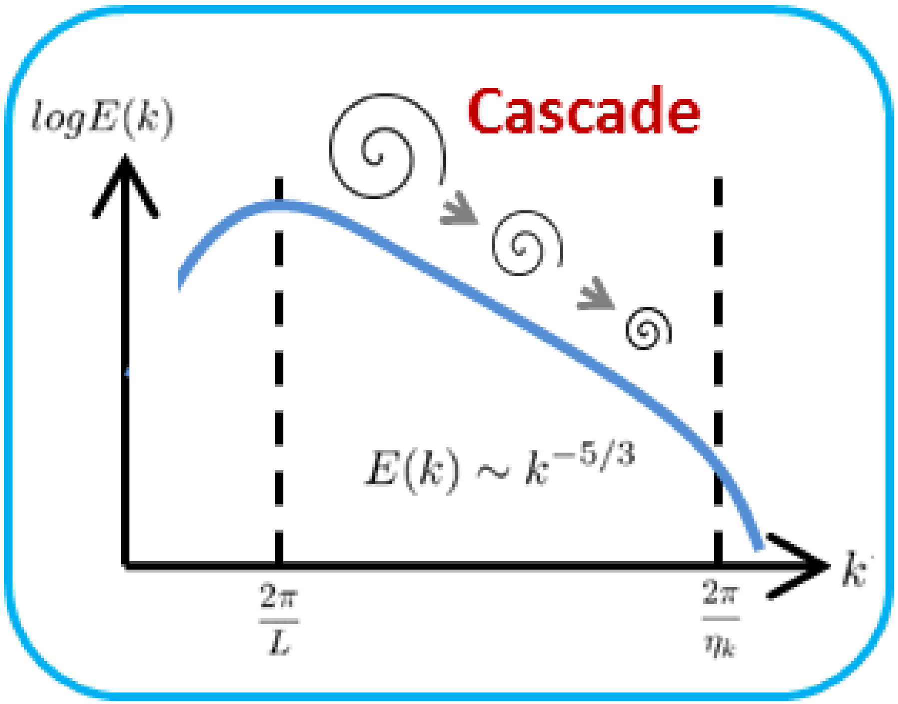
Re — 1775 — 2100 — 2500

Turbulence & mean flows



$$\epsilon = \nu |\nabla u|^2 \neq 0 \text{ as } \nu \to 0$$

Energy cascade



- Eddies spin off other eddies in a Hamiltonian process.
  - Does not involve friction!
  - Hypothesis due to Richardson, Kolmogorov, …
- Implication: viscosity will not enter into the equations

Kolmogorov's similarity hypotheses

- Dimensional analysis
- $E(k) = E(k, \varepsilon, \nu, L)$
  - Kolmogorov length $\eta_K = (\nu^3/\varepsilon)^{1/4}$
- $E(k) = (\nu^2/\eta_K)\, F(k\eta_K, kL)$
  - Complete similarity as $kL \rightarrow \infty$
- $E(k) = (\nu^2/\eta_K)\, F(k\eta_K, \infty)$
  - Complete similarity as $k\eta_K \rightarrow 0$
- $E(k) = F(0, \infty)\, \varepsilon^{2/3}\, k^{-5/3}$

Kolmogorov's similarity hypotheses

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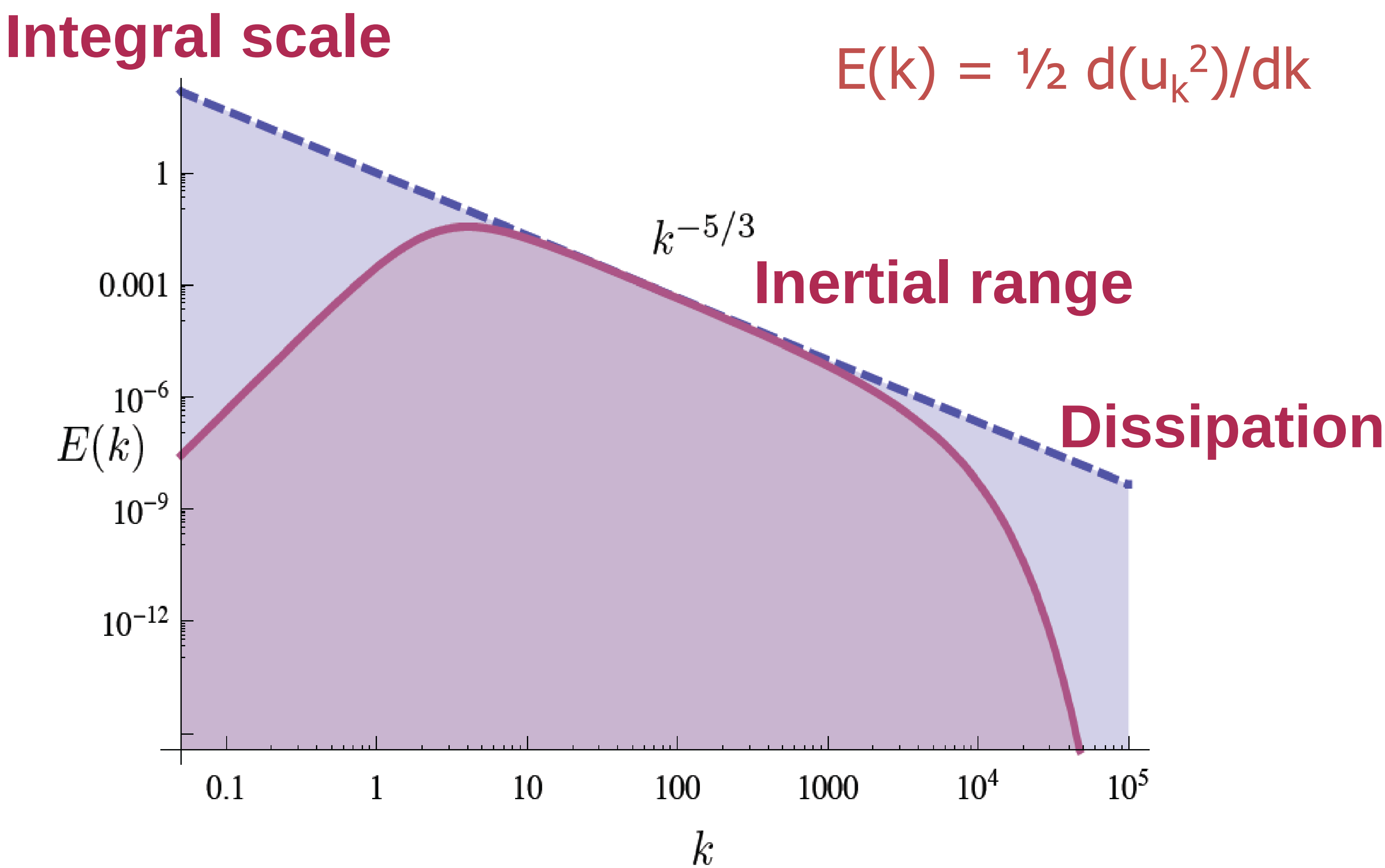
The energy spectrum

$E(k) = ½\, d(u_k^2)/dk$



Kolmogorov's similarity hypotheses

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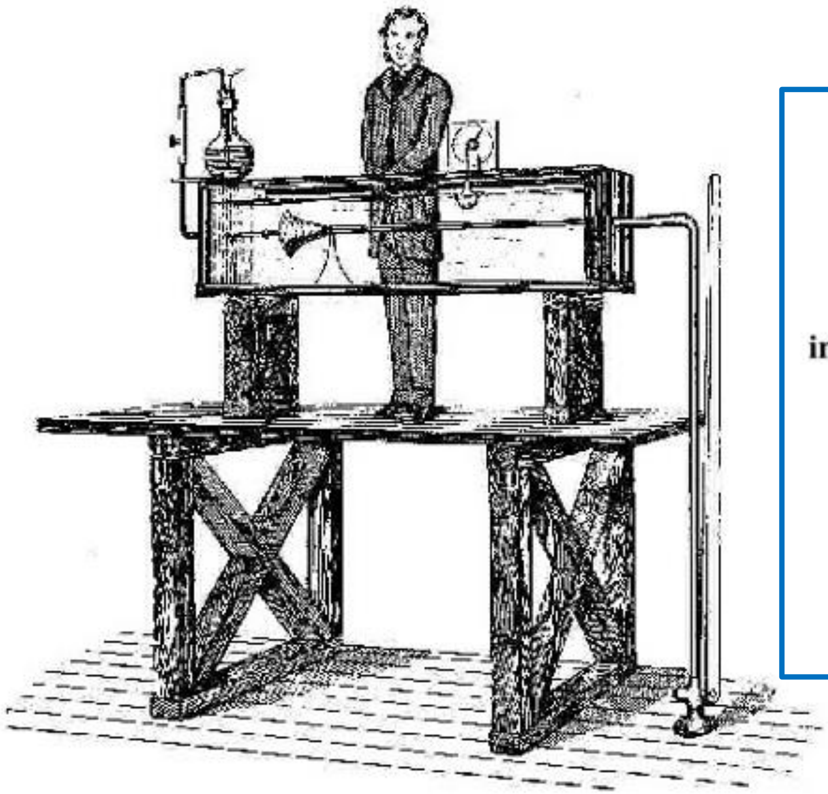
- $E(k) = E(k, \varepsilon, \nu, L)$
  - Kolmogorov length $\eta_K = (\nu^3/\varepsilon)^{1/4}$
- $E(k) = (\nu^2/\eta_K)\, F(k\eta_K, kL)$
  - Incomplete similarity as $kL \rightarrow \infty$
- $E(k) = (\nu^2/\eta_K)\, F_1(k\eta_K)\, (kL)^\eta$
  - Complete similarity as $k\eta_K \rightarrow 0$
- $E(k) = F_1(0)\, \varepsilon^{2/3}\, k^{-5/3}\, (kL)^\eta$

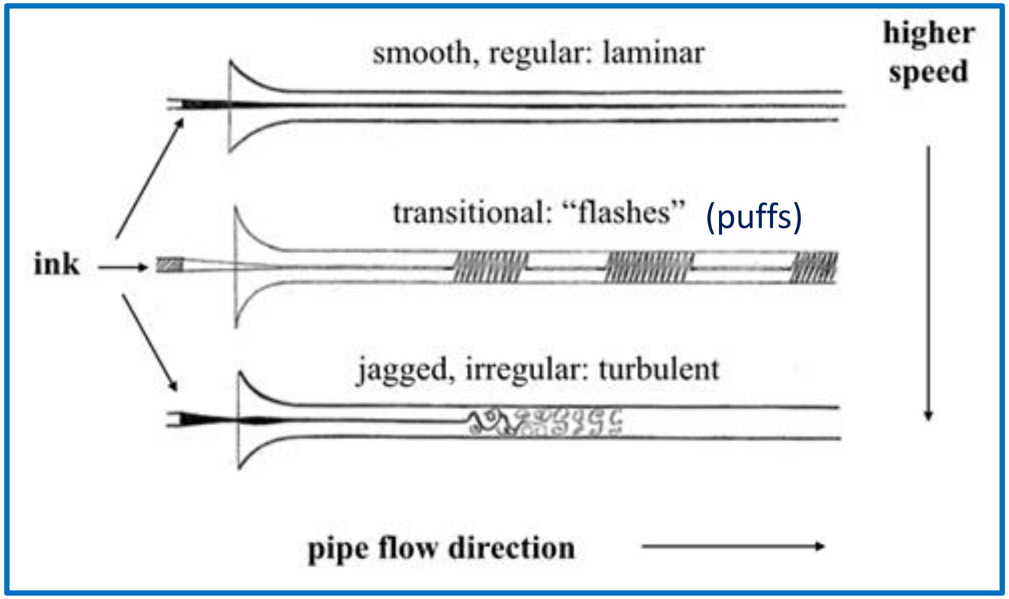
Intermittency exponent

2. Puffs

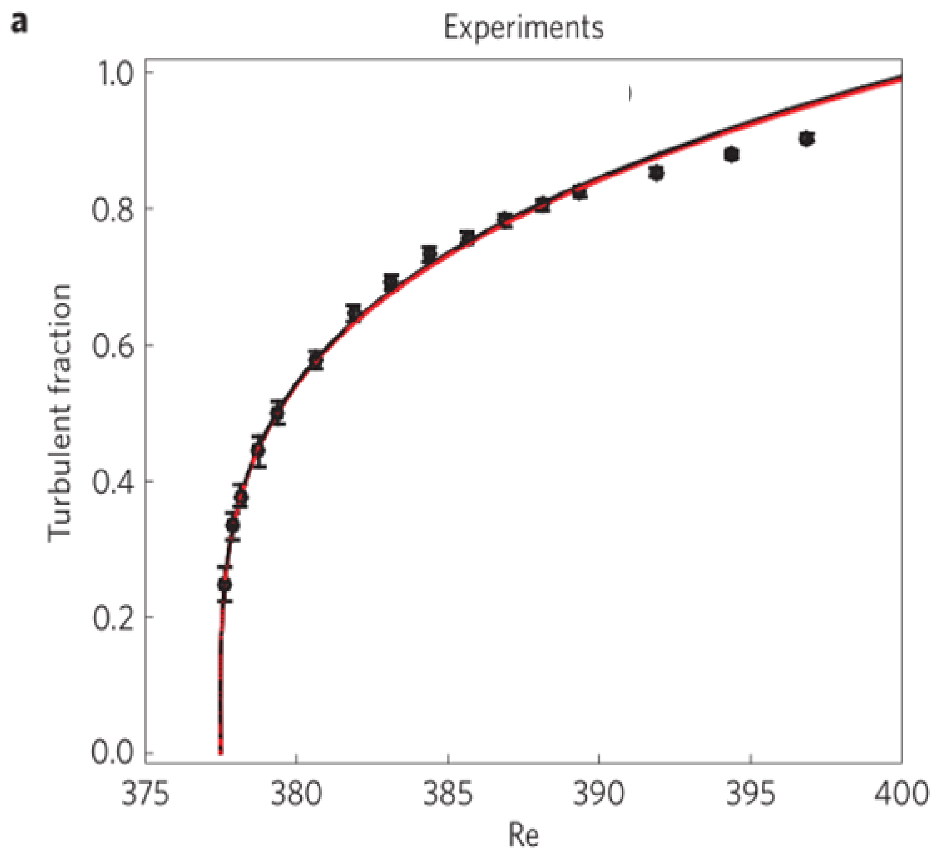
Transitional turbulence in pipe flow: puffs

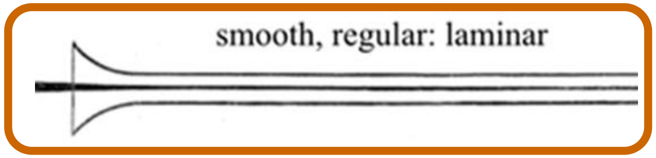
- Reynolds' original pipe turbulence (1883) reports on the transition
- Defined Reynolds' number $\mathrm{Re} = \dfrac{UL}{\nu}$

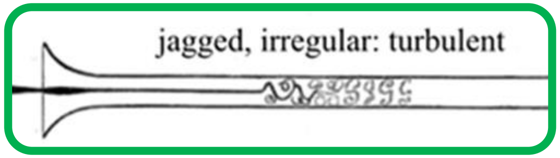




How much turbulence is in the pipe?







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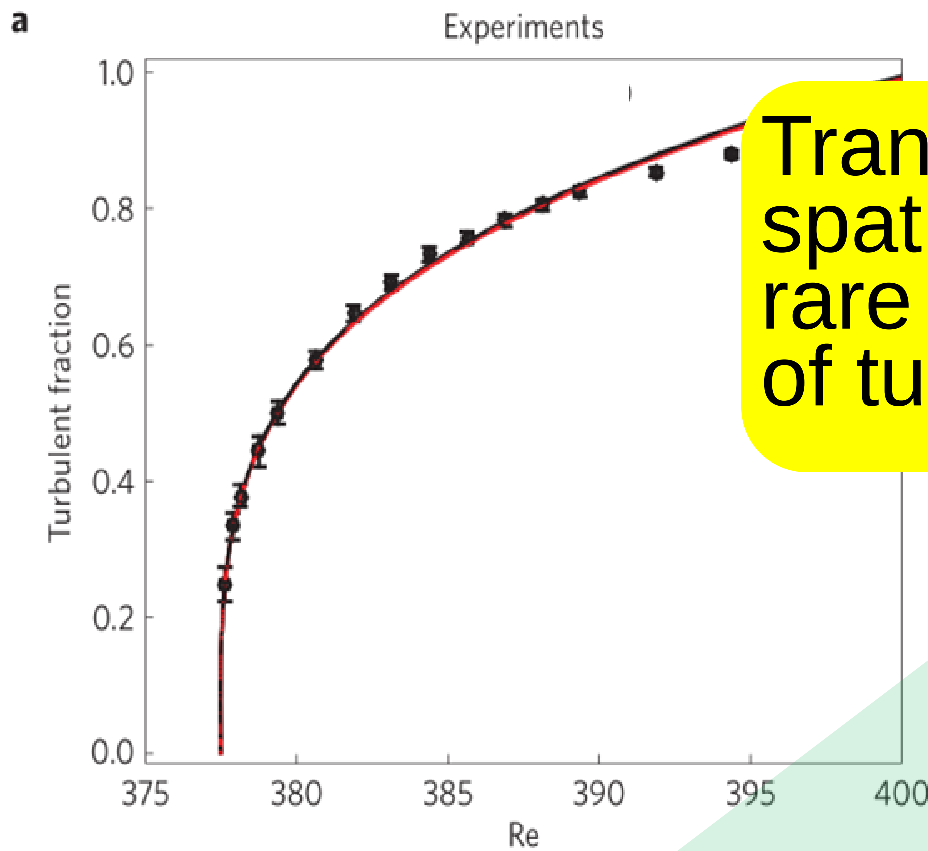
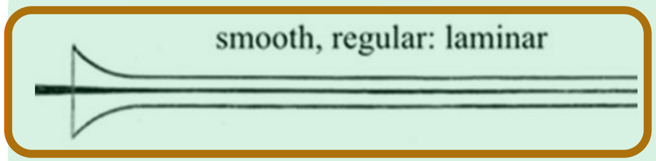
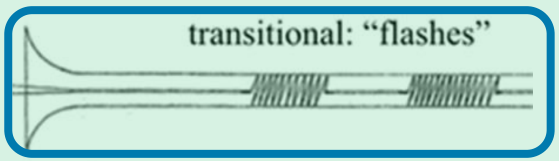
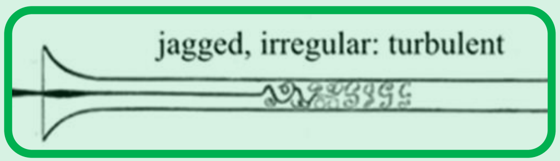


Transition not spatially uniform: rare sharp bursts of turbulence







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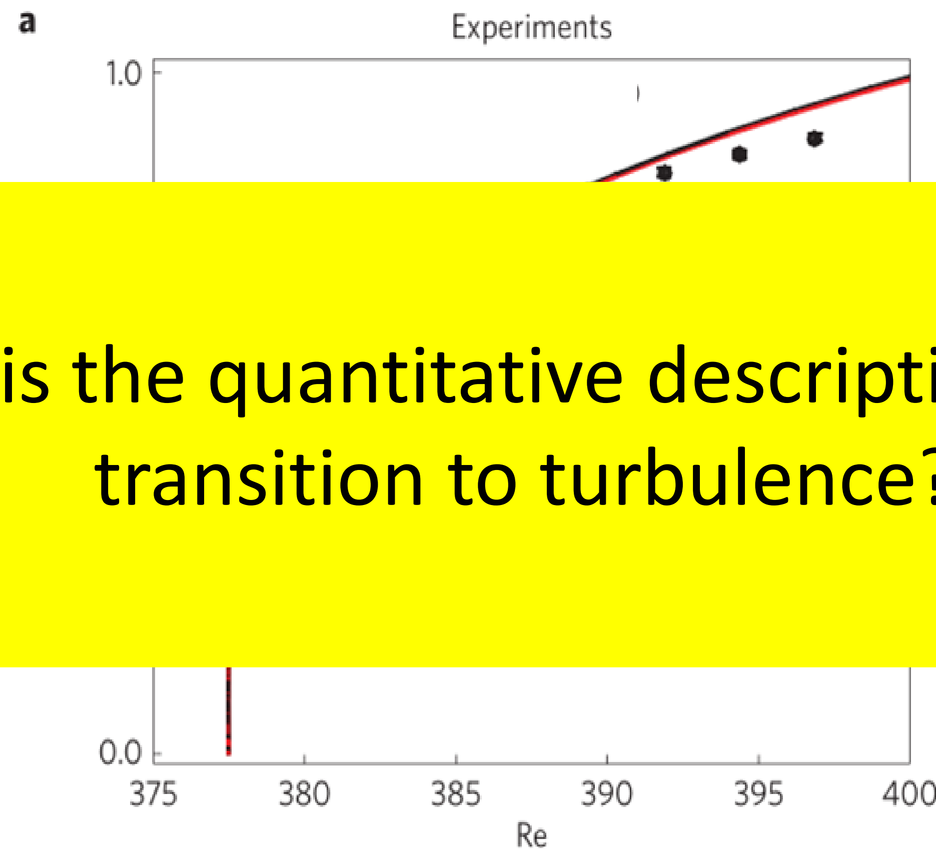
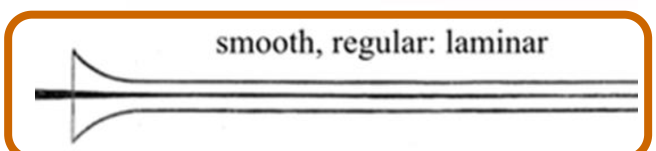
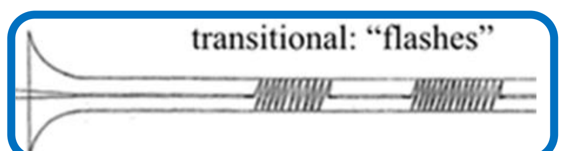
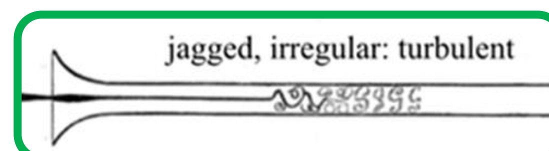


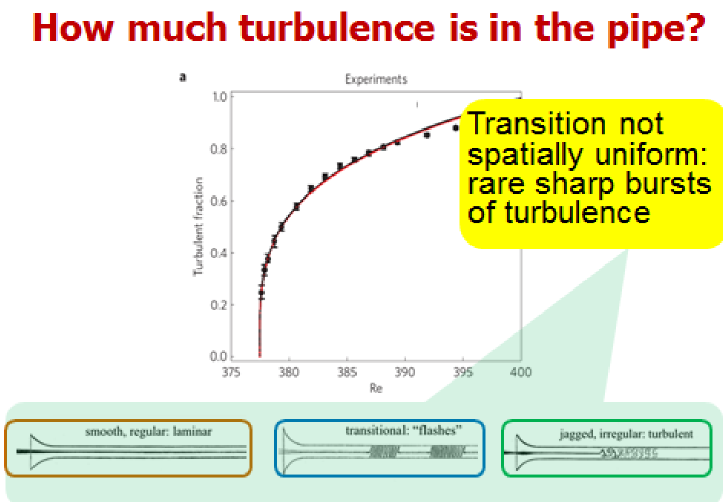
What is the quantitative description of the transition to turbulence?

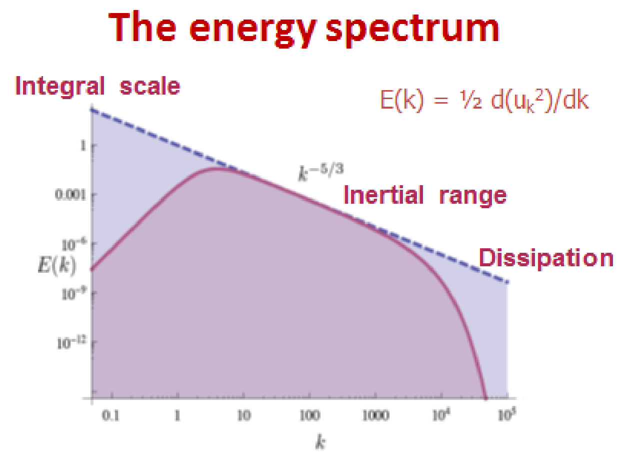






Turbulence & Phase Transitions





- Turbulent fraction as a function of Re is reminiscent of a continuous phase transition

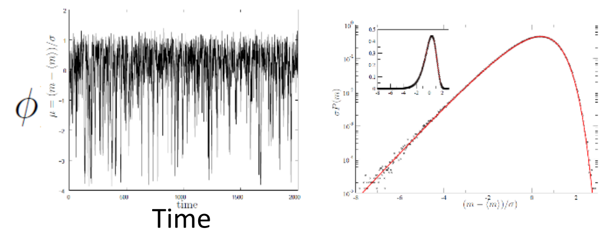
- Strong power-law correlated fluctuations is reminiscent of a continuous phase transition

Why is fully-developed turbulence hard?

Why is turbulence unsolved?

Why are phase transitions hard?

- **Strong interactions + fluctuations** in the order parameter ϕ :
  - Very non-Gaussian
  - Intermittency



- **No usable small parameter!**

Landau free energy

$$\mathcal{H} = \int d^d\mathbf{r} \left[\frac{1}{2}\gamma \left(\nabla\phi\right)^2 + \frac{1}{2} r_0 \phi^2 + \frac{1}{4} u_0 \phi^4 \right]$$

$$t = (T - T_c)/T_c \qquad r_0 \sim t$$

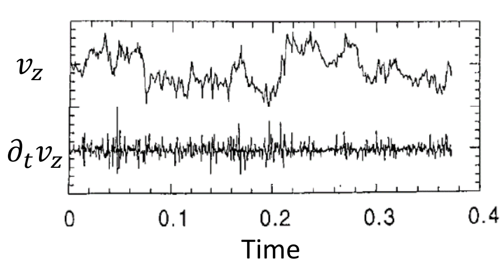
rescaling $\overline{u}_0 \sim u_0 t^{(d-4)/2} \begin{cases} d < 4 : & \overline{u}_0 \to \infty \\ d > 4 : & \overline{u}_0 \to 0 \end{cases}$

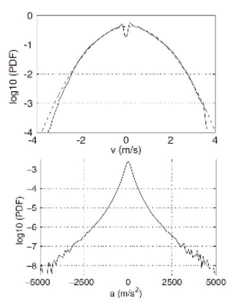
The coefficient of the **interaction** becomes relatively large at **phase transition** $t \to 0$

Why is turbulence unsolved?

Why is turbulence hard?

- **Strong interactions + fluctuations** in the velocity derivatives $\partial_i v_j$:
  - Very non-Gaussian
  - Intermittency: intervals of weak fluctuations interspersed with bursts of strong fluctuations





- **No usable small parameter!**

The Navier-Stokes equation

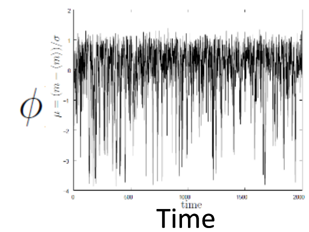
$$\partial_t \mathbf{v} + \boxed{(\mathbf{v}\cdot\nabla)\,\mathbf{v}} = -\frac{1}{\rho}\nabla p + \boxed{\nu}\nabla^2\mathbf{v} + \mathbf{f}$$

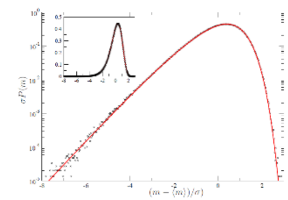
Turbulence: $\mathrm{Re} \to \infty,\ \nu \to 0$

The coefficient of the **nonlinearity** becomes relatively large as the **viscosity** $\nu \to 0$

Why are phase transitions hard?

- **Strong interactions + fluctuations** in the order parameter ϕ :
  - Very non-Gaussian
  - Intermittency





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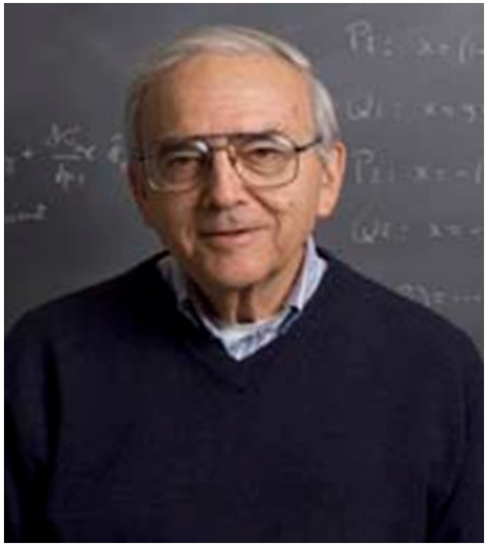
How was critical phenomena solved?



Ben Widom discovered "data collapse" (1965)

Leo Kadanoff explained data collapse, with scaling concepts (1966)

Ken Wilson developed the RG based on Kadanoff's scaling ideas (1970)

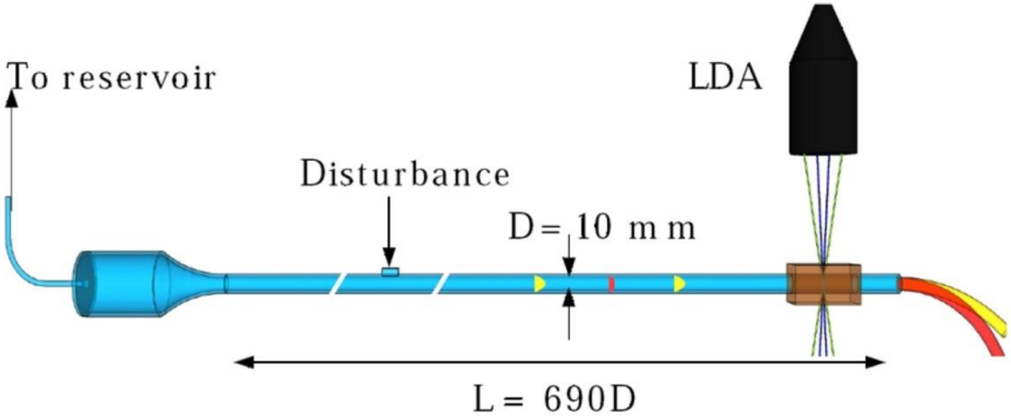
- **Common features**
  - Strong fluctuations
  - Power law correlations

- **Can we solve turbulence by following critical phenomena?**
- **Does turbulence exhibit critical phenomena at its onset?**

Transition to turbulence

Precision measurement of turbulent transition

Q: will a turbulent puff survive to the end of the pipe?



Many repetitions ➔ **Survival probability = P(Re, t)**

Hof et al., *PRL* **101**, 214501 (2008)

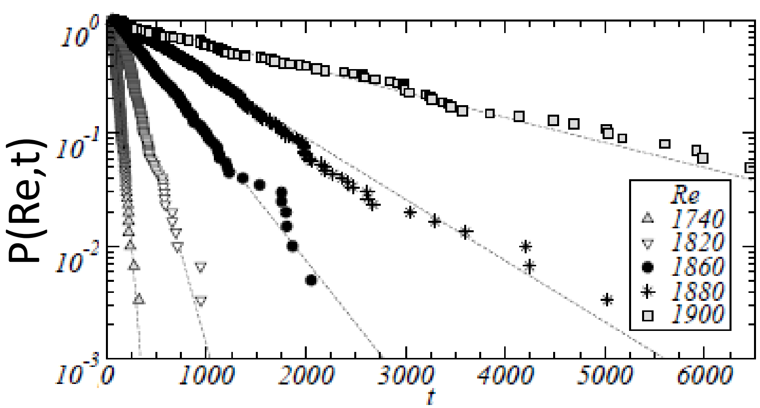
Pipe flow turbulence

Decaying single puff

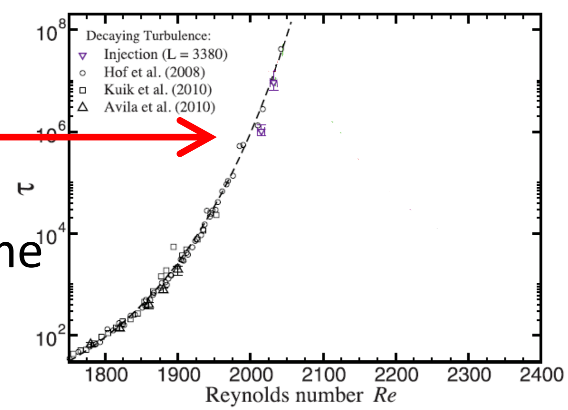
laminar	metastable puffs	spatiotemporal intermittency	expanding slugs

Re 1775 2050 2500

Survival probability $P(\mathrm{Re}, t) = e^{-\frac{t-t_0}{\tau(\mathrm{Re})}}$

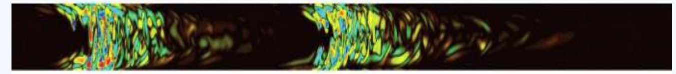


Puff lifetime



Avila *et al.*, (2009) Avila *et al.*, *Science* **333**, 192 (2011) Hof *et al.*, *PRL* **101**, 214501 (2008)

Pipe flow turbulence



Decaying single puff

Splitting puffs

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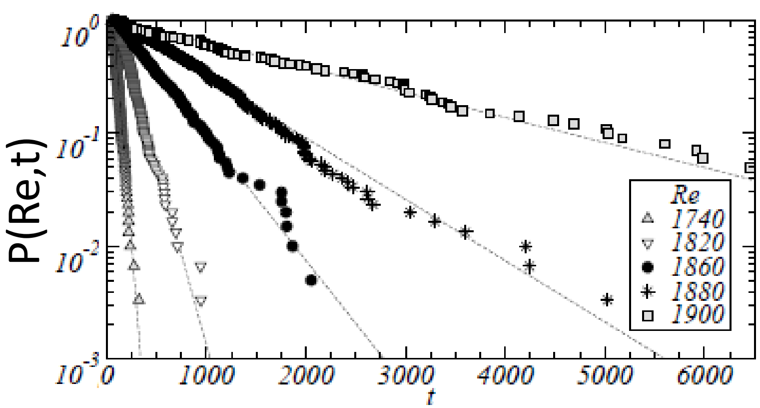

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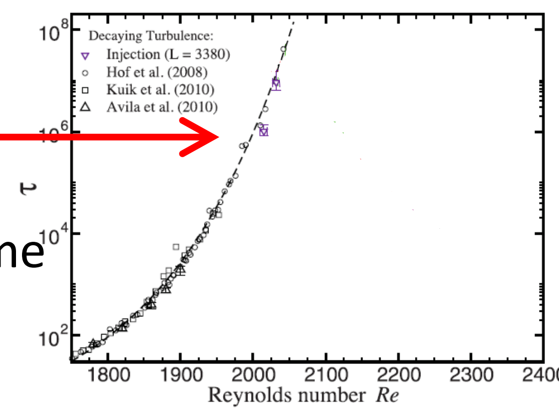

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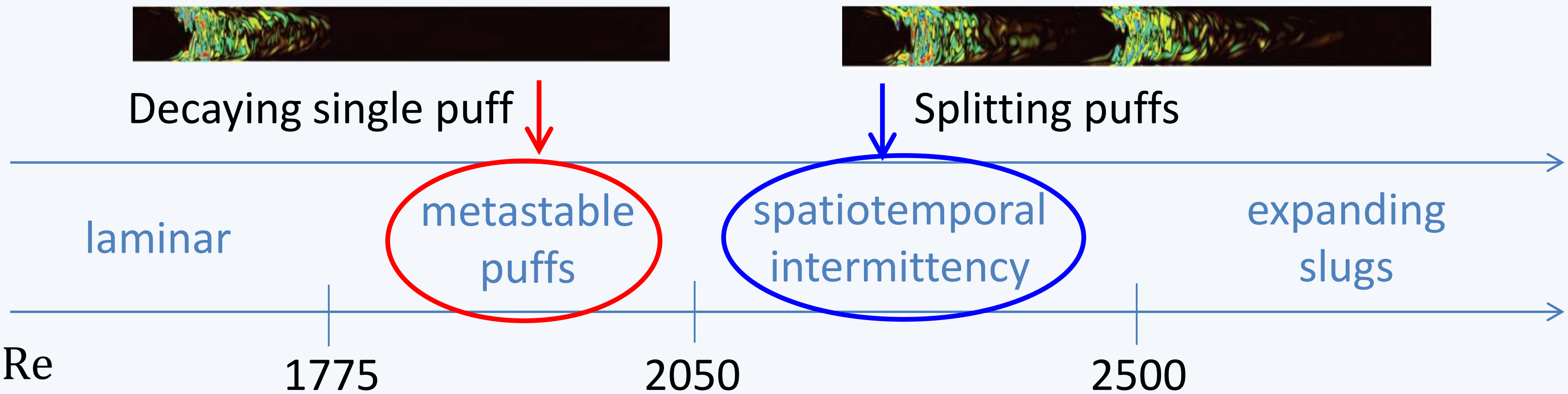
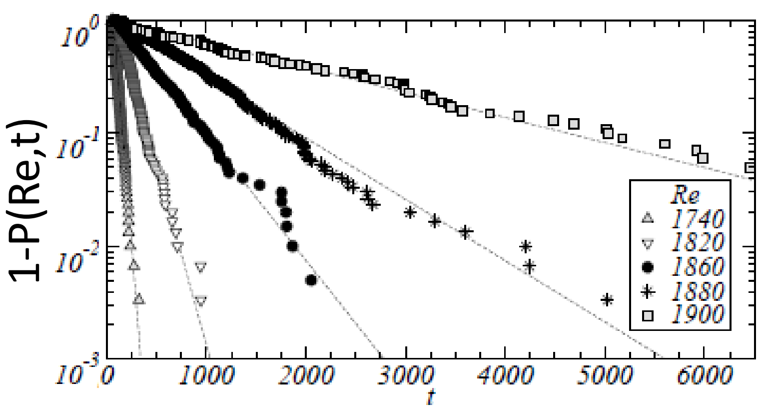
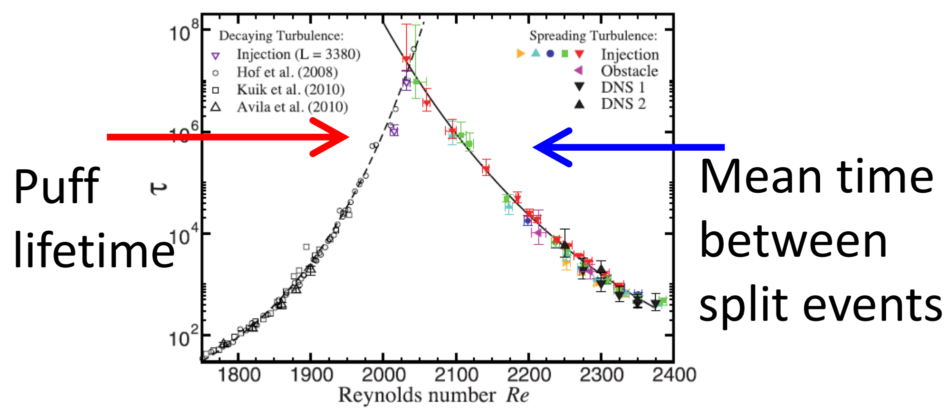
Pipe flow turbulence



Splitting probability $1 - P(\mathrm{Re}, t) = e^{-\frac{t-t_0}{\tau(\mathrm{Re})}}$





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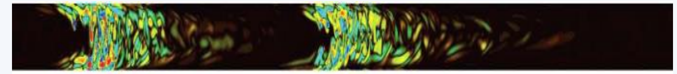
Pipe flow turbulence



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laminar | metastable puffs | spatiotemporal intermittency | expanding slugs

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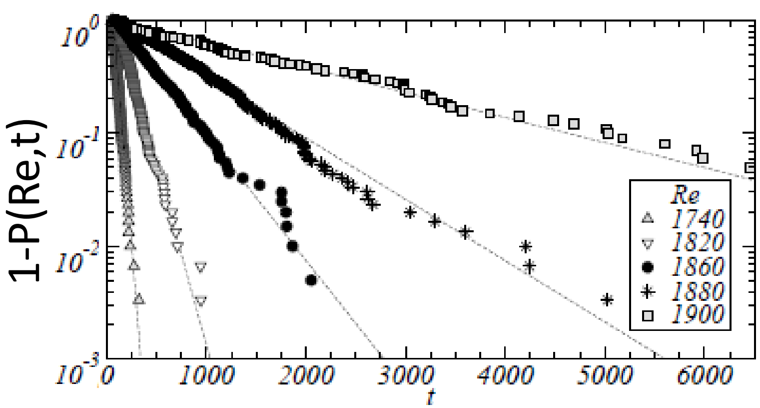

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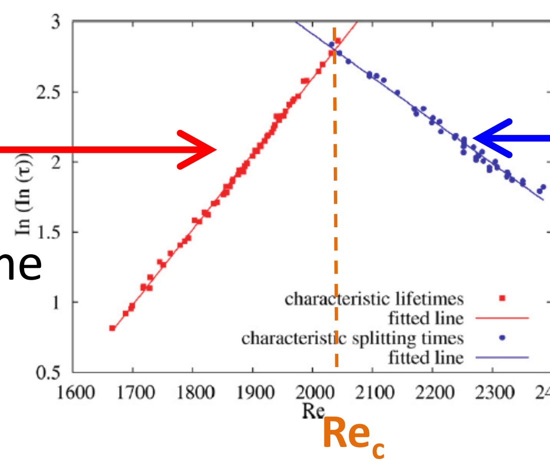
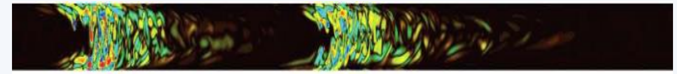


Mean time between split events

Avila *et al.*, (2009) Avila *et al.*, *Science* **333**, 192 (2011) Hof *et al.*, *PRL* **101**, 214501 (2008)

Pipe flow turbulence



Decaying single puff

Splitting puffs

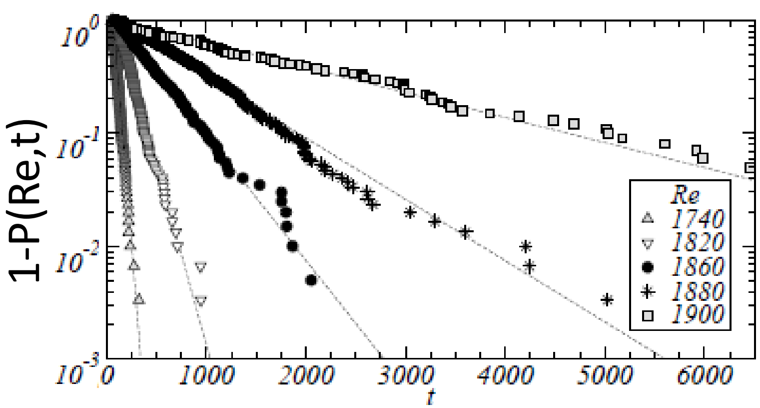
laminar | metastable puffs | spatiotemporal intermittency | expanding slugs

Re

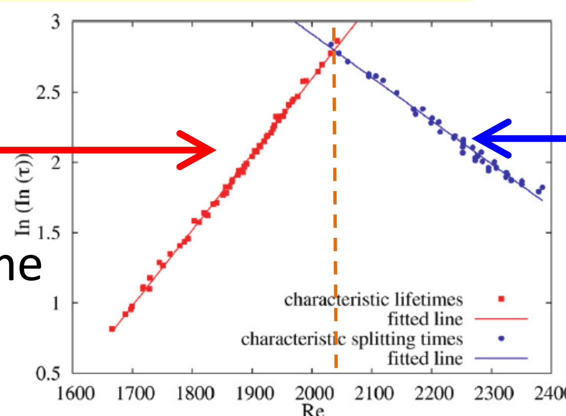
Super-exponential scaling:

$$\frac{\tau}{\tau_0} \sim \exp(\exp \mathrm{Re})$$

Splitting p



Puff lifetime



Re_c

Mean time between split events

Avila *et al.*, (2009) Avila *et al.*, *Science* **333**, 192 (2011) Hof *et al.*, *PRL* **101**, 214501 (2008)

Theory for the laminar-turbulent transition in pipe flow

Logic of modeling phase transitions

Magnets

Electronic structure

↓

Ising model

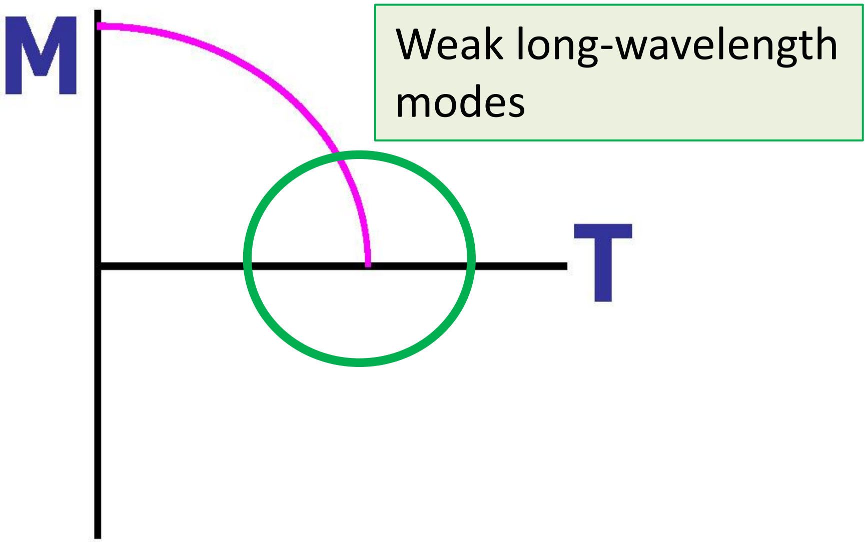
↓

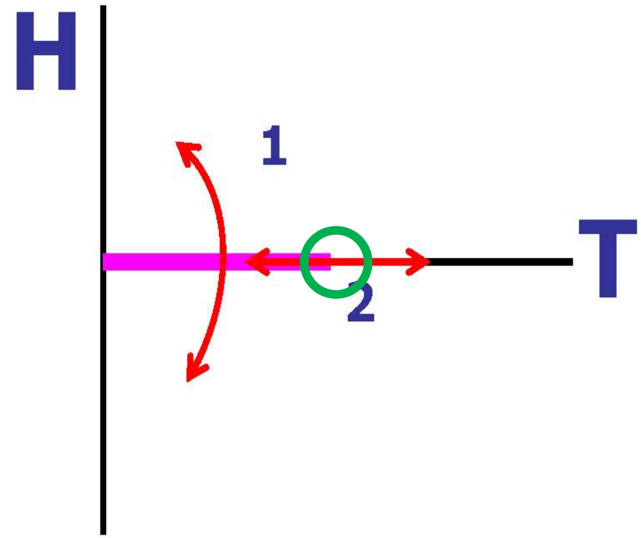
Landau theory

↓

Renormalization group
universality class
(Ising universality class)

Critical phenomena in magnets





$\boxed{M \sim M_0[|T - T_c|/T_c]^\beta}$ for $H = 0$ as $T \to T_c$

Critical isotherm: $\boxed{M \sim H^{1/\delta} \text{ for } T = T_c}$

- Widom (1963) pointed out that both these results followed from a *similarity formula*:

$$\boxed{M(t, h) = |t|^\beta f_M(h/t^\Delta)}$$

where $t \equiv (T - T_c)/T_c$ for some choice of exponent Δ and scaling function $f_M(x)$

Critical phenomena in magnets

$$M(t,h) = |t|^{\beta} f_M(h/t^{\Delta})$$

where $t \equiv (T - T_c)/T_c$ for some choice of exponent Δ and scaling function $f_M(x)$

- **To determine the properties of the scaling function and unknown exponent, we require:**
  - $f_M(z)$ = const. for z = 0
  - This gives $M \sim M_0[|T - T_c|/T_c]^{\beta}$ for T < T_c
  - For large values of z, i.e. non-zero h, and t →0, t dependence must cancel out so that $M \sim H^{1/\delta}$

Thus $f_M(z) \sim z^{1/\delta}, z \to \infty$.
Calculate Δ: t dependence will only cancel out if $\beta - \Delta/\delta = 0$

$$M = |t|^{\beta} f_M(h/|t|^{\beta\delta})$$

- **This data collapse formula connects the scaling of correlations with the thermodynamics of the critical point**

Universality at a critical point

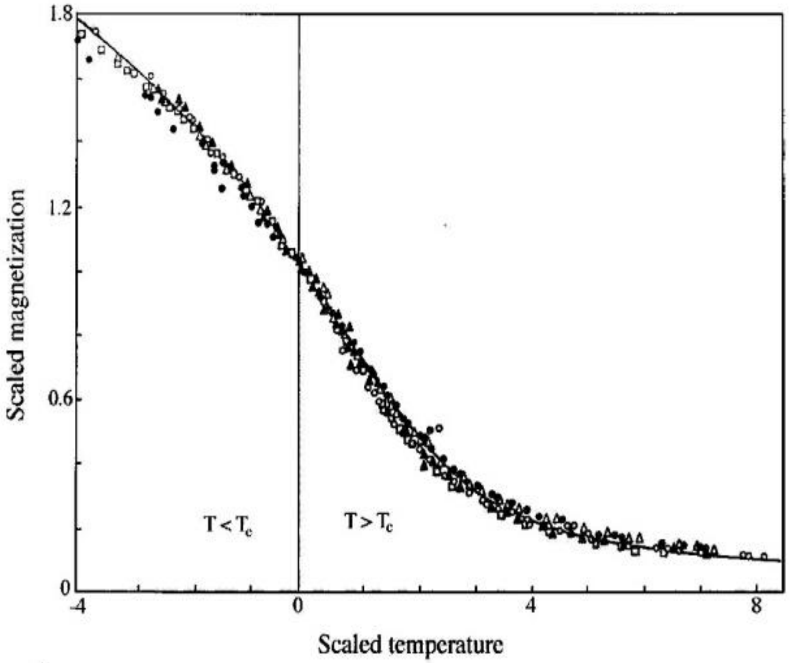


FIG. 1. Experimental *MHT* data on five different magnetic materials plotted in scaled form. The five materials are $CrBr_3$, EuO, Ni, YIG, and Pd_3Fe. None of these materials is an idealized ferromagnet: $CrBr_3$ has considerable lattice anisotropy, EuO has significant second-neighbor interactions. Ni is an itinerant-electron ferromagnet, YIG is a ferrimagnet, and Pd_3Fe is a ferromagnetic alloy. Nonetheless, the data for all materials collapse onto a single scaling function, which is that calculated for the $d = 3$ Heisenberg model [after Milošević and Stanley (1976)].

Stanley (1999)

- Magnetization M of a material depends on temperature T and applied field H
  - M(H,T) ostensibly a function of two variables
- Plotted in appropriate scaling variables get ONE universal curve
- Scaling variables involve critical exponents

A theoretical physics success

A model …

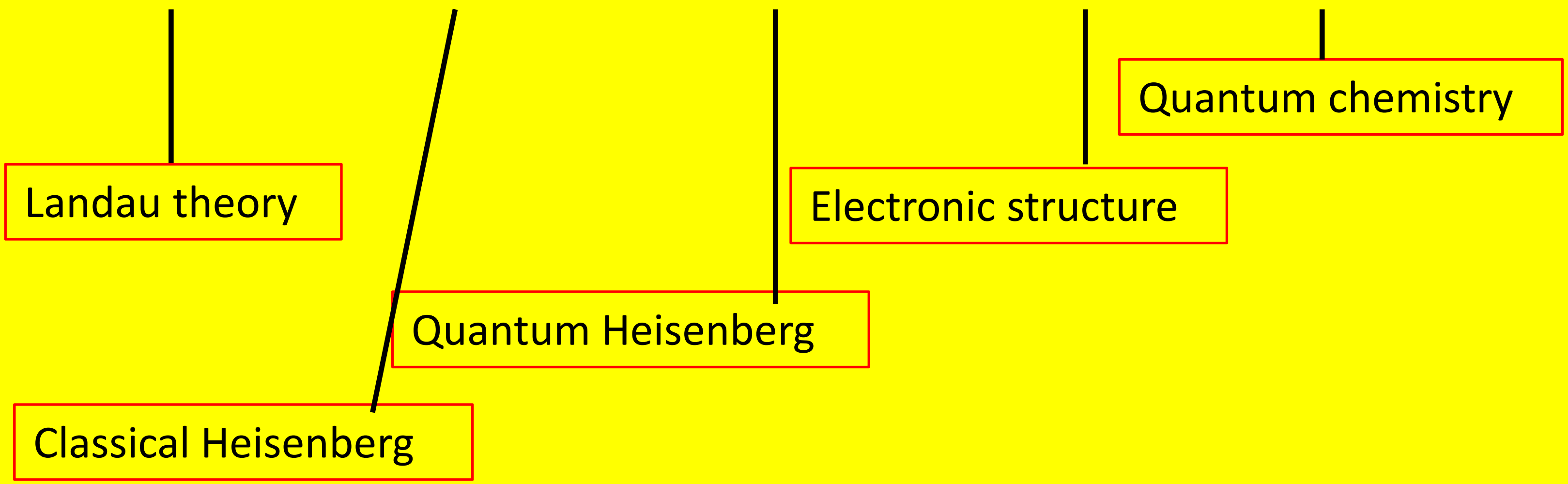
Gives a precise prediction in agreement with experiment!

materials collapse onto a single scaling function, which is that calculated for the $d=3$ Heisenberg model [after Milošević and Stanley (1976)].

Stanley (1999)

A theoretical physics success

A model of a model of a model of a model of a model



Gives a precise prediction in agreement with experiment!

Non-systematic approximations

materials collapse onto a single scaling function, which is that calculated for the $d = 3$ Heisenberg model [after Milošević and Stanley (1976)].

Stanley (1999)

Logic of modeling phase transitions

Magnets	Turbulence
Electronic structure	Kinetic theory
$\downarrow$	$\downarrow$
Ising model	Navier-Stokes eqn
$\downarrow$	$\downarrow$
Landau theory	**?**
$\downarrow$	$\downarrow$
Renormalization group universality class (Ising universality class)	**?**

Logic of modeling phase transitions

Magnets		Turbulence
Electronic structure		Kinetic theory
↓		↓
Ising model		Navier-Stokes eqn
↓		↓
Landau theory	⟷	?
↓		↓
Renormalization group universality class (Ising universality class)	⟷	?

Identification of long-wavelength collective modes at the laminar-turbulent transition

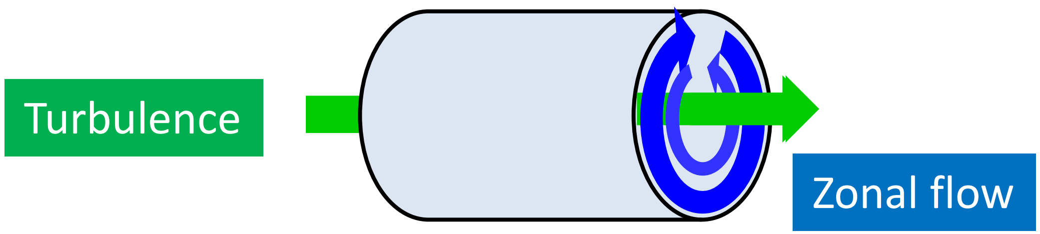
To avoid uncontrolled approximations,
we use **direct numerical simulation** of
the **Navier-Stokes equations**

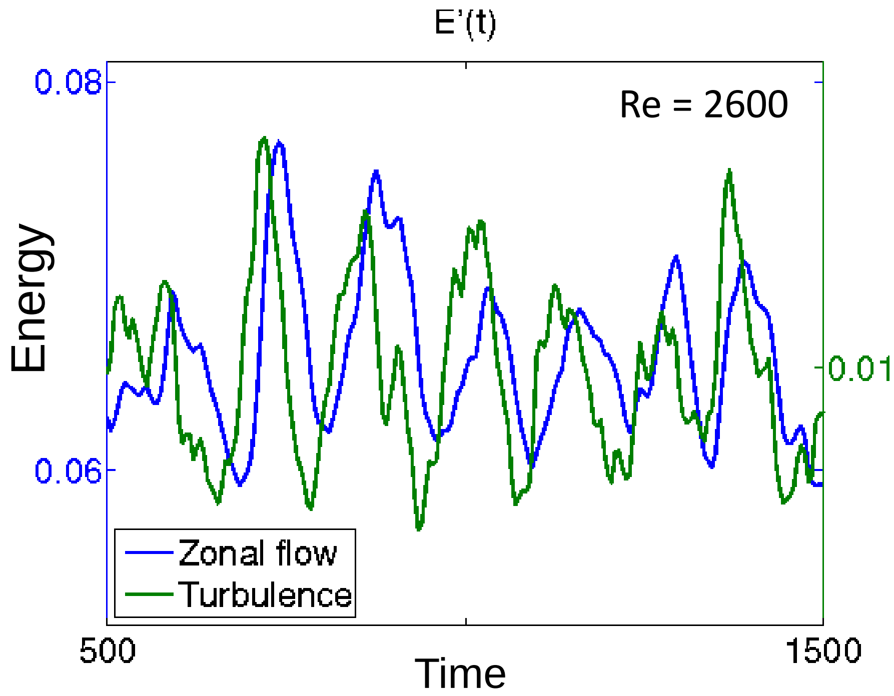
DNS of 3D Navier-Stokes equations

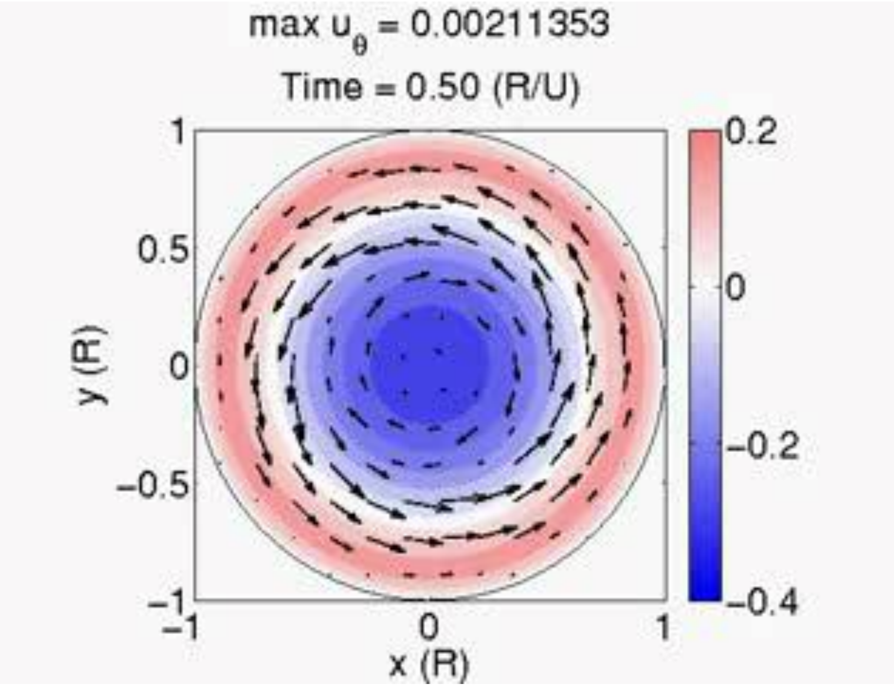
- Use pseudospectral method to deal with both derivatives and nonlinear terms in streamwise direction and azimuthal directions
- Use Chebyshev polynomials to resolve large gradients in the boundary layer, in the radial direction
- 60 grid points in the radial (r) direction,
- 32 Fourier modes in the azimuthal (θ) direction and 128 modes in the axial (z) direction
- The spatial resolutions were chosen such that the resolvable power spectra span over six orders of magnitude.
- The pipe length L is 10 times its diameter D, with periodic boundary conditions in the z direction

Simulation based on the open source code by Ashley Willis: openpipeflow.org

Predator-prey oscillations in pipe flow







Simulation based on the open source
code by Ashley Willis: openpipeflow.org

Decomposition into large & small scales

- Use pseudospectral method to deal with both derivatives and nonlinear terms in streamwise direction and azimuthal directions
- Use Chebyshev polynomials to resolve large gradients in the boundary layer, in the radial direction
- Zonal flow velocity $\boldsymbol{u}_{\mathrm{ZF}} \equiv (\overline{u}_z, \overline{u}_\theta, \overline{u}_r)$, where $\overline{u}_r = 0$
- Zonal flow energy is defined as

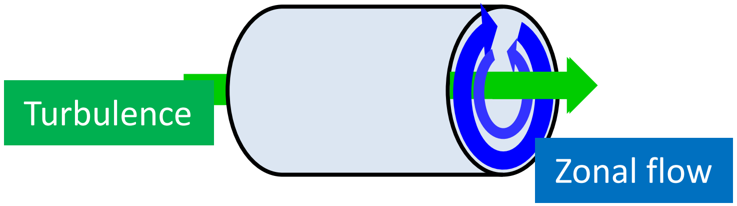
$$E_{\mathrm{ZF}}(t) \equiv (1/2) \int |\tilde{\mathbf{u}}(0,0,r)|^2 \,\mathrm{d}V \quad \text{with} \quad \tilde{\boldsymbol{u}}_{\mathrm{ZF}} \equiv \tilde{\mathbf{u}}(k=0, m=0, r)$$

- Turbulence energy is defined as

$$E_{\mathrm{T}}(t) \equiv (1/2) \sum_{|k|\geq 1, |m|\geq 1} \int |\tilde{\mathbf{u}}(k,m,r)|^2 \,\mathrm{d}V$$

Simulation based on the open source code by Ashley Willis: openpipeflow.org

What drives the zonal flow?

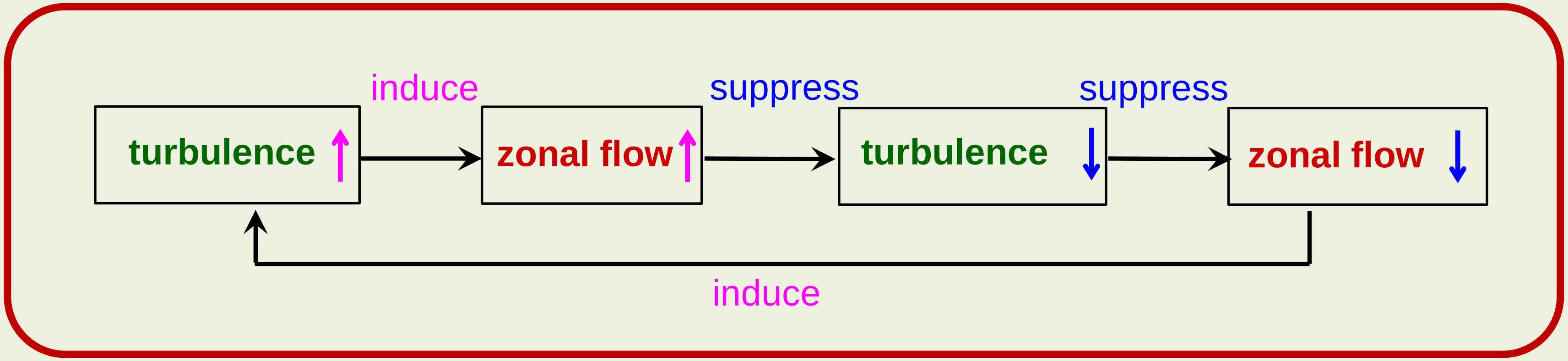


- **Interaction in two fluid model**
  - Turbulence, small-scale (k>0)
  - Zonal flow, large-scale (k=0,m=0): induced by turbulence and creates shear to suppress turbulence

1) Anisotropy of turbulence creates Reynolds stress which generates the mean velocity in azimuthal direction

$$\partial_t \langle v_\theta \rangle = -\partial_r \langle (\widetilde{v}_\theta \cdot \widetilde{v}_r) \rangle - \mu \langle v_\theta \rangle$$

2) Mean azimuthal velocity decreases the anisotropy of turbulence and thus suppress turbulence



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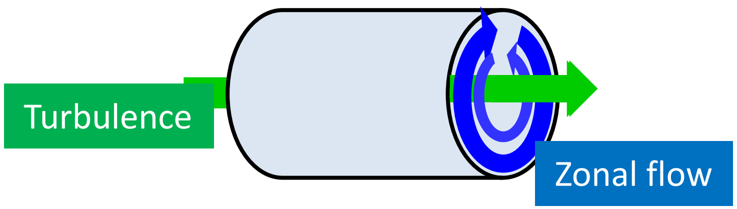

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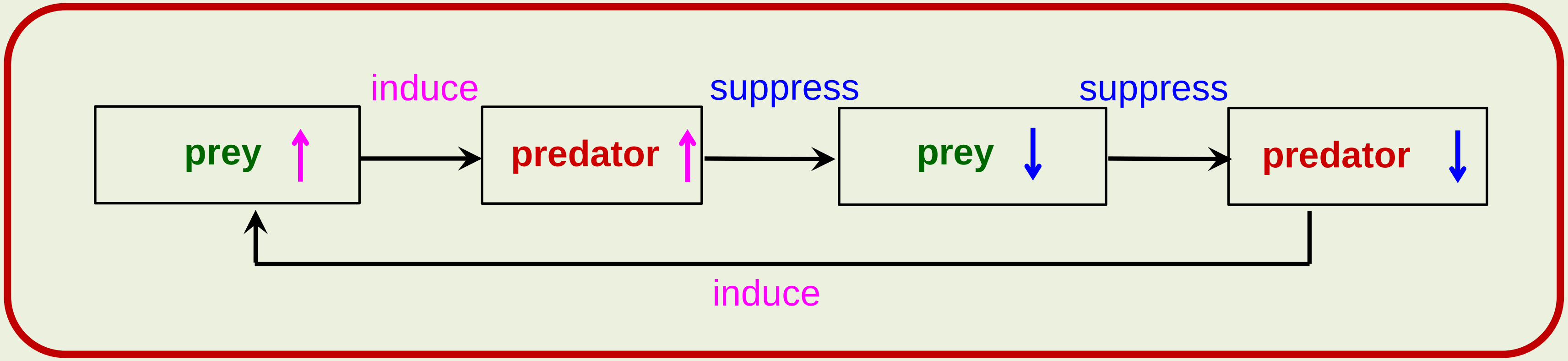
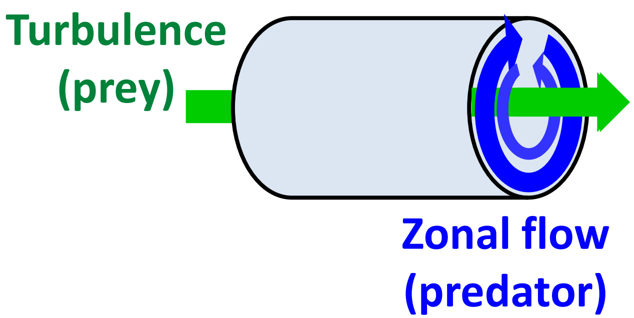
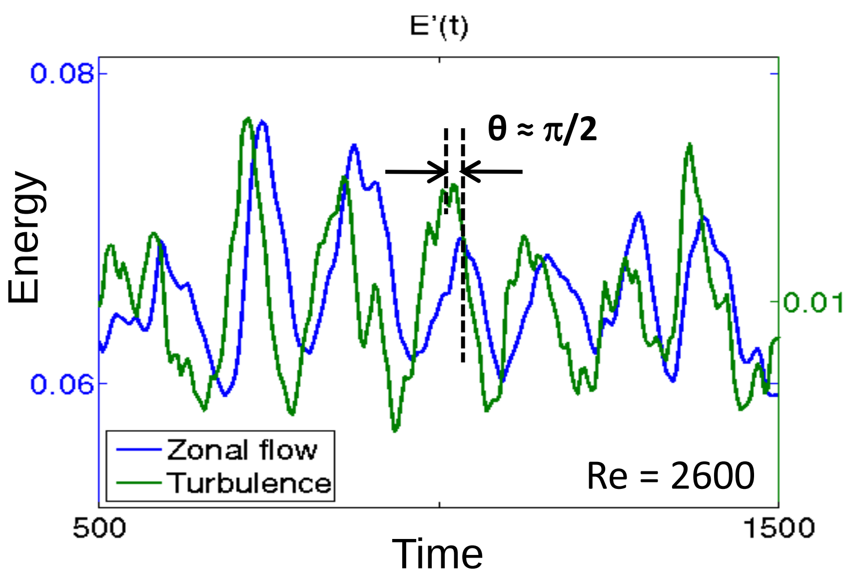


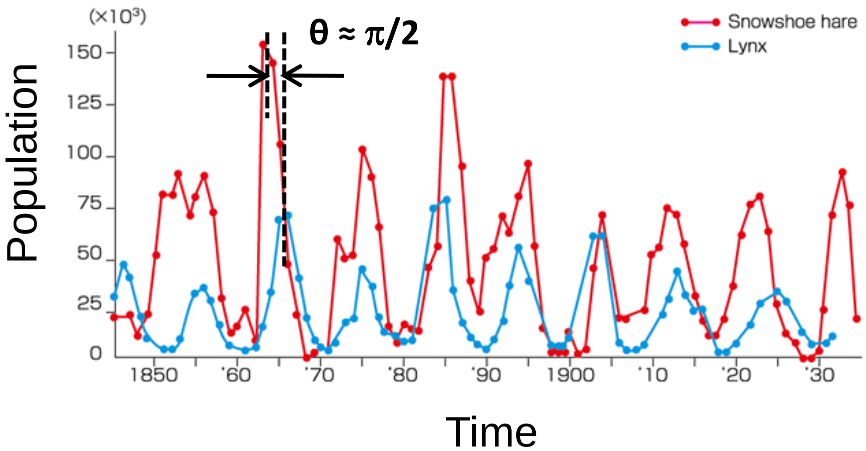
Pipe flow near transition to turbulence

?
=

Predator-prey ecosystem







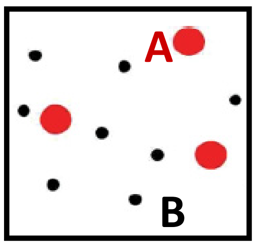
Blackboard interlude 3

- Predator-prey equations description of cyclical behavior is strange, because a deterministic description based on the physics does not work! Let's see this …
- Intrinsically stochastic description is required

Stochastic model of predator-prey dynamics

- **Stochastic individual-level model**
  fluctuations in number → demographic stochasticity that induces **quasi-cycles**

A = predator
B = prey
E = food or
available space

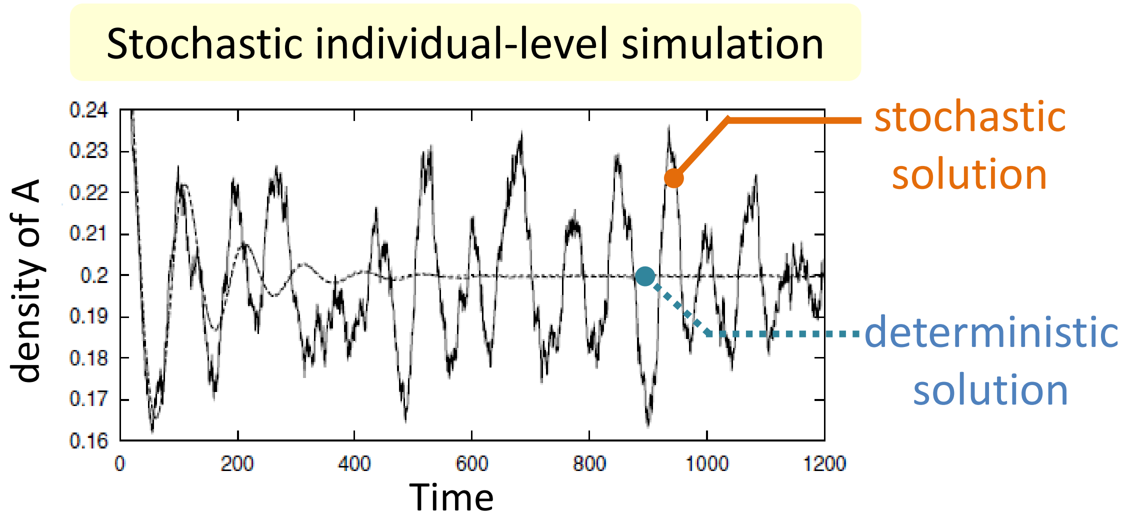


$$B + E \xrightarrow{b} B + B$$

$$B + B \xrightarrow{c} B + E$$

$$A + B \xrightarrow{p} A + A$$

$$A \xrightarrow{d_A} E \qquad B \xrightarrow{d_B} E$$

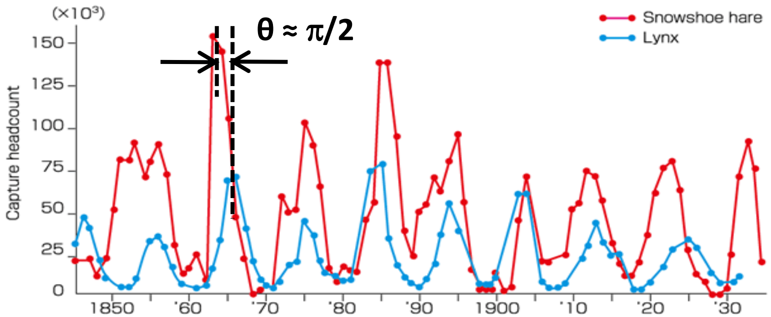


McKane & Newman (2005) Butler & Goldenfeld (2009)

Normal population cycles in a predator-prey system

Resource → Prey → Predator

$\pi/2$ phase shift between prey and predator population



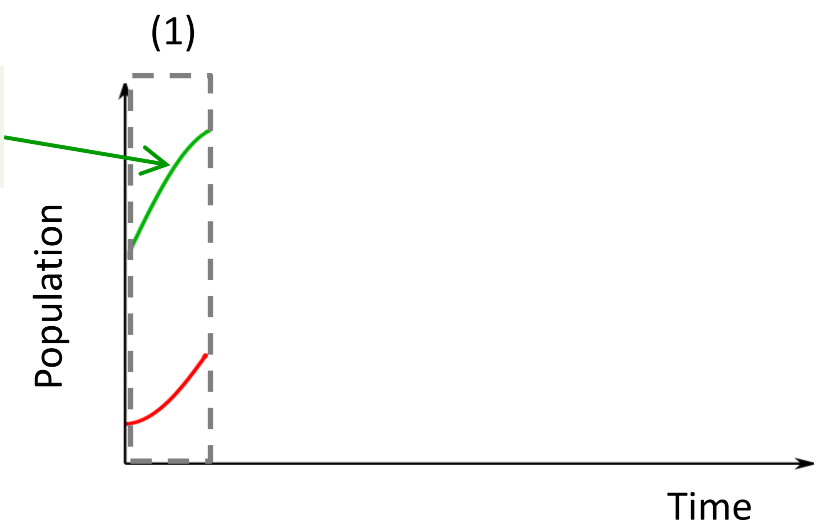
Persistent oscillations
+
Fluctuations

https://interstices.info/jcms/n_49876/des-especes-en-nombre

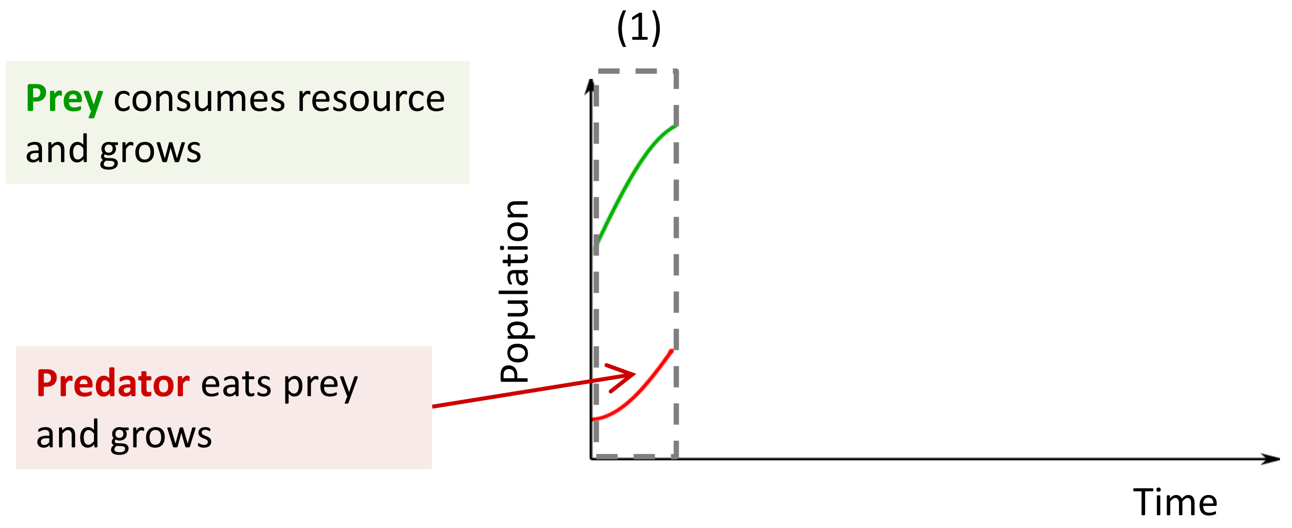
Cartoon picture for normal cycles (π/2 phase shift)

Resource → Prey → Predator

Prey consumes resource and grows

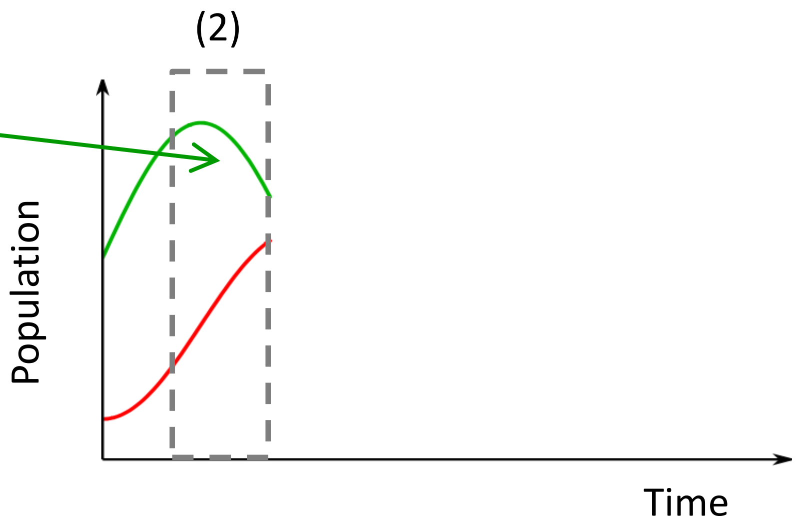


Cartoon picture for normal cycles ($\pi/2$ phase shift)



Cartoon picture for normal cycles (π/2 phase shift)

Prey decreases due to predation by predator



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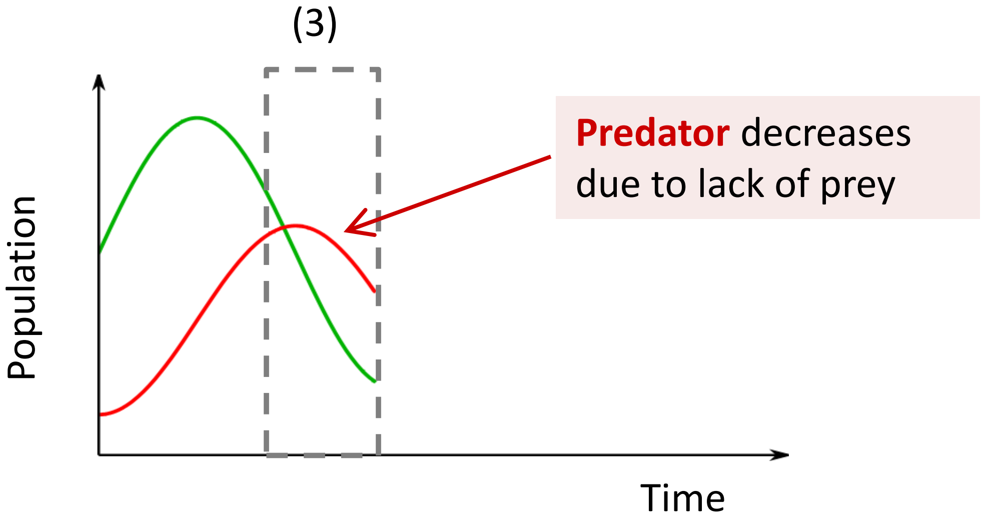

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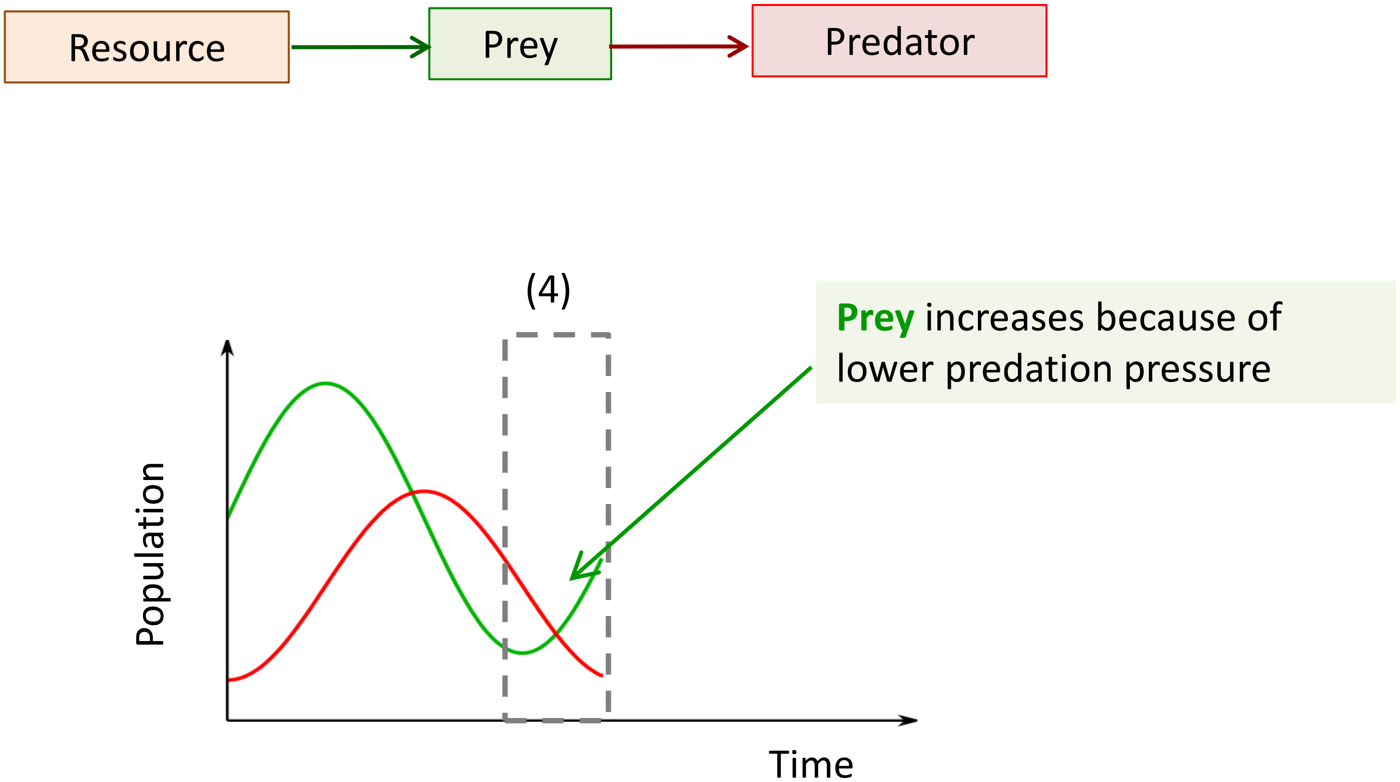

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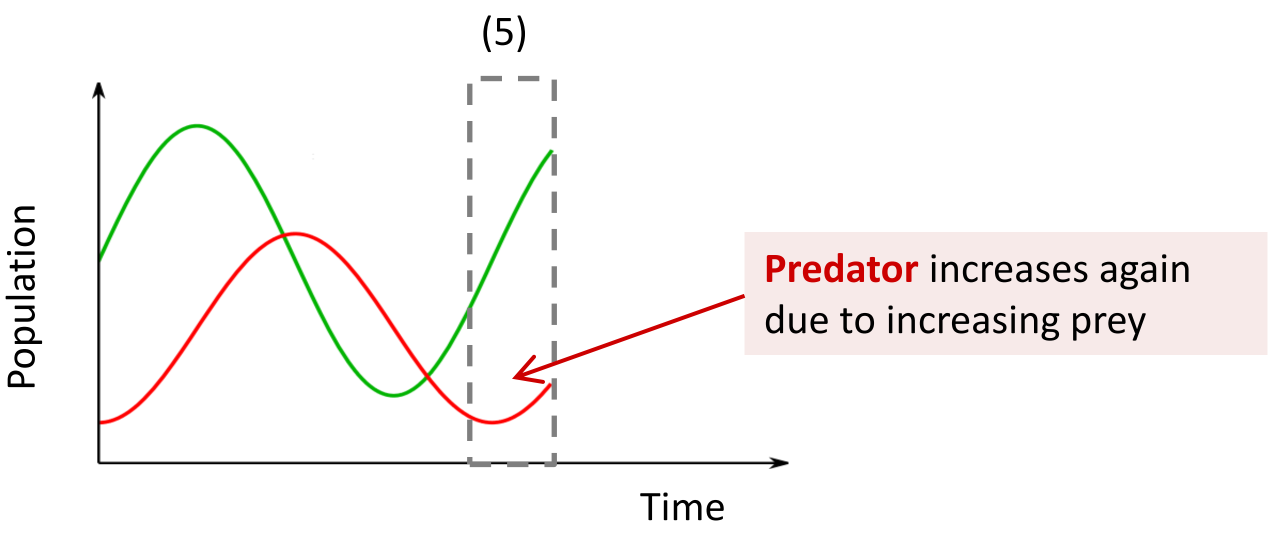

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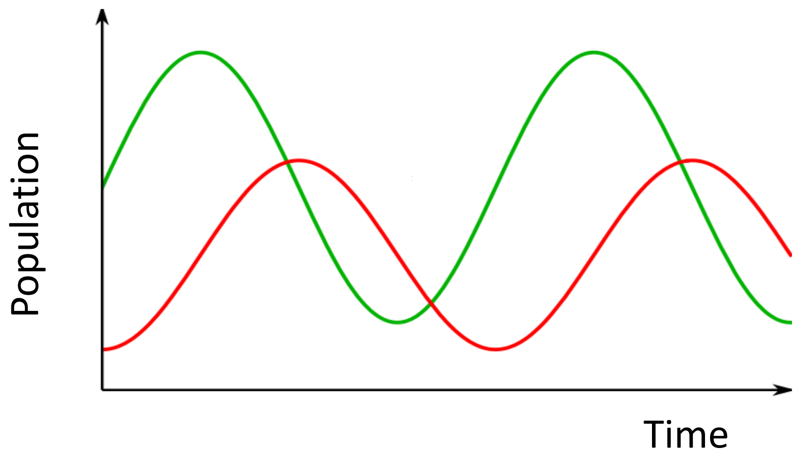
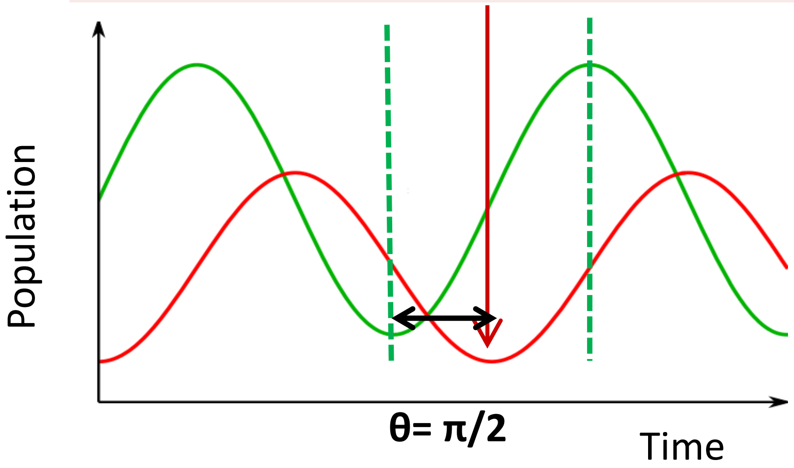


Cartoon picture for normal cycles (π/2 phase shift)

Predator can only start to grow after **prey** grows and before **prey** declines



Phase shift is a quarter period

Lotka-Volterra equations for predator-prey dynamics

- Lotka-Volterra eqn: conventional model for population dynamics
- L-V for prey-predator system:

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  - Reproduction of prey proportional to prey density → exponential growth

$$\frac{du}{dt} = bu$$

Lotka-Volterra equations for predator-prey dynamics

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  - Limited food resource → consider carrying capacity of prey, **K_u**

$$\frac{du}{dt} = bu(1 - \frac{u}{K_u})$$

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  - Predator hunts prey → predation proportional to prey & predator densities

$$\frac{du}{dt} = bu(1 - \frac{u}{K_u}) - puv$$

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  - Predator hunts prey → predation proportional to prey & predator densities
  - Death of predator proportional to predator density

$$\frac{du}{dt} = bu(1 - \frac{u}{K_u}) - puv$$

$$\frac{dv}{dt} = puv - dv$$

u: prey v: predator b: prey metabolic rate
K_u: prey carrying capacity
p: predation rate d: predator death rate

Lotka-Volterra equations for predator-prey dynamics

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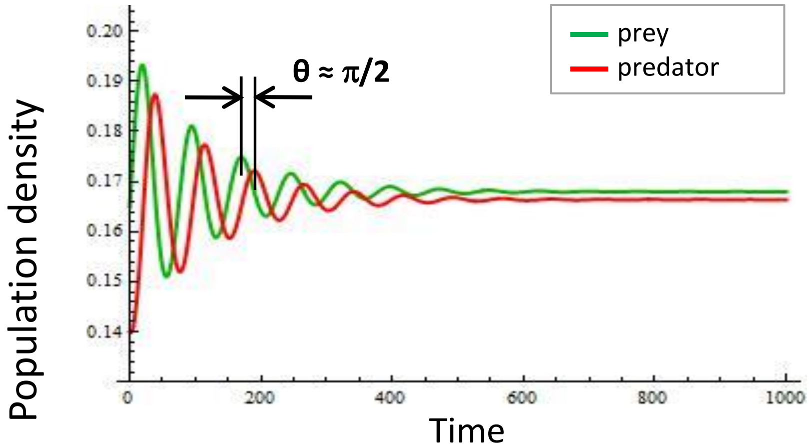


- Predicts π/2 phase shift between prey and predator
- **Problems**: No oscillations → **Contrary to experiments!**

Stochasticity can qualitatively change the predictions of ecological models

Noise can stabilise persistent cycles
in time and patterns in space

T. Newman, A. McKane PRL (2006)

Predator-prey models

Deterministic

prey $$\frac{dx}{dt} = bx - ex^2 - pxy$$

predator $$\frac{dy}{dt} = pxy - dy$$

T. Newman, A. McKane PRL (2006)

Predator-prey models

Deterministic

growth competition predation

prey $$\frac{dx}{dt} = bx - ex^2 - pxy$$

predator $$\frac{dy}{dt} = pxy - dy$$

predation death

T. Newman, A. McKane PRL (2006)

Predator-prey models

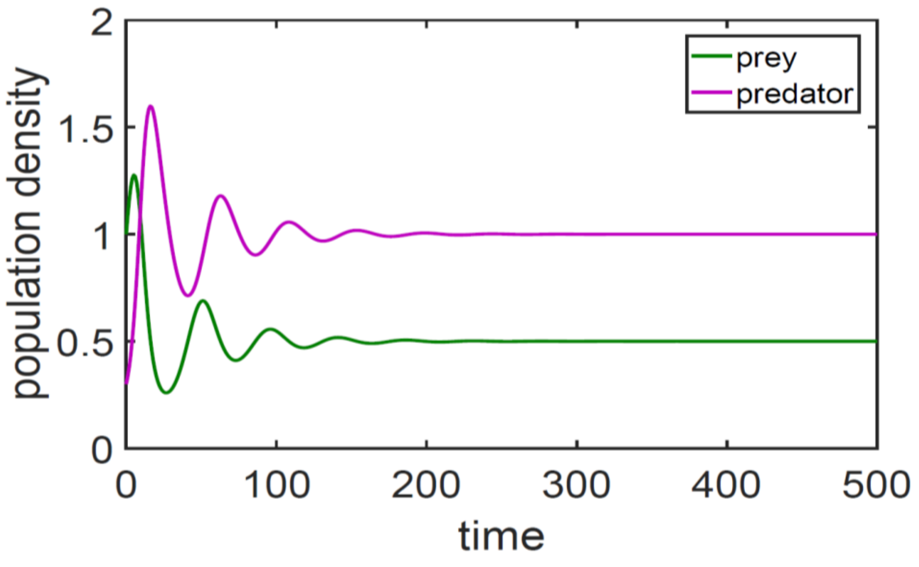
Deterministic

prey $$\frac{dx}{dt} = bx - ex^2 - pxy$$

predator $$\frac{dy}{dt} = pxy - dy$$

Oscillatory damping toward a steady state.

Cycles are not persistent.



T. Newman, A. McKane PRL (2006)

Predator-prey models

Deterministic

prey $$\frac{dx}{dt} = \boxed{bx} - ex^2 - pxy$$

predator $$\frac{dy}{dt} = pxy - dy$$

Stochastic

$$\boxed{X \xrightarrow{b} 2X}$$

$$X + X \xrightarrow{e/V} X$$

$$X + Y \xrightarrow{p/V} 2Y$$

$$Y \xrightarrow{d} \Phi$$

T. Newman, A. McKane PRL (2006)

Predator-prey models

Deterministic

prey $$\frac{dx}{dt} = bx - \boxed{ex^2} - pxy$$

predator $$\frac{dy}{dt} = pxy - dy$$

Stochastic

$$X \xrightarrow{b} 2X \qquad \boxed{X + X \xrightarrow{e/V} X}$$

$$X + Y \xrightarrow{p/V} 2Y \qquad Y \xrightarrow{d} \Phi$$

T. Newman, A. McKane PRL (2006)

Predator-prey models

Deterministic

prey $$\frac{dx}{dt} = bx - ex^2 - \boxed{pxy}$$

predator $$\frac{dy}{dt} = \boxed{pxy} - dy$$

Stochastic

$$X \xrightarrow{b} 2X \qquad X + X \xrightarrow{e/V} X$$

$$\boxed{X + Y \xrightarrow{p/V} 2Y} \qquad Y \xrightarrow{d} \Phi$$

T. Newman, A. McKane PRL (2006)

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predator
$$\frac{dy}{dt} = pxy - \boxed{dy}$$

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Predator-prey models

Deterministic

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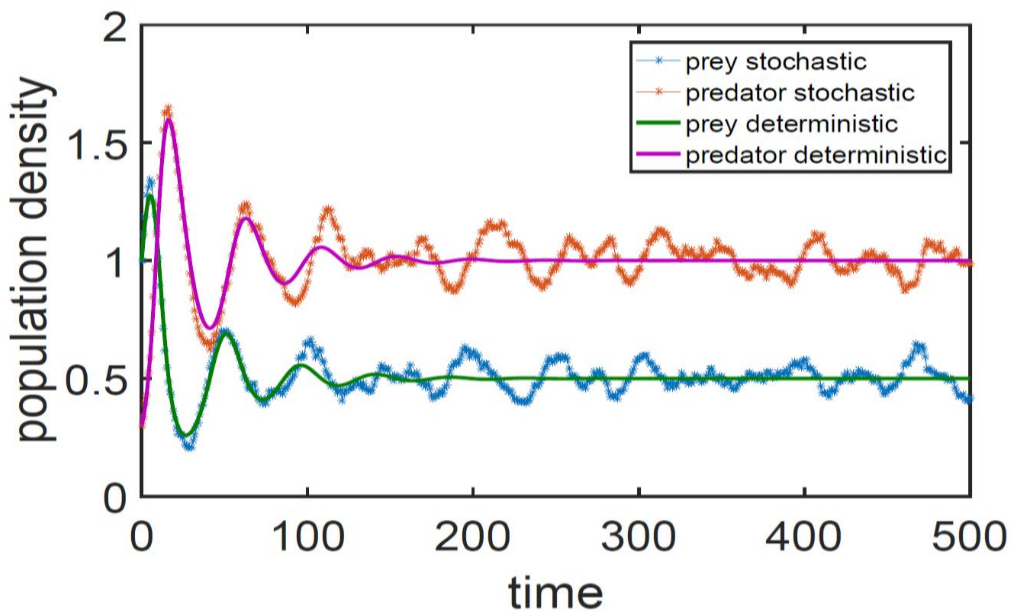
predator $$\frac{dy}{dt} = pxy - \boxed{dy}$$

Noisy persistent
quasi-cycles

Stochastic

$$X \xrightarrow{b} 2X \qquad X + X \xrightarrow{e/V} X$$

$$X + Y \xrightarrow{p/V} 2Y \qquad \boxed{Y \xrightarrow{d} \Phi}$$



T. Newman, A. McKane PRL (2006)

Predator-prey models

Deterministic

prey $\frac{dx}{dt} =$

predator $\frac{dy}{dt} =$

Nois

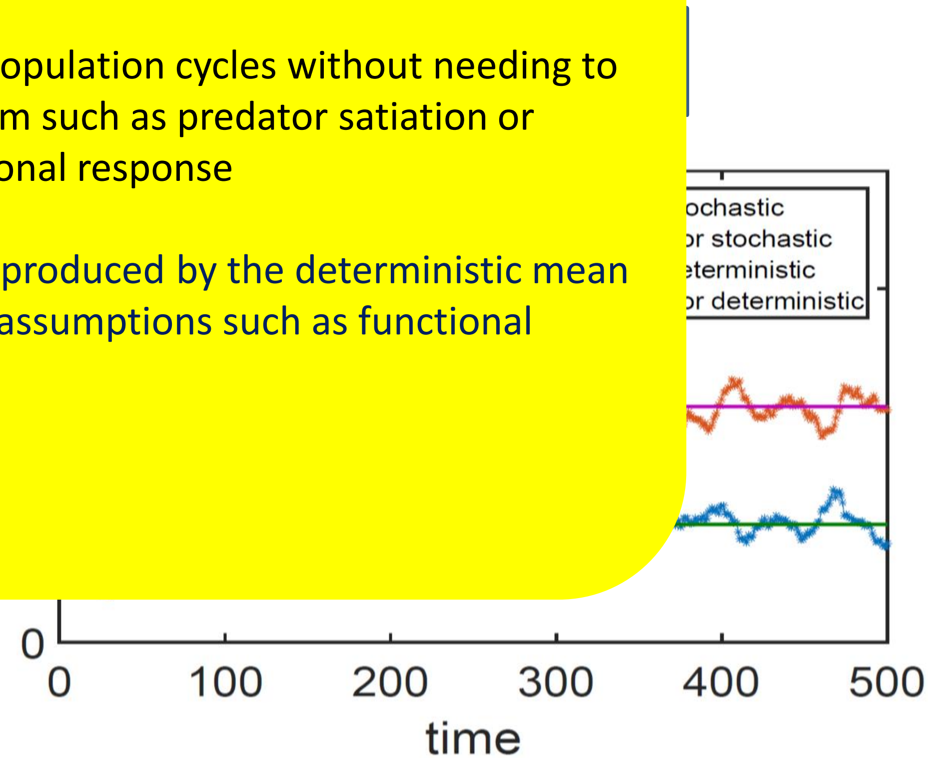
qua

Stochastic

$\xrightarrow{e/V} X$

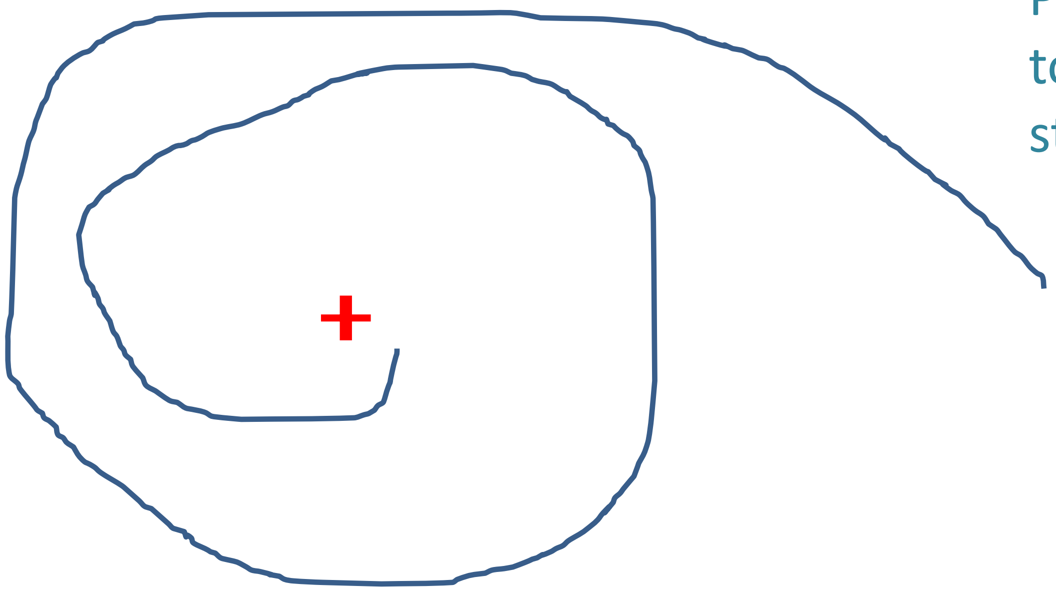
Stochasticity recapitulates the population cycles without needing to invoke additional levels of realism such as predator satiation or other assumptions about functional response

Persistent oscillations are not reproduced by the deterministic mean field theory, without additional assumptions such as functional response



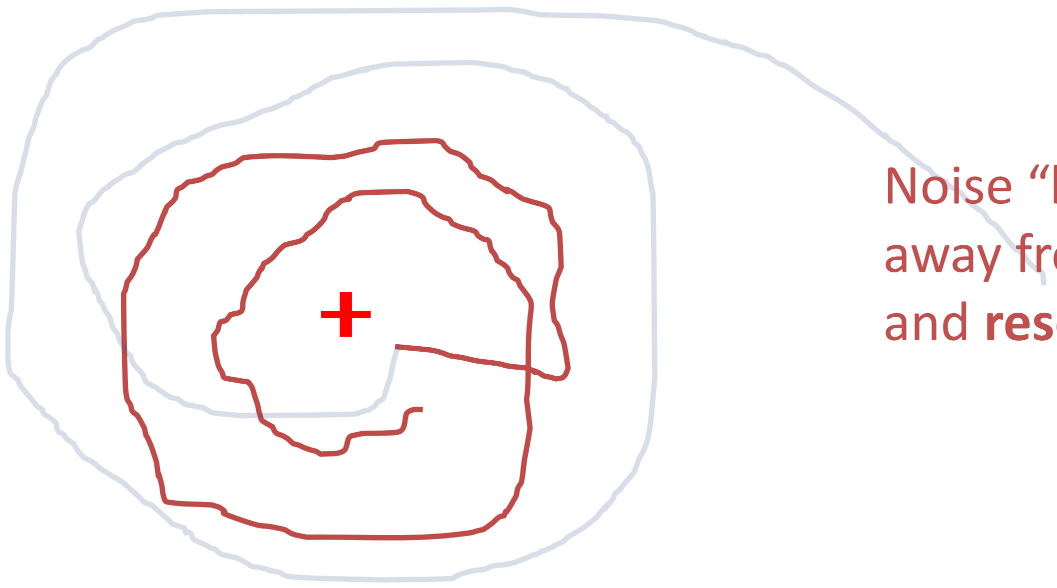
T. Newman, A. McKane PRL (2006)

Noise-induced quasi-cycles



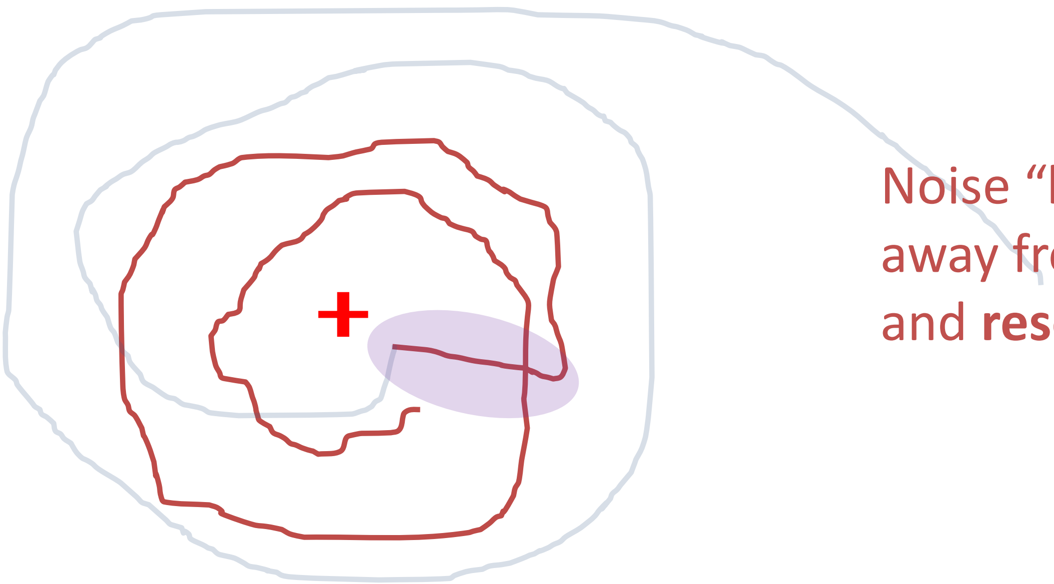
Populations decay toward the steady state.

Noise-induced quasi-cycles



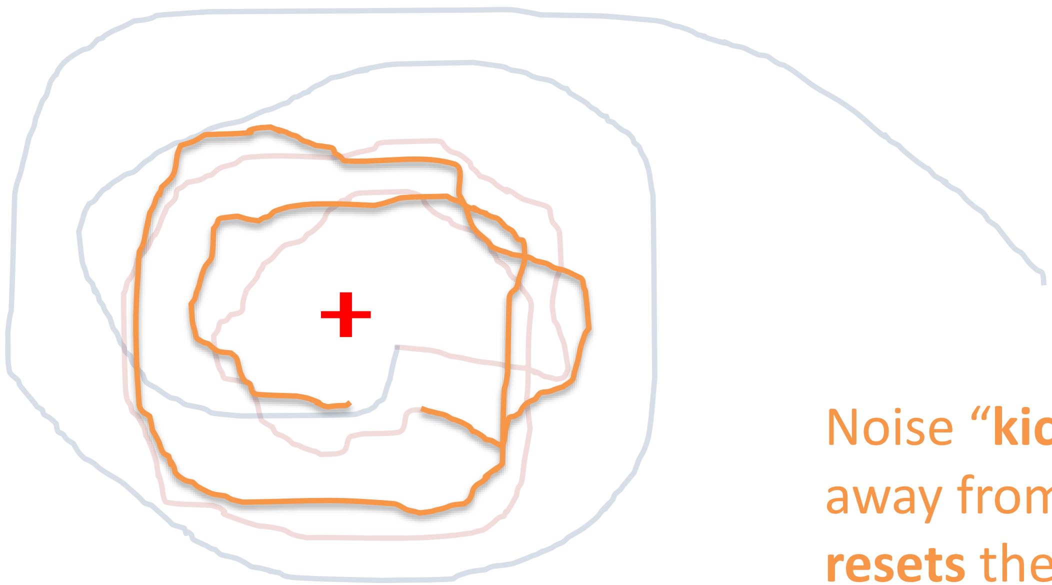
Noise “**kicks**” the trajectory away from the original path, and **resets** the decay.

Noise-induced quasi-cycles



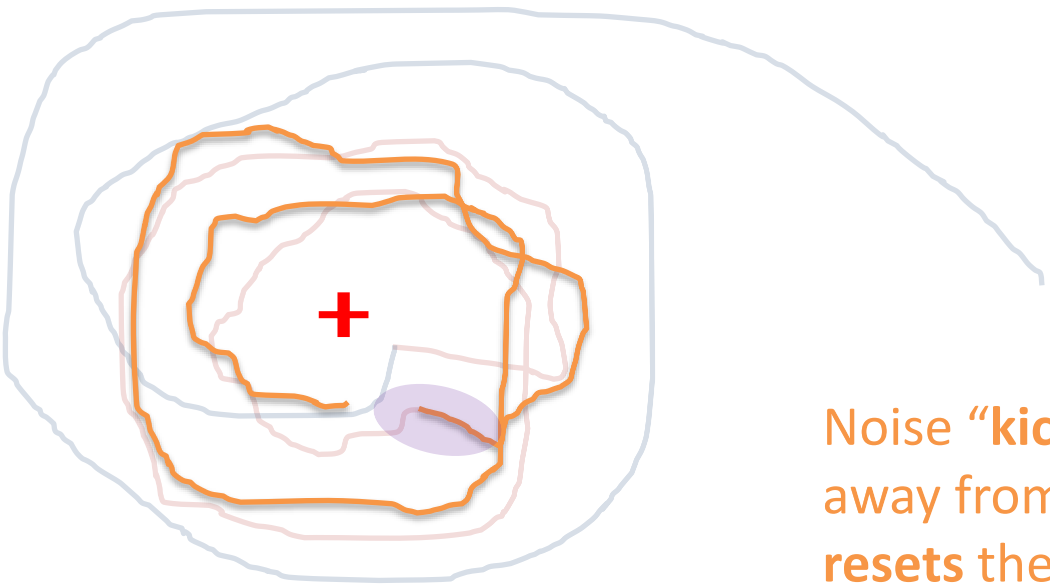
Noise “**kicks**” the trajectory away from the original path, and **resets** the decay.

Noise-induced quasi-cycles



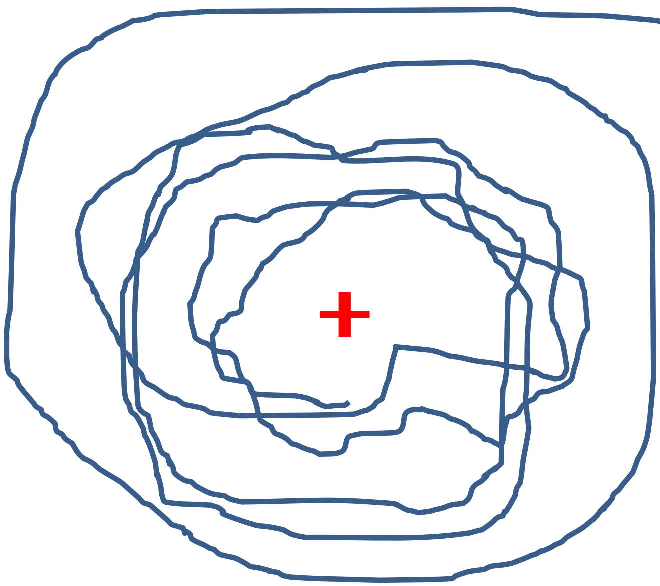
Noise “**kicks**” the trajectory away from the original path, and **resets** the decay, *<u>again</u>*.

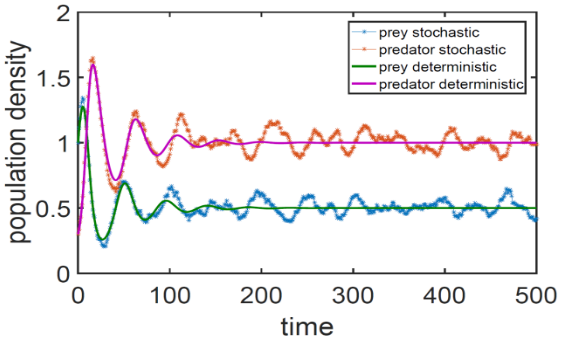
Noise-induced quasi-cycles



Noise “**kicks**” the trajectory away from the original path, and **resets** the decay, *again*.

Noise-induced quasi-cycles

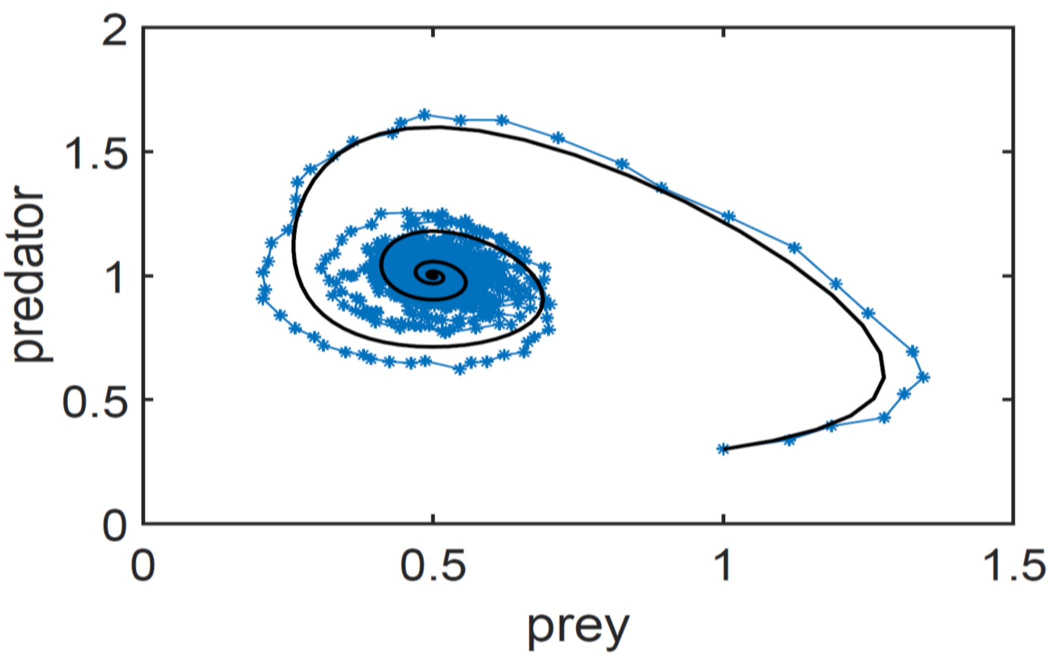




The phase portrait winds around the steady state but never stays at it, corresponding to quasi-cycles in time series.

The winding period is close to the period of stable complex eigenvalue about the fixed point.

Phase portrait



Deterministic

Damped motion toward the fixed point

Stochastic:

Noise stimulates deviation from fixed point, followed by relaxation to fixed point, followed by noise ...

What is the problem with Lotka-Volterra?

Averaging used to derive Lotka-Volterra removes cycles or spatial structure

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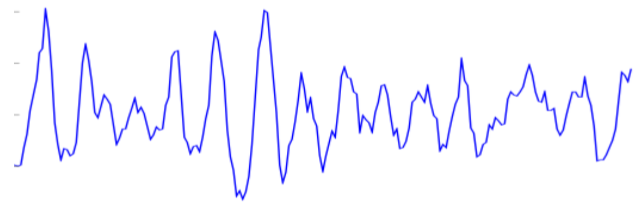

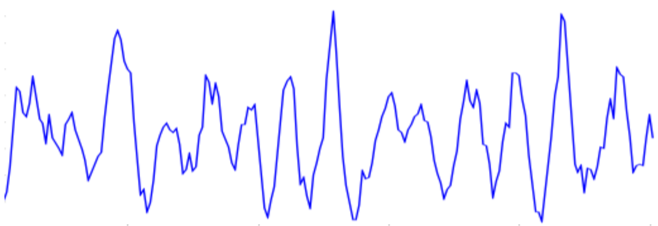

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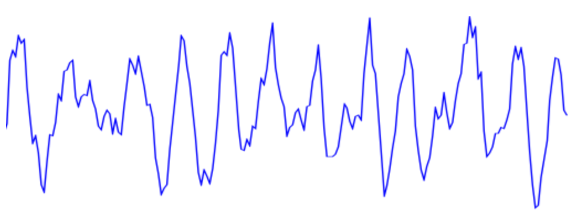

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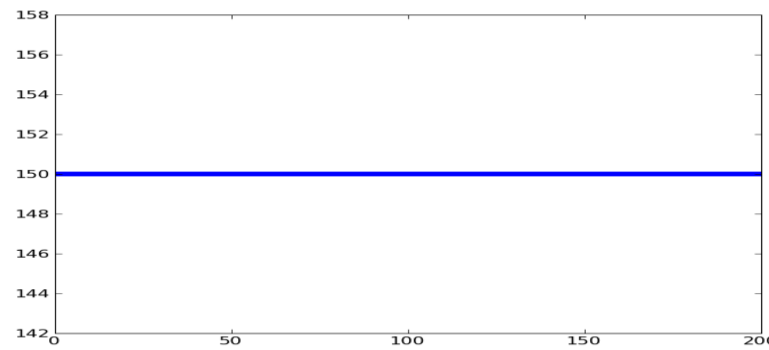


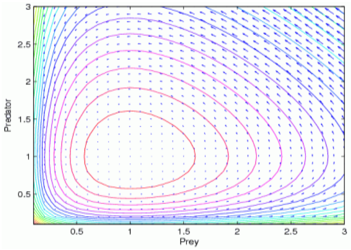
Comparison of Models

- **Lotka-Volterra equations** (Holling type I response function):

$$\frac{du}{dt} = bu - puv$$
$$\frac{dv}{dt} = puv - dv$$

the fixed point is a center
existence of a conserved quantities

Initial cond. dependent

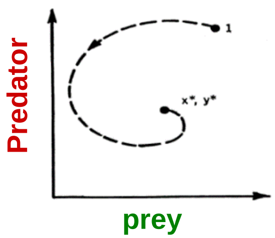


- **Volterra equations:**

$$\frac{du}{dt} = bu(1 - \frac{u}{K_u}) - puv$$
$$\frac{dv}{dt} = puv - dv$$

the fixed point is a stable focus

No persistent oscillations

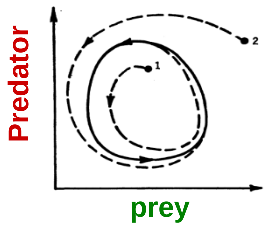


- **Satiation model** (**Holling type II response function**):

$$\frac{du}{dt} = bu(1 - \frac{u}{K_u}) - \boxed{p\frac{uv}{K_s + u}}$$
$$\frac{dv}{dt} = \boxed{p\frac{uv}{K_s + u}} - dv$$

the fixed point is a limit cycle

Persistent oscillations
No fluctuations
Not robust

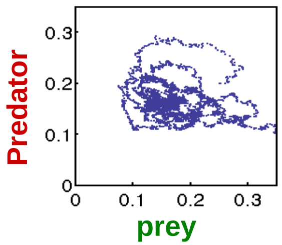


- **Individual level model** (Volterra equations + demographic stochasticity)

$$A \rightarrow \emptyset$$
$$A + B \rightarrow A + A$$
$$B \rightarrow B + B$$

demographic stochasticity drives the trajectory out from the fixed point (a stable focus)

Persistent random oscillations
Generic

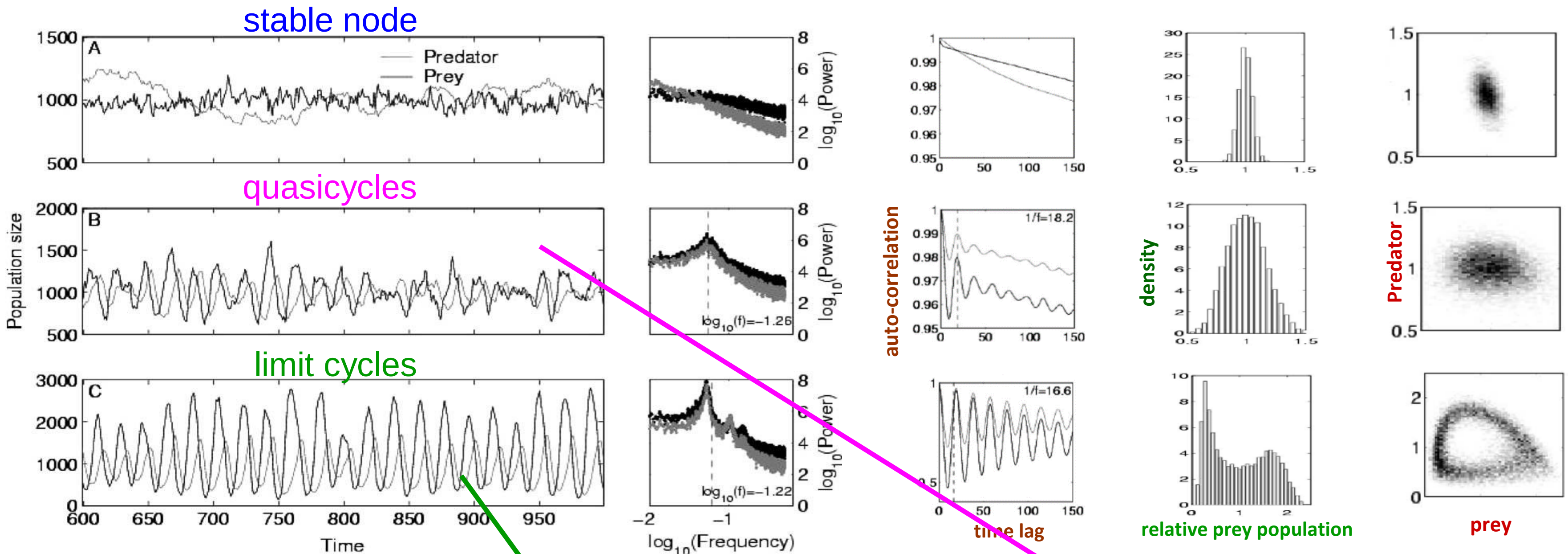


http://www.scholarpedia.org/article/Predator-prey_model

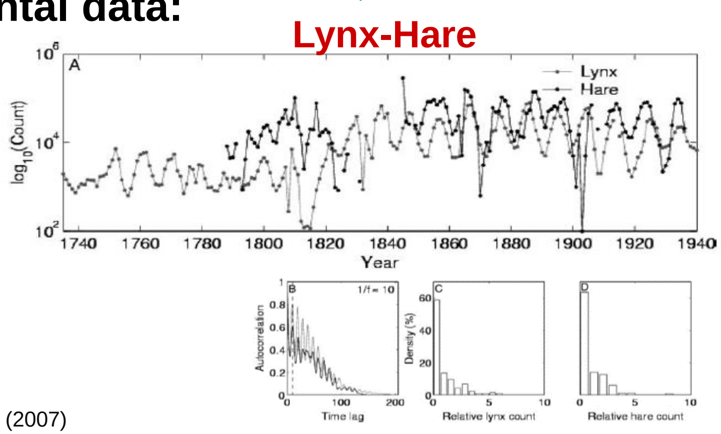
May., *Science* **177**, 4052 (1972)

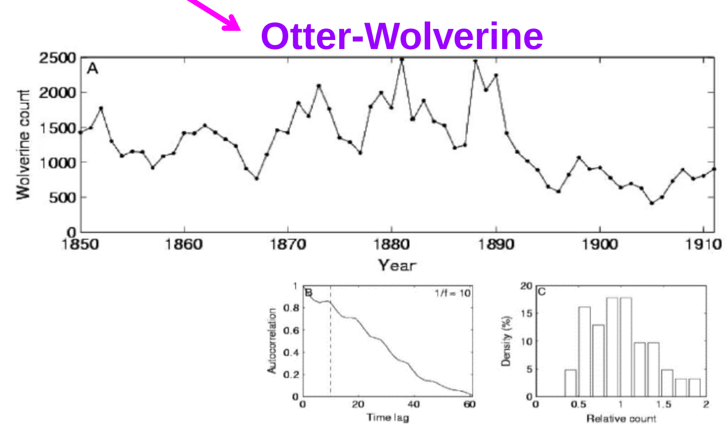
Quasi-cycles vs. Limit cycles in Experiments

- **Qualitative differences predicted by stochastic simulations:**



- **Experimental data:**





Pineda-Krch et al., *Oikos* **116**, 53 (2007)

Landau theory for transitional pipe turbulence is stochastic predator-prey

Bill Wyld (1928-2013)

ANNALS OF PHYSICS: **14:** 143–165 (1961)

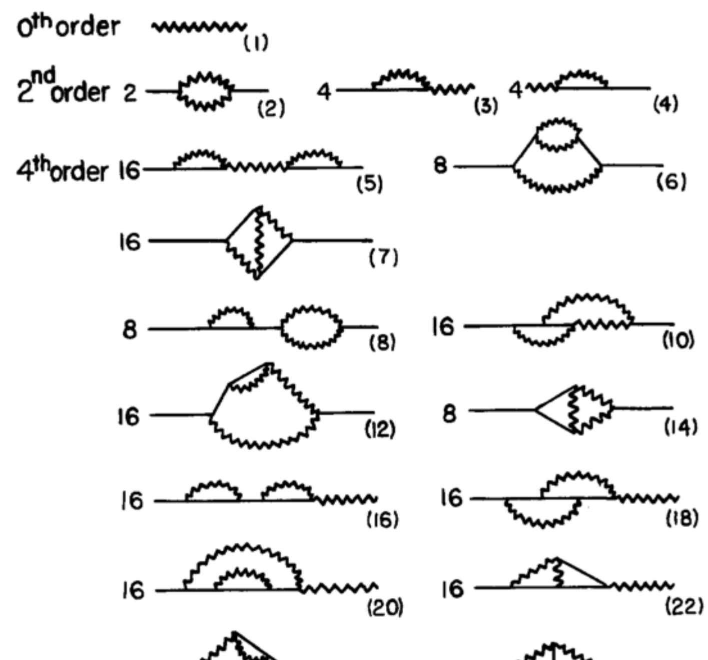
Formulation of the Theory of Turbulence in an Incompressible Fluid

H. W. Wyld, Jr.

Physics Department, University of Illinois, Urbana, Illinois and Space Technology Laboratories, Los Angeles, California

The theory of turbulence in an incompressible fluid is formulated using methods similar to those of quantum field theory. A systematic perturbation theory is set up, and the terms in the perturbation series are shown to be in one to one correspondence with certain diagrams analogous to Feynman diagrams. From a study of the diagrams it is shown that the perturbation series can be rearranged and partially summed in such a way as to reduce the problem to the solution of three simultaneous integral equations for three functions, one of which is the second order velocity correlation function. The equations have the form of infinite power series integral equations, and the first few terms in the power series are derived from an analysis of the diagrams to sixth order. Truncation of the integral equations at the lowest nontrivial order yields Chandrasekhar's equation, and truncation at a higher order yields the equations discussed by Kraichnan.

THEORY OF TURBULENCE



Derivation of predator-prey equations

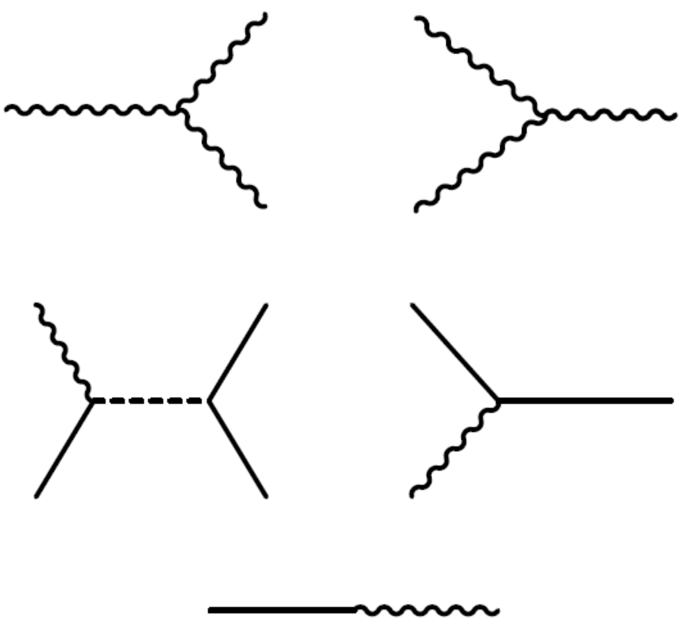
Zonal flow Turbulence

Vacuum = Laminar flow

A = predator B = prey

E = food/empty state

Zonal flow-turbulence



Predator-prey

$$B + E \xrightarrow{b} B + B$$

$$B + B \xrightarrow{c} B + E$$

$$A + B \xrightarrow{p} A + A$$

$$A + B \xrightarrow{p'} A + E$$

$$B \xrightarrow{m} A$$

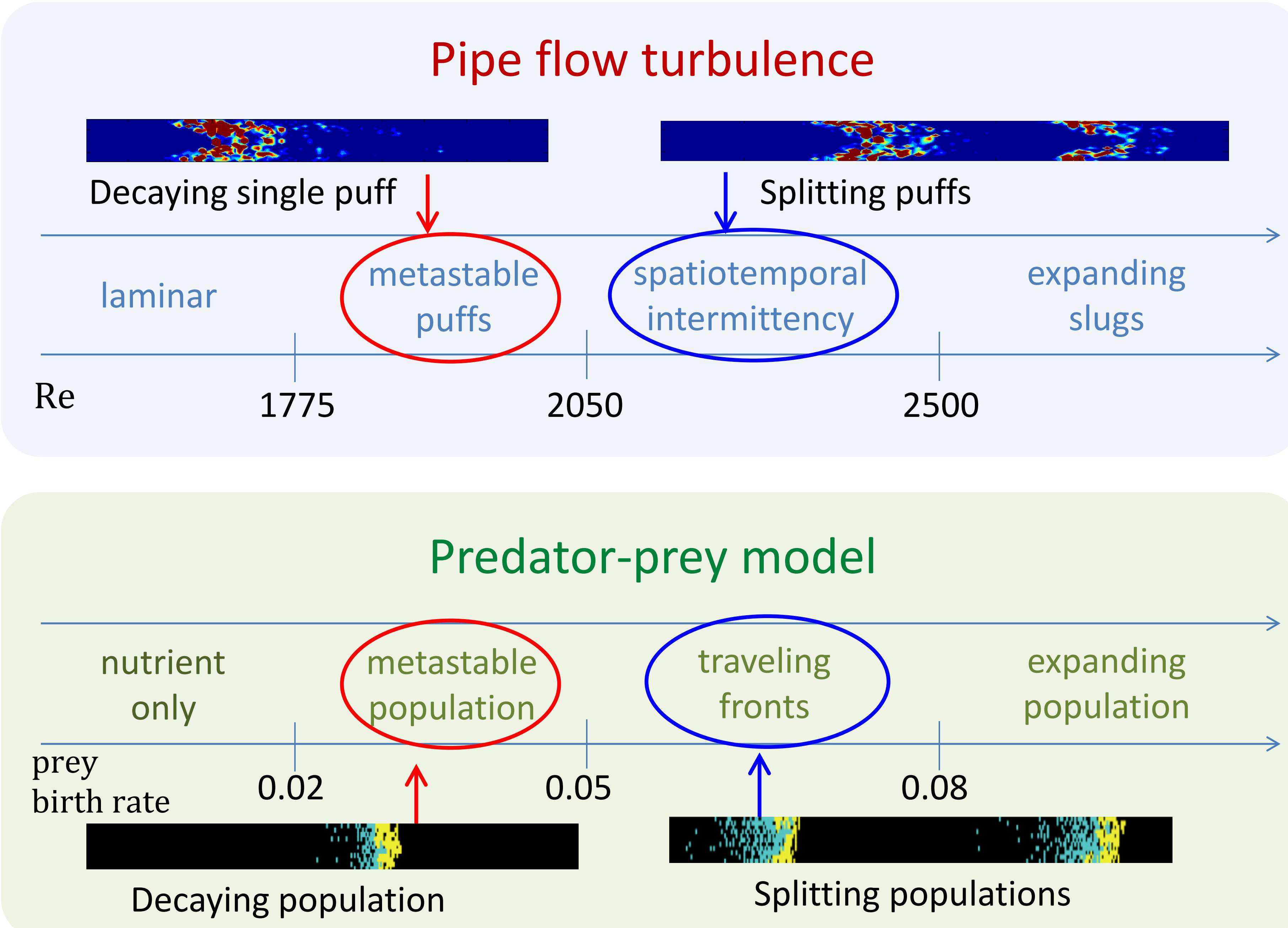
$$A \xrightarrow{d_A} E \quad B \xrightarrow{d_B} E$$

Stochastic predator-prey recapitulates turbulence data

Phase diagram

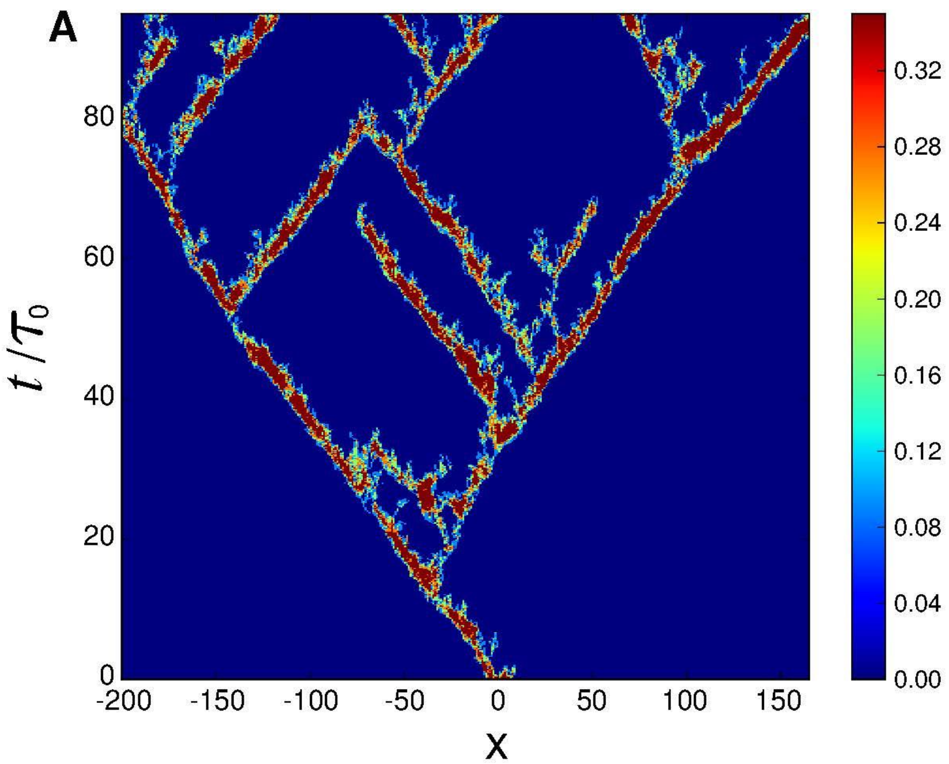
Lifetime statistics

Universality class prediction

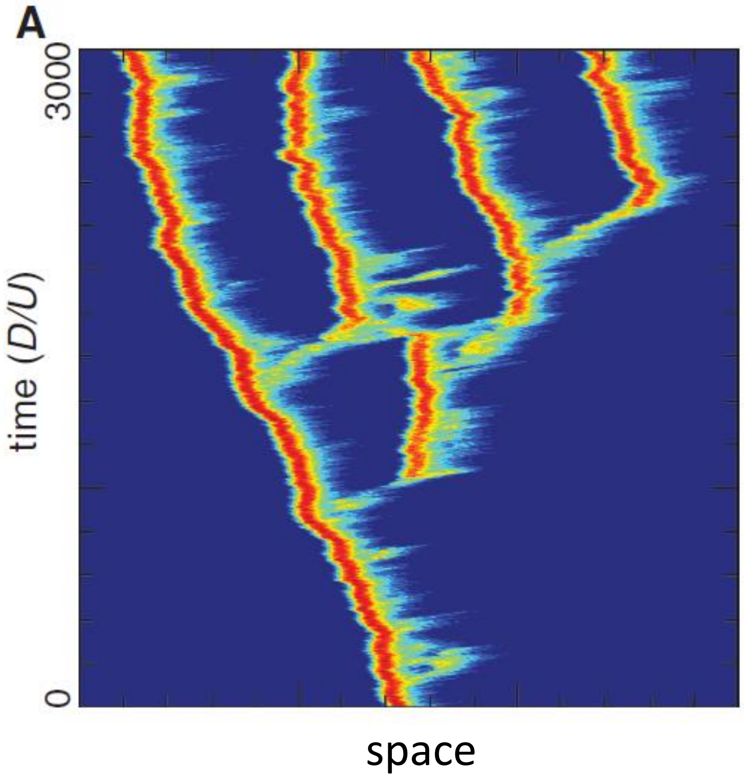


Pipe flow turbulence
Decaying single puff
Splitting puffs
laminar
metastable puffs
spatiotemporal intermittency
expanding slugs
Re
1775
2050
2500
Predator-prey model
nutrient only
metastable population
traveling fronts
expanding population
prey birth rate
0.02
0.05
0.08
Decaying population
Splitting populations

“Puff splitting” in predator-prey systems

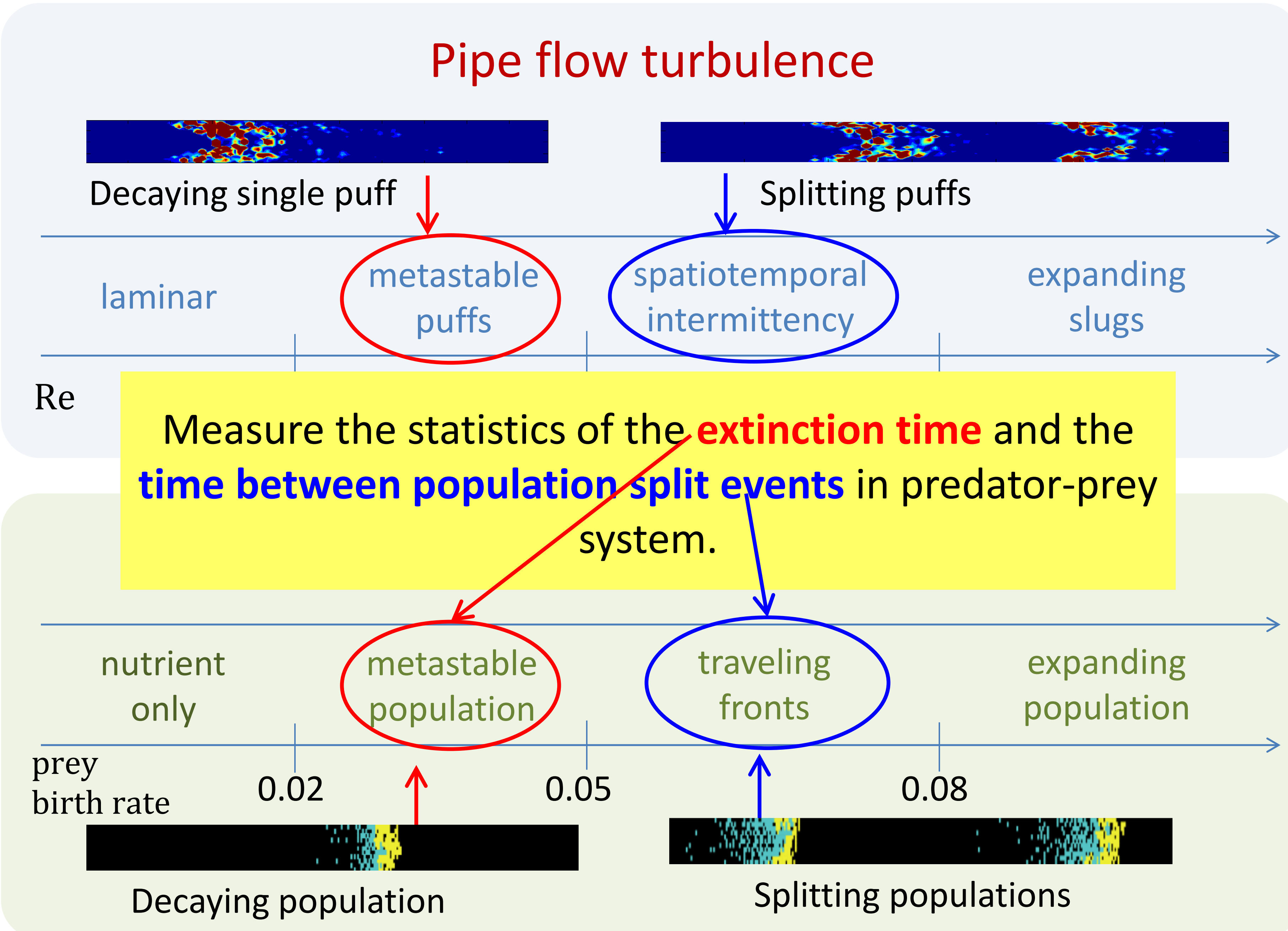


Population-splitting in predator-prey ecosystem



Puff-splitting in pipe turbulence

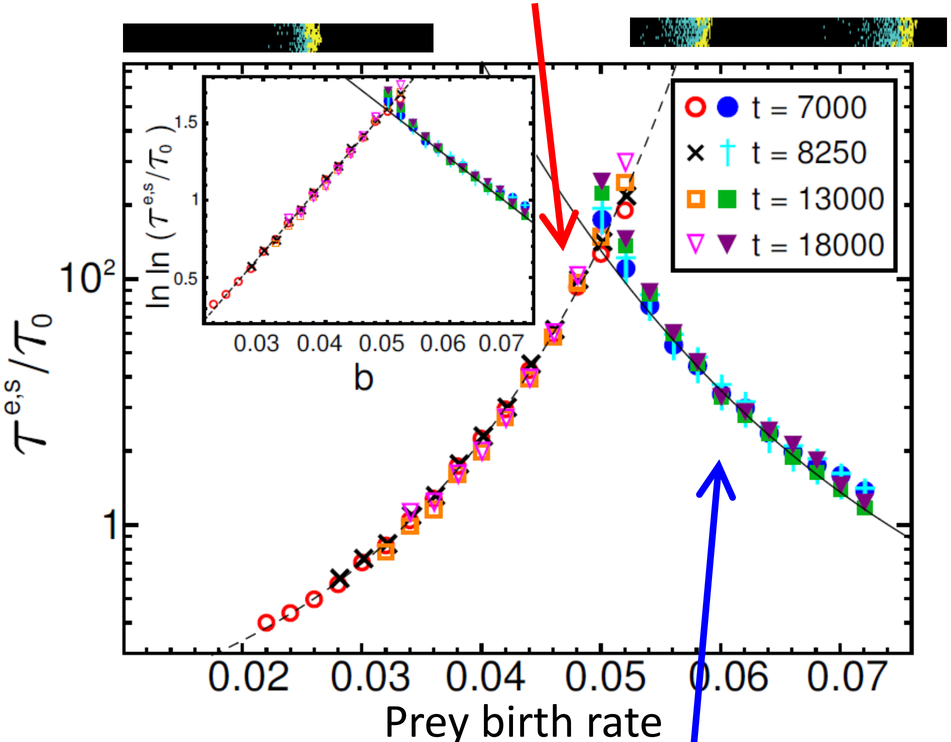
Avila et al., Science (2011)

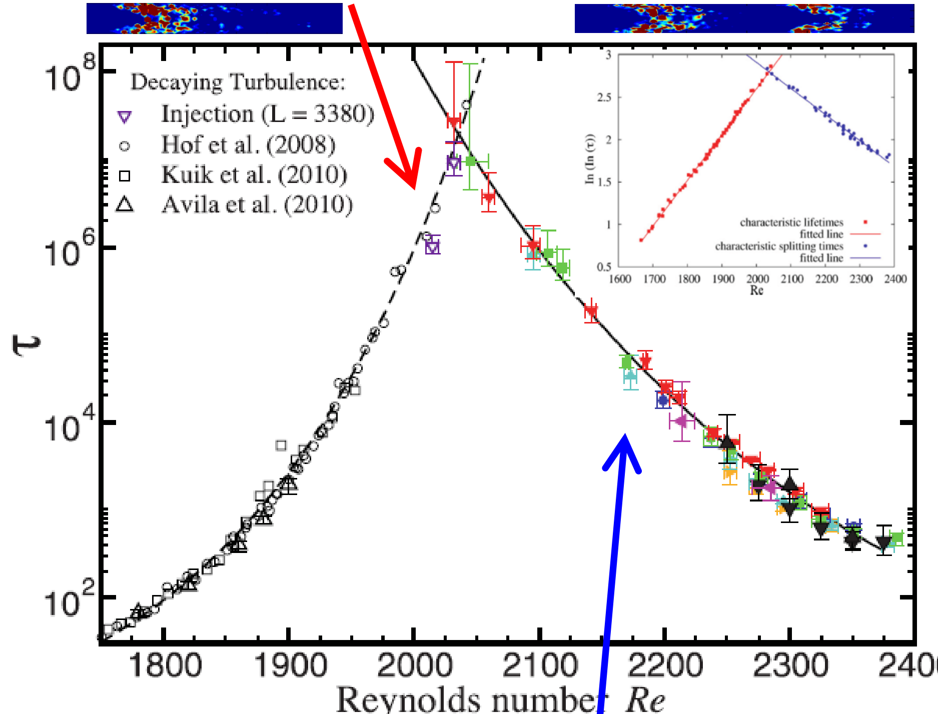

Pipe flow turbulence
Decaying single puff
Splitting puffs
laminar
metastable puffs
spatiotemporal intermittency
expanding slugs
Re
Measure the statistics of the extinction time and the time between population split events in predator-prey system.
nutrient only
metastable population
traveling fronts
expanding population
prey birth rate
0.02
0.05
0.08
Decaying population
Splitting populations

Predator-prey vs. transitional turbulence

Prey lifetime

Turbulent puff lifetime





Mean time between population split events

Mean time between puff split events

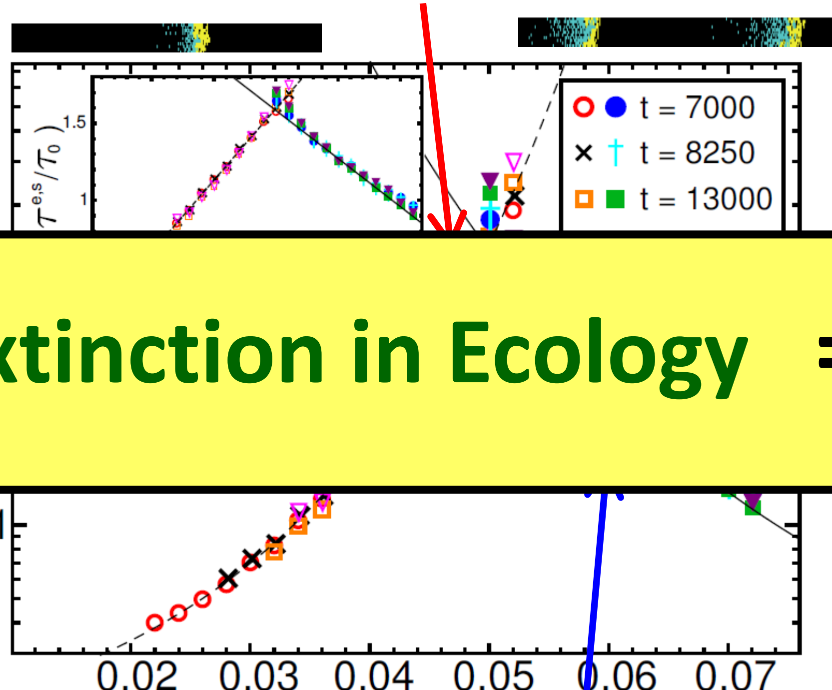
Shih, Hsieh and Goldenfeld
Nature Physics **12**, 245 (2016)

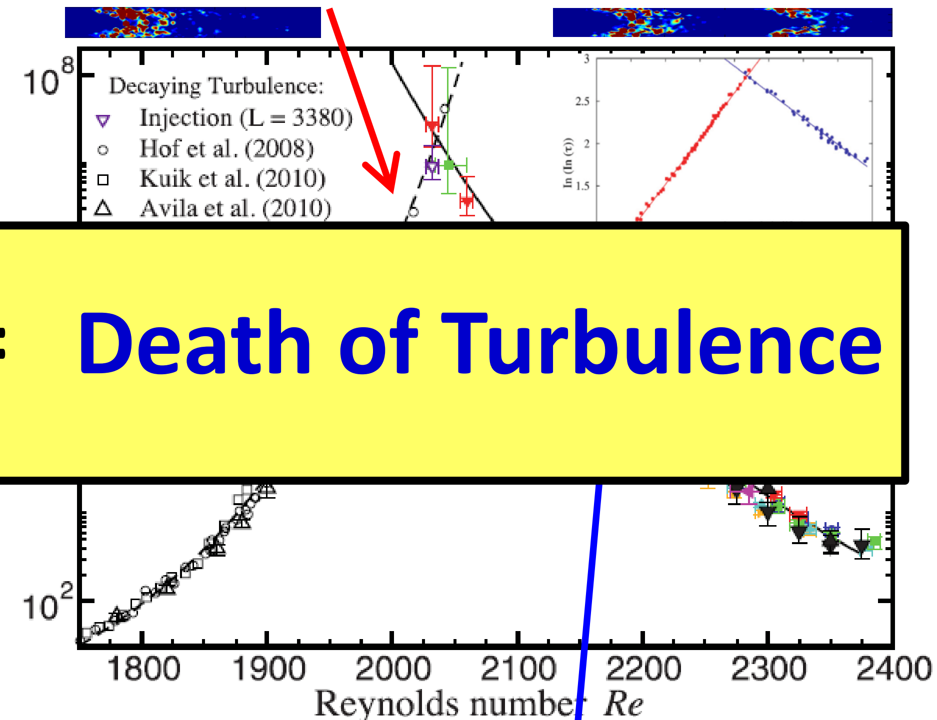
Avila *et al.*, *Science* **333**, 192 (2011)
Song *et al.*, *J. Stat. Mech.* 2014(2**)**, P020010

Predator-prey vs. transitional turbulence

Prey lifetime

Turbulent puff lifetime





Extinction in Ecology = Death of Turbulence

Mean time between population split events

Mean time between puff split events

Shih, Hsieh and Goldenfeld
Nature Physics **12**, 245 (2016)

Avila *et al.*, *Science* **333**, 192 (2011)
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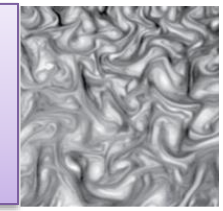
Roadmap: Universality class of laminar-turbulent transition

Universality class

?

(Classical) Turbulence

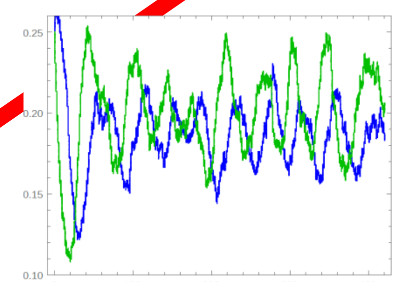


(Boffetta and Ecke, 2012)

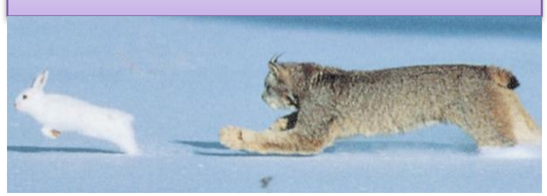
Direct Numerical Simulations of Navier-Stokes

Two-fluid model

Turbulence + Zonal flow

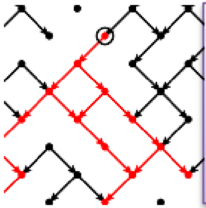


Predator-Prey



(Pearson Education, Inc., 2009)

Roadmap: Universality class of laminar-turbulent transition



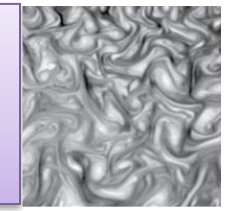
(Wikimedia Commons)

Directed Percolation

?

(Classical) Turbulence

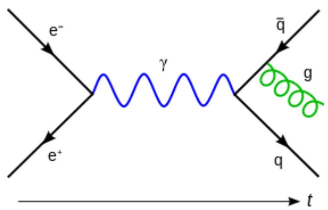
(Boffetta and Ecke, 2012)



Reggeon field theory
(Janssen, 1981)

Direct Numerical Simulations
of Navier-Stokes

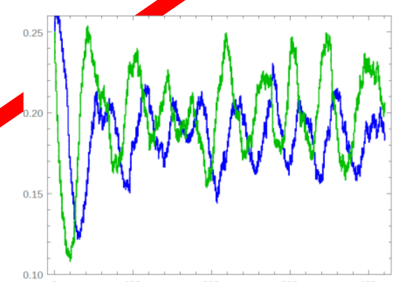
Field Theory



(Wikimedia Commons)

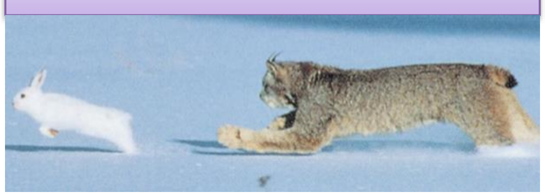
Two-fluid model

Turbulence + Zonal flow



Extinction transition
(Mobilia et al., 2007)

Predator-Prey



(Pearson Education, Inc., 2009)