Geometric Elasticity in Soft Matter Physics
Boulder Summer School

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3 Extra Material

3.1 Noneuclidean Plates - Thin sheets and dimension reduction

We now turn to thin sheets. Let w denote the coordinate through the thickness. The three-dimensional metric induced by a surface may be expanded as

$$g_{ij}(x,w) = \begin{pmatrix} a_{\alpha\beta} - 2wb_{\alpha\beta} + w^2 c_{\alpha\beta} & 0 \\ 0 & 1 \end{pmatrix} + \cdots . \tag{3.1}$$

Similarly, the reference metric may be expanded as

$$\bar{g}_{ij}(x,w) = \begin{pmatrix} \bar{a}_{\alpha\beta} - 2w\bar{b}_{\alpha\beta} + w^2 \bar{c}_{\alpha\beta} & 0 \\ 0 & 1 \end{pmatrix} + \cdots . \tag{3.2}$$

Here $a_{\alpha\beta}$ is the first fundamental form and $b_{\alpha\beta}$ is the second fundamental form of the mid-surface. We now plug in these metrics into the definition of strain, and then into the energy

$$E = \int_\Omega \frac{1}{8} A^{ijkl} \left(g_{ij} - \bar{g}_{ij}\right)\left(g_{kl} - \bar{g}_{kl}\right) \, dV_{\bar{g}} \tag{3.3}$$

After integrating through the thickness, the term that mixes the metric and curvature strains vanishes, and the elastic energy becomes

$$E = \frac{h}{8} \int_\omega A^{\alpha\beta\gamma\delta} \left(a_{\alpha\beta} - \bar{a}_{\alpha\beta}\right)\left(a_{\gamma\delta} - \bar{a}_{\gamma\delta}\right) \sqrt{|\bar{a}|} \, d^2x \tag{3.4}$$

$$+ \frac{h^3}{24} \int_\omega A^{\alpha\beta\gamma\delta} \left(b_{\alpha\beta} - \bar{b}_{\alpha\beta}\right)\left(b_{\gamma\delta} - \bar{b}_{\gamma\delta}\right) \sqrt{|\bar{a}|} \, d^2x + \cdots . \tag{3.5}$$

The first term is stretching and scales as h. The second term is bending and scales as h^3.

Remark: Monge-patch representation

For a nearly flat sheet one may work directly with a configuration in Monge form,

$$\phi(x,y) = \left(x + u_x(x,y),\ y + u_y(x,y),\ w(x,y)\right). \tag{3.6}$$

Then the metric and curvature

$$a_{\alpha\beta} \simeq \delta_{\alpha\beta} + \partial_\alpha u_\beta + \partial_\beta u_\alpha + \partial_\alpha w \, \partial_\beta w, \qquad b_{\alpha\beta} \simeq \partial_\alpha \partial_\beta w. \tag{3.7}$$

Since a and b come from an actual map, the Gauss and Mainardi-Codazzi compatibility conditions are automatic. Minimizing the stretching-bending energy with respect to u_α and w gives the Föppl-von Kármán equations. In the absence of ateral or transverse loads

$$\partial_\beta \sigma^{\alpha\beta} = 0, \quad B\Delta^2 w = \sigma^{\alpha\beta} \partial_{\alpha\beta} w, \tag{3.8}$$

3.1.1 Gauss and Mainardi–Codazzi compatibility

The intrinsic and extrinsic geometries of a surface are not independent. They must satisfy the Gauss equation

$$K_G(a) = \det\left(a^{-1}b\right), \tag{3.9}$$

and the Mainardi–Codazzi equations

$$\nabla_\alpha b_{\beta\gamma} = \nabla_\beta b_{\alpha\gamma}. \tag{3.10}$$

Contrary to the actual fields, the reference fields do not have to satisfy the compatibility conditions, in which case the elastic sheet is frustrated since its reference field would not be simultanously realizable. We distinct between two kinds of incompatibility: The Gauss' incompatibility

$$K_G(\bar{a}) \neq \det\left(\bar{a}^{-1}\bar{b}\right), \tag{3.11}$$

and Mainardi–Codazzi incompatibility

$$\bar{\nabla}_\alpha \bar{b}_{\beta\gamma} \neq \bar{\nabla}_\beta \bar{b}_{\alpha\gamma};. \tag{3.12}$$

Bilayer of fibers. A simple model for a bilayer of fibers, as in Bauhinia seed pods, is

$$\bar{a} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \bar{b} = \begin{pmatrix} 0 & \kappa \\ \kappa & 0 \end{pmatrix}. \tag{3.13}$$

These reference fields violate the Gauss compatibility condition.

Rose petals. In polar coordinates a simplified reference geometry is

$$\bar{a} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \qquad \bar{b} = \begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.14}$$

Then

$$K_G(\bar{a}) = 0, \qquad \det\left(\bar{a}^{-1}\bar{b}\right) = 0, \tag{3.15}$$

so the Gauss equation is satisfied. However,

$$\bar{\nabla}_\alpha \bar{b}_{\beta\gamma} \neq \bar{\nabla}_\beta \bar{b}_{\alpha\gamma}, \tag{3.16}$$

so the Mainardi–Codazzi equations are violated. The symmetric solution may lose stability, and a lower-energy nonsymmetric configuration is obtained.

3.2 Curvature potentials for non-Euclidean plates

The stretching and bending strains are therefore

$$\varepsilon^{\mathrm{st}}_{\alpha\beta} = \frac{1}{2}\left(a_{\alpha\beta} - \bar{a}_{\alpha\beta}\right), \qquad \varepsilon^{\mathrm{bn}}_{\alpha\beta} = \frac{1}{2\sqrt{3}}\left(b_{\alpha\beta} - \bar{b}_{\alpha\beta}\right). \tag{3.17}$$

The elastic energy is

$$E = \frac{1}{2}\int_\Omega \left(hA^{\alpha\beta\gamma\delta}\varepsilon^{\rm st}_{\alpha\beta}\varepsilon^{\rm st}_{\gamma\delta} + h^3 A^{\alpha\beta\gamma\delta}\varepsilon^{\rm bn}_{\alpha\beta}\varepsilon^{\rm bn}_{\gamma\delta}\right) dS. \tag{3.18}$$

Here h is the sheet thickness and $A^{\alpha\beta\gamma\delta}$ is the elastic tensor of the two-dimensional material.

The corresponding stretching stress and bending moment are

$$\sigma^{\alpha\beta} = \frac{\partial E}{\partial \varepsilon^{\rm st}_{\alpha\beta}} = hA^{\alpha\beta\gamma\delta}\varepsilon^{\rm st}_{\gamma\delta}, \qquad m^{\alpha\beta} = \frac{\partial E}{\partial \varepsilon^{\rm bn}_{\alpha\beta}} = \frac{h^3}{2\sqrt{3}}A^{\alpha\beta\gamma\delta}\varepsilon^{\rm bn}_{\gamma\delta}. \tag{3.19}$$

It is useful to define the effective in-plane stress

$$\Sigma^{\alpha\beta} = \sigma^{\alpha\beta} + m^{\mu\beta}s^{\alpha}{}_{\mu}, \qquad s^{\alpha}{}_{\mu} = a^{\alpha\gamma}b_{\gamma\mu}, \tag{3.20}$$

where $s = a^{-1}b$ is the shape operator.

The equilibrium equations are obtained by varying the energy with respect to the configuration ϕ. A virtual displacement can be decomposed into a tangential part and a normal part,

$$\delta\phi = \zeta^\alpha \partial_\alpha \phi + \eta\hat{n}. \tag{3.21}$$

The tangential variation changes the parametrized distances along the surface, while the normal variation bends the surface. The corresponding variations of the fundamental forms have the schematic form

$$\delta a_{\alpha\beta} = \nabla_\alpha\zeta_\beta + \nabla_\beta\zeta_\alpha - 2\eta b_{\alpha\beta}, \tag{3.22}$$

$$\delta b_{\alpha\beta} = \mathcal{L}_\zeta b_{\alpha\beta} + \nabla_\alpha\nabla_\beta\eta - \eta b_{\alpha\gamma}b^{\gamma}{}_{\beta}, \tag{3.23}$$

where $\mathcal{L}_\zeta$ denotes the Lie derivative along the tangential field ζ. After integration by parts, the coefficients of the arbitrary fields ζ^α and η must vanish separately. The coefficient of ζ^α gives the tangential, or in-plane, force balance. The coefficient of η gives the normal force balance, in which bending moments are balanced against the normal projection of the in-plane stresses.

In the notation of geometric plate theory these two bulk equations can be written as

$$0 = (\mathrm{div}_{a,\bar{a}})_\beta\, \Sigma^{\alpha\beta} + (\mathrm{div}_{a,\bar{a}})_\beta\, m^{\mu\beta}\, a^{\gamma\alpha}b_{\gamma\mu}, \tag{3.24}$$

$$0 = \bar{\nabla}_\alpha\left[(\mathrm{div}_{a,\bar{a}})_\beta\, m^{\alpha\beta}\right] - \Sigma^{\alpha\beta}b_{\alpha\beta}. \tag{3.25}$$

The mixed divergence operator is

$$(\mathrm{div}_{a,\bar{a}})_\mu\, T^{\mu\nu} = \bar{\nabla}_\mu T^{\mu\nu} + \left(\Gamma^\nu_{\alpha\beta} - \bar{\Gamma}^\nu_{\alpha\beta}\right)T^{\alpha\beta}. \tag{3.26}$$

The appearance of both a and $\bar{a}$ in this operator is one reason why the fully geometric equations are nonlinear and implicit: the differential operator itself depends on the unknown metric.

For the Föppl–von Kármán scaling, one assumes small stretching strain and small bending strain, with $\bar{b} = 0$, so that the difference $\Gamma - \bar{\Gamma}$ is higher order. To leading order the equilibrium equations reduce to

$$\bar{\nabla}_\beta\sigma^{\alpha\beta} = 0, \tag{3.27}$$

$$\bar{\nabla}_\alpha\bar{\nabla}_\beta m^{\alpha\beta} - \sigma^{\alpha\beta}b_{\alpha\beta} = 0. \tag{3.28}$$

The first equation is the in-plane force balance. The second is the normal force balance. It states that the transverse force generated by gradients of bending moments is balanced by the work done by in-plane stresses on a curved surface.

The equilibrium equations are not yet a closed theory for the fundamental forms a and b, because not every pair (a, b) comes from an embedded surface. They must also satisfy the compatibility equations

$$K_G(a) = \det\left(a^{-1}b\right), \tag{3.29}$$

$$\nabla_\alpha b_{\beta\gamma} - \nabla_\beta b_{\alpha\gamma} = 0. \tag{3.30}$$

The first equation is Gauss' theorem: the intrinsic Gaussian curvature of the metric equals the determinant of the shape operator. The second equation is the Mainardi–Codazzi compatibility condition, expressing the integrability of the normal field. The reference forms $(\bar{a}, \bar{b})$ need not satisfy these conditions. Their failure to do so is precisely the geometric frustration of a non-Euclidean plate. The actual forms (a, b), however, must always satisfy these conditions.

The curvature-potential formulation is based on replacing the direct configuration variables by scalar potentials that solve the two differential compatibility problems. The first scalar is the Airy stress potential χ. In the FvK limit the in-plane equilibrium equation $\bar{\nabla}_\beta \sigma^{\alpha\beta} = 0$ is solved by writing the stress as a double curl,

$$\sigma^{\alpha\beta} = \bar{E}^{\alpha\gamma}\bar{E}^{\beta\delta}\bar{\nabla}_\gamma\bar{\nabla}_\delta\chi, \qquad \bar{E}^{\alpha\beta} = \frac{\epsilon^{\alpha\beta}}{\sqrt{|\bar{a}|}}. \tag{3.31}$$

For a flat reference metric this reduces to the usual expressions

$$\sigma^{xx} = \partial_y^2\chi, \qquad \sigma^{yy} = \partial_x^2\chi, \qquad \sigma^{xy} = -\partial_x\partial_y\chi. \tag{3.32}$$

Thus the stress potential automatically enforces in-plane force balance.

The second scalar is the curvature potential ψ. In the small-strain regime, and locally for a Euclidean or nearly Euclidean reference metric, the Mainardi–Codazzi equation is solved by writing

$$b_{\alpha\beta} = \bar{\nabla}_\alpha\bar{\nabla}_\beta\psi. \tag{3.33}$$

This is the key replacement of the classical height function by an intrinsic curvature-generating field. If the surface is a shallow graph $z = f(x, y)$, then $b_{\alpha\beta} \simeq \partial_\alpha\partial_\beta f$, so that ψ coincides with the height function f. In general, however, ψ is not a height. It is a scalar potential whose Hessian gives the curvature tensor, and the configuration is reconstructed only later from a and b by solving the Weingarten equations.

The Gauss compatibility condition then becomes the stretching equation. In the FvK approximation the metric is written as a small perturbation of the reference metric,

$$a_{\alpha\beta} = \bar{a}_{\alpha\beta} + 2u_{\alpha\beta}. \tag{3.34}$$

Using the isotropic stress–strain relation to express $u_{\alpha\beta}$ in terms of $\sigma^{\alpha\beta}$, and then using the Airy representation of $\sigma^{\alpha\beta}$, the linearized intrinsic curvature of the metric gives

$$\frac{1}{Y}\bar{\Delta}\bar{\Delta}\chi = \det\left(\bar{a}^{-1}b\right) - \bar{K}_G. \tag{3.35}$$

Here Y is the two-dimensional Young modulus and $\bar{K}_G = K_G(\bar{a})$. This equation says that Gaussian curvature is the source of stretching stress. For a non-Euclidean reference metric, $\bar{K}_G$ acts as a prescribed geometric source. Bending can partially screen this source through the extrinsic term $\det(\bar{a}^{-1}b)$.

For an isotropic plate with $\bar{b} = 0$, the bending moment in the FvK regime is

$$m^{\alpha\beta} = B\left[(1-\nu)b^{\alpha\beta} + \nu\bar{a}^{\alpha\beta}b^{\gamma}{}_{\gamma}\right], \qquad B = \frac{Yh^2}{12(1-\nu^2)}. \tag{3.36}$$

Substituting $b_{\alpha\beta} = \bar{\nabla}_\alpha\bar{\nabla}_\beta\psi$ into the normal force balance gives

$$B\bar{\Delta}\bar{\Delta}\psi = \sigma^{\alpha\beta}b_{\alpha\beta}. \tag{3.37}$$

Equivalently,

$$Yh^2\bar{\Delta}\bar{\Delta}\psi = 12(1-\nu^2)\sigma^{\alpha\beta}b_{\alpha\beta}. \tag{3.38}$$

The curvature-potential form of the FvK equations is therefore

$$\boxed{\frac{1}{Y}\bar{\Delta}^2\chi = \det\left(\bar{a}^{-1}\bar{\nabla}\bar{\nabla}\psi\right) - \bar{K}_G} \tag{3.39}$$

$$\boxed{\frac{Yh^2}{12(1-\nu^2)}\bar{\Delta}^2\psi = \sigma^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\psi} \tag{3.40}$$

with

$$\sigma^{\alpha\beta} = \bar{E}^{\alpha\gamma}\bar{E}^{\beta\delta}\bar{\nabla}_\gamma\bar{\nabla}_\delta\chi. \tag{3.41}$$

These equations have the same structure as the classical FvK equations, but the field ψ is interpreted as a curvature potential rather than as a height function.

In flat coordinates, where $\bar{a}_{\alpha\beta} = \delta_{\alpha\beta}$, these equations become

$$\frac{1}{Y}\Delta^2\chi = \psi_{,xx}\psi_{,yy} - \psi_{,xy}^2 - \bar{K}_G, \tag{3.42}$$

$$\frac{Yh^2}{12(1-\nu^2)}\Delta^2\psi = \chi_{,yy}\psi_{,xx} + \chi_{,xx}\psi_{,yy} - 2\chi_{,xy}\psi_{,xy}. \tag{3.43}$$

Using the bracket

$$[f, g] = f_{,xx}g_{,yy} + f_{,yy}g_{,xx} - 2f_{,xy}g_{,xy}, \tag{3.44}$$

this can be written as

$$\frac{1}{Y}\Delta^2\chi = \frac{1}{2}[\psi, \psi] - \bar{K}_G, \qquad \frac{Yh^2}{12(1-\nu^2)}\Delta^2\psi = [\chi, \psi]. \tag{3.45}$$

The advantage of the curvature-potential formulation is that it keeps the scalar-potential structure of FvK theory while removing the need to represent the surface as a single-valued height graph. Once χ and ψ are known, the curvature tensor follows from $b_{\alpha\beta} = \bar{\nabla}_\alpha\bar{\nabla}_\beta\psi$, and the metric follows from the stress–strain relation. The actual configuration is then recovered from the Weingarten equations

$$\partial_\alpha \hat{n} = -b^\mu{}_\alpha \partial_\mu \phi, \qquad \partial_\alpha \partial_\beta \phi = \Gamma^\mu_{\alpha\beta} \partial_\mu \phi + b_{\alpha\beta} \hat{n}. \tag{3.46}$$

Thus the potentials determine the intrinsic and extrinsic geometry first, and the embedding is reconstructed only afterward. This is why the formulation can describe large rotations, multi-valued configurations, and non-Euclidean geometric frustration while retaining the analytic form of the FvK equations.