(1)

Lecture III

Fermi Liquid Theory of Unconventional Superconductivity

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1. Landau Fermi Liquid Theory - Small Parameters
2. Green's Function Approach - Dyson / Gorkov Eqs.
3. Low-Energy Expansion / Resummation
4. Eilenberger's Quasiclassical Transport Equation
5. Applications: Microscopic Theory → GL Equations
   - d-wave + subdominant pairing
   - S=1 E_{1u} pairing ⇒ Broken T-symmetry
   - Impurity suppression of T_c for Unconv. SC.
   - H_{c2} in S=1 & S=0 Superconductors w/ Pauli Limiting

J. Sauls
Lecture 3

(2)

Fermi Liquid Theory of Superconductivity

The theory of superconductivity in strongly interacting Fermi systems that is outlined here was developed largely in the 1960's and early 1970's. It is the theory of the low-energy ($\ll E_f$) physics of metals & superconductors. Some of the key developments were:

1957 Landau — Fermi Liquid Theory

1957 BCS - Pairing Theory

1958 Bogoliubov - Inhom. pairing theory

1958 Migdal - Electron-phonon interaction

1959 Gorkov - Field Theoretical Methods in SC

1959 Abrikosov + Gorkov - SC Alloys

1960-61 { Anderson, Nambu - Gauge invariant pairing; Ambegaokar + Kadanoff " " & Collective Modes }

1962 Anderson & Morel - Unconventional BCS states

1962 Eliashberg - Strong-coupling electron-phonon.

* 1968 Eilenberger - Landau ⊕ BCS unified.

* 1972 Larkin & Ovchinnikov - Non-equilibrium SC

→ 1976 Rainer & Serene - Strong-coupling in ^{3}He

Fermi Liquid Theory

L. Landau (1957–59) developed a phenomenological theory of strongly interacting Fermi systems that is the basis for understanding most metals & superconductors, including strongly correlated metals.

A Fermi Liquid is a system in which

1. dominant excitations obey Fermi Statistics
2. $Q = \pm e$ & $S = \frac{1}{2}\hbar$
3. $f(\vec{p},\vec{R};t)$ = phase space distribution $(\vec{p}-\vec{R})$,

$$\frac{\partial f(\vec{p},\vec{R},t)}{\partial t} + \vec{v}_{\vec{p}}\cdot\vec{\nabla}_{\vec{R}} f - (\vec{\nabla}_{R}\varepsilon_{\vec{p}})\cdot\frac{\partial f}{\partial \vec{p}} = \left(\frac{\partial f}{\partial t}\right)_{\text{collision}}$$

w/ $\vec{v}_{\vec{p}}$ = group velocity $= \dfrac{\partial \xi_{\vec{p}}}{\partial \vec{p}} \simeq v_F\,\hat{p}$

$$\varepsilon_{\vec{p}} = \text{q.p. energy} = \underset{\substack{\shortparallel \\ v_F(p-p_f)}}{\xi_{\vec{p}}} + \delta\varepsilon_{\vec{p}}[\delta f_{\vec{p}}] \; (+\underset{\text{external field.}}{u_{ext}})$$

$$\delta\varepsilon_{\vec{p}} = \int d\vec{p}\,'\, \underset{\text{q.p. interaction energy}}{f_{\vec{p},\vec{p}'}}\, \delta f(\vec{p}\,',\vec{R},t)$$

Landau's quasiparticle theory is valid at low-energies and long-wavelength scales compared to the typical atomic scales.

Energies:	Length scales
$k_B T \ll E_F$	$q \ll p_f/\hbar$
$\hbar\omega \ll E_F$	$\ell = v_F\tau \gg \dfrac{\hbar}{p_f}$
$k_B\Theta_D \ll E_F$	↑ mean scattering time
↑ Typical Normal Metal Energy	

Interactions (mostly Coulomb) lead to:

1). Quasiparticles → Complex, correlated motion of bare electrons

2). Collective Excitations (e.g. phonons, zerosound, spin fluctuations?), which are stable modes of the Fermi Sea.

3). Superconductivity from Bound pair Condensation of q.p.'s.

$$k_B T_c \sim \Delta \ll E_F \quad \& \quad \xi_0 \gg \frac{\hbar}{p_f}$$

Thus, Superconductivity can be described within Fermi Liquid Theory.

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Fermi liquid theory — including BCS pairing is a non-perturbative theory of strongly interacting systems of electrons & ions (or neutral systems like ^3He). However, there are key small parameters that define the precision and range of validity of the theory.

Low-Energy / Long-wavelength Scale Parameters

$$S = \left\{ \begin{matrix} \frac{k_B T}{E_F} & \frac{\hbar\omega_D}{E_F} & \frac{\hbar/\tau}{E_F} & \frac{\Delta}{E_F} \\ \frac{\lambda_F}{\xi_T}, & \frac{\lambda_F}{\ell}, & \frac{\lambda_F}{\xi_0} & \end{matrix} \right\} \ll 1.$$

generic small expansion parameter for classifying interactions & processes in Fermi liquids.

ω_D = Debye frequency

$\lambda_F = \hbar/p_F$ = Fermi wavelength $\approx \mathring{A}$

$\ell = v_F \tau$ = mean free path

$\xi_0 = \hbar v_F/\pi\Delta$ = coherence length (clean)

$\xi_T = \hbar v_F/2\pi k_B T$ = thermal coherence length (clean)

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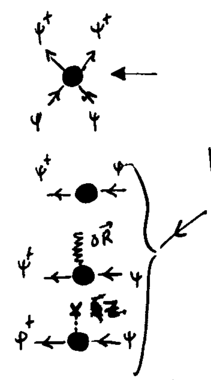
The Fundamental Hamiltonian for Metals is the kinetic energy for the electrons and ions, plus their Coulomb interactions, and when needed relativistic corrections to these interactions.

"High Energy Scale"

$$\mathcal{H} = T_e + T_{ions} + V_{e-e} + V_{Z-Z} + V_{e-Z} \; (+ \dots)$$

$$T_e = \int d\vec{r}\; \psi^\dagger(\vec{r}) \left(\frac{-1}{2m}\nabla^2\right) \psi(\vec{r})$$

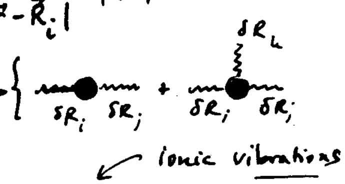
$$T_{ions} = \sum_{i=1}^{N_{ions}}{}' \frac{1}{2M_i} (\delta\dot{\vec{R}}_i)^2$$



$$V_{ee} = \frac{1}{2} \int d\vec{r} \int d\vec{r}\,'\; \psi^\dagger(\vec{r})\, \psi^\dagger(\vec{r}\,')\, \frac{e^2}{|\vec{r} - \vec{r}\,'|}\, \psi(\vec{r}\,')\, \psi(\vec{r})$$

$$V_{eZ} = -\int d\vec{r}\; \psi^\dagger(\vec{r}) \sum_{i=1}^{N_{ions}}{}' \frac{Z_i e^2}{|\vec{r} - \vec{R}_i|}\, \psi(\vec{r})$$

$$V_{ZZ} = \frac{1}{2} \sum_{i \neq j}{}' \frac{(Z_i e)^2}{|\vec{R}_i - \vec{R}_j|} \longrightarrow$$



Lattice Dynamics: $\vec{R}_i = \vec{R}_i^0 + \delta\vec{R}_i$

Static lattice

Electron Green's Functions (Matsubara)

Nambu Spinors: $\Psi(x) \equiv \text{column}\left(\psi_\uparrow, \psi_\downarrow, \psi_\uparrow^+, \psi_\downarrow^+\right)$

$i = 1, 2, 3, 4$

4x4 Nambu Green's Functions:

$$G_{ij}(x,x') = -\left\langle T_\tau \, \Psi_i(x) \, \bar{\Psi}_j(x') \right\rangle \leftarrow Tr\{\rho_{eq} \cdots\}$$

Matrix Form:

$$\hat{G}(x,x') = \begin{pmatrix} G(x,x') & F(x,x') \\ \bar{F}(x,x') & \bar{G}(x,x') \end{pmatrix}$$

$$G(x,x') = -\langle T_\tau \, \psi(x) \, \bar{\psi}(x') \rangle = x \Longleftarrow x'$$

$$F(x,x') = -\langle T_\tau \, \psi(x) \, \psi(x') \rangle = x \Longleftrightarrow x'$$

$$\bar{F}(x,x') = -\langle T_\tau \, \bar{\psi}(x) \, \bar{\psi}(x') \rangle = x \Longrightarrow\!\Longleftarrow x'$$

$$\bar{G}(x,x') = -\langle T_\tau \, \bar{\psi}(x) \, \psi(x) \rangle = x \Longrightarrow x'$$

$x = (\vec{x}, \tau, \alpha)$

NOT essential, simpler "bookkeeping" $\rightarrow$ Translationally Invariant, Homogeneous Equilibrium ("Jellium")

$$G_{ij}(x,x') = G_{ij}(\vec{x}-\vec{x}', \tau-\tau')$$

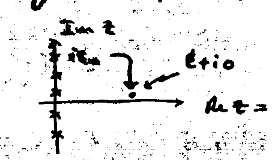
Fourier Representation:

$$G_{ij}(\vec{p}, \varepsilon_n) = \int d\vec{r} \int_0^\beta d\tau \, e^{-i(\vec{p}\cdot\vec{r} - \varepsilon_n \tau)} \, G_{ij}(\vec{r}, \tau)$$

$\varepsilon_n = (2n+1)\pi T$

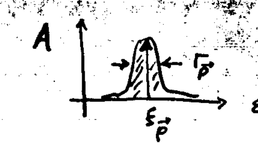
Analytic continuation to the real axis defines the retarded Green's Function, For the diagonal GF,

$$G^R(\vec{p}, \varepsilon) = G(\vec{p}, i\varepsilon_n \rightarrow \varepsilon + i0)$$



The single-particle spectral density is given by

$$\boxed{A(\vec{p}, \varepsilon) = -\frac{1}{\pi} \, \mathrm{Im} \, G^R(\vec{p}, \varepsilon)}$$



The other key physical quantity is the Order parameter, or Equal time pair amplitude,

$$\boxed{F_{\alpha\beta}(\vec{p}) = T \sum_{\varepsilon_n} e^{-i\varepsilon_n \delta} F_{\alpha\beta}(\vec{p}, \varepsilon_n) = \langle a_{\vec{p}\alpha} \, a_{-\vec{p}\beta} \rangle} \quad \delta \rightarrow 0^+$$

The equilibrium theory of superconductivity in strongly interacting systems can be formulated in terms of a stationary functional for the partition function or thermodynamic potential.

$$\Omega[\hat{G}, \hat{\Sigma}] = -\frac{1}{2}\int d^3p/(2\pi)^3 \, Tr_4 \Big[\hat{\Sigma}(\vec{p},\epsilon_n)\, \hat{G}(\vec{p},\epsilon_n) + \ln\Big\{ -\hat{G}_0^{-1}(\vec{p},\epsilon_n) + \hat{\Sigma}(\vec{p},\epsilon_n) \Big\} \Big] + \Phi[\hat{G}]$$

[Luttinger & Ward
De Dominicis & Martin]

$$\hat{G}_0^{-1}(\vec{p},\epsilon_n) = \begin{pmatrix} i\epsilon_n - \xi_{\vec{p}}^{\,0} & 0 \\ 0 & -i\epsilon_n - \xi_{\vec{p}}^{\,0} \end{pmatrix}$$ is the non-interacting propagator ('bare').

$\Phi[\hat{G}]$ is a functional of the full propagator that generates the skeleton expansion for the self-energy,

$$\hat{\Sigma}_{skel}[\hat{G}] = -2 \;\delta\Phi / \delta\hat{G}^{tr}(\vec{p},\epsilon_n)$$

The stationarity conditions define the physical propagator and self-energy, as well as the thermodynamic potential,

$$\Omega(T,\mu,V) = \Omega[\hat{G}_{physical}, \hat{\Sigma}_{physical}]$$

Stationarity Conditions:

(1). $\delta\Omega[\hat{G},\hat{\Sigma}]/\delta\hat{G}^{tr} = 0 \;\Rightarrow\; \boxed{\hat{\Sigma}(\vec{p},\epsilon_n) = \hat{\Sigma}_{skel}[\hat{G}]}$

(2). $\delta\Omega[\hat{G},\hat{\Sigma}]/\delta\hat{\Sigma}^{tr} = 0 \;\Rightarrow\; \boxed{\hat{G}^{-1}(\vec{p},\epsilon_n) = \hat{G}_0^{-1}(\vec{p},\epsilon_n) - \hat{\Sigma}(\vec{p},\epsilon_n)}$

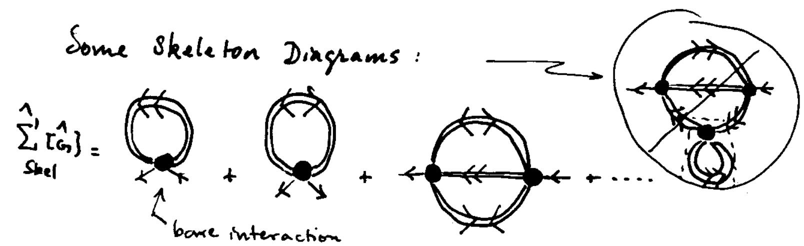
Dyson's Equations

$$\hat{\Sigma}(\vec{p},\epsilon_n) = \begin{pmatrix} \Sigma & \Delta \\ \bar{\Delta} & \bar{\Sigma} \end{pmatrix}$$

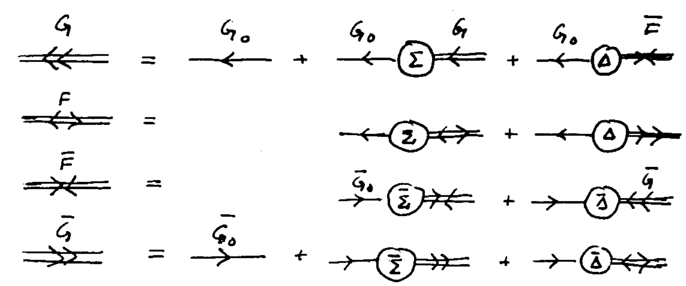
$\bar{\Delta}(\vec{p},\epsilon_n) = \Delta($

$\bar{\Sigma}(\vec{p},\epsilon_n) =$

Some Skeleton Diagrams:



Dyson's Equations → Gorkov's Equations



The Full Many-Body theory for interacting e^- and ions is defined in terms of the bare interactions at the high energy scale, $E_F \sim e^2/a \sim 10^5\,K$. We are generally interested in phenomena at low energies & longer wave lengths. The relevant excitations are not bare electrons or bare ions, but quasiparticles & phonons.

Furthermore, their interactions are not the bare Coulomb interaction, but effective interactions involving virtual excitations of the medium.

Goal: Find an effective theory for the low-energy quasiparticles, pair condensate & their interactions.

Plan:

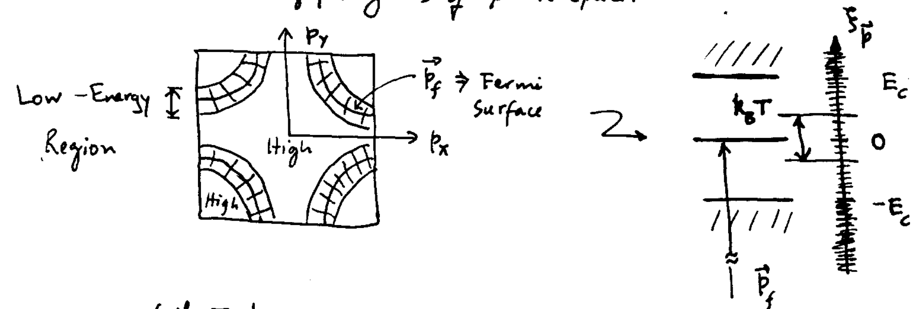
1) Separate the low-energy & high-energy degrees of freedom in the formal perturbation expansion for $\hat{\Sigma}[\hat{G}]$.

2). Re-sum the bare vertices & high-energy virtual processes into block vertices coupled to low-energy propagators.

3). Phase space & Order of Magnitude Estimates: Start from Landau's "guess" for the spectral function & estimate the Low-Energy Self-Energies.

Formulation of the Low-Energy Expansion:

Introduce a formal cutoff, E_c, separating high- & low-energy regions of phase space.



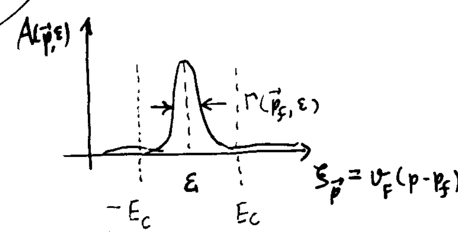
$$\left\{\begin{matrix} k_BT \\ \hbar/\tau \\ \Delta \end{matrix}\right\} \ll E_c \ll E_F \qquad \left\{\begin{matrix} kT/v_F \\ \hbar/\xi_0 \end{matrix}\right\} \ll p_c \ll p_f$$

The key assumption we make is the basis of Landau's Fermi Liquid Theory $\Rightarrow$ Smooth structure for Σ^R

N.B. pairing instability can be regulated

$$\rightarrow \quad \Sigma^R(\vec{p},\varepsilon) = \Sigma'(\vec{p}_f,0) + \vec{\nabla}_p \Sigma'^R\Big|_{p_f} \cdot (\vec{p}-\vec{p}_f) + \partial_\varepsilon \Sigma'^R(\vec{p}_f,0)\cdot\varepsilon + i\gamma(\vec{p})\cdot\varepsilon^2 \ldots$$

Which leads to a spectral function with a quasiparticle resonance at low-energies near the Fermi surface.



Well-defined quasiparticles:

$$\boxed{\begin{aligned} &\varepsilon \simeq \xi_{\vec{p}} = v_F(p-p_f) \\ &\Gamma(\vec{p},\varepsilon) \ll |\xi_{\vec{p}}|^{*} \end{aligned}}$$

Extension for $\Gamma \ll E_F$

Separate the propagators into low- & high-energy propagators:

$$\frac{\hat{G}}{p,\epsilon_n} = \frac{\hat{G}_{low}}{p,\epsilon_n} + \frac{\hat{G}_{high}}{p\,\epsilon_n}$$

$\hat{G}_{low} = \hat{G}$ for $|\epsilon_n| < E_c$ and $|p - p_F| < p_c$

otherwise $\hat{G} = \hat{G}_{high}$.

The key order of magnitude estimates are

$$G_{low} \simeq \frac{1}{i\epsilon_n - \xi_p} \sim \frac{1}{E_F} * \boxed{\frac{1}{\delta}} \quad \text{w/} \quad \delta = \frac{\Delta\varepsilon}{E_F}$$

Small energy denom.

$$G_{high} \approx \frac{1}{E_F} * \boxed{1}$$ ← Large Energy denominator

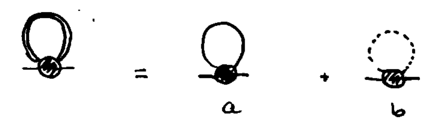
For the bare interaction vertex 

$$\Gamma_{ee} \sim \frac{e^2}{a} \sim E_F * \boxed{1}$$

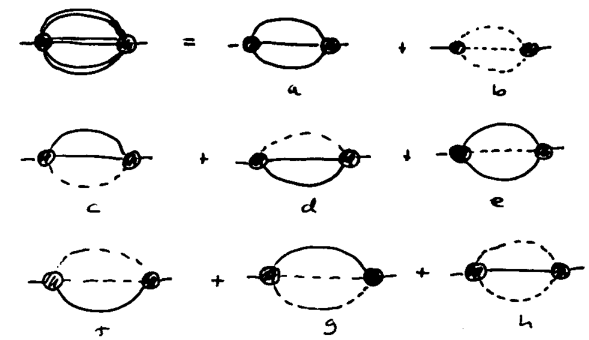
Expand all the Σ_i diagrams in terms of $\hat{G}_{low}$ and $\hat{G}_{high}$, then re-sum the high-energy parts into Block vertices coupled to low-energy propagators.

Low-Energy Self-Energy Diagrams

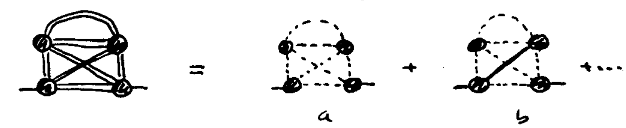
$\hat{\Sigma}^{(1)}$ =



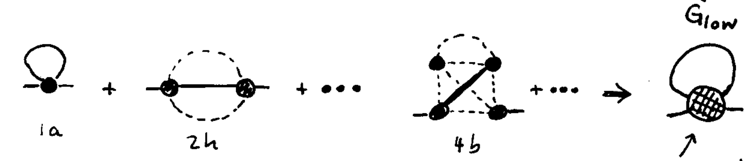
$\hat{\Sigma}^{(2)}$ =



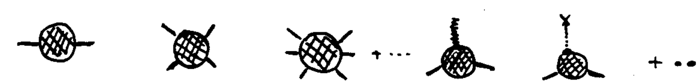
$\hat{\Sigma}^{(4)}$ =



Sum the high-energy intermediate states into Block vertices



We generate renormalized, effective interactions as well as new interaction vertices.



Order of Magnitude Estimates

1. $G_{low} \sim s^{-1}$ $\qquad G_{high} \sim 1$

(Estimates for full G_{low} & G_{high} are preserved)

2. All Bloch vertices are $\sim s^0 = 1$.

Summation of H.E. parts does not introduce new low-energy scales. (The pairing channel is separated out & introduces the scale $\Delta \ll E_F$.)

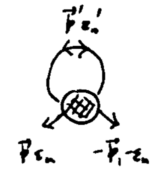
3. Internal Frequency and Momentum integrations generate the following factors:

$$\underset{\varepsilon_n}{T \sum^{low}}{}' \sim s^1 \qquad \int^{low} \frac{d^3p}{(2\pi)^3} \sim N(0) \int dS_F \int^{low} ds_{\vec{p}} \sim s^1$$

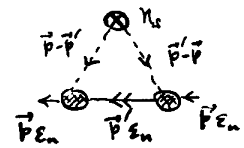
4. Special Phase Space Factors: $\quad s^{N_{ph}}$

The power N_{ph} is determined by to topology of the self energy diagram and represents the # of restrictions of an internal variable to the low-energy region that must be imposed inspite of momentum & energy conservation.

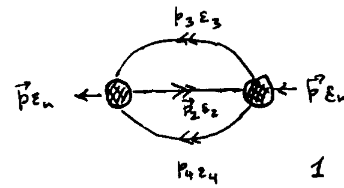
Estimates for Low Energy Diagrams:



$$\sim \underbrace{T \sum_{\varepsilon_n'}{}'}_{s} \times \underbrace{\int \frac{d^3p'}{(2\pi)^3}}_{s} \times \underbrace{V(\vec{p},\vec{p}')}_{1} \times \underbrace{F_{low}(\vec{p}',\varepsilon_n')}_{s^{-1}} \sim \underline{s^1}$$



$$\sim \underbrace{n_s \int \frac{d^3p'}{(2\pi)^3}}_{s \times s} |u(\vec{p}-\vec{p}')|^2 \times \underbrace{G_{low}(\vec{p}'\varepsilon_n)}_{s^{-1}} \sim \underline{s^1}$$



$$\sim \underbrace{T\sum_{\varepsilon_2}}_{s} \times \underbrace{T\sum_{\varepsilon_3}}_{s} \times \cancel{T\sum_{\varepsilon_4}} \underbrace{\int \frac{d^3p_2}{(2\pi)^3}}_{s} \times \underbrace{\int \frac{d^3p_3}{(2\pi)^3}}_{s} \times \underbrace{\int \frac{d^3p_4}{(2\pi)^3}}_{\boxed{s}} \leftarrow N_{ph} = 1$$

$$1 \rightarrow |\Gamma(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4)|^2 \times \delta(\varepsilon_n + \varepsilon_{n_2} - \varepsilon_{n_3} - \varepsilon_{n_4})$$

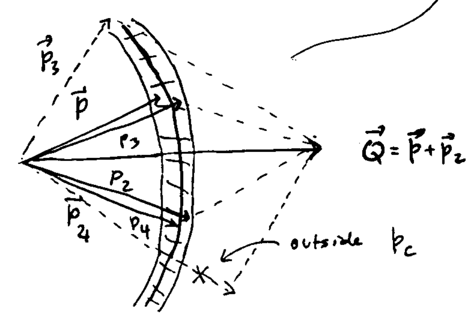
$$\underbrace{G_{low}(\vec{p}_2\varepsilon_2)}_{1/s} \times \underbrace{G_{low}(\vec{p}_3\varepsilon_3)}_{1/s} \times \underbrace{G_{low}(\vec{p}_4\varepsilon_4)}_{1/s} \times \delta(\vec{p}+\vec{p}_2-\vec{p}_3-\vec{p}_4)$$

$\sim \underline{s^2}$

(2D & 3D)

Breakdown of FL Theory in 1D

$\sim \underline{s}$

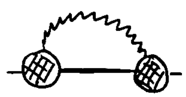


Low-Energy Self-Energies

s^0 Fermi Surface $\left\{ \begin{matrix} \vec{p}_f \\ \vec{v}_f \end{matrix} \right\}$ $(\xi^0_p \to \xi_p)$

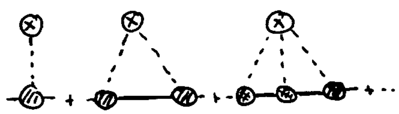
s^1 { Landau Molecular Field — $A(\vec{p}_f, \vec{p}_f{}')$

Electronic Pairing Int. — $V(\vec{p}_f, \vec{p}_f{}')$
(e.g. spin-fluctuation mediated)

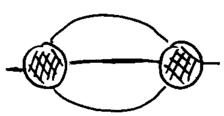


s^1 { Electron-phonon self-energy v_F^*, $1/\tau_{e\text{-}ph}$

Electron-phonon pairing — $\alpha^2 F(\omega)$

s^1 [] Impurity t-matrix $1/\tau_{el}$

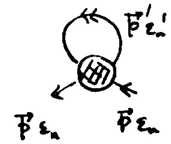
s^1  EM coupling $= \frac{e}{c} \vec{v}_f \cdot \vec{A}\, \hat{\tau}_3$

s^2 

electronic collisions ; $\frac{1}{\tau_{ee}}$ }
electronic strong-coupling } $\Gamma(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)$

$$\hat{\sigma}(\vec{p}_f, \varepsilon_n) \equiv a \cdot \hat{\Sigma}_{low}(\vec{p}_f, \varepsilon_n)\, \hat{\tau}_3$$

Landau's Molecular Field Energy



$$= \hat{\sigma}_L(\vec{p}_f, \cancel{\varepsilon_n}) = \begin{pmatrix} \sigma_L(\vec{p}_f)\mathbf{1} + \vec{h}(\vec{p}_f)\cdot\vec{\sigma} & 0 \\ 0 & \sigma_L(\vec{p}_f) + \vec{h}(\vec{p}_f)\cdot\vec{\sigma}^{tr} \end{pmatrix}$$

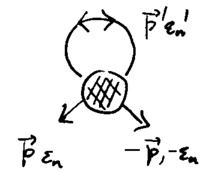
Landau interactions

$$\sigma_L(\vec{p}_f) = T \sum_{\varepsilon_n'}^{low} \int d\vec{p}_f{}'\; A^s(\vec{p}_f, \vec{p}_f{}')\; g(\vec{p}_f{}', \varepsilon_n')$$

$$\vec{h}_L(\vec{p}_f) = T \sum_{\varepsilon_n}^{low} \int d\vec{p}_f{}'\; A^a(\vec{p}_f, \vec{p}_f{}')\; \vec{g}(\vec{p}_f{}', \varepsilon_n) \quad \text{(exchange field)}$$

Dimensionless Interactions: $\begin{cases} A = N_f \sqrt{a_1 a_2 a_3 a_4}\; \Gamma_{p\text{-}h} \\ V = N_f \sqrt{a_1 a_2 a_3 a_4}\; \Gamma_{p\text{-}p} \end{cases}$

Electronic Pairing Energy ("BCS mean field")

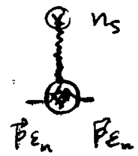


$$= \hat{\Delta}(\vec{p}_f) = \begin{pmatrix} 0 & \Delta(\vec{p}_f)\, i\sigma_y + \vec{\Delta}(\vec{p}_f)\cdot i\vec{\sigma}\sigma_y \\ \tilde{\Delta}\, i\sigma_y + \vec{\tilde{\Delta}}\cdot(i\sigma_y\vec{\sigma}) & 0 \end{pmatrix}$$

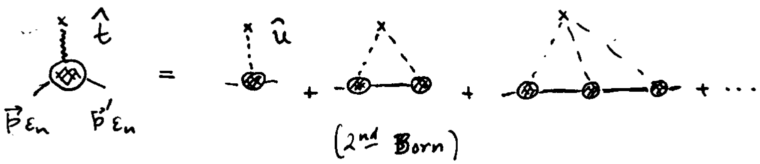
(Spin Singlet) $\Delta(\vec{p}_f) = T \sum_{\varepsilon_n'}^{low} \int d\vec{p}_f{}'\; V_s(\vec{p}_f, \vec{p}_f{}')\; f(\vec{p}_f{}', \varepsilon_n')$

(Spin Triplet) $\vec{\Delta}(\vec{p}_f) = T \sum_{\varepsilon'}^{low} \int d\vec{p}_f{}'\; V_t(\vec{p}_f, \vec{p}_f{}')\; \vec{f}(\vec{p}_f{}', \varepsilon_n')$

Impurity Self-Energy (Random "Alloy")



$$= n_s\,\hat{t}(\vec{p}_f,\vec{p}_f;\varepsilon_n) \equiv \hat{\sigma}_{imp}(\vec{p}_f,\varepsilon_n) \quad \left(\begin{matrix}\sigma_{imp} & \Delta_{imp}\\ \bar{\Delta}_{imp} & \bar{\sigma}_{imp}\end{matrix}\right)$$



$$\hat{t}(\vec{p}_f,\vec{p}_f{}';\varepsilon_n) = \hat{u}(\vec{p}_f,\vec{p}_f{}') + N_f\int d^2p_f''\,\hat{u}(\vec{p}_f,\vec{p}_f'')\,\hat{g}(\vec{p}_f'',\varepsilon_n)\,\hat{t}(\vec{p}_f'',\vec{p}_f{}';\varepsilon_n)$$

Effective Impurity potential:

$$\hat{u}(\vec{p}_f,\vec{p}_f{}') = \begin{pmatrix} u + J\vec{S}\cdot\vec{\sigma} & 0 \\ 0 & u + J\vec{S}\cdot\vec{\sigma}^{tr}\end{pmatrix}$$

non-magnetic potential scattering — spin-exchange scattering

Born: Elastic Scattering (Normal) rate

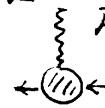
$$\frac{1}{\tau_{el}} = 2\pi\, n_s N_f \langle |u|^2\rangle_{FS}$$

Spin-flip rate (Born)

$$\frac{1}{\tau_s} = 2\pi\, n_s N_f J^2 S^2$$

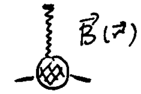
Coupling to a Magnetic Field

Diamagnetic



$$= \underbrace{\frac{e}{c}\,\vec{v}_f}_{\text{Fixed by Gauge covariance}}\cdot\vec{A}(\vec{r})\,\hat{\tau}_3$$

Paramagnetic (Zeeman) coupling



$$= -\hat{\vec{\mu}}_{eff}(\vec{p}_f)\cdot\vec{B}(\vec{r})$$

$$\hat{\vec{\mu}}_{eff}(\vec{p}_f) = \begin{pmatrix}\mu_{eff}(\vec{p}_f)\,\vec{\sigma} & 0\\ 0 & \mu_{eff}(\vec{p}_f)\,\vec{\sigma}^{tr}\end{pmatrix}$$

(For Anisotropic Systems: $\mu\,\vec{\sigma} \to \mu_{ij}(\vec{p}_f)\,\sigma_j$)

Note: $\vec{B}$ in general induces spatial variations of the SC order parameter, which requires generalizing the quasiclassical propagators and self energies to inhomogeneous states.

ξ_p integrated Green's Functions

"Quasiclassical propagators" (Eilenberger, Larkin & Ovchinnikov)

The ξ_p-integrations can be carried out immediately for most leading order diagrams. The low-energy propagators can then be integrated w/r to the displacement away from the Fermi surface,

$\xi_{\vec{p}} = v_F(|\vec{p}| - p_f)$,

quasiparticle weight

$$\boxed{\hat{g}(\vec{p}_f, \varepsilon_n) = \frac{1}{a}\int d\xi_{\vec{p}}\ \hat{\tau}_3\ \hat{G}_{low}(\vec{p}, \varepsilon_n)} = \begin{pmatrix} g & f \\ \bar{f} & \bar{g} \end{pmatrix}$$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in Nambu space

For the Normal state propagator, direct integration gives the simple result,

$$\hat{g}_N(\vec{p}_f, \varepsilon_n) = -i\pi\,\mathrm{sgn}(\varepsilon_n)\,\hat{\tau}_3,$$

which basically says that the normal state DOS is given by the DOS at the Fermi level, N_f, and is energy independent in the low-energy band near the Fermi surface. (Corrections to the DOS at E_f are higher order in S and related to "particle-hole" asymmetry of the excitation spectrum.)

Eilenberger's Transport Equation for $\hat{g}(\vec{p}_f, \vec{R}; \varepsilon_n)$

Generalize the low-energy Dyson Equations to include long-wavelength spatial variations on the scale of $\lambda_F = \hbar/p_f$.

$\vec{R} = \frac{1}{2}(\vec{x} + \vec{x}')$

$$\hat{G}_{low}(\vec{x}, \vec{x}'; \varepsilon_n) \rightarrow \hat{G}_{low}(\vec{p}, \vec{R}; \varepsilon_n)$$

The low-energy Dyson's Equation become to leading order in $(\vec{v}_f \cdot \vec{\nabla}_R / E_f \ll 1)$

(Left Dyson Eq.)
$$\left(i\varepsilon_n \hat{\tau}_3 - a\hat{\Sigma}'_{low}\hat{\tau}_3\right)\frac{1}{a}\hat{\tau}_3\hat{G}_{low} - \left(\xi_{\vec{p}} - \frac{i}{2}\vec{v}_f\cdot\vec{\nabla}_R\right)\frac{1}{a}\hat{\tau}_3\hat{G}_{low} = \hat{1}$$

(Right Dyson)
$$\frac{1}{a}\hat{\tau}_3\hat{G}_{low}\left(i\varepsilon_n\hat{\tau}_3 - a\hat{\Sigma}'_{low}\hat{\tau}_3\right) - \left(\xi_{\vec{p}} + \frac{i}{2}\vec{v}_f\cdot\vec{\nabla}_R\right)\frac{1}{a}\hat{\tau}_3\hat{G}_{low} = \hat{1}$$

$\hat{\Sigma}'_{low}(\vec{p}, \vec{R}; \varepsilon_n)$ is slowly varying w/r $|\vec{p}|$. Subtract Left & Right Dyson Equations & $\xi_{\vec{p}}$-integrate:

$$\boxed{\left[i\varepsilon_n\hat{\tau}_3 - \hat{\sigma}(\vec{p}_f, \vec{R}; \varepsilon_n),\ \hat{g}(\vec{p}_f, \vec{R}; \varepsilon_n)\right] + i\vec{v}_f\cdot\vec{\nabla}_R\,\hat{g}(\vec{p}_f, \vec{R}; \varepsilon_n) = 0}$$

4x4 Matrix Commutator

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Normalization Condition

The left/right subtraction resulted in a loss of the normalization from the original Dyson equations.

G. Eilenberger discovered the proper normalization of the quasiclassical propagators.

From the Normal-state quasiclassical propagator we can immediately see that

$$\hat{g}_N(\vec{p}_f,\varepsilon_n)^2 = -\pi^2\,\hat{1}$$

This normalization condition is more general; it applies to spatially varying states in the superconducting phase,

$$\boxed{\hat{g}(\vec{p}_f,\vec{R};\varepsilon_n)^2 = -\pi^2\,\hat{1}}$$

(There is a further generalization of the normalization condition to non-equilibrium superconducting states obtained by Larkin & Ovchinnikov.)

Qualitatively, $\hat{g}^2 = -\pi^2\hat{1}$ guarantees that the electronic density states integrated over the low-energy band is conserved

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(Homogeneous)

Equilibrium Solution for $\hat{g}(\vec{p}_f,\varepsilon_n)$

Clean Limit : $\hat{\sigma}_{imp} = 0$

$$\hat{\sigma}(\vec{p}_f,\varepsilon_n) = \hat{\Delta}(\vec{p}_f) = \begin{pmatrix} 0 & \Delta(\vec{p}_f) \\ \bar{\Delta}(\vec{p}_f) & 0 \end{pmatrix}$$

Spin-Singlet pairing : $\begin{cases} \Delta(\vec{p}_f) = i\sigma_y\,\Delta(\vec{p}_f) \\ \bar{\Delta}(\vec{p}_f) = i\sigma_y\,\Delta^*(\vec{p}_f) \end{cases}$

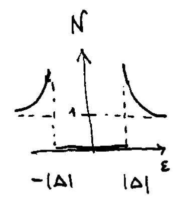
Transport Equation : $\left[i\varepsilon_n\hat{\tau}_3 - \hat{\Delta}(\vec{p}_f)\,,\,\hat{g}(\vec{p}_f,\varepsilon_n)\right] = 0$

&

Normalization : $\hat{g}^2 = -\pi^2\,\hat{1}$

The solution is : $\boxed{\hat{g}(\vec{p}_f,\varepsilon_n) = -\pi\,\dfrac{i\varepsilon_n\hat{\tau}_3 - \hat{\Delta}(\vec{p}_f)}{\sqrt{\varepsilon_n^2 + |\Delta(\vec{p}_f)|^2}}}$

Fermi-surface-resolved DOS : $\hat{g}(\vec{p}_f, i\varepsilon_n \to \varepsilon + i0) = \hat{g}^R(\vec{p},\varepsilon)$



$$\boxed{N(\vec{p}_f,\varepsilon) = -\frac{1}{2\pi}\,\mathrm{Tr}\left[\hat{\tau}_3\,\mathrm{Im}\,\hat{g}^R\right] = \frac{|\varepsilon|}{\sqrt{\varepsilon^2 - |\Delta(\vec{p}_f)|^2}}\,\Theta(\varepsilon^2 - |\Delta(\vec{p}_f)|^2)}$$

Equilibrium Gap Equation:

$$\Delta(\vec p_f) = \int d\vec p_f{}' \, V_s(\vec p_f,\vec p_f{}') \cdot T\sum_{\epsilon_n}^{low} f(\vec p_f{}',\epsilon_n')$$

$$\Delta(\vec p_f) = \int d\vec p_f{}' \, \underline{V_s(\vec p_f,\vec p_f{}')} \left(\pi T\sum_{\epsilon_n}^{E_c} \frac{\Delta(\vec p_f{}')}{\sqrt{\epsilon_n^2+|\Delta(\vec p_f{}')|^2}}\right)$$

Expand the even-parity, singlet interaction in eigenfunctions which form irreducible representations of the symmetry group, G.

$$\boxed{V_s = \sum_{\Gamma}^{irrep} V_\Gamma \sum_{i=1}^{d_\Gamma} \mathcal{Y}_{\Gamma i}(\vec p_f)\, \mathcal{Y}^*_{\Gamma i}(\vec p_f{}')}$$

(invariant) under G.

interaction in channel Γ ($V_\Gamma > 0 \Rightarrow$ Attraction)

orthogonal Basis functions defined on the FS.

The most attractive V_Γ defines the transition temperature, T_c, and symmetry of $\Delta(\vec p_f)$.

(linearized gap equation)

$$1 = V_\Gamma \cdot \pi T_c \sum_{\epsilon_n}^{E_c} \frac{1}{|\epsilon_n|} \Rightarrow \frac{1}{V_\Gamma} = \overbrace{\ln\left(\frac{1.13\,E_c}{T_c}\right)}^{\psi(\frac{1}{2}+\frac{E_c}{4\pi T_c})-\psi(\frac{1}{2})\,\approx}$$

* Use the linearized gap equation to eliminate $(E_c, V_\Gamma) \to T_c$

Regulate the gap equation with

$$\pi T \sum_{\epsilon_n}^{E_c} \frac{1}{|\epsilon_n|} = \psi\left(\tfrac{1}{2} + \tfrac{E_c}{4\pi T}\right) - \psi\left(\tfrac{1}{2}\right) \simeq \ln(1.13\,E_c/T)$$

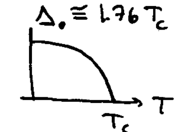
and the W.C. result for T_c.

For a Multi-component O.P. ($d_\Gamma > 1$):

$$\Delta(\vec p_f) = \sum_{i=1}^{d_\Gamma} \Delta_i\, \mathcal{Y}_{\Gamma i}(\vec p_f) \rightarrow \underline{\Delta_i} = \int d\vec p_f\, \mathcal{Y}^*_{\Gamma i}(\vec p_f)\, \Delta(\vec p_f)$$

$$\boxed{\ln(T/T_c)\, \Delta_i = \int d\vec p_f\, \mathcal{Y}^*_{\Gamma i}(\vec p_f) \left\{ \pi T \sum_{\epsilon_n} \left[\frac{\Delta(\vec p_f)}{\sqrt{\epsilon_n^2 + |\Delta(\vec p_f)|^2}} - \frac{\Delta(\vec p_f)}{|\epsilon_n|} \right] \right\} \quad (i = 1 \ldots d_\Gamma)}$$

Ex. S-wave pairing: BCS $\Delta(T)$



$$\ln(T/T_c)\, \Delta = \pi T \sum_{\epsilon_n} \left\{ \frac{\Delta}{\sqrt{\epsilon_n^2 + \Delta^2}} - \frac{\Delta}{|\epsilon_n|} \right\}$$

For $T \lesssim T_c$: $\quad 1/\sqrt{\epsilon_n^2+\Delta^2} \simeq \frac{1}{|\epsilon_n|}\left(1 - \frac{1}{2}\frac{\Delta^2}{\epsilon_n^2} \cdots\right)$

$$(T/T_c - 1)\,\Delta \simeq -\frac{\pi}{2} T \sum_{\epsilon_n} \frac{|\Delta|^2 \Delta}{|\epsilon_n|^3}$$

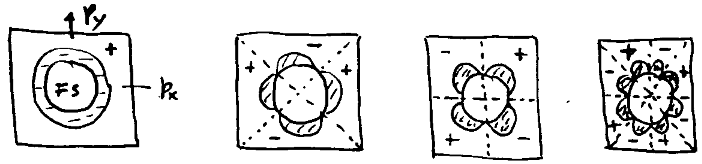
(GL Equation)

$$\Rightarrow \boxed{\alpha(T)\,\Delta + \beta|\Delta|^2\Delta = 0 \quad w/\ \alpha(T) = N_f(T/T_c - 1),\ \beta = \frac{7}{8}\zeta(3)\frac{N_f}{\pi^2 T_c^2}}$$

Spin-Singlet, $d_{x^2-y^2}$ pairing w/ sub-dominant s-wave pairing

In tetragonal symmetry there a 4 even-parity 1D representations, and one 2D representation.

Consider the 1D representations:



A_{1g} ('s-wave') $\quad B_{1g}$ ($d_{x^2-y^2}$) $\quad B_{2g}$ (d_{xy}) $\quad A_{2g}$ ('g-wave')

$\mathcal{Y}_{A_1} = 1 \qquad \mathcal{Y}_{B_1} = p_x^2 - p_y^2 \qquad \mathcal{Y}_{B_2} = p_x p_y \qquad \mathcal{Y}_{A_2} \sim p_x p_y \times (p_x^2 - p_y^2)$

<u>Assume</u> $\quad V_{B_1} > V_{A_1} \quad$ (other channels are repulsive)

$$\boxed{T_{cB_1} > T_{cA_1} > 0}$$

<u>Possible 2nd SC instability</u>

We can derive the GL equations for these coupled $d_{x^2-y^2}$ and s-wave O.P. by solving the Eilenberger Equations perturbatively in Δ.

Expansion parameters: $\begin{cases} \Delta \sim T_c \sqrt{1 - T/T_c} \sim \underline{\underline{\text{small}}} \\ v_F \nabla \sim v_F / \xi(T) \sim T_c \sqrt{1-T/T_c} \sim \underline{\underline{\text{small}}} \end{cases}$

Expansion in $\sqrt{1-T/T_c}$:

$$\hat{g}(\vec{p}_f, \vec{R}; \varepsilon_n) = \underset{(\text{small})^0}{\hat{g}_0} + \underset{(\text{small})^1}{\hat{g}_1} + \hat{g}_2 + \hat{g}_3 + \cdots$$

I. Eilenberger Equation:

$$\left[i\varepsilon_n \hat{\tau}_3 - \hat{\Delta}(\vec{p}_f, \vec{R}),\ \hat{g}(\vec{p}_f, \vec{R}; \varepsilon_n) \right] + i \vec{v}_f \cdot \vec{\nabla} \hat{g} = 0$$

II. Normalization: $\quad \hat{g}^2 = -\pi^2 \hat{1}$

Zeroth Order:

$$\left.\begin{array}{l} [i\varepsilon_n \hat{\tau}_3, \hat{g}_0] = 0 \\ \hat{g}_0^2 = -\pi^2 \hat{1} \end{array}\right\} \Rightarrow \underbrace{\hat{g}_0 = -i\pi\, \text{sgn}(\varepsilon_n)\, \hat{\tau}_3}_{\text{Normal-state}}$$

First Order:

$$[i\varepsilon_n \hat{\tau}_3, \hat{g}_1] - [\hat{\Delta}, \hat{g}_0] + i\partial \hat{g}_0 = 0$$

$$\hat{g}_0 \hat{g}_1 + \hat{g}_1 \hat{g}_0 = 0$$

2nd Order:

$$[i\varepsilon_n \hat{\tau}_3, \hat{g}_2] - [\hat{\Delta}, \hat{g}_1] + i\partial \hat{g}_1 = 0$$

$$\hat{g}_0 \hat{g}_2 + \hat{g}_2 \hat{g}_0 + \hat{g}_1^2 = 0$$

$\vdots$ <u>Need</u> 3rd order as well.

Pair Amplitude to Order $\left(\sqrt{1-T/T_c}\right)^3$:

$$f(\vec{p}_f, \vec{R}; \varepsilon_n) \simeq \frac{\pi}{|\varepsilon_n|} \Delta(\vec{p}_f, \vec{R}) - \frac{\pi}{2|\varepsilon_n|^2} \operatorname{sgn}(\varepsilon_n)\, \vec{v}_f \cdot \vec{\nabla} \Delta$$

$$- \frac{\pi}{2|\varepsilon_n|^3} |\Delta(\vec{p}_f \vec{R})|^2 \Delta(\vec{p}_f, \vec{R})$$

$$+ \frac{\pi}{4} \frac{1}{|\varepsilon_n|^3} (\vec{v}_f \cdot \vec{\nabla}_R)^2 \Delta(\vec{p}_f, \vec{R})$$

Inhomogeneous Gap Equation

$$\boxed{\Delta(\vec{p}_f, \vec{R}) = \int d\vec{p}_f{}' \, V_s(\vec{p}_f, \vec{p}_f{}') * T \sum_{\varepsilon_n}^{\varepsilon_c} \left\{ \frac{\Delta(\vec{p}_f{}', \vec{R})}{|\varepsilon_n|} - \frac{|\Delta(\vec{p}_f{}', \vec{R})|^2}{2|\varepsilon_n|^3} \Delta(\vec{p}_f{}' \vec{R}) + \frac{(\vec{v}_f{}' \cdot \vec{\nabla}_R)^2 \Delta(\vec{p}_f{}' \vec{R})}{4|\varepsilon_n|^3} \right\}}$$

$$\zeta_3(T) = \pi T \sum_{\varepsilon_n} \frac{1}{|\varepsilon_n|^3} = \frac{7\zeta(3)}{4\pi^2 T^2}$$

Amplitudes for $d_{x^2-y^2}$ and S-pairing

$$\Delta(\vec{p}_f, \vec{R}) = \underline{\underline{\Delta_d}}\, \mathcal{Y}_1(\vec{p}_f) + \underline{\underline{\Delta_s}}\, \mathcal{Y}_2(\vec{p}_f)$$

$$\qquad \sim p_x^2 - p_y^2 \qquad\qquad \sim 1$$

Coupled GL equations for $d_{x^2-y^2}$ & s pairing

1) Regulating the ln-divergence as before,
2) ~~and~~ eliminating $V_{A_{1g}} \to T_{cs}$ and $V_{B_{1g}} \to T_c$,

and 3) Carrying out the FS averages gives the GL equations:

I. $$\boxed{\alpha_d \Delta_d + \beta_1 |\Delta_d|^2 \Delta_d + \beta' |\Delta_s|^2 \Delta_d + \beta'' \Delta_s^2 \Delta_d^* - K_1 \nabla^2 \Delta_d - K' (\nabla_x^2 - \nabla_y^2) \Delta_s = 0}$$

II. $$\boxed{\alpha_s \Delta_s + \beta_2 |\Delta_s|^2 \Delta_s + \beta' |\Delta_d|^2 \Delta_s + \beta'' \Delta_d^2 \Delta_s^* - K_2 \nabla^2 \Delta_s - K' (\nabla_x^2 - \nabla_y^2) \Delta_d = 0}$$

$$\alpha_d = N_f (T/T_c - 1) \quad ; \quad \alpha_s = N_f (T/T_{cs} - 1)$$

$$\beta_1 = \frac{7}{8}\zeta(3) \frac{N_f}{\pi^2 T_c^2} \langle |\mathcal{Y}_1|^4 \rangle_{FS} \; ; \; \beta_2 = \frac{7}{8}\zeta(3) \frac{N_f}{\pi^2 T_c^2} \langle |\mathcal{Y}_2|^4 \rangle_{FS}$$

$$\beta' = \frac{7}{4}\zeta(3) \frac{N_f}{\pi^2 T_c^2} \langle |\mathcal{Y}_1|^2 |\mathcal{Y}_2|^2 \rangle = 2\beta''$$

$$K_1 = K_2 = \frac{7\zeta(3)}{32\pi^2 T_c^2} N_f v_f^2 \quad ; \quad K' = K_1 * \langle \mathcal{Y}_1 (\hat{p}_x^2 - \hat{p}_y^2) \mathcal{Y}_2 \rangle_{FS}$$

Some consequences of sub-dominant s-pairing

1. Possible Bulk instability to $d_{x^2-y^2} + is$ with broken T-symmetry below T_{c2}.

$$\Delta\Omega_{GL} = \alpha_d |\Delta_d|^2 + \beta_1 |\Delta_d|^4 + \dots \qquad \begin{pmatrix} \Delta_d = |\Delta_d| e^{i\theta_d} \\ \Delta_s = |\Delta_s| e^{i\theta_s} \end{pmatrix}$$

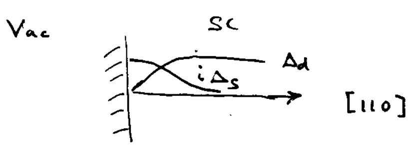
$$+ \alpha_s |\Delta_s|^2 + \beta_2 |\Delta_s|^4$$

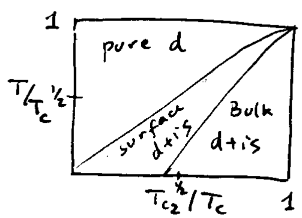
$$+ \beta' |\Delta_d|^2 |\Delta_s|^2 + \beta'' |\Delta_d|^2 |\Delta_s|^2 \cos(2\Delta\theta)$$

$$\boxed{\beta' > 0} \Rightarrow \Delta\theta = \pm\pi/2 \Rightarrow \text{Broken } T \text{ if } |\Delta_s| \neq 0.$$

2^{nd} instability only if $\quad T_c > T_{c2} > T_c\, f(\beta' - \beta'')$

2. Surface instability to $d_{x^2-y^2} \pm is$ for any $T_c > T_{c2} > 0$.





3. Vortex States with complex $d+is$ structure (Franz, et al, '96)

Spin-Triplet, P-wave pairing w/ E_{1u} symmetry

The two-component model for the SC phases of UPt_3 with E_{1u} symmetry is based on the orbital representation $\begin{pmatrix} p_x \\ p_y \end{pmatrix}$,

$$\hat{\Delta}(\vec{p}_f) = \begin{pmatrix} 0 & \vec{\Delta}(\vec{p}_f)\cdot i\vec{\sigma}\sigma_y \\ \vec{\Delta}^*(\vec{p}_f)\cdot i\sigma_y\vec{\sigma} & 0 \end{pmatrix}$$

For general $S=1$ states

$$\hat{\Delta}(\vec{p}_f)^2 = -\begin{pmatrix} |\vec{\Delta}(\vec{p}_f)|^2 + i(\vec{\Delta}\times\vec{\Delta}^*)\cdot\vec{\sigma} & 0 \\ 0 & |\vec{\Delta}|^2 - i(\vec{\Delta}\times\vec{\Delta}^*)\cdot\vec{\sigma}^{tr} \end{pmatrix}$$

Spin Triplet States w/ $i\vec{\Delta}\times\vec{\Delta}^* \neq 0$ are called "non-unitary". $i\vec{\Delta}\times\vec{\Delta}^*$ is proportional to the Cooper pair spin polarization at the point $\vec{p}_f$ on the FS.

In addition, the quasiparticle excitations will have a splitting due to the pair spin polarization,

$E_{\vec{p}\uparrow} \neq E_{\vec{p}\downarrow}$. These states are realized in ^3He. ($\vec{B}\neq 0$)

(Ambegaokar + Mermin)

The E_{1u} model with $\underline{\hat{d}\ \text{real}}$ is an Equal Spin pairing state, which doesn't break $\mathcal{T}$-symmetry in the spin degrees of freedom.

$$\hat{\Delta}(\vec{p}_f)^2 = -|\vec{\Delta}(\vec{p}_f)|^2\,\hat{1}$$

$$\vec{\Delta}(\vec{p}_f) = \hat{d}\,\big(\underbrace{\Delta_1}_{\eta_1}\, \mathcal{Y}_1(\vec{p}_f) + \underbrace{\Delta_2}_{\eta_2}\, \mathcal{Y}_2(\vec{p}_f)\big)$$

$$\mathcal{Y}_1 \sim p_x \qquad \mathcal{Y}_2 \sim p_y$$

We can carry out the GL expansion for this pairing symmetry and derive the material coefficients in the GL functional.

$$\Delta\Omega_{GL}[\vec{\eta}] = \alpha(T)\,|\vec{\eta}|^2 + \beta_1|\vec{\eta}|^4 + \beta_2|\vec{\eta}\cdot\vec{\eta}|^2$$

$$+ K_1\,(\partial_i\eta_j)(\partial_i\eta_j)^* + K_2\,(\partial_i\eta_i)(\partial_j\eta_j)^*$$

$$+ K_3\,(\partial_i\eta_j)(\partial_j\eta_i)^* + K_4\,(\partial_z\eta_i)(\partial_z\eta_i)^*$$

$\beta_2 > 0 \Rightarrow$ Broken $\mathcal{T}$-symmetry

$$\alpha(T) = -N_f\,(1 - T/T_c) \quad ; \quad \boxed{\beta_2 = \tfrac{1}{2}\beta_1} = \frac{7}{8}\zeta(3)\frac{N_f}{16\pi^2 T_c^2}\langle|\mathcal{Y}_+|^4\rangle_{FS}$$

$$\underset{(4)}{K_1} = \langle \mathcal{Y}_1(\vec{p}_f)\, \underset{(z)}{v_{fy}}\, \underset{(z)}{v_{fy}}\, \mathcal{Y}_1(\vec{p}_f)\rangle_{FS} * \frac{7\zeta(3)}{16\pi^2 T_c^2} N_f$$

$$K_2 = K_3 = \langle \mathcal{Y}_1\, v_{fx}\, v_{fy}\, \mathcal{Y}_2(\vec{p}_f)\rangle_F * \quad \text{"}$$

Pair-Breaking By Non-Magnetic Impurities

(non-s-wave)

In $\underline{\text{unconventional}}$ superconductors, non-magnetic impurities are strong pair breakers, just as are magnetic impurities in an s-wave superconductor.

The impurity self-energy (in the Born approximation) for potential scatterers (s-wave for simplicity) is

$$\hat{\sigma}_{imp}(\varepsilon_n) = n_s\left(u_0\,\hat{1} + N_f\,|u_0|^2\langle \hat{g}(\vec{p}_f,\varepsilon_n)\rangle_{FS}\right)$$

For homogeneous equilibrium, then we have Eilenberger's equation,

$$\left[i\varepsilon_n\hat{\tau}_3 - \hat{\Delta}(\vec{p}_f) - \hat{\sigma}_{imp}\,,\ \hat{g}(\vec{p}_f,\varepsilon_n)\right] = 0$$

w/ $\hat{g}^2 = -\pi^2\,\hat{1}$

The $\hat{1}$-term in $\hat{\sigma}_{imp}$ drops out, and if we have $\underline{\text{s-wave pairing}}$ $\hat{\Delta}(\vec{p}_f) = \hat{\Delta}$ independent of $\vec{p}_f$.

$\langle\hat{g}\rangle = \hat{g}$

Thus, we immediately find that $\hat{\sigma}_{imp}$ drops out

$$[\hat{\sigma}_{imp}, \hat{g}] = n_s N_f |u_0|^2 [\hat{g}, \hat{g}] = 0 \quad \left(\begin{matrix}\text{Anderson's}\\ \text{Th.}\end{matrix}\right)$$

However for non-s-wave pairing Anderson's Th fails (even for $s=0$ pairing which is T-symmetric)

$$[\hat{\sigma}_{imp}(\varepsilon_n),\ \hat{g}(\vec{p}_f,\varepsilon_n)] \neq 0$$

Nevertheless, the solution for $\hat{g}$ is given by the same equilibrium form as before

$$\hat{g} = -\pi \frac{i\tilde{\varepsilon}_n \hat{\tau}_3 - \hat{\tilde{\Delta}}(\vec{p}_f,\varepsilon_n)}{\sqrt{\tilde{\varepsilon}_n^{\,2} + |\tilde{\Delta}(\vec{p}_f,\varepsilon_n)|^2}}$$

w/ $$i\tilde{\varepsilon}_n = i\varepsilon_n - \frac{1}{2\tau_{el}} \left\langle \frac{i\tilde{\varepsilon}_n}{\sqrt{\tilde{\varepsilon}_n^{\,2} + |\tilde{\Delta}(p_f,\varepsilon_n)|^2}} \right\rangle_{FS}$$

$$\tilde{\Delta}(\vec{p}_f,\varepsilon_n) = \Delta(\vec{p}_f) + \frac{1}{2\tau_{el}} \cancel{\left\langle \frac{\tilde{\Delta}(\vec{p}_f^{\,\prime})}{\sqrt{\tilde{\varepsilon}_n^{\,2} + |\tilde{\Delta}|}} \right\rangle_{FS}} \to 0$$

The point to note is that the average of $\Delta(\vec{p}_f)$ over the FS vanishes because of a broken symmetry (parity for E_{1u}, reflections for B_{1g}, ...).

Thus, $\left\{ \begin{array}{l} \hat{\tilde{\Delta}} = \Delta(\vec{p}_f) \text{ is unrenormalized} \\ i\tilde{\varepsilon}_n = i\varepsilon_n - \frac{1}{2\tau_{el}} \left\langle \frac{i\tilde{\varepsilon}_n}{\sqrt{\quad}} \right\rangle_{FS} \text{ is renormalized} \end{array} \right.$

⇙ Pair breaking for Δ & T_c.

$$\Delta(\vec{p}_f) = \int d\vec{p}_f^{\,\prime}\, V_t(\vec{p}_f,\vec{p}_f^{\,\prime})\, \pi T \sum_{\varepsilon_n}^{E_c} \frac{\Delta(\vec{p}_f^{\,\prime})}{\sqrt{\tilde{\varepsilon}_n^{\,2} + |\Delta(\vec{p}_f)|^2}}$$

For $\Gamma \neq A_{1g}$, the instability is given by the linearized gap equation,

$$\alpha(T) = N_f \left\{ \frac{1}{V_\Gamma} - \pi T \sum_{\varepsilon_n}^{E_c} \frac{1}{|\varepsilon_n| + 1/2\tau_{el}} \right\}$$

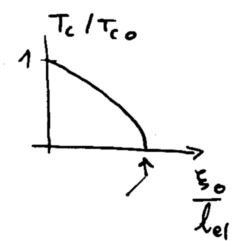
$$\alpha(T) = N_f \left\{ \ln(T/T_{c0}) + \psi\left(\tfrac{1}{2} + \tfrac{1}{4\pi\tau T}\right) - \psi\left(\tfrac{1}{2}\right) \right\}$$

$$\alpha(T_c) = 0 \Rightarrow \boxed{\ln(T_{c0}/T_c) = \psi\left(\tfrac{1}{2} + \tfrac{1}{4\pi\tau_{el} T_c}\right) - \psi\left(\tfrac{1}{2}\right)}$$

This is the Abrikosov-Gor'kov result, but with $1/\tau_s \to \frac{1}{\tau_{el}}$ for potential scattering.

Thus, SC for unconventional pairing is destroyed for relatively weak disorder,

$$l_{el} = v_F \tau_{el} \simeq 3.5\,\xi_0$$



SC transition in a uniform Magnetic Field

The instability temperature (or field) at finite field (temperature $T<T_c$) can be obtained from the linear solution for the pair amplitude in a magnetic field,

$f(\vec{p}_f, \vec{R}; \varepsilon_n)$ obeys the differential equation

$$i\vec{v}_f \cdot \vec{D} f + 2i\tilde{\varepsilon}_n f + \mu_{eff}\left(\vec{\sigma}\cdot\vec{B} f + f \vec{\sigma}^{tr}\cdot\vec{B}\right) = 2i\pi\, \mathrm{sgn}(\varepsilon_n)\, \Delta(\vec{p}_f, \vec{R}; \varepsilon_n)$$

$\vec{D} = \vec{\nabla} + i\frac{2e}{\hbar c}\vec{A}$

Zeeman

For odd-parity, triplet pairing the uppercritical field (largest H_{c2}) is determined by the eigenvalue of

$$\vec{\Delta}(\vec{p}_f, \vec{R}) = N_f \int d\vec{p}_f{}' \, V_t(\vec{p}_f, \vec{p}_f{}') * 2\pi T \sum_{\varepsilon_n}^{E_c} *$$

$$\int_0^\infty ds \, \exp\left\{-2s|\varepsilon_n| - \mathrm{sgn}(\varepsilon_n)\, s\, \vec{v}_f{}' \cdot \vec{D}\right\} * \left\{1 + \left(\cos(2s\mu_{eff} B) - 1\right) \hat{B}\otimes\hat{B}\right\} \cdot \vec{\Delta}(\vec{p}_f{}', \vec{R})$$

$\boxed{\text{Anisotropic Pauli Limiting}}$

Qualitative Effect of the Zeeman Term

Spin-Singlet Superconductors have an upper limit (mean field) for H_{c2} determined by the Zeeman energy; $-\mu_{eff}\, \vec{\sigma}\cdot\vec{B}$.

For $S=0$ this energy is always pairbreaking

However, for Triplet pairing it depends on the relative orientation of $\vec{B}$ and the spin state of the pairs.

For $S=1$ ESP w/ $\vec{d} \parallel \hat{z} \rightarrow S_z = 0$

Thus, $\vec{B} \parallel \hat{z}$ is pairbreaking $\rightarrow$ Pauli Limit.

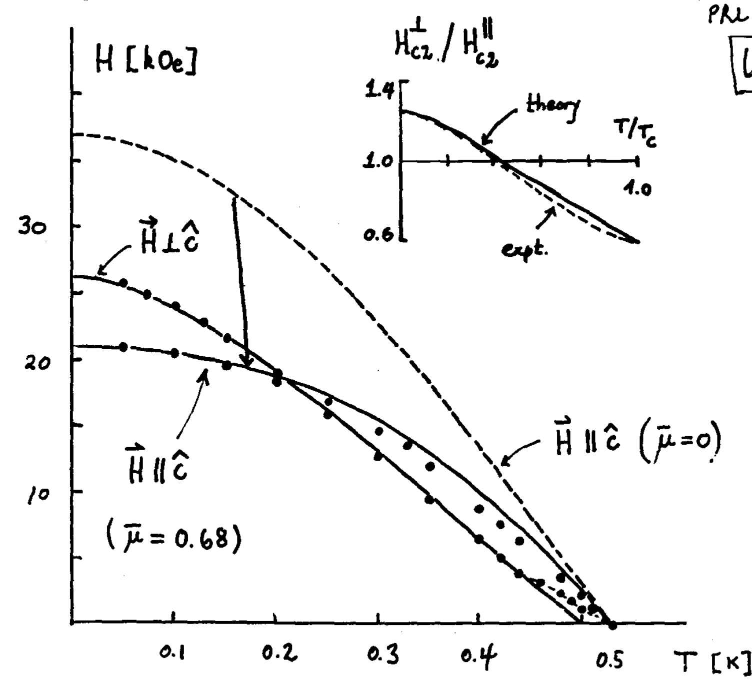
But for $\vec{B} \perp \hat{z}$, we can write

$$\vec{d} = \hat{z} = \underbrace{\tfrac{1}{2}(\hat{z} + i\hat{y})}_{S_x = +\hbar} + \underbrace{\tfrac{1}{2}(\hat{z} - i\hat{y})}_{S_x = -\hbar}$$

$\vec{B} \parallel \vec{x}$ shifts the population of $|\rightrightarrows\rangle$ & $|\leftleftarrows\rangle$ without breaking pairs $\rightarrow$ No Pauli Limit!

C. Choi + JAS
PRL '91

UPt$_3$



: Shivaram, et al. (1986)

$$\xi_{\parallel} / \xi_{\perp} = 1.84$$

$$H_0 = \frac{hc/2e}{\pi \xi_{\perp}^2} = 4.8\ T$$

$$\bar{\mu} = \frac{\mu H_0}{\pi T_c} = 0.68 \rightsquigarrow \mu = 0.34\,\mu_B$$

Strong conflict with M-O model

Order Parameter $\in$ Odd Parity w/ $\boxed{\hat{d} = \hat{c}}$