Lectures on Unconventional Superconductivity

Superfluid ^3He

Heavy Fermion Superconductors

High T_c cuprates

Ruthenates (?)

Organic Metals

(Signatures of Broken Symmetries)

Introduction – Broken Symmetry, Order Parameters

Fermi-Liquid Superconductivity "BCS + Landau + ..." } Thermodynamics, Currents, Disorder

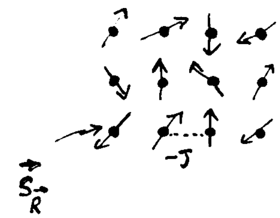
Fermi-Liquid SC-II : Transport in UnSC.

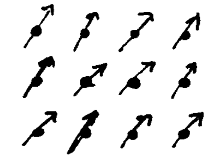
Non-equilibrium Dynamics : { Collective Modes, Vortex Dynamics

Sauls
Lecture 1

Long Range Order – Ferromagnets

Atomic Spins on a LATTICE (3D)





$\underline{T > T_c}$ $\qquad \vec{S} = \sum_{\vec{R}} \vec{S}_{\vec{R}} \qquad$ $\underline{T < T_c}$

$T > T_c$	$T < T_c$
• $\langle \vec{S} \rangle = 0$ $Tr\{e^{-\beta H} \ldots\}$	• $\langle \vec{S} \rangle \neq 0$
• Disordered Phase	• Ordered Phase
• Rotationally Invariant	• Broken Rotational Symmetry
• $G(\vec{R},\vec{R}') \xrightarrow[\lvert\vec{R}-\vec{R}'\rvert \to \infty]{} 0$	• $G(\vec{R},\vec{R}') \longrightarrow \sim \lvert\langle\vec{S}\rangle\rvert^2$

↳ Long-range Order in the Spin-Spin Correlations

$$G(\vec{R},\vec{R}') \equiv \langle \vec{S}(\vec{R}) \cdot \vec{S}(\vec{R}') \rangle \sim \langle \vec{S}(\vec{R}) \rangle \cdot \langle \vec{S}(\vec{R}') \rangle$$

Condensation in Bose Fluids/Gases

③

- One-particle Density Matrix

$$\rho_1(\vec r,\vec r\,') \equiv \langle \psi^\dagger(\vec r)\,\psi(\vec r\,')\rangle$$

$$\psi^\dagger(\vec r) = \sum_{\vec p} \varphi^*_{\vec p}(\vec r)\, a^\dagger_{\vec p}$$

$$\psi(\vec r) = \sum_{\vec p} \varphi_{\vec p}(\vec r)\, a_{\vec p}$$

$$\rho_1(\vec r,\vec r\,') = \sum_{\vec p,\vec p\,'} \varphi^*_{\vec p}(\vec r)\,\varphi_{\vec p\,'}(\vec r\,')\,\langle a^\dagger_{\vec p}\, a_{\vec p\,'}\rangle$$

Translationally Invariant System

$$\langle a^\dagger_{\vec p}\, a_{\vec p\,'}\rangle = \delta_{\vec p,\vec p\,'} * \overbrace{\langle a^\dagger_{\vec p}\, a_{\vec p}\rangle}^{\bar n_{\vec p}}$$

Macroscopic Occupation of the $\vec p = 0$ state ($T < T_{BE}$)

$$\bar n_{\vec p} = N_0\, \delta_{\vec p,0} + \tilde n_{\vec p}\,(1-\delta_{\vec p,0})$$

$N_0 \sim \mathcal{O}(N)$ $\qquad$ $\tilde n_{\vec p} \sim \mathcal{O}(1)$

Long-Range Order below T_{BE}

④

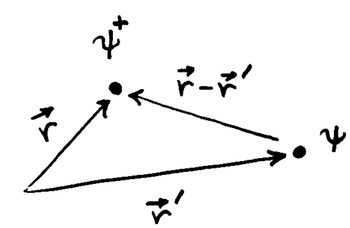
$$\bar n_{\vec p} = N_0\, \delta_{\vec p,0} + \overset{1/(e^{\beta\epsilon_{\vec p}}-1)}{\tilde n_{\vec p}}\,(1-\delta_{\vec p,0})$$

$$\rho_1(\vec r,\vec r\,') = N_0\, \varphi_0^*(\vec r)\,\varphi_0(\vec r\,') + \tilde\rho_1(\vec r,\vec r\,')$$

$$\tilde\rho_1(\vec r,\vec r\,') = \int \frac{d^3p}{(2\pi\hbar)^3}\, e^{-i\vec p\cdot(\vec r-\vec r\,')/\hbar}\, \tilde n_{\vec p}$$

$$\tilde\rho_1(\vec r,\vec r\,') \xrightarrow[|\vec r-\vec r\,'|\to\infty]{} \underline{\underline{0}}\,.$$

$$\boxed{\rho_1(\vec r,\vec r\,') \xrightarrow[|\vec r-\vec r\,'|\to\infty]{} N_0\, \varphi_0^*(\vec r)\,\varphi_0(\vec r\,')}$$



Macroscopically Occupied Single QuantumState

$\Psi \sim \sqrt{\frac{N_0}{V}}$

$$\langle \psi^\dagger(\vec r)\,\psi(\vec r\,')\rangle \xrightarrow[|\vec r-\vec r\,'|\to\infty]{} \Psi^*(\vec r)\,\Psi(\vec r\,')$$

Off-Diagonal Long-Range Order & $\langle \psi \rangle$ (5)

$T=0$ Ground State of the Bose Gas

$$|N\rangle = \frac{1}{\sqrt{N!}} (a_0^\dagger)^N |vac\rangle$$

$$\langle \vec r_1 \cdots \vec r_N | N\rangle \sim \varphi_0(\vec r_1)\cdots\varphi_0(\vec r_N)$$

$$\langle \psi^\dagger(\vec r)\rangle \equiv \langle N+1|\psi^\dagger(\vec r)|N\rangle = \sum_{\vec p} \varphi^*_{\vec p}(\vec r)\,\langle N+1|a^\dagger_{\vec p}|N\rangle$$

$$\downarrow \quad = \sqrt{N+1}\,\varphi_0^*(\vec r) \simeq \sqrt{N}\,\varphi_0^*(\vec r) \equiv \Psi^*(\vec r)$$

$$\langle\psi(\vec r')\rangle \equiv \langle N|\psi(\vec r')|N+1\rangle = \sqrt{N}\,\varphi_0(\vec r') \equiv \underline{\underline{\Psi(\vec r)}}$$

$\underline{T=0}$

$$\rho_1(\vec r,\vec r') = N\,\varphi_0^*(\vec r)\,\varphi_0(\vec r') = \langle\psi^\dagger(\vec r)\rangle\langle\psi(\vec r')\rangle$$

$$= \Psi^*(\vec r)\,\Psi(\vec r')$$

Macroscopic Matter field: $\Psi \sim \sqrt{\frac{N}{V}}$

(6)

Superfluid Helium (^{4}He)

$T_\lambda = 2.17\,K \sim T_{BE}$ (F. London)

Strongly Interacting Bose Liquid

Condensation w/ $n_0 \lesssim O(\frac{N}{V})$.

ODLRO in ^{4}He (Penrose & Onsager)

$$\rho_1(\vec r,\vec r') \longrightarrow \Psi^*(\vec r)\,\Psi(\vec r')$$

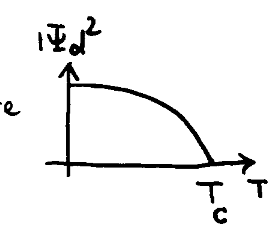
w/ $|\Psi(\vec r)| \simeq \sqrt{n_0} = \Psi_0$

$\underline{\Psi(\vec r)} = \boxed{\text{Order Parameter}} \equiv$ Macroscopic 'wave function' of condensate particles

$$\cong \boxed{\Psi_0}\, e^{i\theta(\vec r)}$$

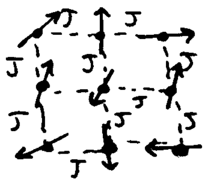
Thermodynamic Variable of State

Dynamical Variable



$\boxed{\hbar\vec\nabla\theta \equiv \vec p_s} \to$ Superflow

Broken (spin) Rotation Symmetry in Ferromagnets



$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j$$ Heisenberg

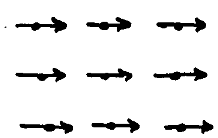
Symmetry: $U[\vec{\theta}] = e^{-i \vec{S}\cdot\vec{\theta}}$

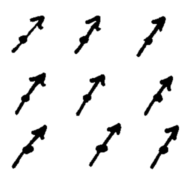
$$\vec{S} = \sum_i \vec{S}_i$$

$$U[\vec{\theta}]\, \vec{S}_i\, U^{+}[\vec{\theta}] = \overleftrightarrow{R}[\vec{\theta}]\cdot \vec{S}_i \Rightarrow \boxed{U \mathcal{H} U^{+} = \mathcal{H}}$$

Ground State (T=0)

$$|\Psi_0\rangle \quad \neq \quad U[\vec{\theta}]\,|\Psi_0\rangle$$





Degeneracy: $E[\Psi_0] = E[U_\theta \Psi_0]$

Parametrizes Degeneracy.

▶ Finite Temperature

$$\langle \vec{S} \rangle = \mathrm{Tr}[\rho \vec{S}] \neq 0 \rightarrow \boxed{\begin{array}{l} U \rho U^{+} \neq \rho \\ U \mathcal{H} U^{+} = \mathcal{H} \end{array}} \quad \begin{array}{l} [\vec{S}, \rho] \neq 0 \\ [\vec{S}, \mathcal{H}] = 0 \end{array}$$

but ,

Broken Gauge Symmetry

T ≠ 0

$$\Psi(\vec{r}) \equiv \langle \psi(\vec{r}) \rangle = \mathrm{Tr}[\rho\, \psi(\vec{r})]$$

Gauge Transformations

$$\psi(\vec{r}) \longrightarrow \psi'(\vec{r}) = U[\phi]\, \psi(\vec{r})\, U^{+}[\phi] = e^{i\phi}\, \psi(\vec{r})$$

$U[\phi] = e^{-i\hat{N}\phi}$ — phase parameter, Generator

$$\hat{N} = \int d\vec{r}\; \psi^{+}(\vec{r})\, \psi(\vec{r})$$

Expect $\quad \Psi(\vec{r}) \xrightarrow{\phi} e^{i\phi}\, \Psi(\vec{r})$

$$\langle \psi(\vec{r}) \rangle = \mathrm{Tr}[\overbrace{U^{+}U}^{1}\, \rho\, \overbrace{U^{+}U}^{1}\, \psi(\vec{r})]$$

$$\downarrow$$

$$= \mathrm{Tr}[\rho_\phi\, \psi(\vec{r})]\, e^{i\phi} \quad \text{w/} \quad \rho_\phi = U \rho U^{+}$$

$*$ $\langle \psi \rangle \neq 0 \Rightarrow [\hat{N}, \rho] \neq 0$

$|\Psi|\, e^{i\theta(\vec{r})}$ — even if $[\mathcal{H}, \hat{N}] = 0$

$\rho_\phi = \rho \Rightarrow$

$[\hat{N}, \rho] = 0$ & $\langle \psi \rangle = 0$

Ensembles w/ Broken Symmetry (η-Ensembles)

■ Spontaneous Magnetization : $\langle \vec{S} \rangle$

► Add an infinitesimal field :

$$\mathcal{H}_\eta = \mathcal{H} - \vec{\eta}\cdot\vec{S} \Longrightarrow \boxed{\rho_\eta \sim e^{-\beta\{\mathcal{H} - \vec{\eta}\cdot\vec{S} - \mu N\}}}$$

$$\langle \vec{S} \rangle = \lim_{\eta \to 0} \frac{1}{Z} \mathrm{Tr}\{ \rho_\eta \vec{S} \} \neq 0 \quad (T < T_c)$$

■ Spontaneous Condensation : $\langle \psi \rangle$

► Introduce a c-number field : "seed condensates"

$$\mathcal{H}_\eta = \mathcal{H} + \frac{1}{2}\int d^3R \, \{ \eta(\vec{R})\psi(\vec{R}) + \eta^*(\vec{R})\psi^\dagger(\vec{R}) \}$$

$$\rho_\eta \sim e^{-\beta\{\mathcal{H} + \frac{1}{2}\int d^3R \{\eta\psi + \eta^*\psi^\dagger\} - \mu N\}}$$

$$\boxed{\langle \psi(\vec{R}) \rangle \equiv \lim_{\eta \to 0} \frac{1}{Z} \mathrm{Tr}\{ \rho_\eta \psi(\vec{R}) \} \neq 0 \rightarrow \text{ODLRO}}$$

ODLRO in Fermion Systems

∘ 1-particle density Matrix

$$\rho_1(\vec{r}\sigma, \vec{r}'\sigma') = \langle \psi^\dagger_\sigma(\vec{r}) \psi_{\sigma'}(\vec{r}') \rangle = \sum_i^{\text{states}} \rho_i \, \varphi_i^*(\vec{r}\sigma) \varphi_i(\vec{r}'\sigma')$$

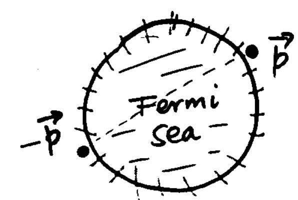
* $\rho_i \leq 1/V$ for all states (Pauli Exclusion)

No ODLRO in ρ_1.

► 2-particle Correlations ($x \equiv (\vec{r}, \sigma)$)

$$\rho_2(x_1, x_2; x_3, x_4) \equiv \langle \psi^\dagger(x_1) \psi^\dagger(x_2) \psi(x_3) \psi(x_4) \rangle$$

• Cooper Instability : Pair formation on the Fermi Surface



Attraction

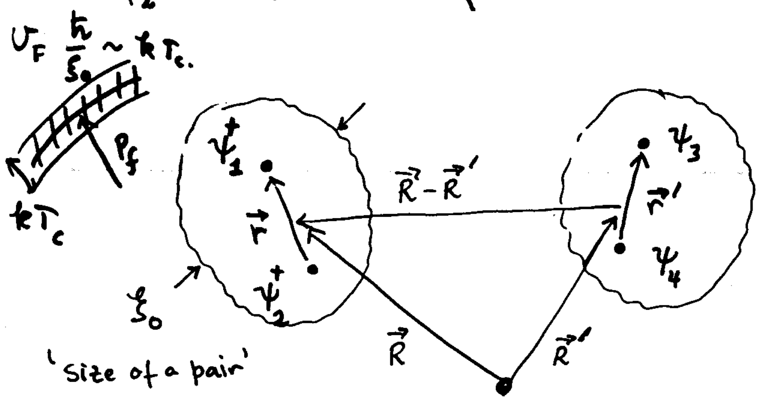
$\rightarrow V(\vec{p}, -\vec{p}) \rightarrow$ Bound State of $a^\dagger_{\vec{p}}$ & $a^\dagger_{-\vec{p}}$

• Superfluidity for Fermions $\leftrightarrow$ ODLRO in the pair channel

ODLRO $\leftrightarrow$ Pair Condensation

- Yang ..
- Gorkov.

$$\rho_2(x_1\, x_2;\, x_3\, x_4) = \left\langle \psi^+(x_1)\, \psi^+(x_2)\, \psi(x_3)\, \psi(x_4) \right\rangle$$



$$\rho_2 \xrightarrow[|\vec{R}-\vec{R}'| \to \infty]{} \Psi^*_{\sigma_1\sigma_2}(\vec{R},\vec{r})\ \underbrace{\Psi_{\sigma_3\sigma_4}(\vec{R}',\vec{r}')}_{\sim \langle \psi(x_3)\,\psi(x_4)\rangle}$$

Center-of-Mass Motion

Internal Orbital State

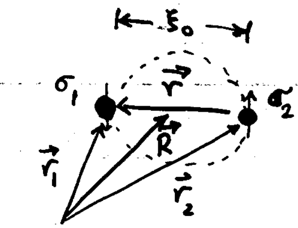
■ $\Psi_{\sigma_1\sigma_2}(\vec{R},\vec{r}) \sim \mathcal{O}\left(\frac{N}{2V}\right)$ (Macroscopically Occupied 2-particle state)

$\sigma_1\sigma_2$: Spin State of Cooper Pairs

Order Parameter = Pair Amplitude

$$\Psi_{\sigma_1\sigma_2}(\vec{R},\vec{r}) \sim \left\langle \psi_{\sigma_1}(\vec{r}_1)\, \psi_{\sigma_2}(\vec{r}_2) \right\rangle$$

$\sigma_1\sigma_2$: Spin projections of the fermions



► Fermion Anti-Symmetry

$$\Psi_{\sigma_1\sigma_2}(\vec{r}_1,\vec{r}_2) = -\Psi_{\sigma_2\sigma_1}(\vec{r}_2,\vec{r}_1)$$

$$\boxed{\Psi_{\sigma_1\sigma_2}(\vec{R},\vec{r}) = -\Psi_{\sigma_2\sigma_1}(\vec{R},-\vec{r})}$$

Ex.: S-wave Pairing ('Conventional' Superconductors)

$L_{pair} = 0$ (isotropic Ψ) $\to$ Rotationally Invariant Pairs

$\downarrow$

$S_{pair} = 0$ (singlet spins) $\to$ Magnetically Inert Pairs

$$\Psi_{\uparrow\downarrow}(\vec{R},\cancel{\vec{r}}) = |\Psi(\vec{R})|\, e^{i\Theta(\vec{R})}$$

Orbital Momentum / CM Representation

$$F_{\sigma_1\sigma_2}(\vec{R},\underset{\sim}{\vec{p}}) = \int d\vec{r}\; e^{-i\underset{\sim}{\vec{p}}\cdot\vec{r}}\; \underbrace{\Psi_{\sigma_1\sigma_2}(\vec{R},\vec{r})}_{\langle \psi_{\sigma_1}(\vec{R}+\frac{\vec{r}}{2})\,\psi_{\sigma_2}(\vec{R}-\vec{r}/2)\rangle}$$

- Homogeneous Equilibrium

$$F_{\sigma_1\sigma_2}(\vec{p}) \simeq \langle a_{\vec{p}\sigma_1}\, a_{-\vec{p}\sigma_2} \rangle$$

- Spin-Matrix Structure

$$\hat{F}(\vec{p}) = \begin{pmatrix} F_{\uparrow\uparrow} & F_{\uparrow\downarrow} \\ F_{\downarrow\uparrow} & F_{\downarrow\downarrow} \end{pmatrix}$$

- Fermion Anti-Symmetry

$$F_{\sigma_1\sigma_2}(\vec{p}) = -F_{\sigma_2\sigma_1}(-\vec{p})$$

Even Parity / Spin Singlet Pairing

$$F_{\sigma_1\sigma_2}(\vec{p}) = +F_{\sigma_1\sigma_2}(-\vec{p}) \qquad \text{(Even parity)}$$

$$\Downarrow$$

$$F_{\sigma_1\sigma_2}(\vec{p}) = -F_{\sigma_2\sigma_1}(\vec{p}) \qquad \text{(Singlet Pairing)}$$

$$F_{\sigma_1\sigma_2}(\vec{p}) = [i\sigma_y]_{\sigma_1\sigma_2} \cdot \boxed{d_0(\vec{p})} \Rightarrow \hat{F}(\vec{p}) = \begin{bmatrix} 0 & d_0 \\ -d_0 & 0 \end{bmatrix}$$

Odd-Parity / Spin Triplet Pairing

$$F_{\sigma_1\sigma_2}(\vec{p}) = -F_{\sigma_1\sigma_2}(-\vec{p}) \qquad \text{(odd parity)}$$

$$\downarrow$$

$$F_{\sigma_1\sigma_2}(\vec{p}) = +F_{\sigma_2\sigma_1}(\vec{p}) \qquad \text{(spin triplet)}$$

$$F_{\sigma_1\sigma_2}(\vec{p}) = [i\vec{\sigma}\sigma_y]_{\sigma_1\sigma_2} \cdot \boxed{\vec{d}(\vec{p})}$$

$$\hat{F}(\vec{p}) = \begin{array}{c} \begin{array}{cc} \uparrow\uparrow & \uparrow\downarrow+\downarrow\uparrow \end{array} \\ \begin{bmatrix} -d_x+id_y & id_z \\ id_z & d_x+id_y \end{bmatrix} \\ \begin{array}{cc} \uparrow\downarrow+\downarrow\uparrow & \downarrow\downarrow \end{array} \end{array}$$

Balian & Werthamer

Spin Rotation of the Pair Amplitudes

$$F_{\alpha\beta}(\vec{p}) = [i\sigma_y]_{\alpha\beta}\, \underset{\text{scalar}}{d_0(\vec{p})} + [i\vec{\sigma}\sigma_y]_{\alpha\beta} \cdot \underset{\text{vector}}{\vec{d}(\vec{p})}$$

Spin Rotations:

$$\psi_\alpha(\vec{r}) \xrightarrow{\vec{\theta}} \left[e^{-i\vec{\theta}\cdot\vec{\sigma}/2}\right]_{\alpha\beta} \psi_\beta(\vec{r})$$

$$\downarrow$$

$$F_{\alpha\beta}(\vec{p}) \rightarrow \left[e^{-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}\right]_{\alpha\gamma} \left[e^{-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}\right]_{\beta\rho} F_{\gamma\rho}(\vec{p})$$

$$= \left[e^{-i\vec{\theta}\cdot\vec{\sigma}/2}\ \hat{F}\ (e^{-i\vec{\theta}\cdot\vec{\sigma}/2})^{tr} \right]_{\alpha\beta}$$

$\sigma_y\vec{\sigma}\sigma_y = -\vec{\sigma}^{tr}$

$$\star \quad \sigma_y\,(e^{-i\vec{\theta}\cdot\vec{\sigma}/2})^{tr} = e^{i\vec{\theta}\cdot\vec{\sigma}/2}\,\sigma_y$$

$$F_{\alpha\beta}(\vec{p}) \xrightarrow{\vec{\theta}} \underbrace{[i\sigma_y]_{\alpha\beta}\, d_0(\vec{p})}_{\text{Rotational Invariant}} + i\underbrace{[e^{-i\vec{\theta}\cdot\vec{\sigma}/2}\,\vec{\sigma}\, e^{i\vec{\theta}\cdot\vec{\sigma}/2}\,\sigma_y]_{\alpha\beta}}_{=\overleftrightarrow{R}_{\vec{\theta}}\cdot(i\vec{\sigma}\sigma_y)}\cdot\vec{d}$$

$$\rightarrow [i\sigma_y]_{\alpha\beta}\, d_0 + [i\vec{\sigma}\sigma_y]_{\alpha\beta}\cdot \left[\overleftrightarrow{R}^{-1}_{\vec{\theta}}\cdot\vec{d}(\vec{p})\right]$$

Spin Vector

Ginzburg-Landau Free Energy Functional

(ala Landau c.a. 1937)

► $F_{\alpha\beta}(\vec{p},\vec{R})$ is a macroscopic Variable of State

$$\Delta\Omega[F_{\alpha\beta}] = -\frac{1}{T}\ln \underset{\text{fixed } F}{Tr_{(F)}} \left\{ e^{-\beta(H-\mu N)} \right\} - \underset{\text{Normal State Free Energy}}{\Omega_0}$$

- States which are slowly varying with respect to $\vec{R}$ on scale ξ_0
- $\Delta\Omega[F]$ is an Energy functional w.r.t. F, & a free-energy w.r.t. microscopic scale

■ Full Free Energy includes fluctuations of F

$$\Delta\Omega = -\beta \ln Tr_F\ e^{-\beta\Delta\Omega[F]}$$

↑ sum over all F configurations

$\star$ GL Theory : $\boxed{\Delta\Omega = \Delta\Omega[F_{min}]}$

$\Delta\Omega[F] \Rightarrow$ Energy functional for a configuration defined by the Order parameter $F_{\alpha\beta}(\vec{p},\vec{R})$.

$$\rho_F = \text{prob. density for } F : \frac{1}{Z_0}\, e^{-\beta\Delta\Omega[F]}$$

Partition function for configurations of F :

$$Z_F = Z_0 * \mathrm{Tr}_F\left\{ e^{-\beta\Delta\Omega[F]} \right\}$$

Landau Theory is the stationary-point approx.

$$\rho_F \to \rho_F^{MAX} = \frac{1}{Z_0} e^{-\beta \underbrace{\Delta\Omega_{min}}_{\Delta\Omega[F_{min}]}}$$

$$\underline{\Delta\Omega = \Delta\Omega[F_{min}]} \qquad w/ \quad \left.\frac{\delta\Delta\Omega}{\delta F}\right|_{F_{min}} = 0 .$$

Expand around F_{min}:

$$\Delta\Omega[F] = \Delta\Omega[F_{min}] + \underbrace{\delta\Delta\Omega[\delta F]}_{\text{Fluctuation Energy Functional}}$$

$$Z_F = \overbrace{Z_0\, e^{-\beta\Delta\Omega[F_{min}]}}^{Z_{Landau}} * \overbrace{\mathrm{Tr}_{\delta F}\left(e^{-\beta\,\delta\Delta\Omega[\delta F]}\right)}^{Z_{Fluc}}$$

Superfluid ^{3}He & "Related Models"

(Weak Spin-Orbit Coupling)

$E_F \approx 1\,K^\circ$

$T_c \approx 10^{-3}\,K^\circ$

$$\xi_0 \simeq \frac{\hbar v_F}{2\pi T_c} \approx \underline{\underline{10^3\,\mathring{A}}} \gg \frac{\hbar}{p_F} \sim \underline{\mathring{A}}$$

$*$ $\vec{s}_1 \overset{a}{\text{---}} \vec{s}_2 \Rightarrow E_{dipole} = \left(\frac{\gamma\hbar}{2}\right)^2 \frac{1}{a^3} \sim \underline{\underline{10^{-7}\,K^\circ}}$

$$G \cong SO(3)_{spin} \times \underbrace{SO(3)_{orbit}}_{\to\, G_{ROT}} \times P \times T \times U(1)$$

space inversion (P); time-reversal (T); gauge ($U(1)$)

Symmetry of Normal ^{3}He (or Normal Metal)

Point Group e.g. D_{6h} for UPt_3

$$\boxed{\begin{array}{c}\text{Unconventional Pairing}\\[6pt] F_{\alpha\beta}(\vec{p},\vec{R}) \xrightarrow{g} F'_{\alpha\beta}(\vec{p},\vec{R}) \neq F_{\alpha\beta}(\vec{p},\vec{R}) \\ \text{for } g \in G/U(1)\end{array}}$$

Representations of G

(Spin-Singlet)

$d(\vec{p},\vec{R}) \Rightarrow$ identity Representation of $SO(3)_{spin}$

$$\hookrightarrow \; = \sum_{lm}^{\text{even-}l} d_{lm}(\vec{R}) \boxed{Y_{lm}(\hat{p})}$$

$d^{(l)} = \{ d_{lm} \}$ – even-l rep. of $SO(3)_{orbit}$

Dim $= (2l+1)$

$$\begin{pmatrix} d^{(0)} \\ \hline d^{(2)}_{+2} \\ d^{(2)}_{+1} \\ d^{(2)}_{0} \\ d^{(2)}_{-1} \\ d^{(2)}_{-2} \\ \hline \vdots \\ l=4 \\ \vdots \end{pmatrix} \begin{array}{l} \leftarrow \text{s-wave, 1 dim, identity rep.} \\ \\ \left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} \text{d-wave, 5-dim Rep.} \\ \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{g-wave, 9-dim, Rep.} \end{array}$$

$\vec{d}(\vec{p},\vec{R}) \Rightarrow$ 3 dim 'Vector' Rep. of $SO(3)_{spin}$

$\downarrow$

$$\vec{d} = \sum_{l,m}^{\text{odd-}l} \vec{d}^{(l)}_{m} \, Y_{lm}(\hat{p})$$

"Vector Rep."

$$\begin{pmatrix} d^{(1)}_{+1} \\ d^{(1)}_{0} \\ d^{(1)}_{-1} \\ \hline \vdots \\ l=3 \\ \vdots \end{pmatrix} \begin{array}{l} \text{— } l=1 \text{, p-wave} \\ \quad \text{3 dim, Rep. of } SO(3)_{orbit} \\ \\ \quad \text{f-wave} \\ \leftarrow \text{7 dim Rep.} \end{array}$$

Spin-Singlet, s-wave SC (conventional)

$$d_0(\vec{R}) \xrightarrow{\alpha} e^{i\alpha} d_0(\vec{R}) \quad : \quad U(1)$$

- Landau-Ginsburg Functional

$$\Delta\Omega[d_0] = \int d^3R \; \underline{\Delta\phi[d_0(\vec{R})]}$$

- $\underset{T \to T_c^-}{d_0 \to 0}$ + $d_0 = 0$ $T > T_c$

- $|T - T_c| \ll T_c$ Expand in d_0

- $$\Delta\phi[d_0(\vec{R})] = \alpha(T)\,|d_0(\vec{R})|^2 + \beta\,|d_0(\vec{R})|^4 + \gamma\,|\vec{\nabla} d_0(\vec{R})|^2$$

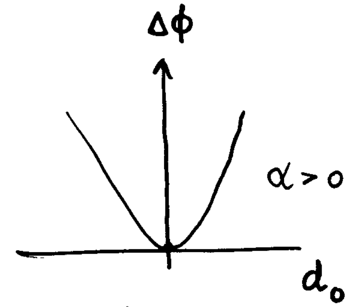
- No odd powers of d_0 → U(1) symm.

Material Parameters:

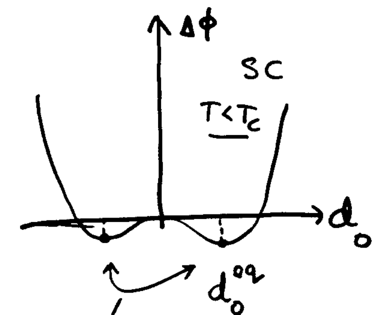
$\alpha(T) = \alpha'\left(\frac{T - T_c}{T_c}\right)$ w/ $\alpha' > 0$

$\beta > 0$ (stability)

$\gamma > 0$ (homogeneous states favored)



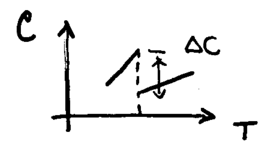
Normal
$T > T_c$



Degeneracy is Continuous

$d_0^{eq} \to d_0^{eq} * e^{i\alpha}$

$$\frac{\partial \Delta\phi}{\partial d_0} = 0 \Rightarrow |d_0(T)|^2 = -\frac{\alpha(T)}{2\beta} \approx (T_c - T) \; ; \; T < T_c$$

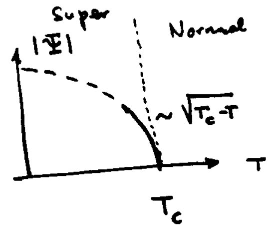


$$\Rightarrow \underset{\sim\sim\sim}{\Delta\phi_{eq} = -\frac{\alpha(T)^2}{4\beta}} \propto -(T - T_c)^2$$

Specific Heat Jump $\Rightarrow$ $\Delta C = -T_c \frac{\partial^2 \phi}{\partial T^2}\Big|_{T_c} = \frac{(\alpha')^2}{2\beta T_c}$

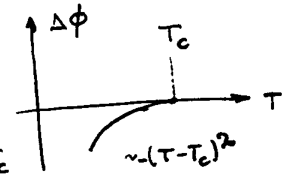
I. Order Parameter

$$\Psi_0 = \begin{cases} 0 & ; T > T_c \\ \sqrt{\frac{\alpha'}{2\beta}}\,(T_c - T)^{\frac{1}{2}} & ; T < T_c \end{cases}$$



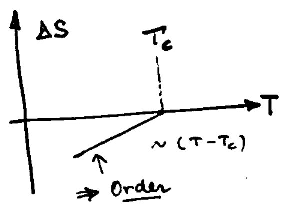
II. Condensation Energy:

$$\Delta\phi = \begin{cases} 0 & ; T > T_c \\ -\frac{1}{2}\left(\frac{\alpha'^2}{2\beta}\right)(T - T_c)^2 & ; T < T_c \end{cases}$$



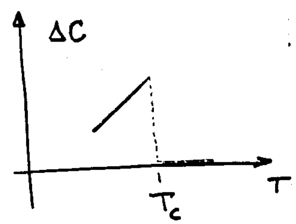
III. Entropy: $\Delta S = -\frac{\partial \Delta\phi}{\partial T}$

$$\Delta S = \begin{cases} 0 & ; T > T_c \\ (\alpha')^2/2\beta\,(T - T_c) & ; T < T_c \end{cases}$$



IV. Specific Heat: $\Delta C = T\,\frac{\partial \Delta S}{\partial T}$

$$\Delta C = \begin{cases} 0 & ; T > T_c \\ (\alpha')^2/2\beta * T & ; T < T_c \end{cases}$$



Non-Uniform Stationary Points
(GL Equation)

$$-\gamma\,\nabla^2 d_0 + \alpha(T)\,d_0(\vec{R}) + 2\beta\,|d_0(\vec{R})|^2\,d_0(\vec{R}) = 0$$

GL coherence length

$$\xi(T) = \sqrt{\frac{\gamma}{|\alpha(T)|}} \sim (T_c - T)^{-\frac{1}{2}}$$

$\gamma \rightarrow$ stiffness

$\xi(T) \Rightarrow$ scale of spatial fluctuations

where $|\alpha(T)|\,|d_0|^2 \sim \gamma\,|\vec{\nabla} d_0|^2$

$$\psi = d_0(\vec{R}) \,/\, |d_0^{eq}|$$

$$\Rightarrow \quad -\xi^2(T)\,\nabla^2\psi - \psi + |\psi|^2\psi = 0$$

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Spin-Triplet, P-wave Pairing

Full Orbital Rotation Group $\Rightarrow$ ^3He

$$d_i(\vec{p}) = A_{ij}\,\underbrace{(\hat{p})_j}$$

Vector w.r.t. Spin Rotations (i)

vector w.r.t. orbital rotations (j)

$$\begin{pmatrix}\hat{p}_x\\ \hat{p}_y\\ \hat{p}_z\end{pmatrix} = \begin{pmatrix} Y_{10} \sim p_z \\ Y_{11} \sim p_x + i p_y \\ Y_{1-1} \sim p_x - i p_y \end{pmatrix}$$

GL Free Energy Functional

1. Construct Scalar invariants
2. $U(1) \rightarrow$ equal # A_{ij} as A^*_{ij}
3. Form scalar products under spin & orbital rotations — Separately

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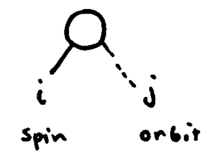
V. Ambegaokar's Construction

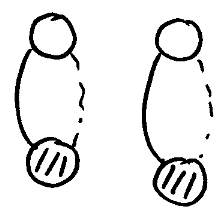
$A_{ij} \longleftrightarrow$ 

$A^*_{ij} \longleftrightarrow$ 

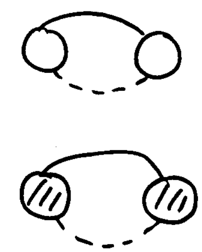
2nd Order:

 $= A_{ij}\,A^*_{ij} = \mathrm{Tr}\,AA^{+}$

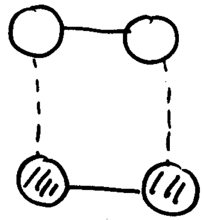
4th Order: (5 independent invariants)



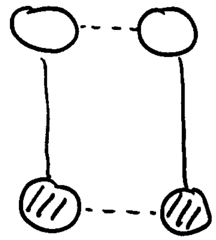
$[\mathrm{Tr}\,AA^{+}]^2$



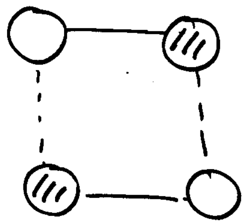
$|\mathrm{Tr}\,AA^{Tr}|^2$



$\mathrm{Tr}\left[(AA^{\dagger})(AA^{\dagger})^{*}\right]$



$\mathrm{Tr}\left[(AA^{Tr})(AA^{Tr})^{*}\right]$



$\mathrm{Tr}\left[(AA^{\dagger})^{2}\right]$

$$\Delta\Phi[A] = \alpha(T)\,\mathrm{Tr}\,AA^{\dagger} + \beta_1 \left|\mathrm{Tr}\,AA^{Tr}\right|^2 + \beta_2 \left[\mathrm{Tr}(AA^{\dagger})\right]^2 + \beta_3\,\mathrm{Tr}\left[(AA^{Tr})(AA^{Tr})^{*}\right] + \beta_4\,\mathrm{Tr}(AA^{\dagger})^2 + \beta_5\,\mathrm{Tr}(AA^{\dagger})(AA^{\dagger})^{*}$$

- Multiple Stationary Points ⇒ ~~Many~~ Nearly Degenerate Phases ⇒ Multiple SC phases

Multiple Superconducting Phases

↓

Multi-Component Order Parameter

$\mathcal{F}_{\alpha\beta}(\vec{p},\vec{R}) \in$ Representation w/ dim >1

- Superfluid ^3He
- Superconducting UPt_3
- Superconducting $(U,Th)Be_{13}$ (?)

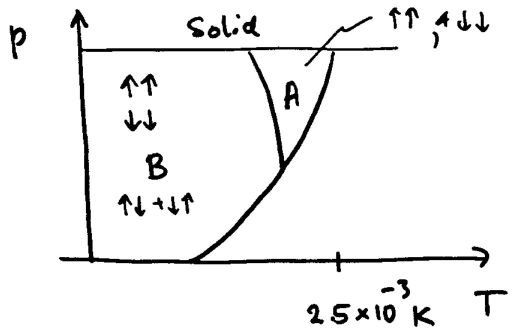
$\mathcal{F}$ is defined by two nearly degenerate Representations

- Surface States of HTC, YBCO (?)

Other candidates for Multiple SC phases: Sr_2RuO_4

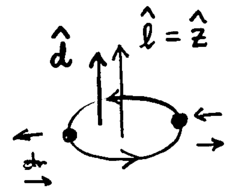
Multiple Superfluid Phases of ^{3}He

(Osheroff Richardson Lee)



A-phase : Equal-Spin-Pairing (ESP)

$$d_3(\vec{p}) = 0$$



Anderson-Brinkman-Morel

$$\vec{d}(\vec{p}) = \Delta(T) \underbrace{\boxed{\hat{d}}}_{\text{Real unit vector}} \underbrace{(p_x + ip_y)}_{\text{orbital } \ell \text{ Momentum}}$$

B-phase : Not ESP, $(\underbrace{J = L + S = 0}) + \hbar$

$$\vec{d}(\vec{p}) = \Delta(T)\ \hat{p}\ ! \rightarrow |\vec{d}(\vec{p})|^2 = \Delta(T)^2$$

Symmetry Breaking in ^{3}He

$$SO(3)_{spin} \times SO(3)_{orbit} \times P \times T \times U(1)$$

A phase : $SO(2)_{spin}$ about $\hat{d}$ $\times$ $U(1)_{gauge-orbit}$

$$p_x + ip_y \xrightarrow[\theta]{Rot} (p_x + ip_y)e^{i\theta} \xrightarrow[-\theta]{gauge} (p_x + ip_y)$$

B phase : $SO(3)_{S-L} \times T$ ← preserved.

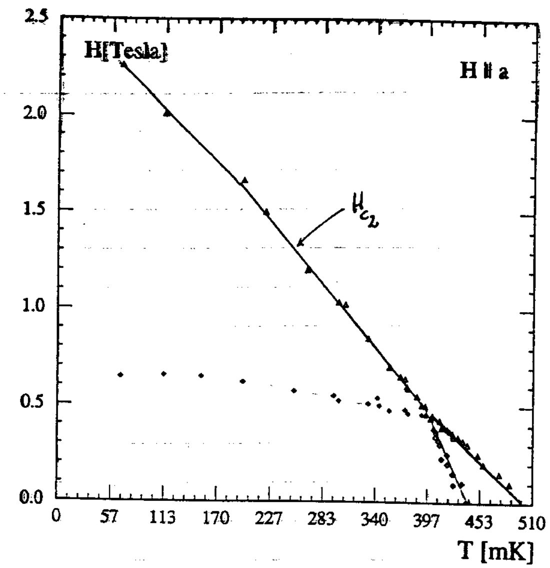
joint Spin-orbit rotations

* Both phases break Relative Spin-Orbit Rot. Symm.

⇒ NMR shifts (Leggett) + Acoustic Faraday Effect

Phase Diagram of UPt_3 ($\vec{H} \perp \hat{c}$) 31

• Sound Velocity Anomalies — S. Adenwalla, et al. PRL 1990.



- Two Meissner phases
- Three Flux phases w/ Tetracritical Point
- ▸ Qualitatively Similar phase diagram for $\vec{H} \parallel \hat{c}$