The Usadel Equations

0) Motivation
1) The Usadel Equations
2) BCS Solution
3) Disorder
4) Proximity Effect Example
5) Maximum Supercurrent

Why Use Usadel Equations?

- BCS-like theory that includes
  - Diffusive motion
  - Interfaces
  - Tunnel junctions
- Practical solution for experimental situations (Mesoscopic length scale)
- Compact Diff. Eq. that can be solved
  - Approximations easy to understand
  - Amenable to numerical solutions
- Energy information (D.O.S.) lends to physical understanding
- Condensate + Excitations treated together
- Includes non-equilibrium S.C.

Usadel Equations

Use E basis (not k)

$\Theta(x, E)$ complex number, describes s.c. state
$\theta = 0$ normal e's
$\theta = \pi/2$ paired e's

$\Delta(x)$ real, local pair energy

$\varphi(x)$ real phase, usual phase order parameter

Usadel:

$$\underbrace{\frac{\hbar D}{2}\frac{\partial^2\theta}{\partial x^2}}_{\text{Diffusive}} + \underbrace{iE\sin\theta}_{\text{Energy of excitation}} - \underbrace{\left[\frac{\hbar}{\tau_{SF}} + \frac{\hbar D}{2}\left(\frac{\partial\varphi}{\partial x} + \frac{2e}{\hbar}A_x\right)^2\right]\sin\theta\cos\theta}_{\text{pair breaking}} + \underbrace{\Delta(x)\cos\theta}_{\text{pairing interaction}} = ($$

$D = \sigma/ne^2$

normal, $\theta = 0$

$\rightarrow \theta = \pi/2$

Gap:

$$\Delta(x) = n_0 V_{eff}\int_0^{\hbar\omega_D} dE \tanh\frac{E}{2k_BT}\,\mathrm{Im}\left(\sin\Theta(x,E)\right)$$

Interface:

$$\sigma_r\frac{\partial\Theta_r}{\partial x} = \sigma_\ell\frac{\partial\Theta_\ell}{\partial x} = \frac{G_{int}}{A}\sin(\Theta_\ell - \Theta_r)$$

[note: $\frac{\partial\theta}{\partial x} = 0$ at insulator]

Physical Quant.'s

D.O.S. $n(x,E) = n_0\,\mathrm{Re}(\cos\theta)$

$$j_s = \frac{\sigma}{e}\int_{-\infty}^{\infty} dE \tanh\frac{E}{2k_BT}\,\mathrm{Im}\left(\sin^2\theta\right)\left(\frac{\partial\varphi}{\partial x} + \frac{2e}{\hbar}A_x\right.$$

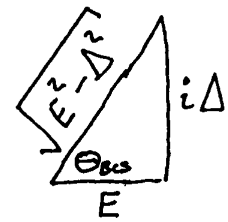
$$+ \ldots\left(j_{qp},\ j_{Andreev},\ j_{Josephson},\ j_{qp}\right)$$

BCS Solution

$\frac{\partial^2\theta}{\partial x^2} \to 0$, no pair breaking

$$i E \sin\theta_{BCS} + \Delta \cos\theta_{BCS} = 0$$

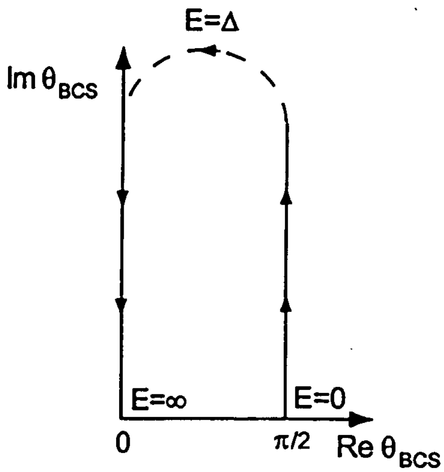
$$\tan\theta_{BCS} = \frac{i\Delta}{E}$$

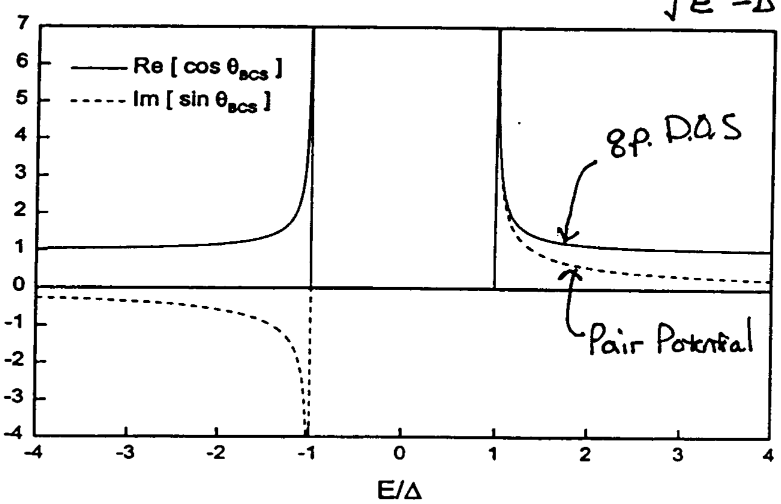


$$\Rightarrow \quad \theta_{BCS} = \begin{cases} \frac{\pi}{2} + i\tanh^{-1}\frac{E}{\Delta}, & |E| < \Delta \\ i\tanh^{-1}\frac{\Delta}{E}, & |E| > \Delta \end{cases}$$

$$\cos\theta_{BCS} = \frac{|E|}{\sqrt{E^2-\Delta^2}}$$

$$\sin\theta_{BCS} = \frac{i\Delta}{\sqrt{E^2-\Delta^2}}$$





$$\Delta = n_0 V_{eff} \int_0^{\hbar\omega_D} dE \tanh\frac{E}{2k_BT}\left(\frac{\Delta}{\sqrt{E^2-\Delta^2}}\,\Theta(E-\Delta)\right)$$

$\Rightarrow$ BCS gap equation!

$$I_s = \frac{\sigma}{e}\left(\vec{\nabla}\varphi - \frac{2e}{\hbar}\vec{A}\right)\int_{-\infty}^{\infty} dE \tanh\frac{E}{2k_BT}\,\mathrm{Im}\left(\frac{\Delta^2}{\Delta^2-E^2}\right)$$

$$\downarrow$$

$$\frac{\pi}{2}\Delta\left[\delta(E-\Delta) - \delta(E+\Delta)\right]$$

$$= \frac{\sigma\pi\Delta}{e}\tanh\frac{\Delta}{2kT}\left(\vec{\nabla}\varphi - \frac{2e}{\hbar}\vec{A}\right)$$

Disorder

Impurities

$D = \sigma / ne^2$, impurities decrease D

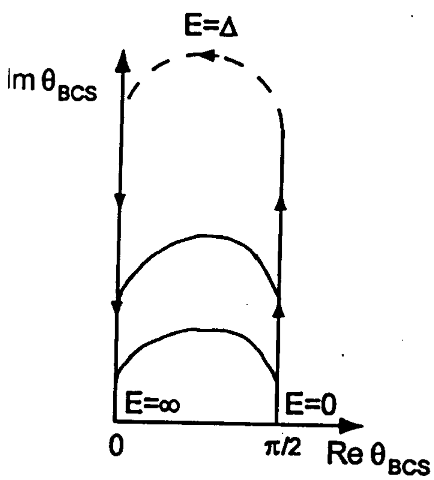
For homogenious films, $\frac{\partial^2 \theta}{\partial x^2} = 0 \Rightarrow$ BCS solution

No change in T_c !

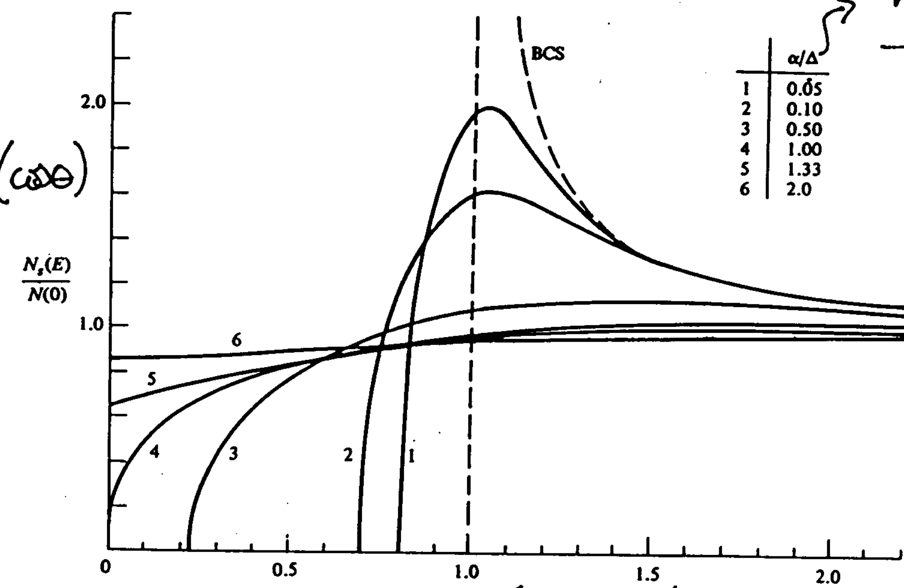
Magnetic Impurities

$$i E \sin\theta - \frac{\hbar}{\tau_{SF}} \sin\theta\cos\theta + \Delta\cos\theta = 0$$

Solve 4th order polynomial of $\sin\theta$



Re(cosθ)



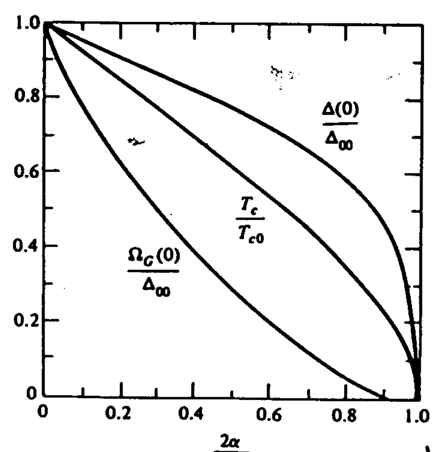
$\alpha/\Delta \rightarrow \frac{\hbar/\tau_{SF}}{\Delta}$

$\xi \quad E/\Delta$

Solve Δ equation,

Near T_c :

$$\boxed{(T_c - T) = 0.107 \frac{\Delta^2}{k_B T_c} + 0.80 \frac{\hbar}{\tau_{SF}}}$$

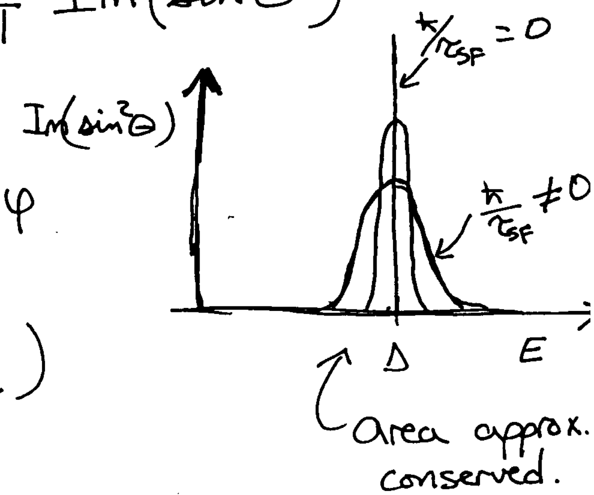


$\frac{2\alpha}{\Delta_\infty} \rightarrow 2 \frac{\hbar/\tau_{SF}}{\Delta}$

Supercurrent

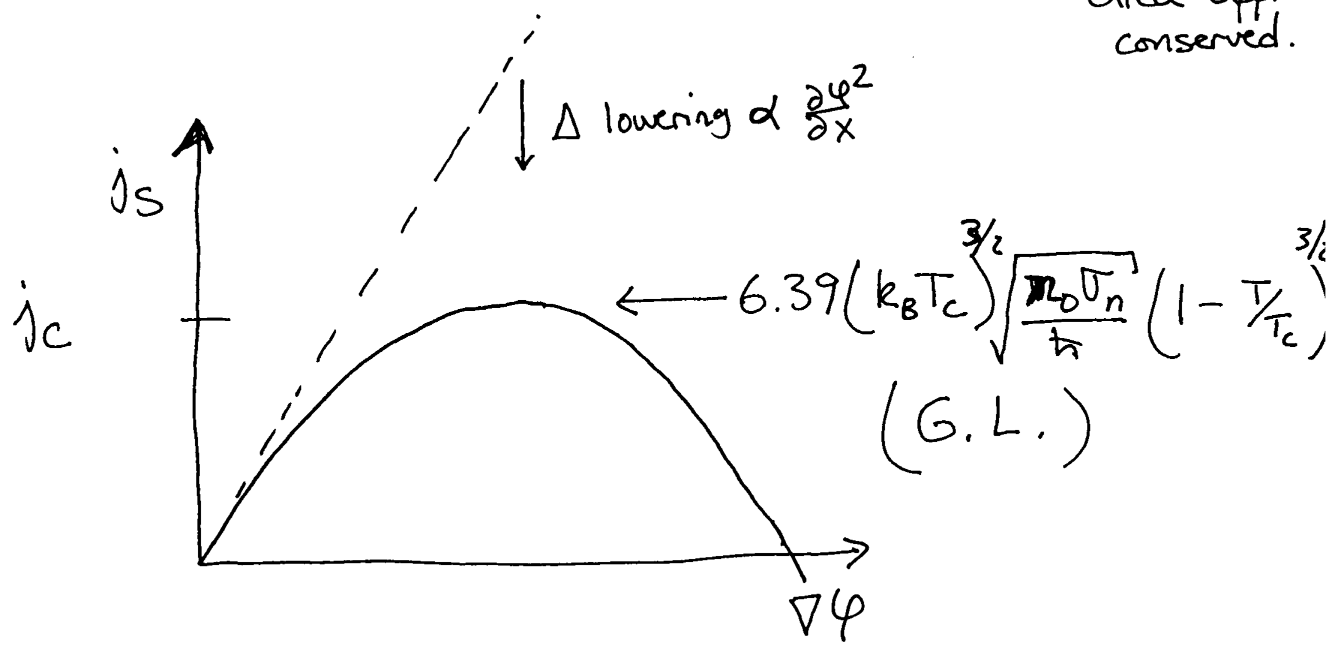
Supercurrent is pair breaking

$$I_s = \frac{\sigma}{e} \nabla\varphi \int_{-\infty}^{\infty} dE \tanh\frac{\Delta}{2k_BT} \, \mathrm{Im}(\sin^2\theta)$$



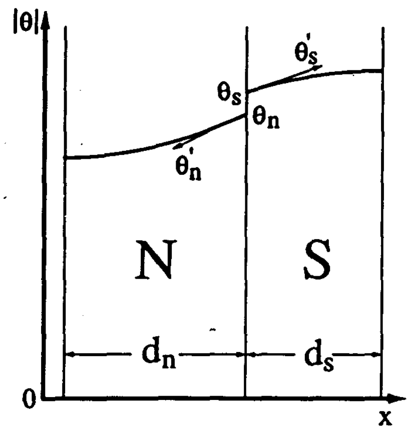
$$\simeq \frac{\sigma}{e} \pi \Delta \tanh\frac{\Delta}{2k_BT} \nabla\varphi$$

$$\propto \Delta^2 \nabla\varphi \quad (\text{for } T \to T_c)$$



Proximity Effect

NS bilayer
1-D.



T_c calculation:

* $\Delta \to 0$ at T_c
linearize around $\theta = 0$

$$(1) \quad \frac{\hbar D_s}{2} \frac{\partial^2 \theta_s}{\partial x^2} + iE\theta_s + \Delta = 0$$

$$(2) \quad \frac{\hbar D_n}{2} \frac{\partial^2 \theta_n}{\partial x^2} + iE\theta_n = 0$$

$$(3) \quad \sigma_s \frac{\partial \theta_s}{\partial x} = \sigma_n \frac{\partial \theta_n}{\partial x} = \frac{G_{int}}{A}(\theta_n - \theta_s)$$

* For thin film
$\theta_{s,n}$ approx. constant
small parabolic terms

$$(1) \rightarrow \quad \theta_s' = -d_s \frac{\partial^2 \theta_s}{\partial x^2} = d_s \frac{iE\theta_s + \Delta}{\hbar D_s/2}$$

$$(2) \rightarrow \quad \theta_n' = d_n \frac{\partial^2 \theta_n}{\partial x^2} = d_n \frac{-iE\theta_n}{\hbar D_n/2}$$

With 2 eq's of (3), can solve for θ_s.

Proximity Effect - Solutions

$$\Theta_s = \frac{i\Delta}{E}\left[1 - \frac{d_n n_n}{d_n n_n + d_s n_s}\,\frac{1}{1 + iE/\tau}\right]$$

$$\tau = \frac{G_{int}}{A}\,\frac{h}{4\pi e^2}\left(\frac{1}{d_n n_n} + \frac{1}{d_s n_s}\right)$$

Gap Equation:

$$\frac{1}{n_s V_{eff}} = \int_0^{\hbar\omega_D} \frac{dE}{E}\left[1 - \left(\frac{d_n n_n}{d_n n_n + d_s n_s}\right)\left(\frac{1}{1 + E^2/\tau^2}\right)\right] \tanh\frac{E}{2k_B T_c}$$

① Thin limit, $d_n, d_s \to 0$; $1/\tau \to 0$

$$\frac{1}{n_s V_{eff}\underbrace{\left(\frac{n_s d_s}{n_s d_s + n_n d_n}\right)}} = \int_0^{\hbar\omega_D} \frac{dE}{E}\tanh\frac{E}{2k_B T_c}$$

Fracton of states in S layer

② Moderately Thick Film; ($1/\tau$ cuts-off E, not $\hbar\omega_D$)

$$\frac{T_c}{T_{co}} = \left(\frac{k_B T_{co}}{1.13\,\tau}\right)^{\frac{d_n n_n}{d_s n_s}}$$

[$T_{co} \equiv T_c$ of S layer]

Prefactor Term depends on interface conductance (which describes strength of N-S coupling)

Landauer Conductance Formula:

$$G_{int} = 2\,t\,N_{ch}\frac{e^2}{h}\,, \qquad N_{ch} = \frac{A}{(\lambda_f/2)^2}$$

Expect $t \lesssim 1$ for good interface.

$$\frac{T_c}{T_{c0}} = \left(\frac{d_s}{\xi_0}\,\frac{1}{t}\,\frac{1}{1+\frac{d_s n_s}{d_n n_n}}\right)^{\frac{d_n n_n}{d_s n_s}}$$

[Valid for ( ) $\lesssim 0.8$]

↑ intrinsic coherence length
MoCu: ~ 1.2 μm

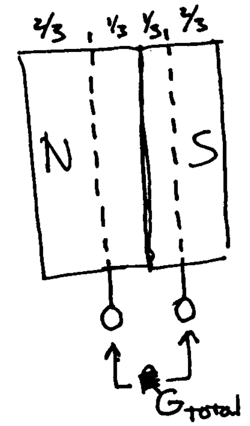
For MoCu, $t = 0.21$
AlAg, $t = 0.12$

③ Thick films; N + S high ρ

Analytic + Numerical calculations:

$$\frac{1}{t} \rightarrow \frac{1}{t} + \frac{1}{3}\,\frac{d_n}{\sigma_n}\,\frac{2\frac{e^2}{h}}{(\lambda_f/2)_N^2} + \frac{1}{3}\,\frac{d_s}{\sigma_s}\,\frac{2\frac{e^2}{h}}{(\lambda_f/2)_s^2}$$

$$\tau \propto \frac{G_{total}}{A}$$



T_c dependent of total ⊥ conductivity – describes coupling of states in N + S.

Experimental Implications:

1) Prediction of T_c as change parameters

! 2) Interface quality very important.

Numerical Solution of Usadel Equations

* E variable $\quad 0 < E \lesssim 10\Delta \quad$ linear spacing of E

$\quad 10\Delta < E < \hbar\omega_D \quad$ log spacing

Algorithm

Initial guess of $\Delta(x)$

Solve for $\theta(x, E)$:

* Start with $E = \hbar\omega_D$

Linearized eq's at $\theta = 0$

Solve Linear D.E. for $\theta(x, E)$

Next (lower) energy:

* Linearized eq's at $\theta = \theta(x, E_{last})$

Solve Linear D.E. for $\theta(x, E)$

$|\theta(x, E) - \theta(x, E_{last})|$ small? no / yes

Compute Δ from gap equation + $\theta(x, E)$

$|\Delta(x)_{new} - \Delta(x)_{old}|$ small? no

Done

Easy to solve linear D.E.'s via (sparse) Matrix Inversion techniques.

J_c for Bilayers

Motivation:

Need large j_c for TES x-ray detectors

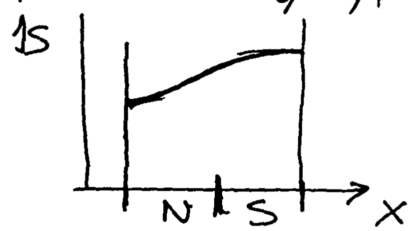
$P_{\substack{bias\\TES}} \sim E_{x\text{-ray}}/\tau_{fall}$

↳ limited by j_c of TES bilayer, make as large as possible given total thickness.

Numerical solution finds

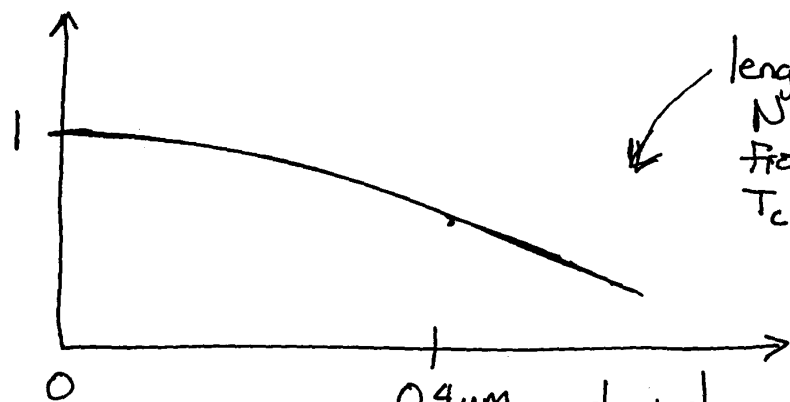
Low energy, $|E| \sim \Delta$ Θ constant over S+N

high energy, $|E| \gg \Delta$ Θ larger in S.



$\dfrac{J_c|_{num}}{j_c|_{G\text{-}L}}$

Use T_c, $\langle \rho \rangle$ of bilayer for $J_c|_{G\text{-}L}$



$d_n + d_S$ (Bilayer Thickness)

length scale where N starts to decouple from S – that is $T_c \simeq T_{co}$

J_c of magnetically suppressed films:

W, 120 mK suppressed to 60 mK by $\hbar/\tau_{SF}$

J_c still given by G-L formula (Even though smeared gap).