Dorsey

ATD 7/4/00 Boulder Lectures - Lecture 2. ①

Vortex Structure in Type-II Superconductors

What is a vortex? The simplest explanation is that it is a filament of magnetic flux which threads the superconductor, with the flux quantized in units of $\phi_0 = \frac{hc}{2e} = 2.07 \times 10^{-7}\ \mathrm{G\,cm^2}$.
A more precise definition might be that a vortex is a zero of the SC order parameter ψ, such that the phase θ is undefined.

Vortices are interesting and important:

1. Vortex "matter" has a plethora of interesting phases, accessed by changing the applied magnetic field (the field changes the intervortex separation).
2. The transport properties in the mixed state are determined by vortex motion; a vortex moving with a velocity $\vec{v}$ produces an electric field $\vec{E} = -\vec{v} \times \vec{B}$. For vortices in superconductors the velocity is approximately at right angles to the applied current:

$$\underbrace{\eta}_{\text{friction coefficient}} \vec{v} = \hat{\phi}_0 \times \underbrace{\vec{J}}_{\text{transport current}} \quad ; \quad \hat{\phi}_0 = \underbrace{\phi_0}_{\text{flux quantum}} \hat{z}$$

$$\therefore \vec{E} = -\frac{(\hat{\phi}_0 \times \vec{J})}{\eta} \times \vec{B} = \frac{\phi_0 B}{\eta} \vec{J} = \rho \vec{J}; \quad \rho = \frac{\phi_0 B}{\eta}.$$

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Reduce ρ by pinning the vortices.

3. Magnetic properties are determined by the vortex structure and pinning - relaxation, hysteresis.

Imaging methods: Bitter decoration, neutron scattering, electron holography, magneto-optics, STM, scanning probes. Also μSR, NMR (probe local fields).
Let's start with the microscopic and work up to a macroscopic description of vortices.

Microscopic Theory of Vortices

I'll only sketch how this is done. We start with electrons with an attractive BCS interaction an we make the pairing approximation (a generalized Hartree-Fock approximation), leaving us with the effective Hamiltonian

$$H_{eff} = \int d^3x \left\{ \sum_\sigma \psi_\sigma^+(\vec{x}) \left[\frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 - \mu \right] \psi_\sigma(\vec{x}) + \Delta(\vec{x})\, \psi_\uparrow^+(\vec{x})\, \psi_\downarrow^+(\vec{x}) + \Delta^*(\vec{x})\, \psi_\downarrow(\vec{x})\, \psi_\uparrow(\vec{x}) \right\} \qquad (1)$$

The pairing field $\Delta(\vec{x})$ must be determined self-consistently:

$$\Delta(\vec{x}) = -V \langle \psi_\downarrow(\vec{x}) \psi_\uparrow(\vec{x}) \rangle \qquad (2)$$

with $V>0$ the attractive pairing potential

The Hamiltonian is quadratic in the field operators; we can diagonalize it with a unitary (Bogoliubov) transformation:

$$\psi_\uparrow(\vec{x}) = \sum_n \left[\gamma_{n\uparrow} u_n(\vec{x}) - \gamma^+_{n\downarrow} v^*_n(\vec{x}) \right]$$
$$\psi_\downarrow(\vec{x}) = \sum_n \left[\gamma_{n\downarrow} u_n(\vec{x}) + \gamma^+_{n\uparrow} v^*_n(\vec{x}) \right] \qquad (3)$$

Recall that

$$\{ \psi^+_\sigma(\vec{x}), \psi_{\sigma'}(\vec{x}') \} = \delta_{\sigma\sigma'} \delta(\vec{x}-\vec{x}')$$
$$\{ \psi_\sigma(\vec{x}), \psi_{\sigma'}(\vec{x}') \} = \{ \psi^+_\sigma(\vec{x}), \psi^+_{\sigma'}(\vec{x}') \} = 0. \qquad (4)$$

If $\{u_n, v_n\}$ are complete and orthonormal, then the γ's also satisfy fermion anti-commutation rules:

$$\{ \gamma_{m,\sigma}, \gamma^+_{n,\sigma'} \} = \delta_{mn} \delta_{\sigma\sigma'}$$
$$\{ \gamma_{m,\sigma}, \gamma_{n,\sigma'} \} = \{ \gamma^+_{m,\sigma}, \gamma^+_{n,\sigma'} \} = 0. \qquad (5)$$

Now we want H_{eff} to be diagonal:

$$H_{eff} = E_g + \sum_{n,\sigma} \epsilon_n \gamma^+_{n,\sigma} \gamma_{n,\sigma}$$

with ϵ_n the excitation energy and E_g.
For this to occur $\{u, v\}$ must satisfy the Bogoliubov-de Gennes equations

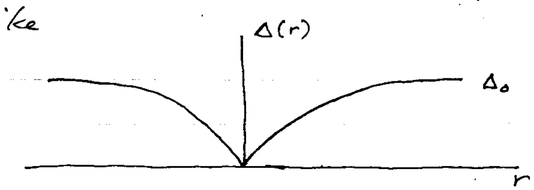
$$\begin{pmatrix} H_0 & \Delta \\ \Delta^* & -H_0^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \epsilon \begin{pmatrix} u \\ v \end{pmatrix} \qquad (6)$$

with $$H_0 = -\frac{\hbar^2}{2m} \left(\vec{\nabla} + i \frac{e\vec{A}}{c} \right)^2 - \mu \qquad (7)$$

Scheme:

1. Chose a $\Delta(\vec{x})$ which has the vortex structure built into it. Usually one makes a singular gauge transformation
$$\Delta(\vec{x}) \to \Delta(\vec{x}) e^{i\theta}, \quad u(\vec{x}) \to u(\vec{x}) e^{i\theta/2}$$
$$v(\vec{x}) \to v(\vec{x}) e^{-i\theta/2}$$

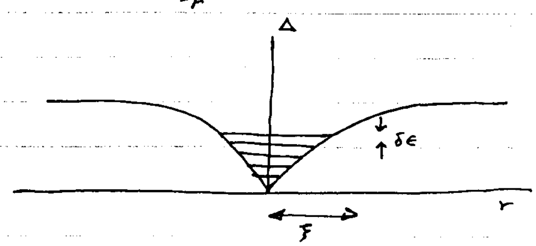
so that θ is absorbed into $\vec{A}$. Then $\Delta(\vec{x})$ is choosen to be real, and looks like



Guess an initial $\Delta(r)$.

2. Solve the BdG equations.
Can label the states by k_z (along the field) and an "angular momentum" quantum number μ, such that $\mu = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2} \ldots$ (needed so that u and v pick up a minus sign after one complete turn around the vortex).
There are both bound states ($\epsilon < \Delta_0$) and scattering states ($\epsilon > \Delta_0$). For $\epsilon \ll \Delta_0$, we have (Caroli, de Gennes, Matricon 1964)

$$\epsilon_\mu \sim \mu \frac{\Delta^2}{E_F}$$



$$\text{Energy} \sim \frac{\hbar^2}{2m\xi^2} \; ; \quad \xi \sim \frac{\hbar v_F}{\Delta_0}$$

$$\therefore \; \delta\epsilon \sim \frac{\Delta_0^2}{E_F}$$

- Small number for low T_c $\Rightarrow$ quasicontinuum
- $\sim 0.5\,\Delta$ in high T_c (?)

3. Calculate $\Delta(\vec{x})$ from self-consistency requirement:

$$\Delta(\vec{x}) = V \sum_n u_n(\vec{x}) V_n^*(\vec{x}) \left[1 - 2f(\epsilon_n)\right] \qquad (8)$$

Fermi function

4. Repeat 2 using the new $\Delta(\vec{x})$, until convergence is achieved.

Notes

1. The existence of a quasi-continuum justifies the notion of a "normal" core.
2. Can probe {u, v} as well as the ϵ_n's using an STM (first done by H.F. Hess et al in 1988 in $NbSe_2$), which measures the local density of states.
3. For a vortex lattice the bound states will broaden into bands (→ Zlatko).
4. Matters are more complicated in d-wave superconductors. It is believed that due to the nodes of the d-wave order parameter there are no bound states (Wang + MacDonald, Franz + Tesanovic). Interesting recent work by UC Berkeley group on BSCCO.

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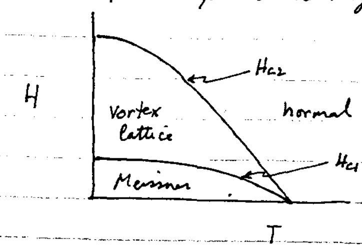
Vortices in GL Theory.

If we're not interested in the excited states, but only the order parameter and field, then we can simplify by using GL theory. Last time we discussed H_{c2}, the onset of superconductivity in the bulk. The point of first vortex entry is H_{c1}. The mean-field phase diagram looks like:



$$H_{c2} = \sqrt{2}\,\kappa\, H_c = \frac{\phi_0}{2\pi \xi^2}$$

To calculate H_{c1} we first need the line energy for a vortex. The free energy per unit length is

$$\frac{F}{L} = \frac{H_c^2}{4\pi}\lambda^2 \int d^2x \left[\frac{1}{\kappa^2}\left|(\vec{\nabla}+i\kappa\vec{A})^2\psi\right|^2 - |\psi|^2 + \frac{1}{2}|\psi|^4 + (\vec{\nabla}\times\vec{A})^2\right] \quad (9)$$

We need to solve the GL equations and substitute the result into the free energy — a difficult task. Instead we'll cheat and assume that $\psi = e^{i\theta}$ (amplitude = 1) so that

$$\frac{F}{L} \approx \frac{H_c^2}{4\pi}\lambda^2 \int d^2x \left[\frac{1}{\kappa^2}(\vec{\nabla}\theta + \kappa\vec{A})^2 - \frac{1}{2} + (\vec{\nabla}\times\vec{A})^2\right]. \quad (10)$$

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Now $\vec{\nabla}\theta = \hat{e}_\theta/\rho$ in polar coordinates; this will be a more singular contribution to F than $\vec{A}$, so we'll drop $\vec{A}$ (and absorb its effect in cut-offs):

$$\frac{F}{L} \approx \frac{H_c^2\lambda^2}{4\pi}\cdot\frac{1}{\kappa^2}\int d^2x\,(\vec{\nabla}\theta)^2;$$

Note: vortices in ^4He – $\lambda\to\infty$, λ replaced by system size.

$$\int d^2x\,(\vec{\nabla}\theta)^2 = 2\pi\int_\xi^\lambda \rho\, d\rho\,\frac{1}{\rho^2} = 2\pi \ln\kappa. \quad (11)$$

Pulling all of this together, we have

$$\frac{F}{L} = \frac{\phi_0^2}{16\pi^2\lambda^2}\left[\ln\kappa + C\right] \equiv \epsilon_0 = \text{line energy}. \quad (12)$$

with C a constant of order 1. The field of first entry is determined by looking at the Gibbs potential:

Note: $F = F(T,B)$, $G = G(T,H)$

$\frac{BH}{4\pi} = n\,\frac{\phi_0 H}{4\pi} \Rightarrow H$ is like chemical potential for vortices

$$G = n\epsilon_0 - \frac{BH}{4\pi}, \quad n = \#\text{ vortices/area} = \frac{B}{\phi_0}.$$

$$\therefore G = B\left(\frac{\epsilon_0}{\phi_0} - \frac{H}{4\pi}\right). \quad (13)$$

This becomes <0 when $H > H_{c1}$, with

$$\boxed{H_{c1} = \frac{4\pi}{\phi_0}\epsilon_0 = \frac{\phi_0}{4\pi\lambda^2}\left[\ln\kappa + C\right]} \quad (14)$$

Recall that $H_c = \frac{\phi_0}{\sqrt{2}\,2\pi\xi\lambda}$, $H_{c2} = \frac{\phi_0}{2\pi\xi^2}$

$$\therefore \sqrt{H_{c1}H_{c2}} \simeq H_c\,\left[\ln\kappa + C\right]^{\frac{1}{2}}. \quad (15)$$

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Lattice Structure, $H \lesssim H_{c2}$.

Recall that H_{c2} is determined from a linear analysis. Unfortunately this tells us nothing about the structure of the vortex state for $H < H_{c2}$ — for this we need to perform a weakly nonlinear analysis. The details are rather complicated, but clearly discussed in de Gennes' books. The result is that the free energy, to lowest order in $H_{c2}-B$, is (in conventional units)

$$F = \frac{B^2}{8\pi} - \frac{1}{8\pi}\frac{(H_{c2}-B)^2}{1+(2\kappa^2-1)\beta_A} \qquad (16)$$

where the "Abrikosov parameter" β_A is defined as

$$\beta_A \equiv \frac{\langle|\psi|^4\rangle_v}{\langle|\psi|^2\rangle_v^2}, \quad \langle\cdots\rangle_v = \text{Volume average.} \qquad (17)$$

The lowest free energy occurs for the smallest value of β_A. To determine β_A, we use a linear combination of our harmonic oscillator basis states:

$$\psi(x,y) = \sum_n C_n\, e^{inky}\, e^{-(x+x_n)^2/2\ell^2} \qquad (18)$$

$$x_n = nk\ell^2.$$

$\left.\begin{matrix} C_{n+2}=C_n \\ C_1 = iC_0 \end{matrix}\right\}$ triangular C_n's equal $\Rightarrow$ square

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A detailed analysis shows that β_A is minimized when the C_n's are chosen so that the vortices form a triangular lattice:

$\beta_A = 1.16$ triangular $\leftarrow$ this wins. Δ
$\phantom{\beta_A = {}}1.18$ square. $\square$

Notes

1. There is a very small free energy difference between Δ and $\square$. Additional terms in the GL theory (fourfold anisotropy, say) or surfaces might favor $\square$ over Δ. In fact, a triangular to square transition may occur in YBCO and in the borocarbides, as well as in the ruthenates.
2. Going beyond the weakly nonlinear analysis requires numerical work.

London model.

In the London model we ignore the spatial variation of the amplitude of the order parameter, setting it equal to a constant — we have point vortices. This is equivalent to taking $\kappa \to \infty$ in the GL equations:

$$-\frac{1}{\kappa^2}\left(\vec{\nabla} + i\kappa\vec{A}\right)^2 \psi - \psi + |\psi|^2 \psi = 0 . \qquad (19)$$

$$\nabla \times \vec{\nabla} \times \vec{A} = -\frac{1}{2i\kappa}\left[\psi^* \nabla\psi - \psi\nabla\psi^*\right] - |\psi|^2 \vec{A} \qquad (20)$$

Taking $\kappa \to \infty$, we can drop the first term in (19), so that

$$-\psi + |\psi|^2 \psi \approx 0 \Rightarrow \psi = e^{i\theta} \qquad (21)$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = -\frac{1}{\kappa}\nabla\theta - \vec{A} \qquad (22)$$

$$\Rightarrow \quad \vec{\nabla} \times \vec{\nabla} \times \vec{h} = -\vec{h}$$

or, in conventional units,

$$\lambda^2 \vec{\nabla} \times \vec{\nabla} \times \vec{h} + \vec{h} = 0 . \qquad (23)$$

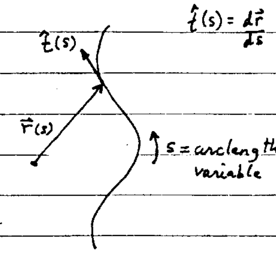
This approximation breaks down when $x = O(\frac{1}{\kappa})$ $(x \sim \xi)$ — then we need to solve the full GL equations and match them onto the London model (the far-field or "outer" solution). The net effect is to

put a δ-function on the RHS:

$$\lambda^2 \vec{\nabla} \times \vec{\nabla} \times \vec{h} + \vec{h} = \phi_0 \hat{t}(s) \delta^{(2)}[\vec{x} - \vec{r}(s)] \qquad (24)$$

$\hat{t}(s) = \frac{d\vec{r}}{ds}$

For a straight vortex,



$K_0(x) \sim \ln(\frac{1}{x})$

$\sim \frac{e^{-x}}{\sqrt{x}}$

$$h_z(r) = \frac{\phi_0}{2\pi\lambda^2} K_0(r/\lambda) \qquad (25)$$

↑ Bessel function of imaginary argument

The vortices interact via pairwise interactions (only in the London limit)

$$\frac{U}{L} = \sum_{i<j} \frac{\phi_0^2}{8\pi\lambda^2} K_0\left(|\vec{r}_i - \vec{r}_j|/\lambda\right) \quad \text{(straight vortices)} \qquad (26)$$

length $\longrightarrow L$

This energy is minimized when the $\vec{r}_i$ form a Δ lattice.

For bent vortices, we have

$$\vec{h}(\vec{x}) = \frac{\phi_0}{4\pi\lambda^2} \int_C ds\, \hat{t}(s) \frac{e^{-|\vec{x} - \vec{r}(s)|/\lambda}}{|\vec{x} - \vec{r}(s)|} \qquad (27)$$

$$U = \sum_{i<j} \frac{\phi_0^2}{(4\pi)^2\lambda^2} \int_{C_i} ds \int_{C_j} ds'\, \hat{t}_i(s) \cdot \hat{t}_j(s') \frac{e^{-|\vec{r}_i(s) - \vec{r}_j(s')|/\lambda}}{|\vec{r}_i(s) - \vec{r}_j(s')|} \qquad (28)$$

To get the total energy we need to add the line energy to this:

$$E_{Line} = \epsilon_0 \sum_i \int_{C_i} ds \qquad (29)$$

We could also use this model to describe vortices in He^4, by letting $\lambda \to \infty$ and recognizing that

$$\frac{\phi_0^2}{4\pi\lambda^2} = \frac{4\pi^2\hbar^2}{m^*}|\psi_0|^2. \qquad (30)$$

Then for a collection of vortex loops,

$$E[\{\vec{r}_i(s)\}] = \sum_i \epsilon_0 \oint_{C_i} ds$$

$$+ \sum_{i<j} \frac{4\pi^2\hbar^2|\psi_0|^2}{m^*} \oint_{C_i} ds \oint_{C_j} ds' \frac{\hat{t}_i(s)\cdot\hat{t}_j(s')}{|\vec{r}_i(s) - \vec{r}_j(s')|} \qquad (31)$$

(see lectures by Sudbo).

Vortex Dynamics

Vortex motion in superconductors is heavily overdamped; recall that

$$\eta\vec{v} = \vec{J}\times\hat{\phi}_0 = \frac{c}{4\pi}\phi_0(\vec{\nabla}\times\vec{h})\times\hat{t} \qquad (32)$$

so for the i^{th} vortex,

$$\eta\vec{v}_i = \eta\frac{\partial\vec{r}_i(s,t)}{\partial t} = \frac{c}{4\pi}\phi_0(\vec{\nabla}_i\times\vec{h})\times\hat{t}_i = -\frac{\delta E}{\delta\vec{r}_i}. \qquad (33)$$

i.e.,

$$\eta\frac{\partial\vec{r}_i}{\partial t} = -\frac{\delta E[\{\vec{r}_i\}]}{\delta\vec{r}_i}. \qquad (34)$$

How could we calculate η? Start from the time-dependent Ginzburg-Landau equations:

$$\Gamma^{-1}\left(\frac{\partial}{\partial t} + i\frac{e^*}{\hbar}\underset{\text{scalar potential}}{\Phi}\right)\psi = -\frac{\delta F}{\delta\psi^*} \qquad (35)$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \frac{4\pi}{c}\left[\vec{J}_s + \vec{J}_n\right] \qquad (36)$$

$$\vec{J}_n = \sigma\vec{E} = \sigma\left[-\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right] \qquad (37)$$

The friction coefficient η is determined by the fields generated in the core of the vortex. Use matched asymptotics to connect the "core" (inner) physics to the far-field physics (outer). For details, see ATD, Phys. Rev. B 1992.