Dorsey

TD 6/30/00 Boulder lectures – Lecture 1 (1)

Introduction to Ginzburg-Landau Theory

Interesting and important physical phenomena can occur on widely different length- and time-scales. We need to tailor the physical model to the scale of the problem which we are considering.

Microscopic (length ~ 1 nm, time ~ 1 ps)

Pairing mechanism	Many-body theory - BCS(?)
Electronic structure	Collective coordinates
	Numerical methods

Mesoscopic (?) (lengths ~ 0.1 μm, time ~ 1 ns)

Vortex structure	Ginzburg-Landau theory
Inhomogeneous states	London model
(surfaces, junctions)	numerical methods
Nucleation	techniques - match asymptotic expansions

Macroscopic (?) (lengths ~ 10 μm, time ~ 1 μs)

Flux lattice melting	London model / elasticity theory
Collective pinning	Bean model / constitutive relations
Hysteresis / critical state	Monte Carlo simulations

(2)

GL theory serves as the go-between, connecting microscopic and macroscopic behaviors. It also gives us a convenient caricature of the SC state, without detailed a priori knowledge of the microscopic mechanism.

In principle, it is possible to connect the different models:

BCS theory

↓ Gor'kov ; Gor'kov + Eliashberg (TDGL)

GL theory

↓ Large-κ · matched asymptotics

London model

↓ long wavelengths, average over many vortices

Elasticity theory

↓ Disorder

Bean (critical state) model.

What can we learn from GL theory? It's useful for properties on length scales greater than the coherence length. GL theory focusses on the order parameter (condensate), and doesn't address quasiparticle properties (excitations).

Here are some phenomena which can be profitably treated using GL theory:

Vortex structure (especially near H_{c2})
Surface/interface phenomena:
- superheating/cooling
- N/S junctions
- weak links

Nucleation
Resistive state in SC wires (phase slips)
Fluctuation phenomena (static and dynamic)

Ingredients for GL Theory.

1. Order parameter $\psi(\vec{x}) \sim \langle C_\uparrow C_\downarrow \rangle$.
For the simplest s-wave superconductors, $\psi(\vec{x})$ is a complex function:

$$\psi(\vec{x}) = |\psi(\vec{x})|\, e^{i\theta}$$

"Unconventional" superconductors require more complicated order parameters.

2. Expansion of free energy difference between normal and SC phases (near T_c) (general feature of Landau theory):

$$F = F_{n_0} + \int d^d x \left[\frac{\hbar^2}{2m^*} |\vec{D}\psi|^2 + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{h^2}{8\pi} \right] \qquad (1)$$

F_{n_0} = free energy of normal phase

m^* = mass of Cooper pair

$\alpha = \alpha_0 \left(T/T_c - 1 \right)$

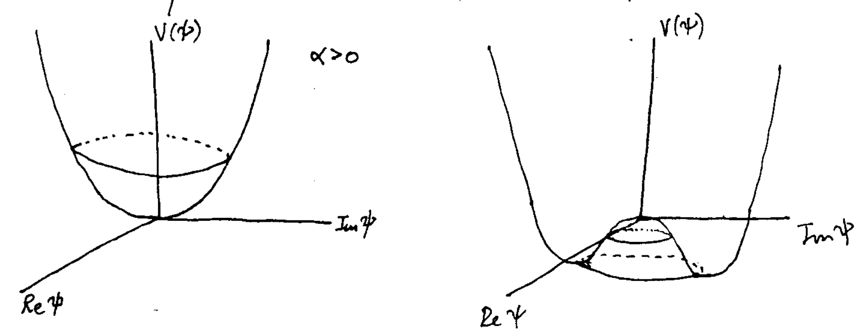
β = constant > 0

$\vec{h}$ = magnetic field $= \vec{\nabla} \times \vec{A}$,
$\vec{A}$ = vector potential.

$\vec{D} = \vec{\nabla} + \frac{ie^*}{\hbar c}\vec{A}$ = "covariant" derivative

e^* = charge of Cooper pair.

Effective potential $V(\psi) = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4$ "sombrero"



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3. $F = F[\psi, \psi^*, \vec{A}]$. Look for solutions which minimize F. Calculus of variations:

n- ange- ations

$$\frac{\delta F}{\delta \psi^*} = 0 \Rightarrow -\frac{\hbar^2}{2m^*}\left(\vec{\nabla} + \frac{ie^*}{\hbar c}\vec{A}\right)^2 \psi + \alpha\psi + \beta|\psi|^2\psi = 0 \qquad (2)$$

$$\frac{\delta F}{\delta \vec{A}} = 0 \Rightarrow \vec{\nabla}\times\vec{\nabla}\times\vec{A} = \frac{4\pi}{c}\vec{J}_s, \quad \text{(Ampère's Law)}$$

$$\vec{J}_s = -\frac{e^*\hbar}{2m^* i}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) - \frac{e^{*2}}{m^* c}|\psi|^2\vec{A} \qquad (3)$$

Boundary conditions:

le: this a stationary ion. To show it is a num, we to examine F.

$$\hat{n}\cdot\left(-i\hbar\vec{\nabla} + \frac{e^*\vec{A}}{c}\right)\psi = 0 \qquad (4)$$

$$\hat{n}\times(\vec{h} - \vec{H}) = 0$$

applied field.

Length scales: specialize to one-dimension in zero applied field. Then the first Ginzburg-Landau equation (GL1) is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \alpha\psi + \beta|\psi|^2\psi = 0. \qquad (5)$$

If there are no currents, then we can also choose ψ to be real. There are two constant solutions: $\psi_0 = 0$, or $\psi_0 = \sqrt{\frac{-\alpha}{\beta}}$ (for $\alpha < 0$)

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Note: if ψ is uniform, then the free energy difference (per unit volume) is
$f - f_{n_0} = -\frac{|\alpha|^2}{2\beta} \equiv -\frac{H_c^2}{8\pi} \Rightarrow \frac{|\alpha|^2}{\beta} = \frac{H_c^2}{4\pi}$

If we let

$$\psi = \psi_0 f, \quad \psi_0 = \sqrt{\frac{|\alpha|}{\beta}} \qquad (6)$$

$$\Rightarrow -\frac{\hbar^2}{2m|\alpha|}\frac{d^2 f}{dx^2} - f + f^3 = 0. \qquad (7)$$

We see that there is a characteristic length ξ, the coherence (or correlation) length:

$$\xi = \sqrt{\frac{\hbar^2}{2m|\alpha|}} \propto (T_c - T)^{-1/2}. \qquad (8)$$

The coherence length is the characteristic scale for spatial variations of ψ.

Now let's examine GL2. Let's assume that $T < T_c$ and that the fields are weak, so that the amplitude of ψ is a constant, $|\psi_0|$:

$$\psi = |\psi_0|\, e^{i\theta(\vec{x})} \qquad (9)$$

$$\therefore \vec{J}_s = -\frac{e^{*2}}{m^* c}|\psi_0|^2\left[\vec{A} - \frac{\hbar c}{e^*}\vec{\nabla}\theta\right] \qquad (10)$$

Defining $\vec{Q} \equiv \vec{A} - \frac{\hbar c}{e^*}\vec{\nabla}\theta \qquad (11)$

then $\vec{\nabla}\times\vec{Q} = \vec{\nabla}\times\vec{A}$, so that

$$\vec{\nabla}\times\vec{\nabla}\times\vec{Q} = -\frac{4\pi e^{*2}}{m^* c^2}|\psi_0|^2 \vec{Q} \qquad (12)$$

so we see that there is a second length scale λ, so that

$$\vec{\nabla}\times\vec{\nabla}\times\vec{Q} = -\frac{1}{\lambda^2}\vec{Q}, \quad \lambda = \sqrt{\frac{m^* c^2}{4\pi e^{*2}|\psi_0|^2}} = \sqrt{\frac{m^* c^2 \beta}{4\pi e^{*2}|\alpha|}}. \qquad (13)$$

λ is the magnetic penetration depth, and is the characteristic length scale for spatial variations of $\vec{A}$ (and $\vec{h} = \vec{\nabla}\times\vec{A}$). Since $|\alpha| \propto T_c - T$, so that $\lambda \propto (T_c - T)^{-1/2}$.

The ratio of λ to ξ is temperature independent near T_c;

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e Appendix C

$$\kappa \equiv \frac{\lambda(T)}{\xi(T)} = \sqrt{\frac{2m^{*2}c^2\beta}{4\pi e^{*2}\hbar^2}} \qquad (14)$$

κ determines the behavior of the solutions to the GL equations. We can best see this by rewriting the GL equations in dimensionless units:

$\phi_0 = \frac{\hbar c}{e^*}\cdot 2\pi = \frac{hc}{e^*}$

$$\psi = \sqrt{\frac{|\alpha|}{\beta}}\,\psi', \quad \vec{x} = \lambda\vec{x}', \quad \vec{A} = \frac{\hbar c}{e^*\xi}\vec{A}' = \frac{\phi_0}{2\pi\xi}\vec{A}', \qquad (15)$$

$$\vec{h} = \frac{\phi_0}{2\pi\xi\lambda}\vec{h}' = \sqrt{2}\,H_c\,\vec{h}'. \quad \left(\text{Recall } \frac{H_c^2}{8\pi} = \frac{|\alpha|^2}{2\beta}\right).$$

$$\boxed{-\frac{1}{\kappa^2}\left(\vec{\nabla} + i\kappa\vec{A}\right)^2\psi - \psi + |\psi|^2\psi = 0,} \qquad (17)$$

$$\boxed{\vec{\nabla}\times\vec{\nabla}\times\vec{A} = -\frac{1}{2i\kappa}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - |\psi|^2\vec{A},} \qquad (18)$$

$$\boxed{\hat{n}\cdot(\vec{\nabla} + i\kappa\vec{A})\psi\big|_s = 0, \quad \hat{n}\times(\vec{h} - \vec{H})\big|_s = 0.}$$

The free energy is

$$F - F_{n_0} = \frac{H_c^2}{4\pi}\lambda^d \int d^d x \left[\frac{1}{\kappa^2}|(\vec{\nabla} + i\kappa\vec{A})\psi|^2 - |\psi|^2 + \frac{1}{2}|\psi|^4 + (\vec{\nabla}\times\vec{A})^2\right]. \qquad (19)$$

So we see that the solutions of the GL equations can be characterized by

1. κ
2. Geometry of the domain (sample)
3. External magnetic field.

Solve using

1. Perturbation theory ($\frac{1}{\kappa}$ as a small parameter, use matched asymptotics).
2. Numerical methods (finite difference schemes + relaxational algorithms; use "link" variables

$$U = e^{i\kappa\int\vec{A}\cdot d\vec{\ell}} \qquad (20)$$

rather than $\vec{A} \Rightarrow$ "lattice gauge theory". Also finite element methods.) See Appendix A.

Two Types of Superconductor.

As you probably know, the behavior of the superconductor is radically different depending upon the value of κ; for $\kappa < \frac{1}{\sqrt{2}}$ we have type-I superconductors, and for $\kappa > \frac{1}{\sqrt{2}}$ we have type II superconductors. Let's recall the difference, by considering nucleation in the bulk of the superconductor in a magnetic field H. Since we're looking for the onset of superconductivity, we can solve the linearized version of GL 1 (let's work in conventional units):

$$-\frac{\hbar^2}{2m^*}\left(\vec{\nabla} + \frac{ie^*}{\hbar c}\vec{A}\right)^2 \psi + \alpha\psi = 0. \qquad (21)$$

Choose a gauge such that $\vec{A} = (0, Hx, 0)$, so that

$$-\frac{\hbar^2}{2m^*}\frac{\partial^2 \psi}{\partial x^2} - \frac{\hbar^2}{2m^*}\left(\frac{\partial}{\partial y} + \frac{ie^*}{\hbar c}Hx\right)^2 \psi - \frac{\hbar^2}{2m^*}\frac{\partial^2 \psi}{\partial z^2} = -\alpha\psi. \qquad (22)$$

The solutions will be plane waves in the y and z directions, so take

$$\psi(x,y,z) = e^{ik_y y} e^{ik_z z} \phi(x)$$

so that

$$-\frac{\hbar^2}{2m^*}\frac{d^2\phi}{dx^2} + \frac{1}{2} m \omega_c^2 (x + x_0)^2 \phi = \epsilon\, \phi \qquad (23)$$

$$\omega_c = \frac{e^* H}{m^* c} = \text{cyclotron frequency} \qquad (24)$$

$$x_0 = \frac{\hbar c}{e^* H} k_y = \ell^2 k_y, \quad \ell = \sqrt{\frac{\hbar c}{e^* H}} = \text{magnetic length} \qquad (25$$

$$\epsilon = -\alpha - \frac{\hbar^2 k_z^2}{2m}. \qquad (26)$$

This is the Schrödinger equation for a harmonic oscillator centered at $-x_0$, of energy ϵ. The eigenvalues are

$$\epsilon_n(x_0) = \hbar\omega_c\left(n + \tfrac{1}{2}\right) \leftarrow \text{independent of } x_0! \text{ highly degenerate.} \qquad (27)$$

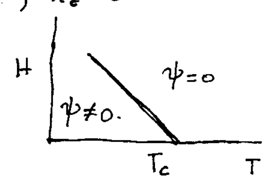
Recalling that $\alpha(T) = \alpha_0 (T - T_c)$, then a transition to a superconducting state occurs at a temperature T_{c2}:

$$\epsilon_n(x_0) = \hbar\omega_c\left(n + \tfrac{1}{2}\right) = -\alpha_0\left(T_{c2} - T_c\right) - \frac{\hbar^2 k_z^2}{2m} \qquad (28)$$

$$\therefore\ T_{c2} - T_c = -\frac{1}{\alpha_0}\left[\hbar\omega_c\left(n + \tfrac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m}\right]. \qquad (29)$$

The <u>highest</u> T_{c2} occurs when $n = 0$, $k_z = 0$:

$$T_{c2} - T_c = -\frac{1}{\alpha_0}\cdot\frac{\hbar\omega_c}{2} = -\frac{1}{\alpha_0}\,\frac{\hbar e^* H}{2m^* c} \qquad (30)$$



The eigenfunction for this lowest Landau level is

$$\psi_0 = \underset{\substack{\uparrow \\ \text{normalization}}}{A}\, e^{i x_0 y/\ell^2}\, e^{-(x+x_0)^2/2\ell^2} \qquad (31)$$

Let's rewrite (30) in terms of α:

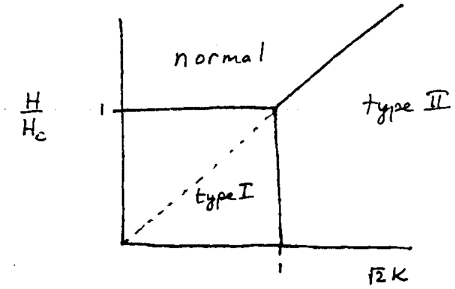
$$|\alpha| = \frac{\hbar e^* H}{2m^* c} \Rightarrow \boxed{H_{c2} = \frac{2m^*|\alpha|}{\hbar^2}\cdot\frac{\hbar c}{e^*} = \frac{\phi_0}{2\pi\xi^2}} \qquad (32)$$

Recall that

$$H_c = \frac{\phi_0}{2\pi\xi\lambda}\,\frac{1}{\sqrt{2}}\,. \qquad (33)$$

$$\therefore \quad \boxed{H_{c2} = \sqrt{2}\,\kappa\, H_c} \qquad (34)$$

If $\kappa > \frac{1}{\sqrt{2}}$, $H_{c2} > H_c \Rightarrow$ transition into an inhomogeneous (vortex) state.
If $\kappa < \frac{1}{\sqrt{2}}$, $H_{c2} < H_c \Rightarrow$ transition into the uniform (Meissner) state; H_{c2} becomes the classical spinodal for the superconductor.
Can summarize as follows:



Notes.

1. Can also show that the surface tension of an interface between the normal and superconducting phases is positive for $\kappa < \frac{1}{\sqrt{2}}$ and negative for $\kappa > \frac{1}{\sqrt{2}}$. The GL equations become self-dual at $\kappa = 1/\sqrt{2}$. See Appendix B.

2. Type I materials are usually elemental (Al, Sn, ...) – they have longer coherence lengths (less scattering). The exception is Nb, which is right on the borderline.

3. Type II materials are alloys – disorder reduces ξ without significantly changing the superfluid density (hence λ), so they have larger values of κ. The cuprate (high T_c) superconductors have $\kappa \sim 50-100$!

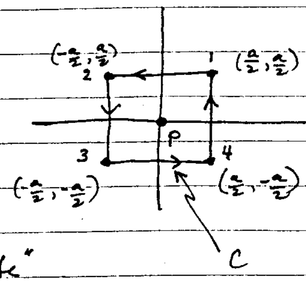
4. GL theory can be extended to treat SCs with more complicated order parameters and/or anisotropy.

5. Under some circumstances we can include dynamic effects (time dependent Ginzburg-Landau)

$$\left.\begin{aligned} \Gamma^{-1}\left(\frac{\partial}{\partial t} + i\frac{e^*\Phi}{\hbar}\right)\psi &= -\frac{\delta F}{\delta \psi^*} \\ \sigma\frac{\partial \vec{A}}{\partial t} &= -\frac{\delta F}{\delta \vec{A}} \end{aligned}\right\} \text{relaxational dynamics.} \qquad (35)$$

Appendix A – Link variables and discretization

We want to represent $\vec{\nabla}\times\vec{A}$ as a line integral. From Stokes' theorem,



$$\oint_C \vec{A}\cdot d\vec{l} = \int_A (\vec{\nabla}\times\vec{A})\cdot\hat{n}\,dS \qquad (36)$$

Let the contour C be the perimeter of the "plaquette" shown in the figure. Then $\hat{n}$ is a unit normal pointing out of the page. Now take the area A to zero (i.e., take $a\to 0$); then

$$\int_A (\vec{\nabla}\times\vec{A})\cdot\hat{n}\,dS \approx (\vec{\nabla}\times\vec{A})\cdot\hat{n}\;a^2 \qquad (37)$$

with a^2 the area of the plaquette, and where the derivative is evaluated at the center of the plaquette (point P). Thus we have

$$(\vec{\nabla}\times\vec{A})\cdot\hat{n} = \lim_{a\to 0}\frac{1}{a^2}\oint_C \vec{A}\cdot d\vec{l} \qquad (38)$$

which can be taken as a definition of $\vec{\nabla}\times\vec{A}$.

Next, consider

$$U(\vec{x}_2,\vec{x}_1) = e^{i\kappa\int_{\vec{x}_1}^{\vec{x}_2}\vec{A}\cdot d\vec{l}} \qquad (39)$$

where the line integral is along a straight path.

The U's are our "link" variables. By multiplying them together we can construct $\vec{\nabla}\times\vec{A}$:

$$\begin{aligned} U(1\to 2) &= e^{i\kappa\int_1^2\vec{A}\cdot d\vec{l}} \\ U(2\to 3) &= e^{i\kappa\int_2^3\vec{A}\cdot d\vec{l}} \\ U(3\to 4) &= e^{i\kappa\int_3^4\vec{A}\cdot d\vec{l}} \\ U(4\to 1) &= e^{i\kappa\int_4^1\vec{A}\cdot d\vec{l}} \end{aligned} \qquad (40)$$

$$\therefore\; U(1\to 2)\,U(2\to 3)\,U(3\to 4)\,U(4\to 1) = e^{i\kappa\oint_C\vec{A}\cdot d\vec{l}}$$

$$\approx e^{i\kappa a^2(\vec{\nabla}\times\vec{A})\cdot\hat{n}}$$

$$\approx 1 + i\kappa a^2(\vec{\nabla}\times\vec{A})\cdot\hat{n} + \ldots \qquad (41)$$

$$\therefore\; (\vec{\nabla}\times\vec{A})\cdot\hat{n} \approx \frac{UUUU-1}{i\kappa a^2}\,. \qquad (42)$$

We can also use the link variables to calculate the covariant derivatives. Consider

$$\partial_\mu\left[e^{i\kappa\int_{\vec{x}_1}^{\vec{x}}\vec{A}\cdot d\vec{l}}\,\psi(\vec{x})\right] = e^{i\kappa\int_{\vec{x}_1}^{\vec{x}}\vec{A}\cdot d\vec{l}}\left[\partial_\mu\psi + i\kappa A_\mu\psi\right]$$

$$\therefore\; (\partial_\mu + i\kappa A_\mu)\psi = U^{-1}\partial_\mu[U\psi] \qquad (43)$$

When we discretize, this becomes $U^{-1}\Delta_\mu[U\psi]$ with Δ_μ the discrete derivative.

The free energy can be written entirely in terms of ψ and U. The resulting

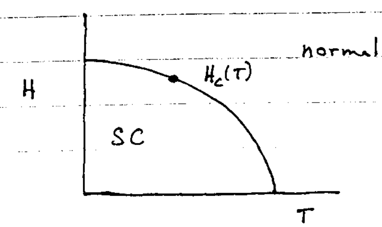
Euler-Lagrange equations can be discretized and solved using relaxational methods.

The link variables are advantageous since the curl of $\vec{A}$ and the covariant derivative are obtained by multiplication. The equations are also manifestly gauge-invariant, since the combination $e^{i\kappa\int^{x}\vec{A}\cdot d\vec{l}}\,\psi$ enters, so changes in the phase of ψ are compensated by $\kappa\int^{x}\vec{A}\cdot d\vec{l}$. More details can be found in

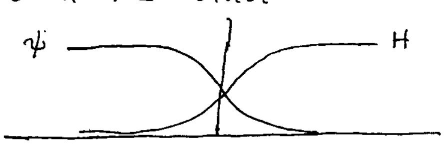
H. Frahm, S. Ullah, and A. T. Dorsey, Phys. Rev. Lett. 66, 3067 (1991).

Appendix B Surface Tension for the Normal-Superconducting Interface.

If we adjust the applied field so that $H = H_c$, we have two-phase equilibrium. There should exist an interface solution such that



$\psi = 0$ and $H = H_c$ on one side, and $\psi = \sqrt{-\frac{\alpha}{\beta}}$ and $H = 0$ on the other:



Let's search for such one-dimensional solutions. In dimensionless units, we have the GL equations, with $\psi = f e^{i\theta}$, $q = A_y + \frac{1}{\kappa}\partial_x\theta$

$$\frac{1}{\kappa^2}\frac{d^2 f}{dx^2} - q^2 f + f - f^3 = 0 \qquad (44)$$

$$\frac{d^2 q}{dx^2} - f^2 q = 0 \qquad (45)$$

$$f(\infty) = 0, \quad q'(+\infty) = \frac{1}{\sqrt{2}} \quad (H_c \text{ in conventional units})$$

$$f(-\infty) = 1, \quad q'(-\infty) = 0 .$$

It turns out that there is a unique solution of these equations for every value of κ.

Under some condition it may occur that these equations simplify – there could be a reduction in order of the equations. If this occurs, an inspired guess might be that

$$\boxed{\frac{dq}{dx} = \frac{1-f^2}{\sqrt{2}}} \qquad (46)$$

which certainly satisfies the boundary conditions. It must also satisfy (44) and (45). Let's differentiate (46) w.r.t. x:

$$\frac{d^2 q}{dx^2} = -\sqrt{2}\, f \,\frac{df}{dx} . \qquad (47)$$

This is only compatible with (45) when

$$\boxed{\frac{df}{dx} = -\frac{1}{\sqrt{2}} fq.} \qquad (48)$$

Differentiate this w.r.t. x:

$$\frac{d^2f}{dx^2} = -\frac{1}{\sqrt{2}} q \frac{df}{dx} - \frac{1}{\sqrt{2}} f \frac{dq}{dx}$$

$$= \frac{1}{2} q^2 f - \frac{1}{2} f(1-f^2)$$

$$\therefore 2\frac{d^2f}{dx^2} - q^2 f + f - f^3 = 0. \qquad (48)$$

This is equivalent to Eq. (44) when $\kappa = \frac{1}{\sqrt{2}}$! Therefore, at $\kappa = \frac{1}{\sqrt{2}}$ the GL equations reduce to 2 coupled first order equations. We can further simplify by writing

$$f = e^{-u}, \qquad (50)$$

so that (48) becomes

$$u' = \frac{1}{\sqrt{2}} q.$$

$$\therefore u'' = \frac{1}{\sqrt{2}} q' \overset{(46)}{=} \frac{1}{2}(1-e^{-2u})$$

$$\therefore \boxed{u'' + \frac{1}{2}(e^{-2u}-1) = 0.} \qquad (51)$$

This is an integrable 2nd order ODE — in principle the problem is solved by quadratures (Bogolmonyi).

The Gibbs potential for the interface solution is

$$G = F - \frac{BH}{4\pi}$$

$$= \frac{H_c^2}{4\pi} \lambda A \int_{-\infty}^{\infty} dx \left[\frac{1}{\kappa^2} f'^2 + q^2 f^2 - f^2 + \frac{1}{2} f^4 + (q')^2 - \sqrt{2}\, q' \right] \qquad (52)$$

The surface energy $\sigma = G/A - \int dx \left(-\frac{H_c^2}{8\pi}\right)$ ← energy of uniform SC phase

dimensionless units, $\sigma = \frac{H_c^2}{4\pi} \lambda \sigma'$, with

$$\sigma' = \int_{-\infty}^{\infty} dx \left[\frac{1}{\kappa^2} f'^2 + q^2 f^2 - f^2 + \frac{1}{2} f^4 + \underbrace{q'^2 - \sqrt{2}\, q' + \frac{1}{2}}_{(q' - \frac{1}{\sqrt{2}})^2} \right] \qquad (53)$$

To further simplify we need a useful identity. Consider

$$\frac{d}{dx}\left[\frac{1}{\kappa^2} f'^2 - q^2 f^2 + f^2 - \frac{1}{2} f^4 + q'^2 \right]$$

$$= 2f'\left[\frac{1}{\kappa^2} f'' - q^2 f + f - f^3 \right]$$

$$+ 2q'\left[q'' - f^2 q \right]$$

$$\overset{(44),(45)}{=} 0. \qquad (54)$$

Using the boundary conditions, we then have

$$\frac{1}{\kappa^2} f'^2 - q^2 f^2 + f^2 - \frac{1}{2} f^4 + q'^2 = \text{constant} = \frac{1}{2}.$$

$$\boxed{q^2 f^2 - f^2 + \frac{1}{2} f^4 = \frac{1}{\kappa^2} f'^2 + q'^2 - \frac{1}{2}.} \qquad (55)$$

Substituting into (53) to eliminate $q^2 f^2 - f^2 + \frac{1}{2} f^4$, we have

$$\sigma' = \int_{-\infty}^{\infty} dx \left[\frac{2}{\kappa^2} f'^2 + 2 q'^2 - \sqrt{2}\, q' \right]. \qquad (56)$$

This is a useful expression, as it shows that the surface energy depends on gradients of f and q - the surface energy is associated with the "bending" of the order parameter and vector potential.

Finally, return to Eq. (53). Integrate f'^2 by parts:

$$\sigma' = \int_{-\infty}^{\infty} dx \left[-\frac{1}{\kappa^2} f f'' + q^2 f^2 - f^2 + \frac{1}{2} f^4 + \left(q' - \frac{1}{\sqrt{2}} \right)^2 \right]$$

$$\overset{\text{use (44)}}{=} \int_{-\infty}^{\infty} dx \left[-\frac{1}{2} f^4 + \left(q' - \frac{1}{\sqrt{2}} \right)^2 \right]. \qquad (57)$$

But at $\kappa = \frac{1}{\sqrt{2}}$, $q' - \frac{1}{\sqrt{2}} = -f^2/\sqrt{2}$

$$\boxed{\therefore\ \sigma' = \int_{-\infty}^{\infty} dx \left[-\frac{1}{2} f^4 + \frac{1}{2} f^4 \right] = 0\ !} \qquad (58)$$

So the surface tension vanishes at $\kappa = \frac{1}{\sqrt{2}}$. With more work, one can show that near $\kappa = \frac{1}{\sqrt{2}}$,

$$\sigma' \approx 0.388 \left(\frac{1}{2\kappa^2} - 1 \right)$$ (A T Dorsey, Ann Phys 233, 248 (1994)),

so $\sigma' > 0$ for $\kappa < \frac{1}{\sqrt{2}}$ and $\sigma' < 0$ for $\kappa > \frac{1}{\sqrt{2}}$.

Appendix C - Anisotropic GL Theory

In an effective mass model we have

$$F = \int d^3x \left[\sum_{i=1}^{3} \frac{\hbar^2}{2m_i} \left| \left(\partial_i + \frac{ie^*}{\hbar c} A_i \right) \psi \right|^2 + \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 + \frac{1}{8\pi} (\vec{\nabla} \times \vec{A})^2 \right]. \qquad (59)$$

The coherence lengths are

$$\xi_i = \sqrt{\frac{\hbar^2}{2m_i |\alpha|}} \qquad (60)$$

and the penetration depths are

$$\lambda_i = \sqrt{\frac{m_i c^2 \beta}{4\pi e^{*2} |\alpha|}}. \qquad (61)$$

Notice that $\xi_i \lambda_i$ is *independent* of the effective mass.

Now

$$\frac{H_c^2}{8\pi} = \frac{|\alpha|^2}{2\beta} = \frac{1}{8\pi} \frac{\phi_0^2}{2 \cdot (2\pi)^2 \lambda_i^2 \xi_i^2} \qquad (62)$$

for any $\lambda_i \xi_i$; i.e.,

$$H_c = \frac{\phi_0}{\sqrt{2}\ 2\pi\ \lambda_i \xi_i}. \qquad (63)$$