Quantum Mechanics of Vortices: Duality and Fractionalization

with Matthew Fisher (ITP, UCSB)
Chetan Nayak (UCLA)
T. Senthil (ITP)

Outline

1. Vortices and Duality for *Bosons*
2. Conventional physics from unconventional Vortex field theory
3. Fractionalization, vortex pairing, and spin liquids
4. Z_2 gauge theory
5. Fermionic quasiparticles

References:

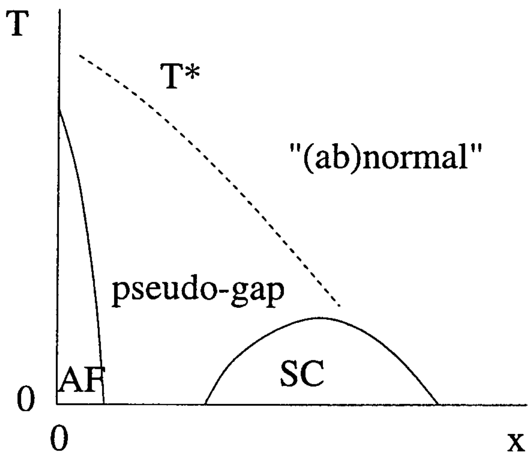
1. L. Balents, M. P. A. Fisher and C. Nayak, Int. J. Mod. Phys. B, (1998);
2. L. Balents, M. P. A. Fisher, and C. Nayak, Phy. Rev. B **60**, 1654 (1999);
3. L. Balents, M. P. A. Fisher, and C. Nayak, Phys. Rev. B, **61**, 6307 (2000);
4. M. P. A. Fisher, cond-mat/9806164
5. T. Senthil and M. P. A. Fisher, cond-mat/9910224

Particle-Wave Dualism

- At the heart of quantum mechanics are uncertainty relations, which prescribe the competition between the wave and particle nature of matter
  - Momentum-position $[x, p] = i$
  - Number-phase $[n, \phi] = i$
- Usually take particle viewpoint:
  - Superconductor = condensate of Cooper pairs
  - Superfluid = condensate of Helium atoms
  - Solid Helium = array of Helium atoms
- In "wave" picture, the phase is fundamental, and the "quanta" are vortices
  - Superconductor/superfluid = vortex vacuum
  - Solid Helium = ???
- Advantages of wave viewpoint:
  - Emphasizes delocalization of charge: strongly quantum view
  - Convenient for studying vortices
  - We will see that vortex variables provide a simple means to study exotic fractionalization
- Disadvantage of wave viewpoint:
  - Vortex Hamiltonian is not microscopically known
  - Hard to do precise quantitative calculations

Motivations

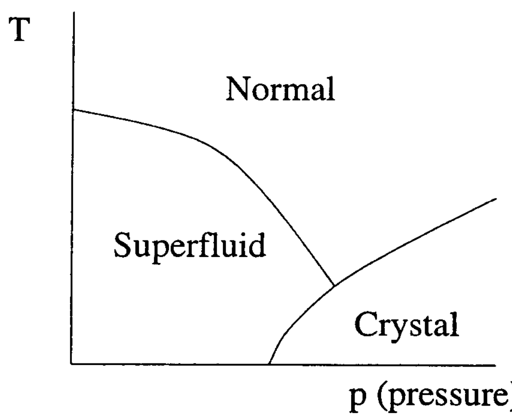
- Cuprates – Phase Diagram:



  - Quantum Antiferromagnets ⇒ "Spin Liquid"?
    = Mott Insulator with other than an even number of electrons (e.g. odd) per unit cell without any broken symmetries
    = Non-magnetic ground state of a half-integer spin model without any broken symmetries.
  - Pseudo-gap = strange insulating state
    * apparent gap for quasiparticle excitations *outside* the superconducting state
    * quasiparticle peaks are not "sharp"
  - Non-Fermi Liquid behavior in "normal" state
- Theoretical – why vortices?
  - Want to describe "(ab)normal" states in which quasiparticle *spectrum* is very similar to that in superconductor.
  - Most significant change in quasiparticle spectrum on departing the superconductor is *width* of spectral features.
  - Pursue quantum analogs of Kosterlitz-Thouless transition.
  - Only truly accepted two-dimensional state with "anomalous" excitations is the Laughlin liquid in FQHE – vortices pay a crucial role

Superfluids

- Consider bosonic superfluid
- Dualism evident from Kosterlitz-Thouless transition (c.f. Girvin)
  - $T < T_{KT}$: No mobile vortices $\Rightarrow$ superfluidity = Cooper pairs mobile
  - $T > T_{KT}$: Lots of vortices $\Rightarrow$ normal fluid = Cooper pairs diffuse
- *example*: ^{4}He



Q: why is the S-C boundary vertical at T=0? What is its shape?

Uncertainty principle: $[N, \phi] = i$

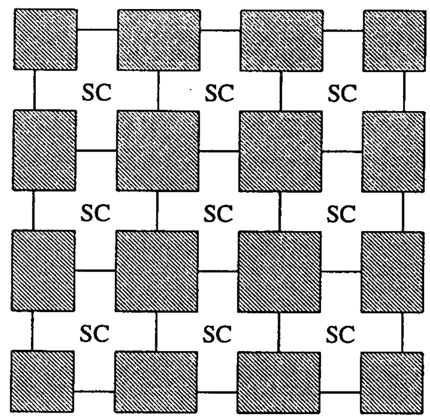
- Superfluid: bosons are *condensed* $\rightarrow$ ϕ is fixed, density N is spread out
- Crystal: bosons are *localized* $\rightarrow$ N is fixed, ϕ is uncertain

Crystal is indeed a state of strongly fluctuating phase

- Goal: develop a dual view of the quantum superfluid-insulator transition
  - Focus on phases, not critical phenomena

Quantum phase model

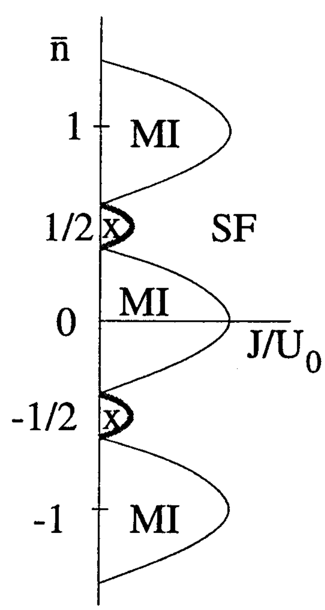
- Developed for Josephson junction arrays:



- Hamiltonian:

$$H = \sum_{ij} \frac{U_{ij}}{2}(n_i - \overline{n})(n_j - \overline{n}) - \sum_{\langle ij \rangle} J \cos(\varphi_i - \varphi_j)$$

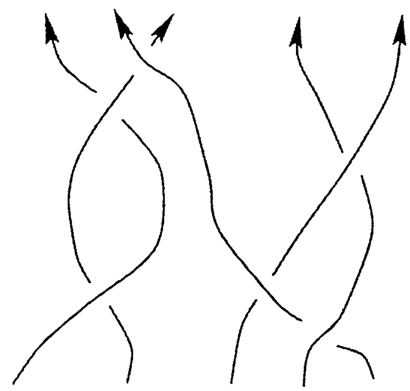
- Commutation relation: $[\varphi_i, n_j] = i\delta_{ij} \Rightarrow e^{-i\varphi}|n\rangle = |n+1\rangle$.
- Schematic Phase Diagram (nearest-neighbor interactions):



Duality

- There are many formulations of duality (c.f. Sudbø)
  - Lattice (Villain) duality
  - Here: Continuum "field theory" duality
- Physical basis: Feynmann world-line formulation of QMs:

$$Z = \int e^{-\int \mathcal{L}}$$



- Lagrangian:

$$\mathcal{L} = T - V = \frac{m}{2\overline{n}}|\vec{J}|^2 - \frac{(n-\overline{n})^2}{2\kappa_0}$$

- Continuity equation:

$$\partial_t n + \vec{\nabla}\cdot\vec{J} = 0$$

- Solution:

$$J^\mu = (n, \vec{J}) = \epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda/2\pi \equiv \partial \times a/(2\pi)$$

- Dual "electromagnetism"

$$\mathcal{L}_{\text{em}} = \frac{1}{8\pi^2\kappa_0}\left[|\vec{e}|^2 - (b-\overline{n})^2\right] - A_{\text{ext}}\cdot\partial\times a/2\pi$$

  - "electric" field $e_i = v_c^{-1}[\partial_i a_0 - \partial_0 a_i] = 2\pi\sqrt{\kappa_0 m/\overline{n}}\epsilon_{ij}J_j$
  - "magnetic" field $b = \epsilon_{ij}\partial_i a_j = 2\pi n$

Duality - 2

- Charge discreteness:

$$\int d^2\mathbf{r}\, n(\mathbf{r}) = \frac{1}{2\pi}\int d^2\mathbf{r}\, \vec{\nabla}\times\vec{a} = N$$

- Flux quantization! $\mathcal{L} = \mathcal{L}_{\rm em} + \mathcal{L}_\Phi$:

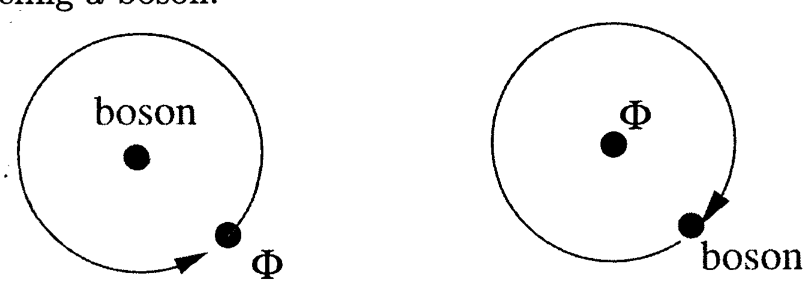
$$\mathcal{L}_\Phi = \frac{1}{2}|(\partial_t - ia_0)\Phi|^2 - \frac{v_c^2}{2}|(\vec{\nabla} - i\vec{a})\Phi|^2 - V_\Phi(|\Phi|)$$

- Ginzburg-Landau potential

$$V_\Phi(X) = -rX^2 + uX^4$$

  - Want $r > 0$ to make lines discrete but fluctuations could still drive system into "disordered" phase of Ginzburg-Landau model.
  - Detailed form of $\mathcal{L}_\Phi$ (e.g. V_Φ) determined by short-distance physics of interaction potential of bosons and lattice potential

- Physical meaning of Φ
  - $\mathcal{L}_\Phi$ looks like the Lagrangian for a relativistic "charged" particle (and anti-particle!) in a "magnetic field" b.
  - The original charges (bosons), appear as fluxes, since $b(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}')$ for a boson located at $\mathbf{r}'$.
  - The Φ particles thus pick up an Aharonov-Bohm phase of $\oint \vec{a}\cdot d\vec{l} = 2\pi$ upon encircling a boson:



  $\Rightarrow$ Boson phase winds by 2π upon encircling Φ particle
  - Φ particles are vortices!

Dual Formulation

- World-lines
  - Lagrangian $\mathcal{L} = T - V = \frac{m}{2\overline{n}}|\vec{J}|^2 - \frac{(n-\overline{n})^2}{2\kappa_0}$
  - Gauge field $J^\mu = (n, \vec{J}) = \epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda / 2\pi \equiv \partial \times a/2\pi$
- Dual theory
  - Lagrangian $\mathcal{L}_{\mathrm{dual}} = \mathcal{L}_{\mathrm{em}} + \mathcal{L}_\Phi$
  - Dual "electromagnetism"

$$\mathcal{L}_{\mathrm{em}} = \frac{1}{8\pi^2\kappa_0}\left[|\vec{e}|^2 - (b-\overline{n})^2\right] - A_{\mathrm{ext}} \cdot \partial \times a/2\pi$$

  - Vortices:

$$\mathcal{L}_\Phi = \frac{1}{2}|(\partial_t - i a_0)\Phi|^2 - \frac{v_c^2}{2}|(\vec{\nabla} - i\vec{a})\Phi|^2 - V_\Phi(|\Phi|)$$

  - Fields:
    * "electric" field $e_i = v_c^{-1}[\partial_i a_0 - \partial_0 a_i] = 2\pi\sqrt{\kappa_0 m/\overline{n}}\epsilon_{ij}J_j$
    * "magnetic" field $b = \epsilon_{ij}\partial_i a_j = 2\pi n$
    * Ginzburg-Landau potential $V_\Phi(X) = -rX^2 + uX^4$
    * "dual phase" $\Phi = |\Phi|e^{i\theta}$
- Table

Original	Dual
$\cdot$ charge ± 1 boson	$\odot$ $\Delta\theta = \pm 2\pi$ dual vortex $\int b\, d^2\mathbf{r} = \pm 2\pi$
$\odot$ $\Delta\varphi = \pm 2\pi$ vortex	$\cdot$ dual charge ± 1 Φ boson
current J^μ	flux $(\partial \times a)^\mu / 2\pi$
logarithmic vortex interaction	logarithmic 2D Coulomb potential
superfluid	insulator (dielectric)
insulator (crystal)	superconductor (vortex lattice)
Meissner effect	charge gap

Superfluid phase

- Fundamental properties of superfluid phase are:
  - Meissner effect: external electromagnetic flux (rotation) is expelled
  - Off-Diagonal Long-Range Order (ODLRO) – how is this recovered in dual picture?
  - Vortex excitations cost finite core energy and logarithmic cost for long-range currents
- Expect to recover these in *disordered* phase of GL theory = vortex vacuum $r = -m^2 < 0 \Rightarrow$ can "integrate out" massive Φ field
- Meissner effect:
  - (electro)magnetic response of superfluid:
    * $\mathcal{L}_{\text{sf}} = -\frac{\kappa_0 v_c^2}{2}|\vec{\nabla}\varphi - \vec{A}|^2$.
    * Equation of motion: $\varphi = \vec{\nabla} \cdot \vec{A}/\nabla^2$

$$\Rightarrow \mathcal{L}_{\text{eff}} = -\frac{\kappa_0 v_c^2}{2}|P_T \vec{A}|^2$$

  - dual response :
    * $\mathcal{L}_{\text{dual}} = \frac{1}{8\pi^2\kappa_0}|\vec{e}|^2 - \vec{A} \cdot v_c \vec{e}/2\pi$
    * equation of motion: $\vec{e} = -2\pi v_c \kappa_0 P_T \vec{A}$

$$\Rightarrow \mathcal{L}_{\text{eff}} = -\frac{\kappa_0 v_c^2}{2}|P_T \vec{A}|^2 \qquad \text{identical to superfluid response!}$$

- Vortex excitations:
  - Vortex spectrum can be read off from $\mathcal{L}_{\Phi}$:

$$\omega_v^2(k) = m^2 + v_c^2 k^2$$

    $\Rightarrow$ minimum energy "gap" m for vortex
  - Vortices experience $2+1$–dimensional fictitious Coulomb force due to "electromagnetic" gauge field a
    * *logarithmic* Coulomb force in two dimensions!

Vortex dynamics

- For low-energies, $vk \ll m$, have non-relativistic limit. Recall

$$m\frac{d\vec{u}}{dt} = q(\vec{E} + \frac{\vec{u}}{c} \times B\hat{z})$$

- Rewrite:
  - $q \to 1$
  - no normal fluid $\Rightarrow \vec{E} \to \hat{z} \times \vec{J} = \overline{n}\hat{z} \times \vec{V}_s$
  - $B/c \to \overline{n}$

$$m\frac{d\vec{u}}{dt} = \overline{n}\hat{z} \times (\vec{V}_s - \vec{u})$$

- In superfluid literature, the total force is known as Magnus force.
- In superconductivity, the first and second terms are called Hall and Lorentz forces, respectively.
- Additional dissipative and non-Galilean invariant terms arise from other degrees of freedom (normal fluid, quasiparticles) and coupling to the lattice.
  - Can study these phenomena via current-current couplings: $\mathcal{L}_{\text{int}} = -gj_n \cdot \partial \times a$.

Insulating phase

- Fundamental properties of insulator
  - charge gap (commensurate boson density)
  - dielectric electromagnetic response (commensurate)
  - spontaneous spatial periodicity
  - single-particle gap
- Expect insulator has *lots* of vortices (c.f. Girvin: vortices disrupt phase)
  - At $T = 0$, these should *Bose condense* $\langle\Phi\rangle = \Phi_0 \neq 0$
  - Note: needn't have non-zero *vorticity*, since $V = \mathrm{Im}(\Phi^*(\partial_t - ia_0)\Phi) = 0$ is consistent with $\Phi = 0$
- Charge gap: consider extreme case of $\overline{n} \in \mathcal{Z}$ in QPM
  - Zero dual flux $\overline{b} = 0 \Rightarrow$ Meissner state
  - chemical potential $\rightarrow$ dual external field h
    $\Rightarrow$ Meissner effect is charge gap $\overline{b}(h < h_c) = 0$!
- Dielectric response: in dual SC, choose Φ_0 real gauge
  - $\mathcal{L}_{\text{eff}} = \frac{1}{2}M^2(a_0^2 - v_c^2|\vec{a}|^2) - A_{\text{ext}} \cdot \partial \times a/2\pi$
  - integrate out $a^\mu \Rightarrow$
    $\mathcal{L}_A = \frac{\epsilon_0}{2}|\vec{E}|^2 - \frac{B^2}{2\mu_0} \qquad \epsilon_0 = (2\pi M v_c)^{-2}, \mu_0 = (2\pi M)^2$
- Spatial periodicity: $\overline{b} = \overline{n} \Rightarrow$ Abrikosov lattice (minimal unit cell = one boson)
- Single-particle gap:
  - Consider single-boson correlation function $C(\tau) = \langle a(0,\tau)a^\dagger(0,0)\rangle$
  - In imaginary time ($\tau = it$), expect $C(\tau) \sim e^{-\Delta_1|\tau|}$, with $\Delta_1 =$ SPG.
  - In dual picture, $C(\tau) = e^{-S_{\text{eff}}(1\,\text{vortex})}$
    - flux confinement $\Rightarrow \Delta_1 =$ dual fluxon core energy
- Appears that $\langle\Phi\rangle \neq 0$ is an XY *order parameter* for the insulator
  - Does this mean that there is a finite-temperature phase transition separating the insulator from the "normal" state? (assume the density $\overline{n} \in \mathcal{Z}$)

Conventional vs. unconventional insulators

- Most fundamental property of insulator is *charge* quantization
  - c.f uncertainty principle
  - Formally, there exists a excitation with a given momentum and the minimal (but non-zero) charge : "quasiparticle"
  - e.g. in band insulator, quasiparticles have charge e
- In ordinary Bose solid, expect the quasiparticles are *bosons* with the fundamental (single) boson charge
  i.e. the smallest boson number is 1 helium atom!
- How to see this in vortex theory?
  - Charged excitations are dual *vortices* in Φ:
    – $\langle\Phi\rangle = \Phi_0 e^{i\theta(\mathbf{r})}$
  - London $a^\mu = \partial_\mu \theta \Rightarrow N = (2\pi)^{-1} \int b\, d^2\mathbf{r} = (2\pi)^{-1} \oint \vec{\nabla}\theta \cdot d\vec{\ell} \in \mathcal{Z}$
  - $\Delta\theta = \pm 2\pi$ vortex vortices carry boson number ± 1
- What if vortices condensed in *pairs*? unbroken Ising symmetry
  - Formally, $\Phi_2 = \langle\Phi^2\rangle \neq 0$ but $\langle\Phi\rangle = 0$
  - Energetics? Come to this question of "mechanism" later – focus now on phenomenology (c.f. GL theory of SC is independent and more general than BCS theory)
  - Gauge invariance "halves" the dual flux quantum:
    $\mathcal{L}_2 = \kappa_\mu |(\partial_\mu - 2a_\mu)\Phi_2|^2 + \cdots$
  - New topological defects $\Phi_2 = \Phi_0 e^{i\theta}$
    – fractional charge $N = \frac{1}{4\pi} \oint \vec{\nabla}\theta \cdot d\vec{\ell} = \pm 1/2$ "chargons"
- General "Dirac quantization condition": $Q_c Q_v = \frac{Q}{e} \frac{\phi}{(hc/e)} = 1$
  - FQHE Laughlin $Q_v = 3 \Rightarrow Q_c = 1/3$
  - BCS $Q_c = 2 \rightarrow Q_v = 1/2$ ($\phi_0^{sc} = hc/2e$)

Spin-1/2 XXZ Model

- Hamiltonian :

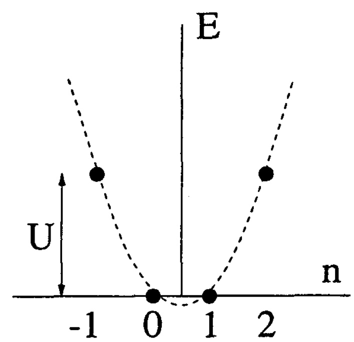
$$H = \sum_{\langle ij \rangle} \left[-J_{xy} \left(S_i^x S_j^x + S_i^y S_j^y \right) + J_z S_i^z S_j^z \right]$$

- Hard-core bosons:
  - ops: $S^z = a^\dagger a - 1/2, \qquad S^+ = (S^x + iS^y) = a^\dagger, \qquad S^- = a$
  - Hamiltonian:

$$H = \sum_{\langle ij \rangle} -\frac{1}{2} J_{xy}(a_i^\dagger a_j + a_j^\dagger a_i) + J_z(a_i^\dagger a_i - 1/2)(a_j^\dagger a_j - 1/2)$$

- Rotor approximation:

$$a \to e^{i\varphi}$$
$$a^\dagger \to e^{-i\varphi}$$
$$a_i^\dagger a_i \to n_i$$



  - Hamiltonian:

$$H = \sum_{\langle ij \rangle} -J_{xy} \cos(\varphi_i - \varphi_j) + J_z(n_i - 1/2)(n_j - 1/2) + U \sum_i n_i(n_i - 1)$$

  - Zero magnetization $\Rightarrow \overline{n} = 1/2$

- Duality:
  - Lattice: vortices sit on *plaquettes*
  - $\overline{n} = 1/2 \Rightarrow \pi -$ dual flux per site (dual plaquette)
  - Fully-frustrated XY gauge model: $H = H_{xy} + H_g$

$$H_{\text{xy}} = -\sum_n \tilde{J}_n \sum_{\langle ij \rangle} \cos[n(\theta_i - \theta_j - a_{ij})] + \tilde{U} \sum_i V_i^2$$

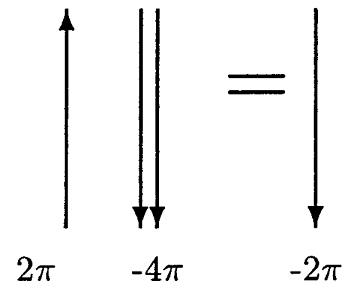
$$H_{\text{g}} = K \sum_{\Box} |\nabla \times a_{ij} - \pi|^2 + \frac{1}{4\pi\kappa_0} \sum_{\langle ij \rangle} e_{ij}^2$$

Spin-1/2 XXZ Model – Phases

- Vortex vacuum: $\langle e^{in\theta_i}\rangle = 0$
  - superfluid $\langle a\rangle = \psi \neq 0 \leftrightarrow \langle S^-\rangle = \psi$ XY magnet
- Single-vortex condensate $\langle e^{i\theta}\rangle \neq 0$
  - "insulator" $\langle a\rangle = 0$
  - Dual vortex lattice $n(\mathbf{r}) \sim (-1)^{x+y} \Rightarrow S_i^z \sim (-1)^{i+j}$ Ising antiferromagnet
- Double-vortex condensates? $\langle e^{2i\theta}\rangle \neq 0$
  - XY paramagnet $\langle a\rangle = 0$
  - Dual flux quantum *halved* $\Rightarrow 2\pi$ flux/plaquette $\Rightarrow$ Spin liquid $\langle S^z\rangle = 0$
  - vortex-vortices carry spin-1/2: "spinons"
- Real phase diagram of XXZ model?
  - $J_{xy} > J_z$: XY (anti-)ferromagnet
  - $J_z > J_{xy}$: Ising anti-ferromagnet
  - $J_{xy} = J_z$: Heisenberg antiferromagnet ($SU(2)$)
- Additional interactions are needed to access spin liquid!
  - will return to these later if there is time

What happens to the $\pm 2\pi$ vortex in the spin liquid?

- Apparently, $\pm 4\pi$ vortices cannot "screen" $\pm 2\pi$ vortex



- Treat vortex pairing by "BCS" theory

$$\mathcal{L} \sim \kappa_\mu |(\partial_\mu - i a_\mu)\Phi|^2 + m^2|\Phi|^2 - u\left[\Phi_2(\Phi^*)^2 + \Phi_2^*\Phi^2\right] + \cdots$$

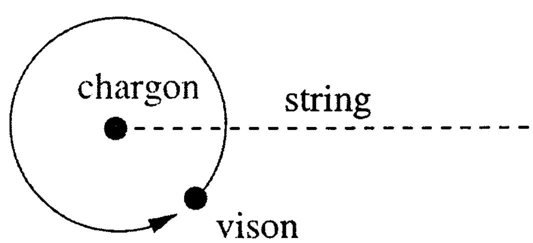
  - Similar to Bogoliubov deGennes Hamiltonian
  - "Pair field" Φ_2 mixes vortices and anti-vortices!

- "Neutralize" the $\pm 2\pi$ vortices:
  - Amplitude and phase: $\Phi_2 = |\Phi_2| e^{i\theta_2}$
  - Rescale $\Phi = \tilde{\Phi} e^{i\theta_2/2}$
  - Important: exponential factor is *double-valued* for "chargon" $\oint \vec{\nabla}\theta_2 \cdot d\vec{\ell} = \pm 2\pi$

- After tranformation, get *Ising* fields $\tilde{\Phi} = \Phi_a + i\Phi_b$

$$\mathcal{L} \sim \kappa_\mu |(\partial_\mu - i j_\mu)\tilde{\Phi}|^2 + m^2|\tilde{\Phi}|^2 - u|\Phi_2|(\Phi^2 + (\Phi^*)^2)$$
$$\mathcal{L} = \kappa_\mu[(\partial_\mu \Phi_a)^2 + (\partial_\mu \Phi_b)^2] + (m^2 \pm u|\Phi_2|)\Phi_{a/b}^2 + \text{ints.}$$

- $\Phi_{a/b}$ represent (real) massive *Ising* excitations
  - "Ising vortex" = vison
  - vison and chargon have statistical interactions!

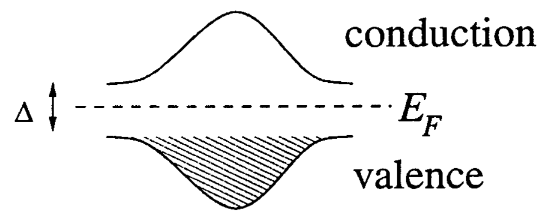


Quasiparticles and vortices in s-wave superconductors

- There are *many* interesting issues (c.f. Franz+Tesanovic) regarding spectra of quasiparticles with *static* vortices within the Bogoliubiv deGennes approach
  - Quasiparticle states of vortex and STM spectra
  - "Zero modes" of vortices in unconventional SCs (e.g. $p_x + ip_y$)
  - Excitation spectra of vortex lattice
  - Vortex damping by quasiparticle motion in unconventional SCs
  - Localization of quasiparticle states by disorder
- Focus here on *general structure*
  - What are the fundamental properties of quasiparticles, viewed as *elementary excitations* of a SC
    - statistics, interactions with vortices?
    - spectral functions
  - Try to extend the dual description of *insulators* as vortex condensates to understand the nature of insulating quasiparticles

From band insulator to SC

- Consider a 2D system of electrons in a periodic potential with 2 electrons per unit cell and *attractive* interactions
- *strong* crystal potential and *weak* attraction ⇒ band insulator:



  - weak interactions cannot destroy band gap
  - excitations:
    * electrons : $Q = \pm e$, $s = 1/2$, fermion
    * excitons: $Q = 0$, $s = 0, 1$, boson
    * pairs: $Q = \pm 2e$, $s = 0, 1$, boson
- Increased attractive interaction ⇒ lower energy of pairs
- Get transition to *superconductor* (could analyze using BCS theory)
- How does this work in *reverse* using *duality*
  - Cooper pairs = bosons ⇒ can use duality
  - Elementary ($2\pi = hc/2e$) vortex condensation
    * Indeed gives insulator
    * Gives charge of 1 boson/unit cell = 2 electrons/unit cell
    * Charge excitations have $Q = \pm 2e$ (single boson)
    * Where are the electrons?? ($Q = \pm e$,$s = 1/2$)
  - If condensed $4\pi = hc/e$ vortices, get $Q = \pm e$ "chargons"
  - But for spin, need to consider *quasiparticles*

Z_2 gauge theory

- Spin-1/2 quantum XY model (rotors):

$$H = -J \sum_{\langle ij \rangle} \cos(\varphi_i - \varphi_j) + U \sum_i (n_i - 1/2)^2$$

- Split the boson:
  - Spin operator $S_i^+ = e^{i\varphi_i} \equiv \left(e^{i\phi_i}\right)^2 \equiv \left(b_i^\dagger\right)^2$
  - Number operator $N_i = 2n_i$

$$H = -J \sum_{\langle ij \rangle} \cos(2\phi_i - 2\phi_j) + \frac{U}{4} \sum_i (N_i - 1)^2$$

  - Constraint N_i is even

- Path integral $Z = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\, (e^{-\epsilon H})^{\beta/\epsilon}$
  - Partition function $Z = \int \frac{d\phi_i(\tau)}{2\pi} \sum_{N_{i\tau}}' e^{-S}$
  - Action
    $S = \sum_{\langle ij \rangle \tau} -\epsilon J \cos(2\phi_i - 2\phi_j) + \frac{U\epsilon}{4} \sum_{i\tau} (N_{i\tau} - 1)^2 + iN_{i\tau}(\phi_{i\tau+\epsilon} - \phi_{i\tau})$

- Implement constraint: $\sum_{N_{i\tau}}' = \sum_{N_{i\tau}} \frac{1}{2} \sum_{\sigma_{i\tau}=\pm 1} e^{iN_{i\tau}\frac{\pi}{2}(1-\sigma_{i\tau})}$

- Split cosine: $e^{J\epsilon \cos(2\phi_i - 2\phi_j)} \approx \sum_{\sigma_{ij}=\pm 1} e^{t_{x/y}\sigma_{ij}\cos(\phi_i - \phi_j)}$, $\qquad t_{x/y} \approx \sqrt{2J\epsilon}$

- Sum boson numbers:
  $\sum_{N_{i\tau}} e^{-\frac{U\epsilon}{4}(N_{i\tau}-1)^2} e^{iN_{i\tau}[\frac{\pi}{2}(1-\sigma_{i\tau})+\phi_{i\tau+\epsilon}-\phi_{i\tau}]} \approx e^{-S_B^{(i\tau)}} e^{t_\tau \sigma_{i\tau}\cos(\phi_{i\tau+\epsilon}-\phi_{i\tau})}$
  - Berry's phase $S_B^{(i\tau)} = -i[\frac{\pi}{2}(1-\sigma_{i\tau}) + \phi_{i\tau+\epsilon} - \phi_{i\tau}]$, $\qquad t_\tau \approx 1/\epsilon U$

- Final gauge theory $Z \propto \int [d\phi_{i\tau}/2\pi] \sum_{\sigma_{i\mu}=\pm 1} \exp(-S_{Z_2})$

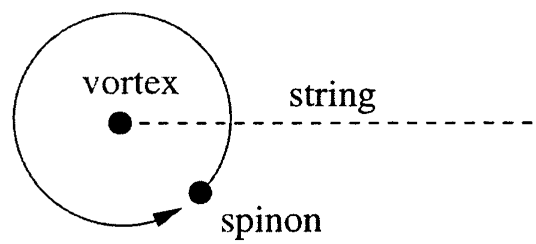
$$S_{Z_2} = -\sum_{i\mu} t_\mu \sigma_{i\mu} \cos(\phi_{i+\mu} - \phi_i) + S_B - K \sum_{\Box} \prod_{\Box} \sigma_{ij}$$

Quasiparticles and vortices in s-wave superconductors

- BCS reduced Lagrangian: $\mathcal{L} = \mathcal{L}_{\rm qp} + \mathcal{L}_{\varphi} + \mathcal{L}_{\rm int}$
  - Quasiparticles: $\mathcal{L}_{\rm qp} = c^\dagger_\alpha [i\partial_t - \nabla^2 + U_{\rm p}(\mathbf{r}) - \mu] c_\alpha$
  - Phase: $\mathcal{L}_\varphi = \frac{\kappa_\mu}{2}(\partial_\mu \varphi)^2$
  - Interaction: $\mathcal{L}_{\rm int} = |\Delta| e^{i\varphi} c_\uparrow c_\downarrow + \text{h.c.}$
- Spinons: $f_\alpha = e^{i\varphi/2} c_\alpha$

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_\varphi + \frac{g}{2} J_\mu \partial_\mu \varphi$$

  - spinon lagrangian $\mathcal{L}_f = f^\dagger_\alpha [i\partial_t - \nabla^2 + U_{\rm p}(\mathbf{r}) - \mu] f_\alpha + |\Delta|(f_\uparrow f_\downarrow + \text{h.c.})$
  - "quasiparticle current" $J_0 = f^\dagger_\alpha f_\alpha$, $\vec{J} = i f^\dagger_\alpha \vec{\nabla} f_\alpha + \text{h.c.}$
  - *bare* "Doppler shift" interaction $g = 1$
- Statistical interactions



  - ordinary flux $\pm hc/2e$ vortex: $\frac{1}{2}\oint \nabla\varphi \cdot d\ell = \pm\pi$
  - spinon ψ obeys antiperiodic boundary conditions around vortex
  - spinons and vortices are relative semions
- $\pm hc/2e$ vortex condensation $\Rightarrow$ confinement