Lecture II

- Reentrance problems in perturbation theory : need to sum all orders.

- Resummation technique:
  - solution by maximum section
  - solution by matrix diagonalization

- Tuning T_c suppression with other parameters
  - magnetic impurities $\frac{1}{\tau_S}$ ✓
  - magnetic field H ✓
  - width in wires a ✓
  - electromagnetic environment
  - spin-orbit scattering

Quick Summary of Lecture I

- Disorder in metals causes electrons to move diffusively, so Coulomb interaction less effectively screened.

- In equation for T_c,

$$\underline{T_c = 1.13\, \omega_D \exp[-1/N(0)[\lambda - \mu^*]]}$$

find that DoS $N(0)\downarrow$, pseudoptl $\mu^*\uparrow$.

- Simple-minded predictions for 2D films

$$\ln\left(\frac{T_c}{T_{c0}}\right) \simeq \begin{cases} -\frac{R_\square}{R_0}\left\{\ln^3\left(\frac{1}{2\pi T_c\tau}\right) + \ln^2\left(\frac{1}{2\pi T_c\tau}\right)\right\} & \text{(featureless potential)} \\ -\frac{R_\square}{R_0}\left\{\ln^2\left(\frac{1}{2\pi T_c\tau}\right)\ln(D\varkappa_2^2\tau) + \ln^3\left(\frac{1}{2\pi T_c\tau}\right)\right\} & \text{(screened Coulomb)} \end{cases}$$

- Need honest calculation to check if this is true.

Reentrance Problems in PT

- Basic PT predicts exponential decay of T_c with $R_\square$, so T_c never suppressed to zero:

$$\ln\left[\frac{T_c(R_\square)}{T_{c0}}\right] = -\frac{1}{3}\frac{R_\square}{R_0}\ln^3\left(\frac{1}{2\pi T_{c0}\tau}\right)$$

- "Self-consistent" version involves solution of cubic eqn $x = \frac{1}{3}t(x+\beta)^3$

$$\Rightarrow x = -\beta + \frac{2}{\sqrt{t}}\sin\left[\frac{1}{3}\sin^{-1}\left(\frac{3\sqrt{t}\beta}{2}\right) + \frac{2\pi n}{3}\right] \qquad n = 0, 1, 2$$

- There are 3 real solutions, but only $n=0$ version gives $T_c(R_\square = 0) = T_{c0}$, and so this is physical correct sol^n [$n=1 \Rightarrow T_c(R_\square=0) = 0$
  $n=2 \Rightarrow T_c(R_\square=0) = \infty$]

- The $n=0$ and $n=1$ solutions meet at the reentrance point $t = 4/9\beta^2$ $x = \beta/2$ beyond which no physical soln exists.

- For $\beta = 6$ this is $R_\square \simeq 2k\Omega$, $T_c/T_{c0} \sim 0.05$

- Reentrance means still can't get to zero T_c.

Raffy et al $T_c(R_\square)$ Data

PRB28 6607 (1983)

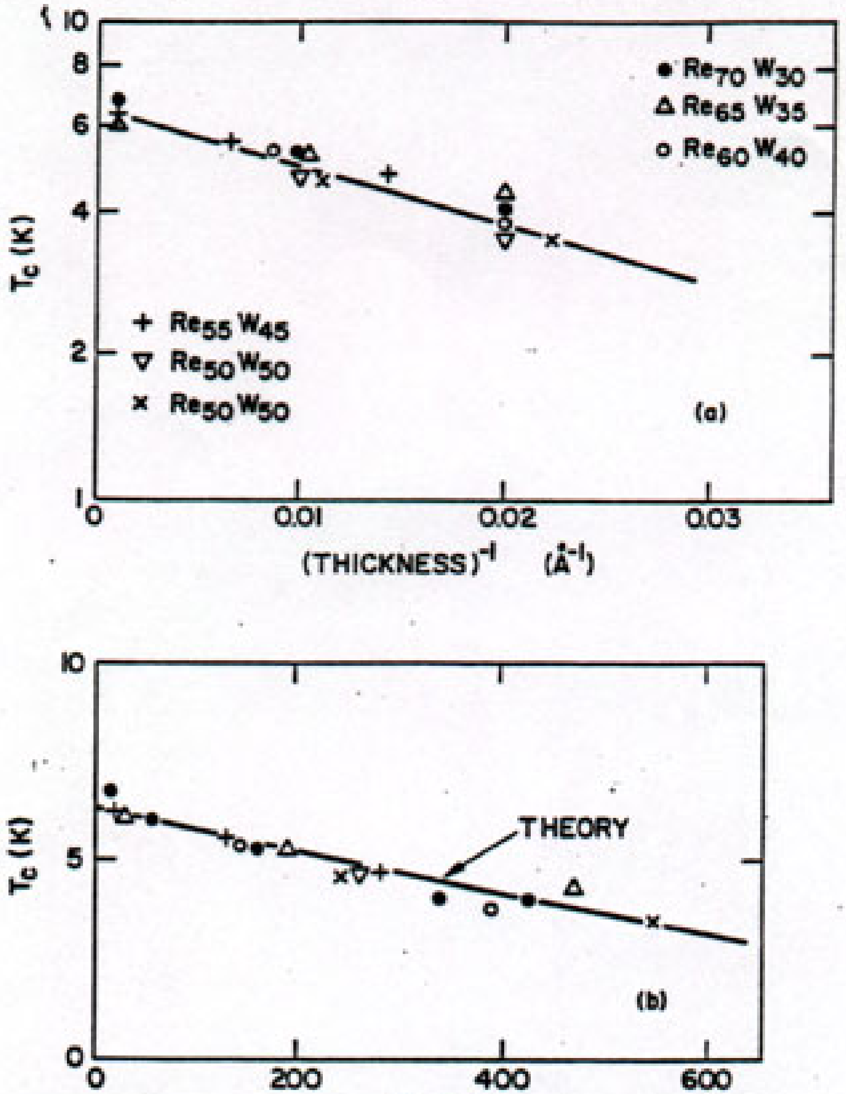


FIG. 1. Superconducting transition temperature T_c plotted as a function of the inverse of the thickness for a variety of alloy compositions. The solid line is a best fit to the data. Data replotted as a function of sheet resistance $R_\square$. Theory curve is that of Maekawa or Fukuyama (Ref. 5).

Graybeal + Beasley

PRB29 4167 (1984)

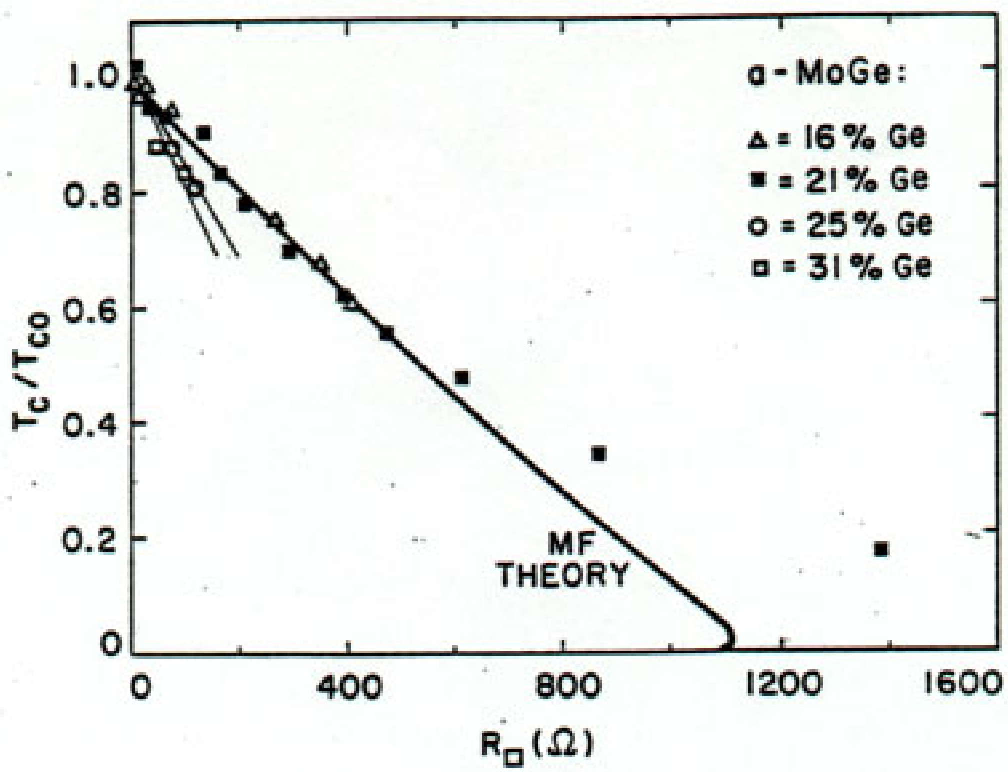
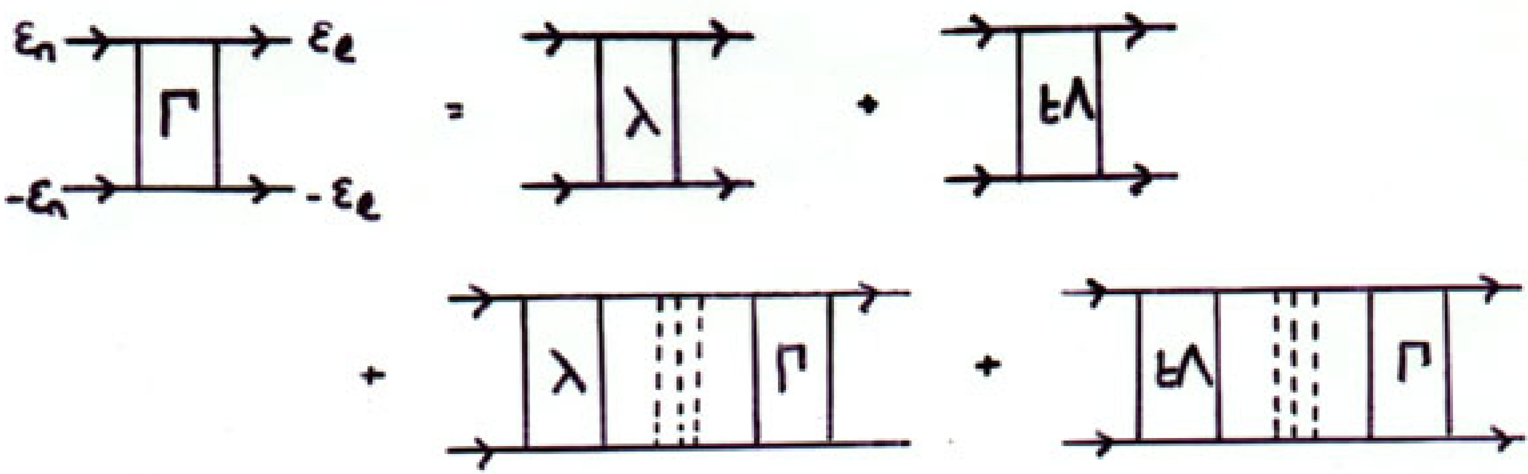
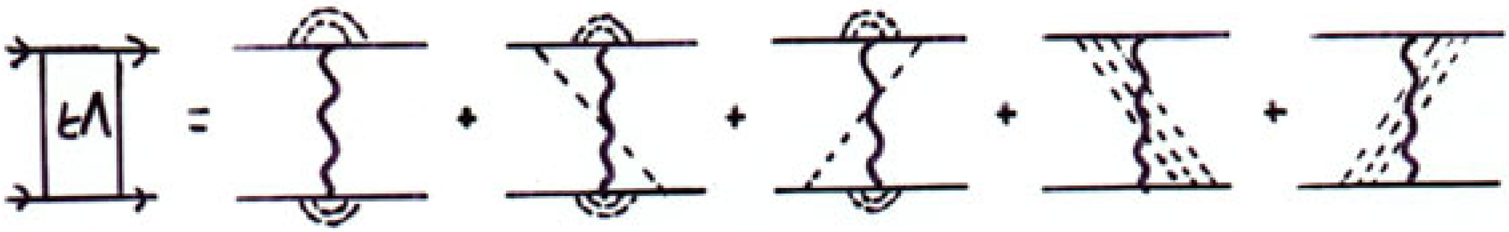


FIG. 2. (T_c/T_{c0}) vs $R_\square$ for various compositions, with fit to theory. Values of T_{c0} are 7.6, 7.2, 6.8, and 5.8 K for increasing Ge content.

Beyond Perturbation Theory

- Problem with PT is that phonon and Coulomb interactions treated differently.
- Cancellation of low-q singularity in Coulomb potential means we can fix this:





$$\Gamma(\varepsilon_n, \varepsilon_\ell) = \left[-\lambda + t\Lambda(\varepsilon_n, \varepsilon_\ell)\right] - 2\pi T \sum_{m=0}^{M} \left[-\lambda + t\Lambda(\varepsilon_n, \varepsilon_m)\right] \frac{\Gamma(\varepsilon_m, \varepsilon_\ell)}{|\varepsilon_m|}$$

$$\Lambda(\varepsilon_n, \varepsilon_\ell) = \ln\left[\frac{1}{(\varepsilon_n + \varepsilon_m)\tau}\right] \quad (d=2)$$

[Oreg + Finkelstein PRL 83 191 (1999)]

Solution By Maximum Section Method

- Continuous version of Γ eqn is:

$$\Gamma(\varepsilon,\varepsilon') = f(\varepsilon,\varepsilon') - \int_T^{1/\tau} \frac{d\varepsilon_1}{\varepsilon_1} f(\varepsilon,\varepsilon_1)\,\Gamma(\varepsilon_1,\varepsilon')$$

$$f(\varepsilon,\varepsilon') = -\lambda + t\ln\left[\frac{1}{(\varepsilon+\varepsilon')\tau}\right]$$

- Define $\Gamma(\omega) = \Gamma(\omega,\omega)$, the effective magnitude of Γ at energy scale ω. If we consider $\omega = T$ then $\Gamma(\omega)$ is given by:

$$\Gamma(\omega) = f(\omega) - \int_\omega^{1/\tau} d\varepsilon_1\, f(\omega,\varepsilon_1)\frac{1}{\varepsilon_1} f(\varepsilon_1,\omega)$$

$$+ \int_\omega^{1/\tau} d\varepsilon_1 \int_\omega^{1/\tau} d\varepsilon_2\, f(\omega,\varepsilon_1)\frac{1}{\varepsilon_1} f(\varepsilon_1,\varepsilon_2)\frac{1}{\varepsilon_2} f(\varepsilon_2,\omega) + \cdots$$

$$= f(\omega) - Q(\omega)$$

- Consider N-th term in $Q(\omega)$:

$$Q_N(\omega) = (-1)^{N+1}\int_\omega^{1/\tau} d\varepsilon_1 \ldots d\varepsilon_N\, f(\omega,\varepsilon_1)\frac{1}{\varepsilon_1} f(\varepsilon_1,\varepsilon_2)\frac{1}{\varepsilon_2}\cdots\frac{1}{\varepsilon_N} f(\varepsilon_N,\omega)$$

Split integral into N regions with j-th region

$$\omega < \varepsilon_j < \varepsilon_1, \varepsilon_2 \ldots \varepsilon_{j-1}, \varepsilon_{j+1} \ldots \varepsilon_N$$

[Sudakov, Doklady 1 662 (1956)]

- Consider terms to left and right of $d\varepsilon_j$ and use approximation $f(\varepsilon,\varepsilon') = f(\max[\varepsilon,\varepsilon'])$:

$$Q_N^{Lj}(\omega) = (-1)^j \int_\omega^{1/T} d\varepsilon_1 \ldots d\varepsilon_{j-1}\, f(\varepsilon_j,\varepsilon_1)\frac{1}{\varepsilon_1} f(\varepsilon_1,\varepsilon_2)\cdots \frac{1}{\varepsilon_{j-1}} f(\varepsilon_{j-1},\varepsilon_j)$$

$$Q_N^{Rj}(\omega) = (-1)^{N-j+1} \int_\omega^{1/T} d\varepsilon_{j+1} \ldots d\varepsilon_N\, f(\varepsilon_j,\varepsilon_{j+1})\frac{1}{\varepsilon_{j+1}} \cdots \frac{1}{\varepsilon_N} f(\varepsilon_N,\varepsilon_j)$$

where we noted $f(\omega,\varepsilon_1) = f(\varepsilon_j,\varepsilon_1)$
$f(\varepsilon_N,\omega) = f(\varepsilon_N,\varepsilon_j)$

- The N different j values just give the N different terms with $N+1$ f's in integral

$$\int_\omega^{1/T} \frac{d\varepsilon}{\varepsilon}\Gamma(\varepsilon)^2$$

$$\Rightarrow \quad \Gamma(\omega) = f(\omega) - \int_\omega^{1/T} \frac{d\varepsilon}{\varepsilon}\Gamma(\varepsilon)^2$$

$$\Rightarrow \quad \Gamma(\omega) = -\lambda + t\ln\left(\frac{1}{\omega T}\right) - \int_\omega^{1/T} \frac{d\varepsilon}{\varepsilon}\Gamma(\varepsilon)^2$$

- Define variable $z = \ln(1/\omega T)$ to get

$$\frac{d\Gamma}{dz} = t - \Gamma^2$$

- Integrating this gives

$$\ln\left(\frac{1}{\omega\tau}\right) = \frac{1}{2\sqrt{t}} \ln\left|\frac{\Gamma+\sqrt{t}}{\Gamma-\sqrt{t}}\right| + \text{const}$$

- We know limits : $\omega = \frac{1}{\tau} \Rightarrow \Gamma = -\lambda$

  $\omega = T_c \Rightarrow \Gamma = -\infty$

$$\Rightarrow \ln\left(\frac{1}{T_c\tau}\right) = \frac{1}{2\sqrt{t}} \ln\left(\frac{\lambda+\sqrt{t}}{\lambda-\sqrt{t}}\right)$$

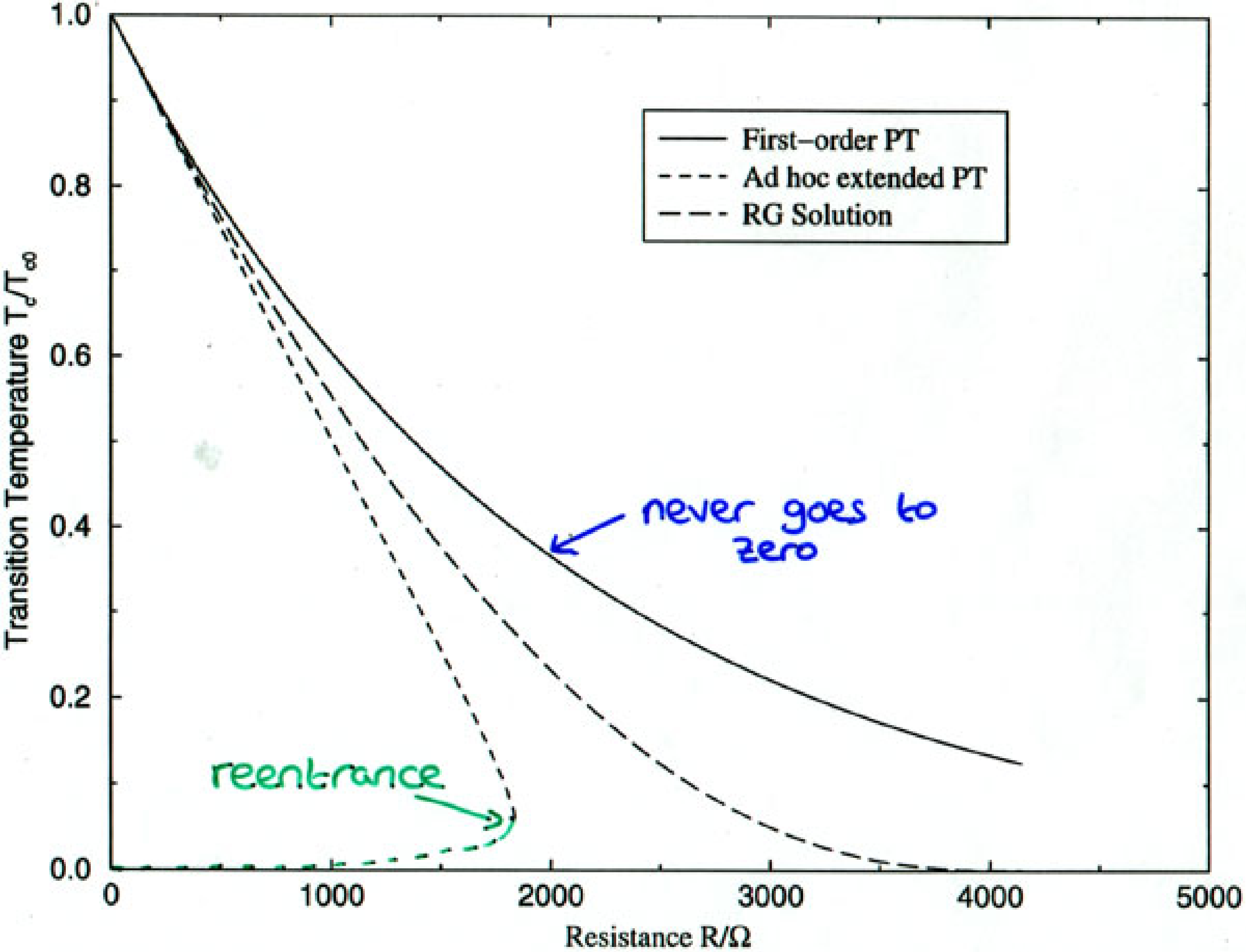
- When $t=0$, $T_c = T_{c0} \Rightarrow \ln\left(\frac{1}{T_{c0}\tau}\right) = \frac{1}{\lambda}$

$$\ln\left(\frac{T_c}{T_{c0}}\right) = \beta - \frac{1}{2\sqrt{t}} \ln\left(\frac{1+\beta\sqrt{t}}{1-\beta\sqrt{t}}\right) \qquad \beta = \ln\left(\frac{1}{T_{c0}\tau}\right)$$

- Expanding as a power series in t, the leading term $-\frac{1}{3}t\beta^3$ reproduces P.T.

- T_c goes smoothly to zero at $t = 1/\beta^2$. For $\beta = 6 \Rightarrow R_\square \approx 4.5\,k\Omega$.

Why We Need Non-Perturbative Approach



Fitting of Graybeal + Beasley Data to Finkel'stein $T_c(R_\square)$

JETP Lett 45 46 (1987)

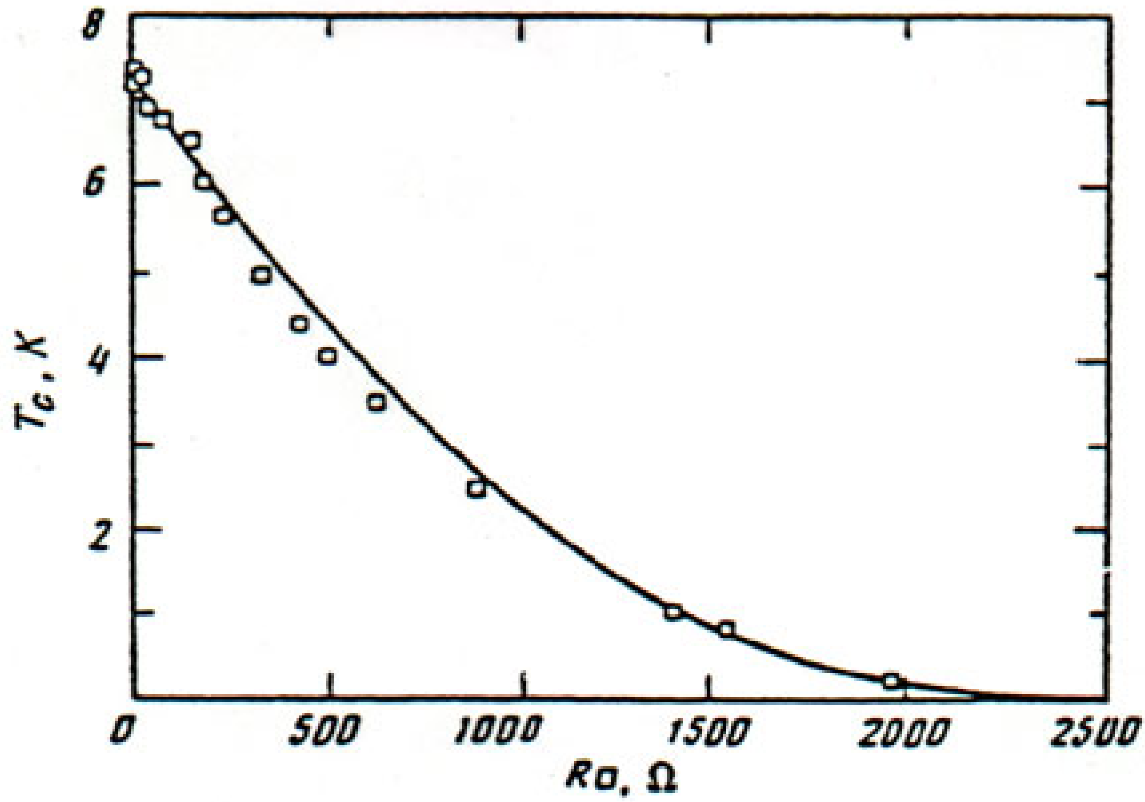


Fig. 14. Suppression of superconductivity in amorphous $Mo_{79}Ge_{21}$ [42]. The solid line is a theoretical fit with Eq. (13).

Aside: "RG" View of Superconductivity

- Take pair amplitude $\Gamma(\omega)$ at large energy scales and integrate out high energies until Γ diverges at T_c:

$$\frac{d\Gamma}{d\omega} = -\frac{\Gamma^2}{\omega} \quad \Rightarrow \Gamma^{-1} = A + \ln\omega$$

- First start with Coulomb $\Gamma(\varepsilon_F) = \mu$ and integrate down to $\omega = \omega_D$

$$\Gamma(\omega_D) = \mu^* = \frac{\mu}{1 + \mu \ln(\varepsilon_F/\omega_D)}$$

- At this energy add in phonons so total $\Gamma(\omega_D) = \mu^* - \lambda = -\lambda^*$. Integrating gives

$$\Gamma(\omega) = -\frac{\lambda^*}{1 - \lambda^* \ln(\omega_D/\omega)}$$

$$\Rightarrow \underline{T_c = \omega_D e^{-1/\lambda^*}}$$

- Magnetic impurities give eqn

$$\frac{d\Gamma}{d\omega} = -\frac{\Gamma^2}{\omega + \frac{1}{\tau_s}} \quad \Rightarrow \quad T_c\left(\tfrac{1}{\tau_s}\right) - T_{c0} = -\frac{1}{\tau_s}$$

Solution by Matrix Diagonalization

- Eqn for Γ is a matrix eqn with rows and columns labelled by Matsubara frequency:

$$\hat{\Gamma} = [-\lambda\hat{1} + t\hat{\Lambda}] - [-\lambda\hat{1} + t\hat{\Lambda}]\hat{\varepsilon}^{-1}\hat{\Gamma}$$

where $\Gamma_{nm} = \Gamma(\varepsilon_n, \varepsilon_m)$; $\Lambda_{nm} = \Lambda(\varepsilon_n, \varepsilon_m)$; $1_{nm} = 1$ and $\varepsilon_{nm} = (n + \frac{1}{2})\delta_{nm}$. Solving for Γ with $I_{nm} = \delta_{nm}$

$$\hat{\Gamma} = \{\hat{I} + [-\lambda\hat{1} + t\hat{\Lambda}]\hat{\varepsilon}^{-1}\}^{-1}[-\lambda\hat{1} + t\hat{\Lambda}]$$

$$= \hat{\varepsilon}^{-\frac{1}{2}}\{\hat{I} + \hat{\varepsilon}^{-\frac{1}{2}}[-\lambda\hat{1} + t\hat{\Lambda}]\hat{\varepsilon}^{-\frac{1}{2}}\}^{-1}\hat{\varepsilon}^{\frac{1}{2}}[-\lambda\hat{1} + t\hat{\Lambda}]$$

- When the matrix in brackets has a zero eigenvalue, $\hat{\Gamma}$ is singular and we are at T_c.

- Rank of matrix $M = \frac{1}{2\pi T\tau}$ depends upon T. Start at $M_0 = \frac{1}{2\pi T_{c0}\tau}$, find lowest eigenvalue and increase M by one until eigenvalue changes sign. Then $T_c/T_{c0} = M_0/M$.

- Fast + adaptable using Lanczos on Pentium PC.

Effect of Magnetic Impurities

- Extend Γ eqn to include magnetic impurities

$$\Gamma(\varepsilon_n,\varepsilon_\ell) = \left[-\lambda + t\Lambda(\varepsilon_n,\varepsilon_\ell)\right] - \pi T \sum_{m=-(M+1)}^{M} \left[-\lambda + t\Lambda(\varepsilon_n,\varepsilon_m)\right] \frac{\Gamma(\varepsilon_m,\varepsilon_\ell)}{|\varepsilon_m| + 1/\tau_s}$$

$$\Lambda(\varepsilon_n,\varepsilon_m) = \begin{cases} \ln\left[\dfrac{1}{(|\varepsilon_n| + |\varepsilon_m|)\tau}\right] & \varepsilon_n\varepsilon_m < 0 \\ \ln\left[\dfrac{1}{(|\varepsilon_n| + |\varepsilon_m| + 2/\tau_s)\tau}\right] & \varepsilon_n\varepsilon_m > 0 \end{cases}$$

- Matrices double in size to rank 2M due to breaking of time-reversal invariance.

- Magnetic impurities affect both mean field and perturbative terms. Hard to approximate by differential eqn unless we ignore 2nd effect

$$\Rightarrow \frac{d\Gamma}{d\omega} = \frac{\Gamma^2}{\omega + \frac{1}{\tau_s}} - \frac{t}{\omega}$$

- Experimentally measure pair-breaking rate per impurity

$$\alpha(R_\square) = \frac{T_c(R_\square, 0) - T_c(R_\square, 1/\tau_s)}{1/\tau_s}$$

Theory predicts $\alpha(R_\square)$ roughly constant.

MAGNETIC IMPURITIES EXPT

[CHERVENAK + VALLES PRB51 11977 (1995)]

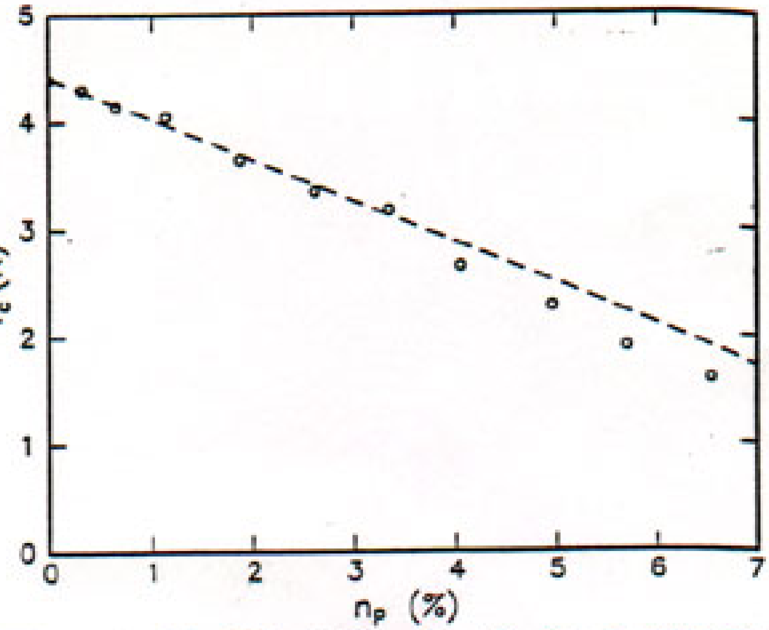


FIG. 1. A 10-Å, 800 – Ω $Pb_{0.9}Bi_{0.1}$/Ge film in which T_c is suppressed by incrementing the thickness of the Gd top layer. The dashed line suggests the regime of linear suppression of T_c with n_P.

LINEAR SUPPRESSION OF T_c BY $1/\tau_s$

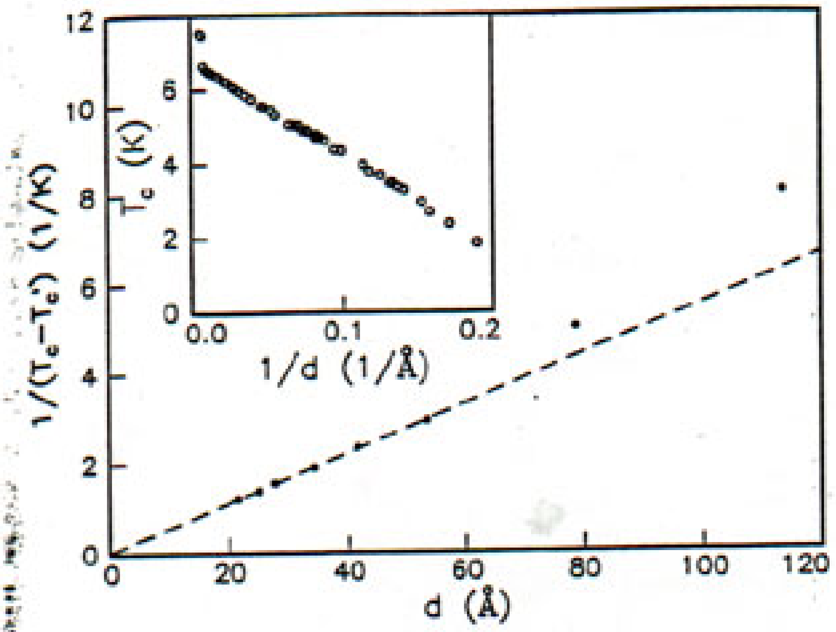
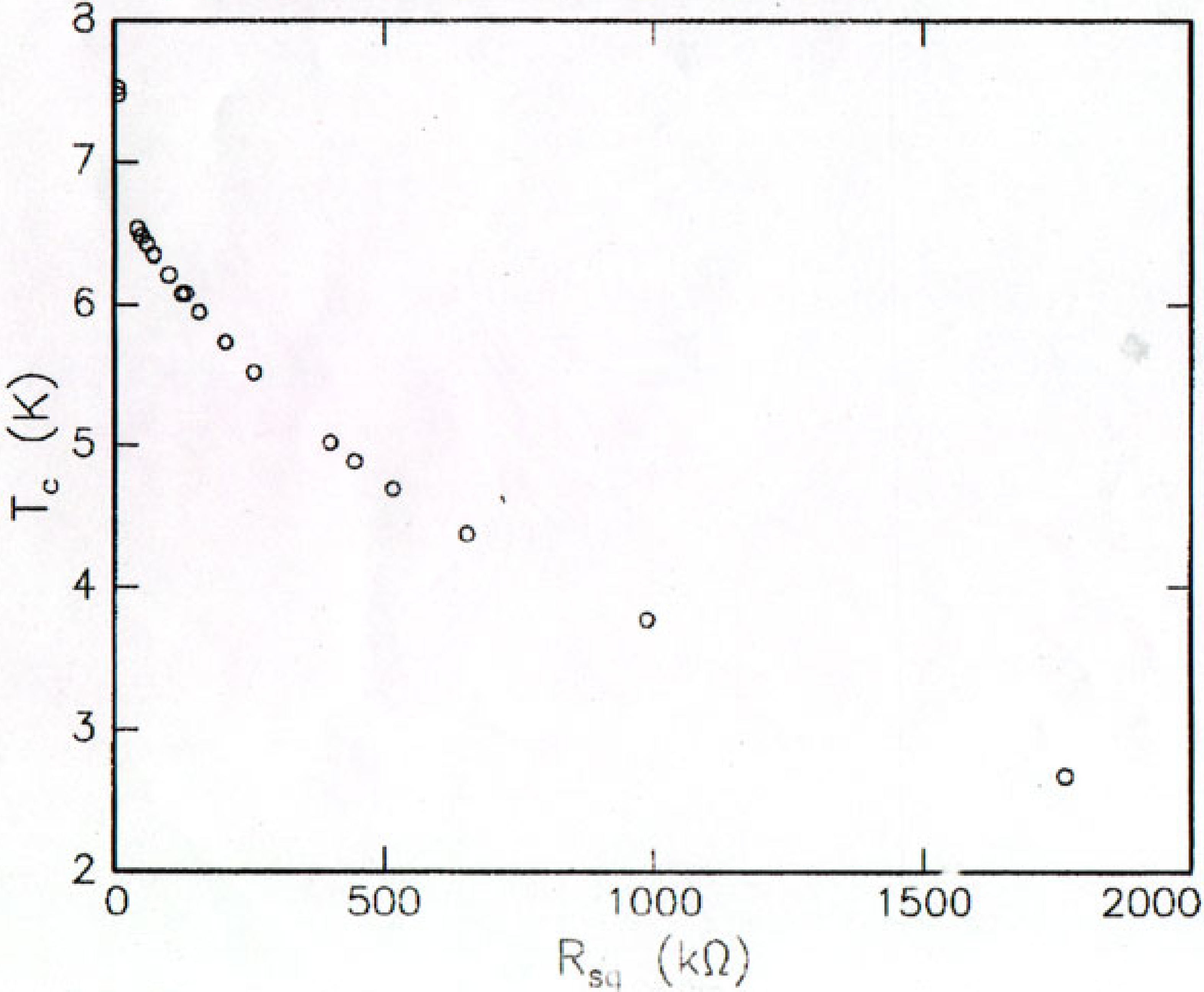


FIG. 2. Inverse T_c shift vs film thickness of PbBi. The deviation from the dashed line indicates reduced pair-breaking strength in the thickest films. Inset: Data from three experimental runs showing $T_c \propto \frac{1}{d}$ in PbBi films.

LINEAR SUPPRESSION OF T_c BY $1/d$

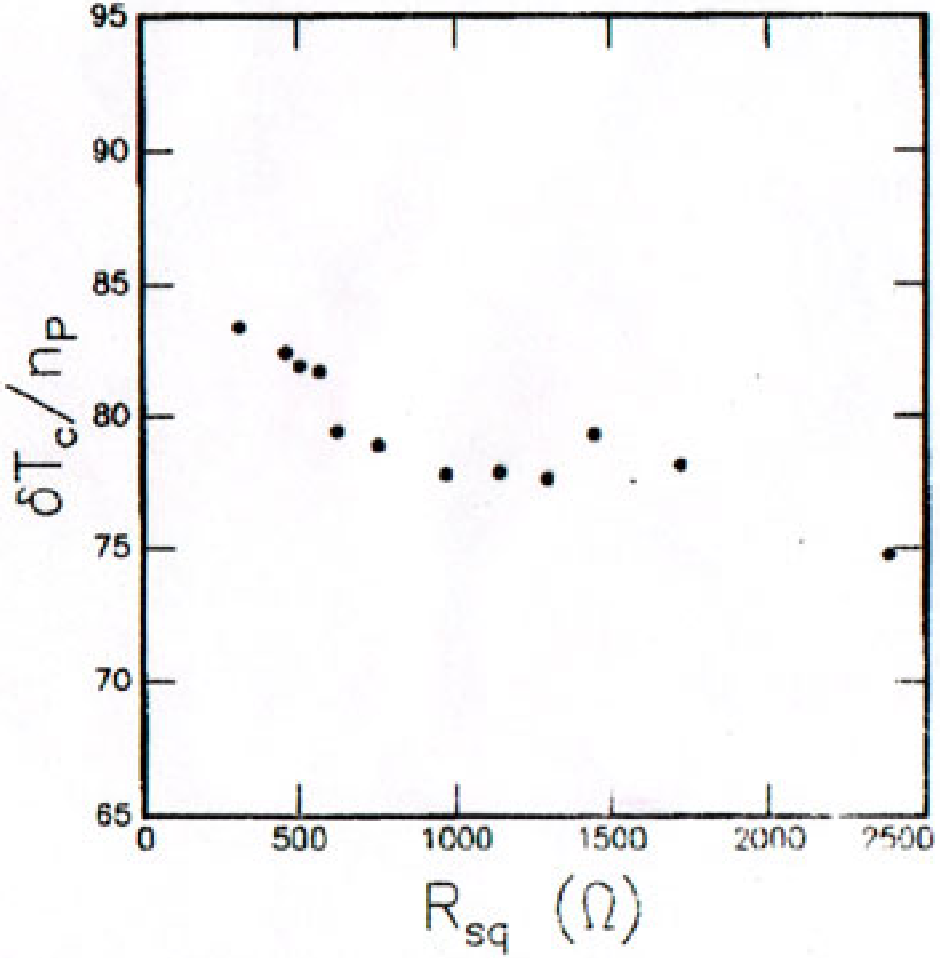
MAGNETIC IMPURITIES EXPT

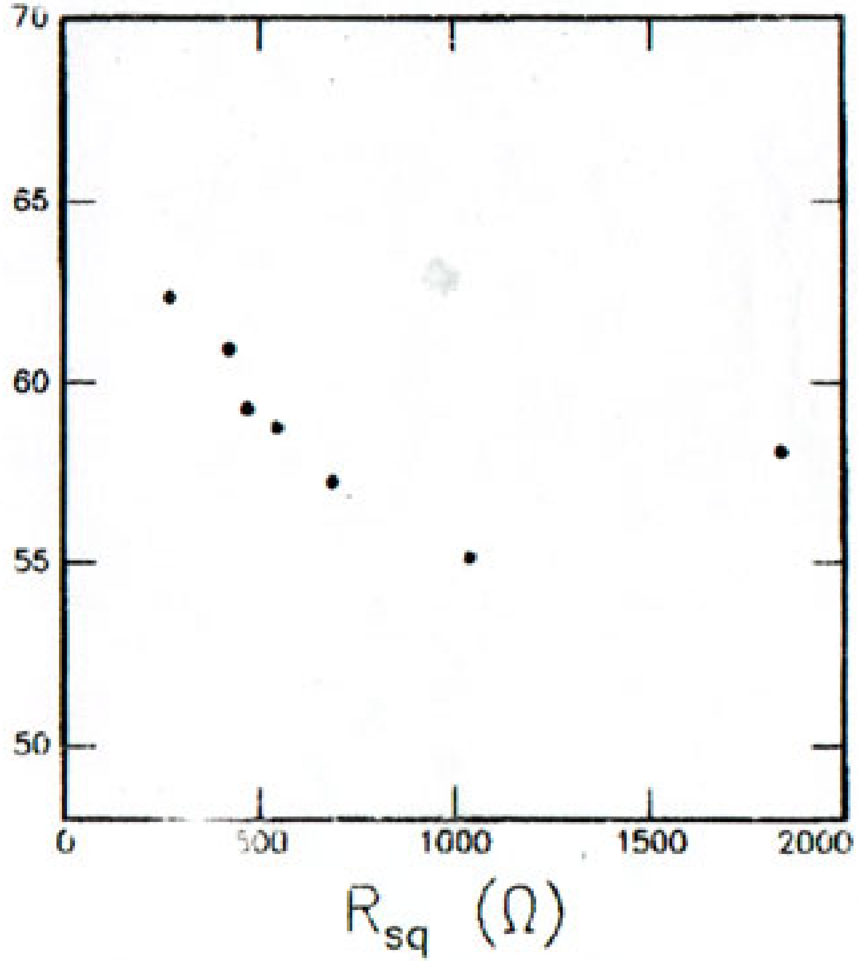
SUPRRESSION OF T_c BY $R_\square$



MAGNETIC IMPURITIES EXPT

PAIR-BREAKING RATE PER IMPURITY
VS $R_\square$ OF FILM





First Order P.T. : Magnetic Field

- First done by Maekawa, Ebisawa and Fukuyama, J Phys. Soc. Jpn. 52 1352 (1983) :

$$\ln\left(\frac{T_c}{T_{c0}}\right) = \Psi\left(\tfrac{1}{2}\right) - \Psi\left(\tfrac{1}{2} + \frac{1}{2\pi T_c \tau_H}\right) + R_{HF} + R_V$$

$$R_{HF} = -\frac{t}{2}\ln^2\left(\frac{1}{2\pi T_c \tau}\right) - t\ln\left(\frac{1}{2\pi T_c \tau}\right)\left[\Psi(t) - \Psi\left(\tfrac{1}{2} + \frac{1}{2\pi T_c \tau_H}\right)\right]$$

$$R_V = -\frac{t}{3}\ln^3\left(\frac{1}{2\pi T_c \tau}\right) - t\ln^2\left(\frac{1}{2\pi T_c \tau}\right)\left[\Psi(t) - \Psi\left(\tfrac{1}{2} + \frac{1}{2\pi T_c \tau_H}\right)\right]$$

- Basic idea: magnetic fields scramble the phase coherence needed for superconductivity and the localization processes that suppress superconductivity : WHO WINS ?.

- Above formula predicts upturn in $H_{c2}(T)$ at low T as $R_\square$ increased i.e. localization loses.

- We find effect disappears if sums in perturbation theory performed carefully.

‖ NO UPTURN IN $H_{c2}(T)$ PREDICTED BY LOCALIZATION THEORY ‖

Effect of Magnetic Fields

- We extend Γ eqn to include magnetic fields by calculating Γ at finite external momentum and setting $Dq^2 = 2/\tau_H = 2DeH$:

$$\Gamma(\varepsilon_n, \varepsilon_\ell) = [-\lambda + t\Lambda(\varepsilon_n, \varepsilon_\ell)] - \pi T \sum_{m=-(M+1)}^{M} [-\lambda + t\Lambda(\varepsilon_n, \varepsilon_m)] \frac{\Gamma(\varepsilon_m, \varepsilon_\ell)}{|\varepsilon_m| + \frac{1}{\tau_H}}$$

$$\Lambda(\varepsilon_n, \varepsilon_m) = \begin{cases} \ln\left[\frac{1}{(|\varepsilon_n| + |\varepsilon_m|)\tau}\right] & \varepsilon_n \varepsilon_m > 0 \\ \ln\left[\frac{1}{(|\varepsilon_n| + |\varepsilon_m|)\tau}\right] + \frac{2}{(|\varepsilon_n| + |\varepsilon_m|)\tau_H} & \varepsilon_n \varepsilon_m < 0 \end{cases}$$

- Matrices double in size to rank 2M due to breaking of time-reversal invariance.

- Find NO POSITIVE CURVATURE IN $H_{c2}(T)$.

Graybeal + Beasley $H_{c2}(T)$ Data

PRB29 4167 (1984)

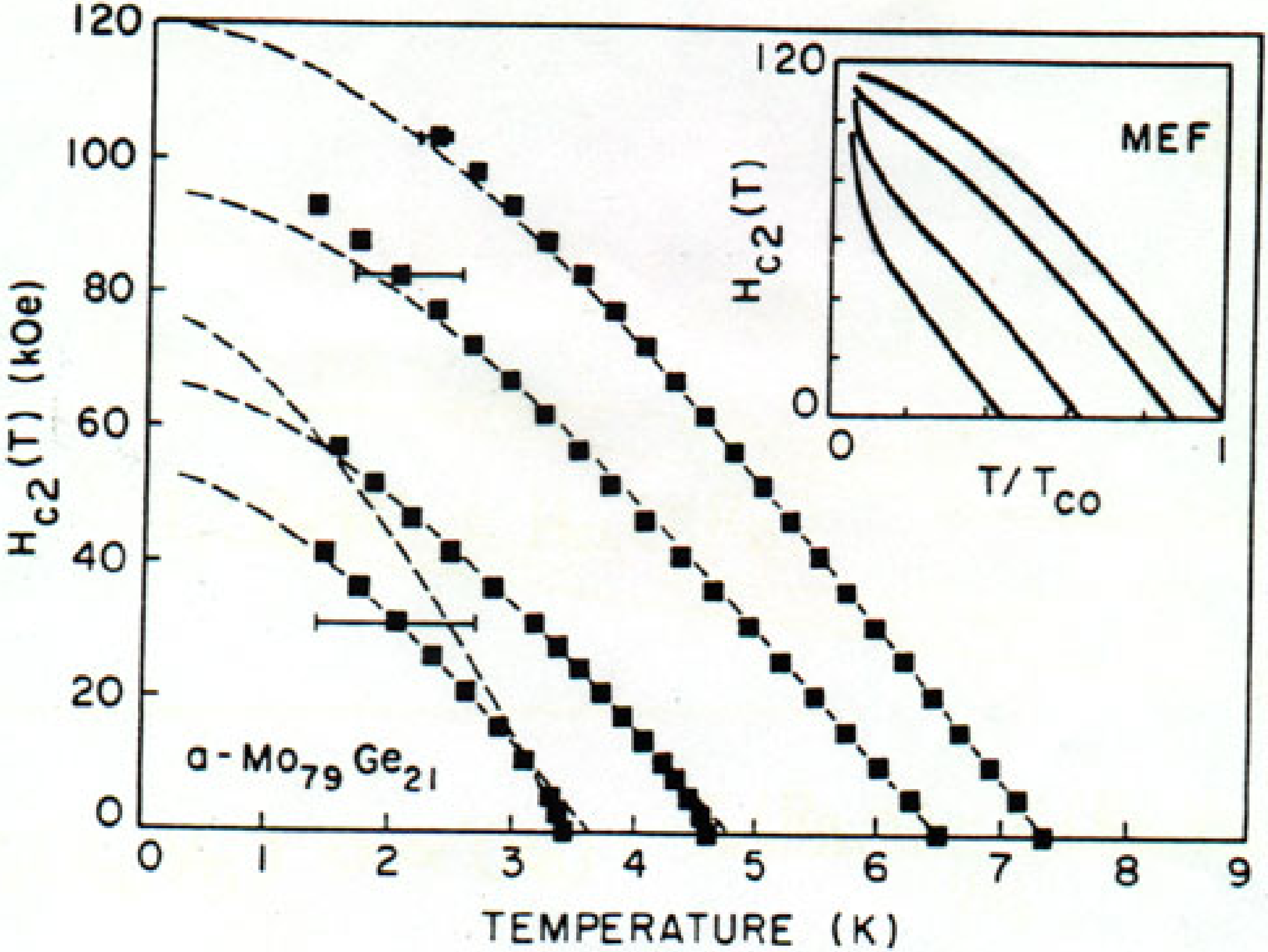
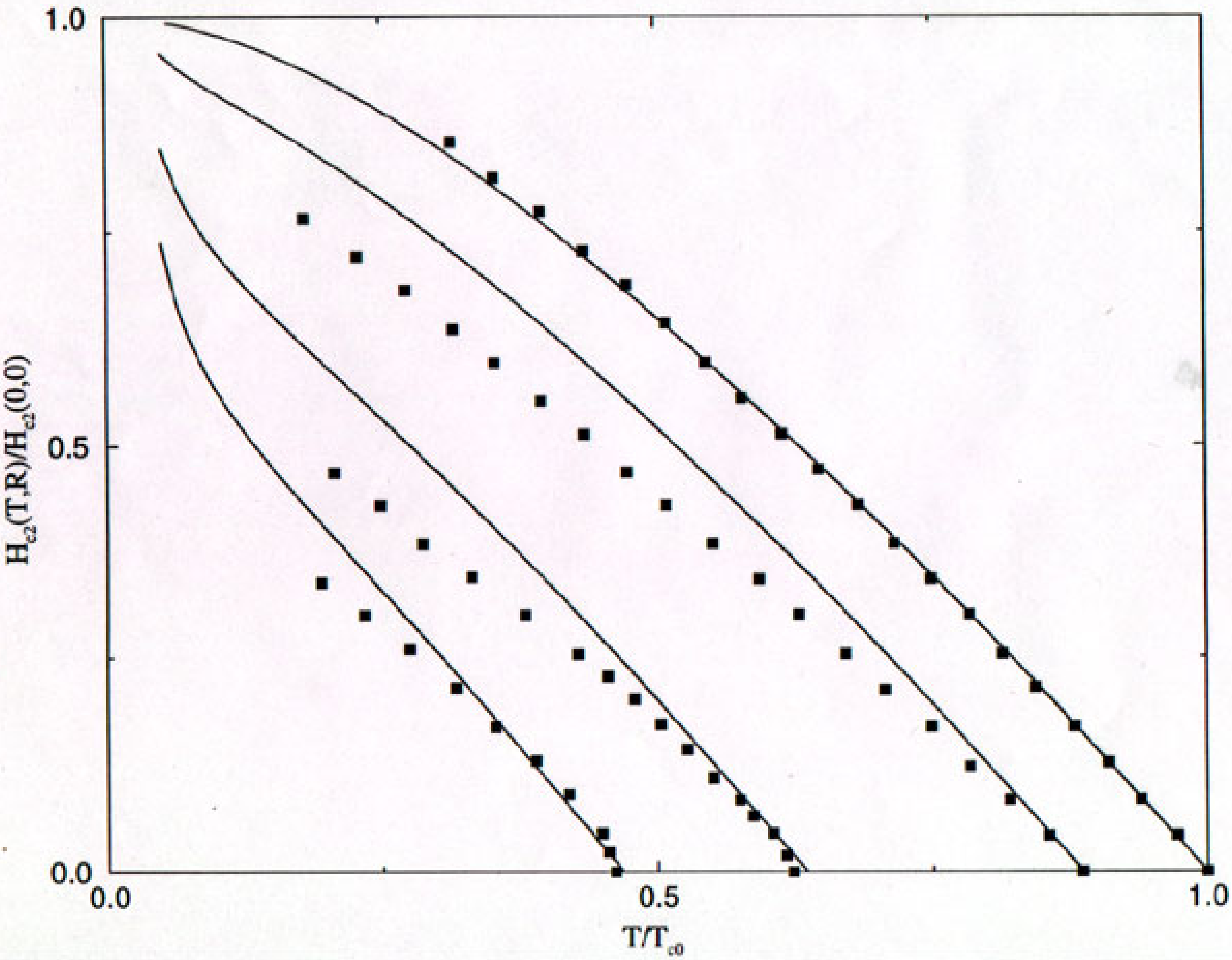


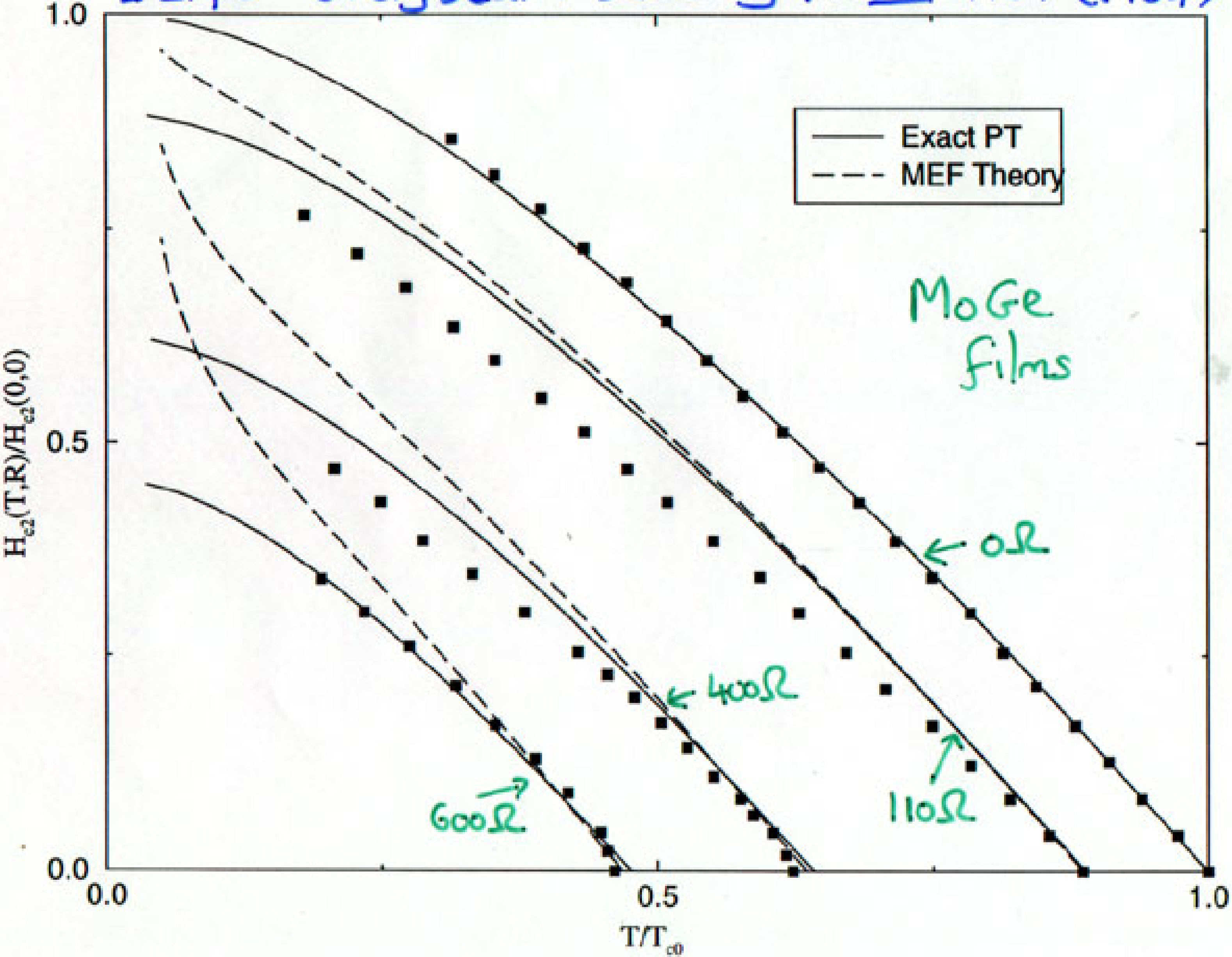
FIG. 3. Upper critical-field data, with GLAG fits (dashed lines). Inset shows MEF theory for same samples.

Fit of Graybeal and Beasley H_{c2} Data to MEF Theory



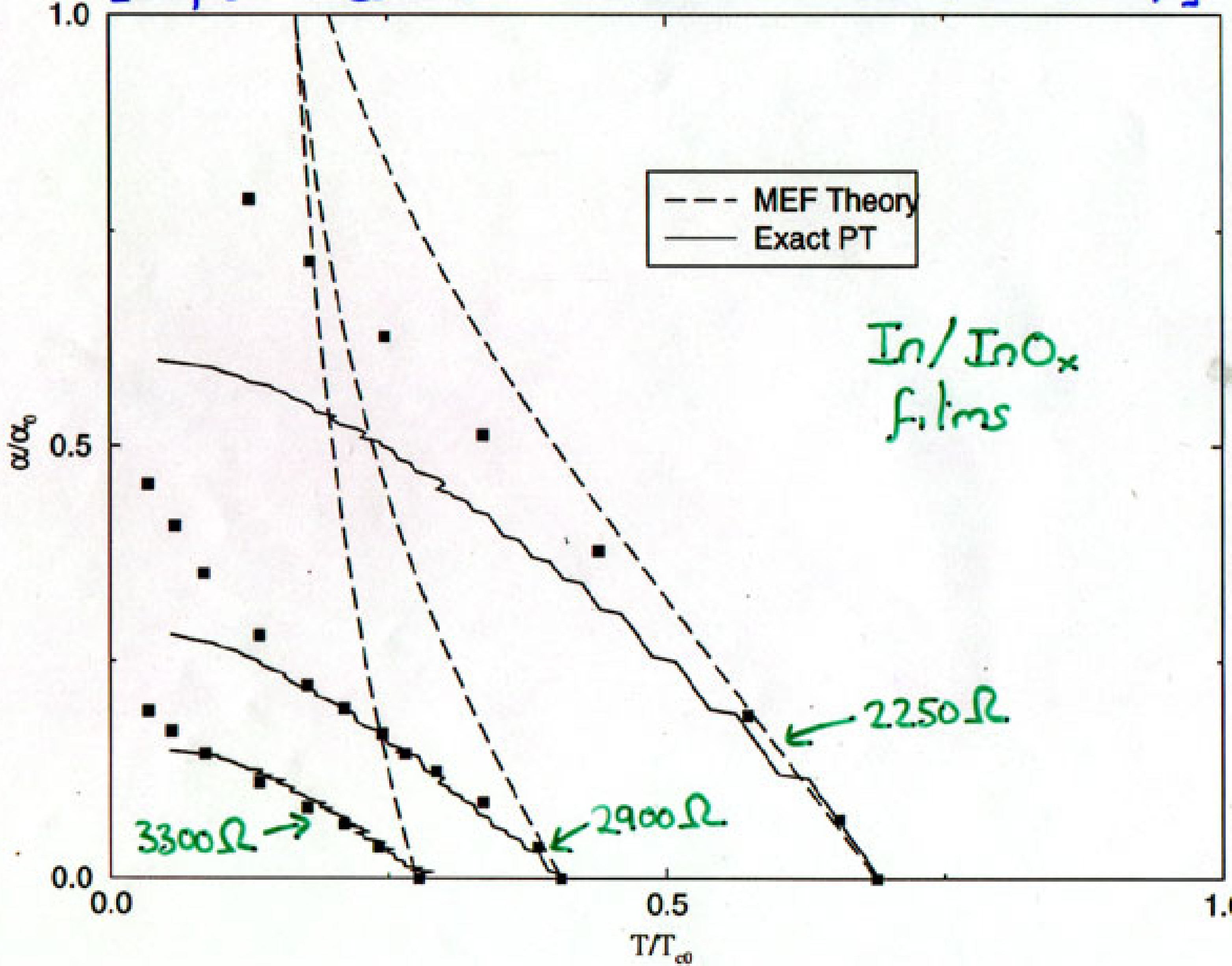
Fit of H_{c2} Data to Exact PT and MEF Theory

[Expt : Graybeal + Beasley PRB29 4167 (1984)]



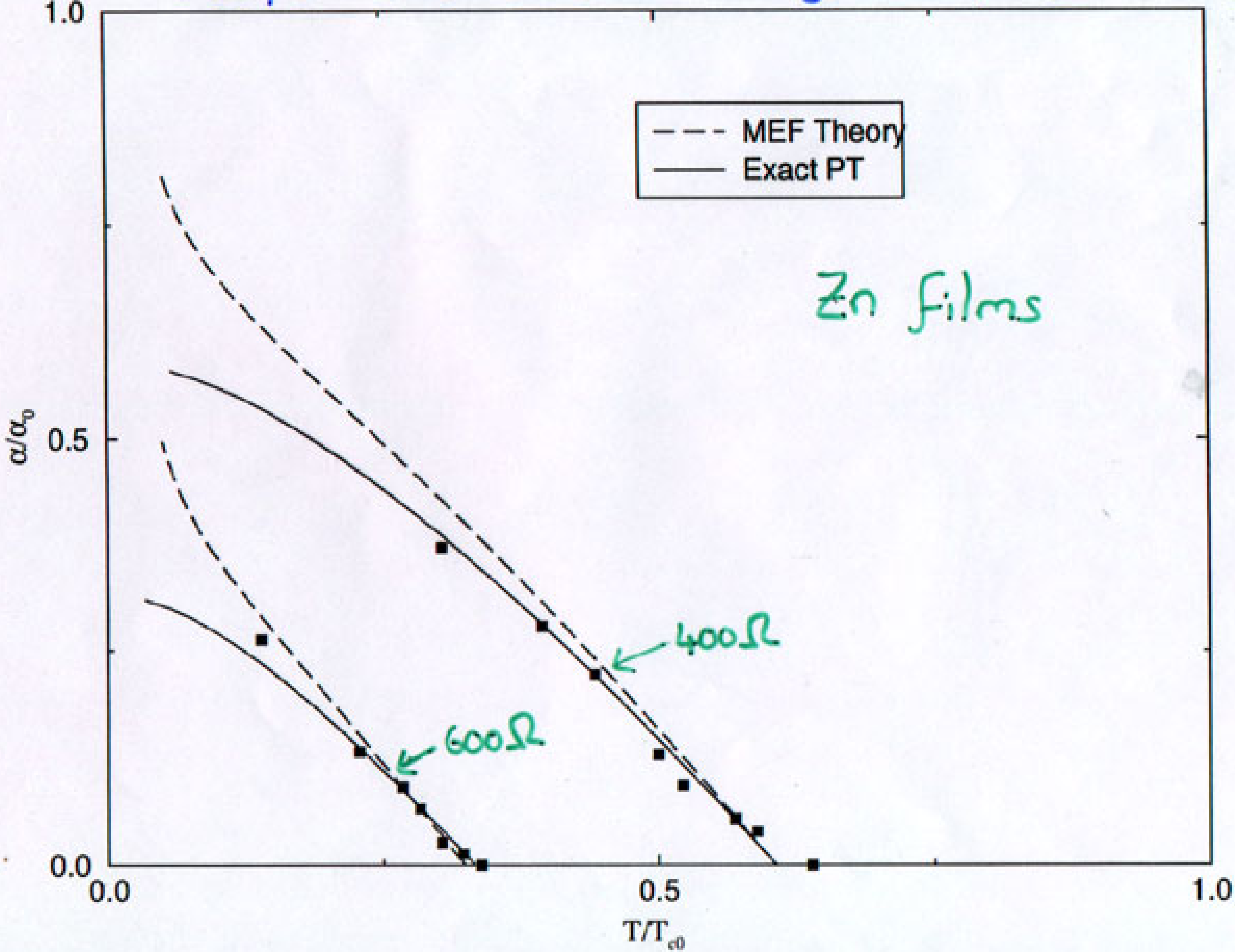
Fit of H_{c2} Data to Exact PT and MEF Theory

[Expt: Hebard + Paalanen PRB30 4063 (1984)]



Fit of H_{c2} Data to Exact PT and MEF Theory

[Expt : Okuma et al J. Phys. Soc. Jpn. 52 3269 (1983)]



T_c Suppression in Quasi-1D Wires

- Pb wires of length $L \sim 2\mu m$ and widths $a \sim 1000 - 150 Å$. Find decreasing width further suppresses T_c.

- Theoretically new feature is that transverse momentum sum cannot be replaced by an integral:

$$\Lambda(\varepsilon_n, \varepsilon_m) = \frac{4\pi D}{aL} \sum_q \frac{1}{Dq^2 + |\varepsilon_n| + |\varepsilon_m|}$$

$$= \frac{4\pi D}{a} \sum_{q_a} \int_0^{Dq_L^2 = \frac{1}{\tau}} \frac{2\,dq_L}{2\pi} \frac{1}{Dq_L^2 + Dq_a^2 + 2\pi T(n+m+1)}$$

$$= \frac{4}{a} \sum_{q_a} \frac{1}{\sqrt{q_a^2 + \frac{2\pi T}{D}(n+m+1)}} \tan^{-1}\left[\frac{1/\sqrt{D\tau}}{\sqrt{q_a^2 + \frac{2\pi T}{D}(n+m+1)}}\right]$$

- Find that result is very sensitive to boundary conditions:

$q_a = \frac{2\pi p}{a}$ $\quad p = 0, \pm 1, \pm 2 \ldots$ (periodic)

$q_a = \frac{2\pi p}{a}$ $\quad p = \pm 0, \pm 1, \pm 2 \ldots$ (Oreg + Finkel'stein)

$q_a = \frac{\pi p}{a}$ $\quad p = 0, 1, 2 \ldots$ (zero gradient) ✓

- Zero gradient boundary conditions are correct since these correspond to zero supercurrent flowing out of sides of wire.

- An additional problem is that $T_c(R_\square)$ data on 2D films doesnt fit well to theory: quantity to consider is T_c suppression of wires relative to films:

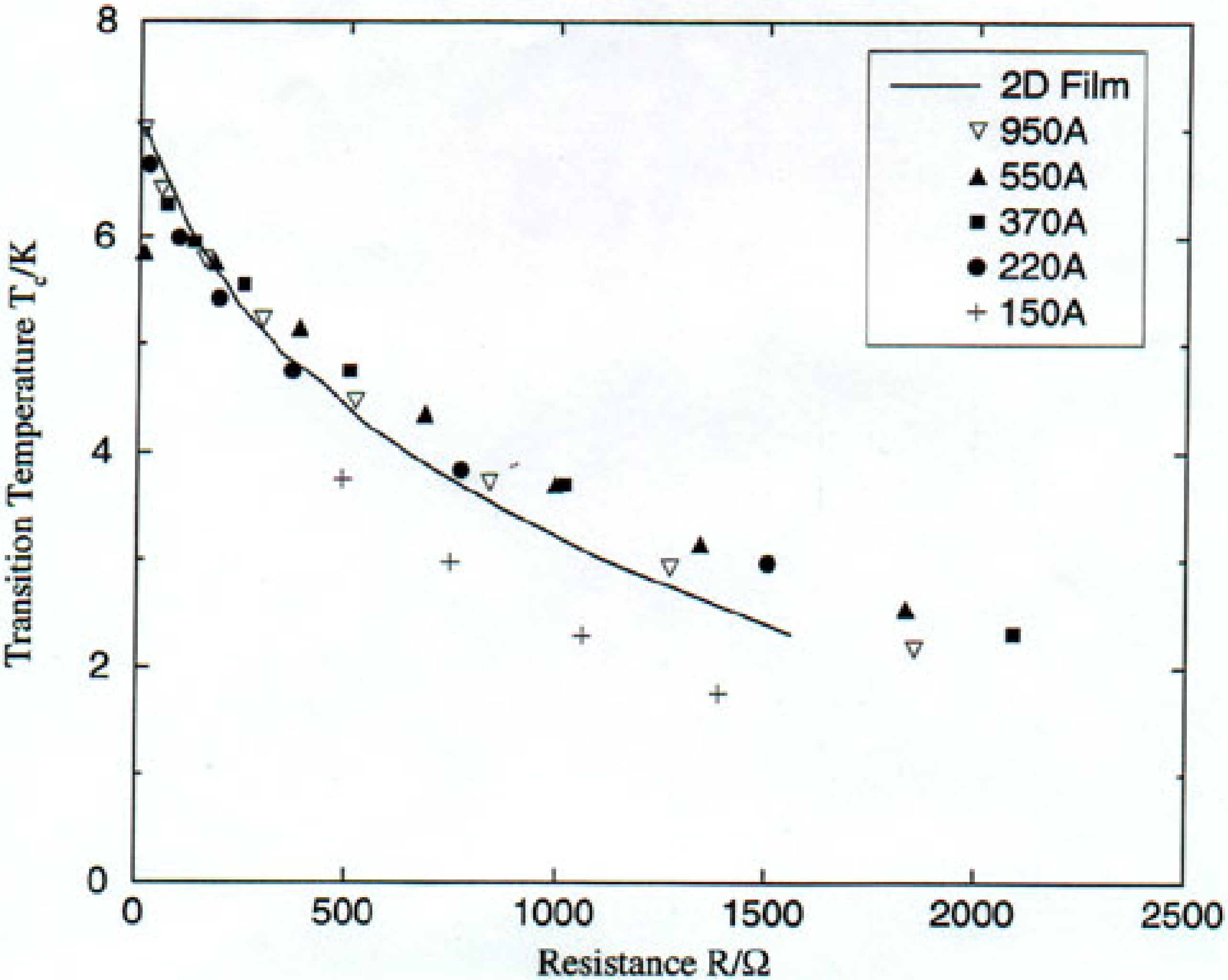
$$\underline{\underline{[T_c(a, R_\square)/T_c(a,0)] / [T_c(2D, R_\square)/T_c(2D,0)]}}$$

- 2D -> 1D crossover occurs when width a is of same size as thermal length $L_T = [2\pi D/T]^{1/2} \sim 300 A^\circ$.

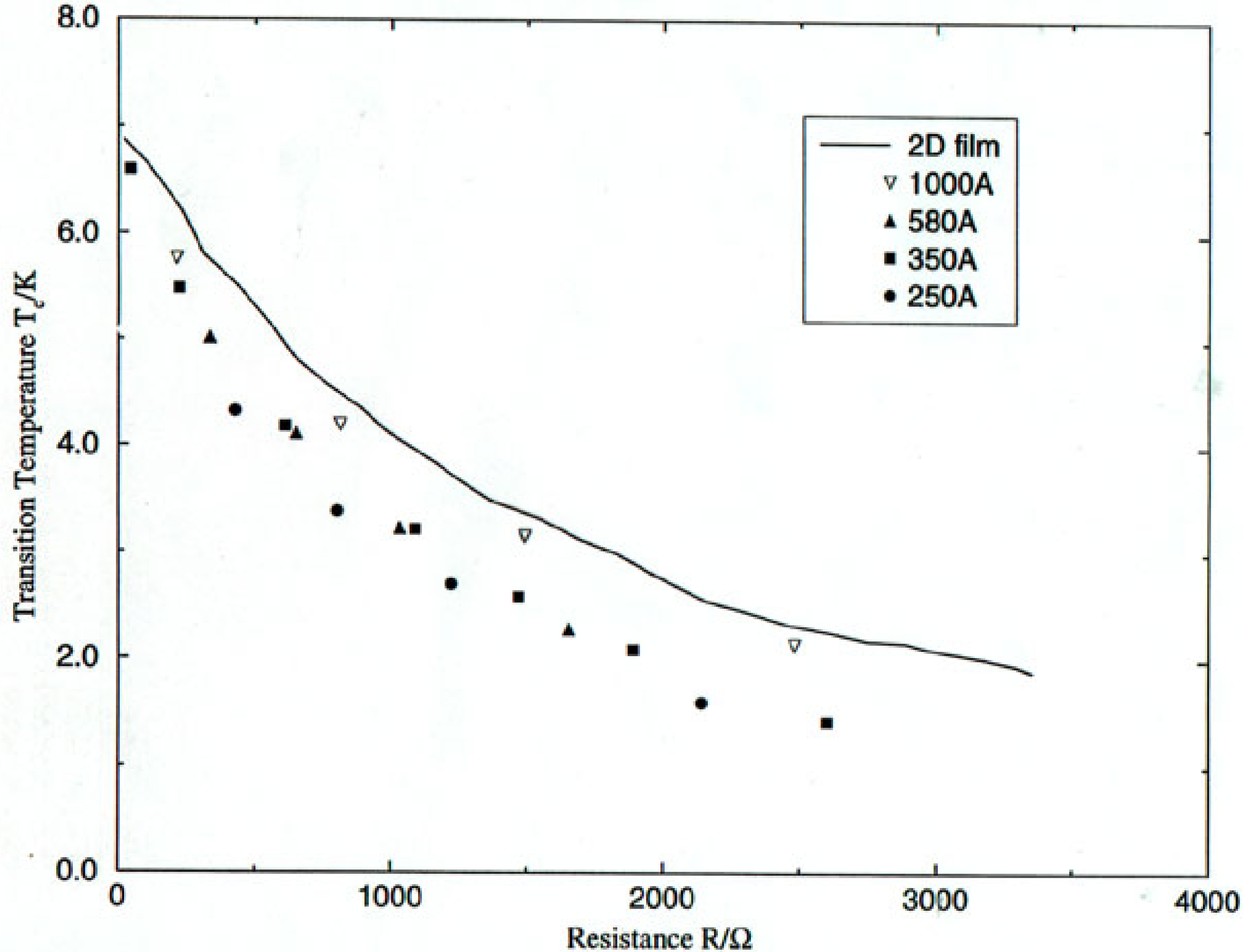
- Get qualitative but not quantitative agreement between theory + expt.

T_c vs Normal State $R_\square$ for 2D Pb Films + Wires

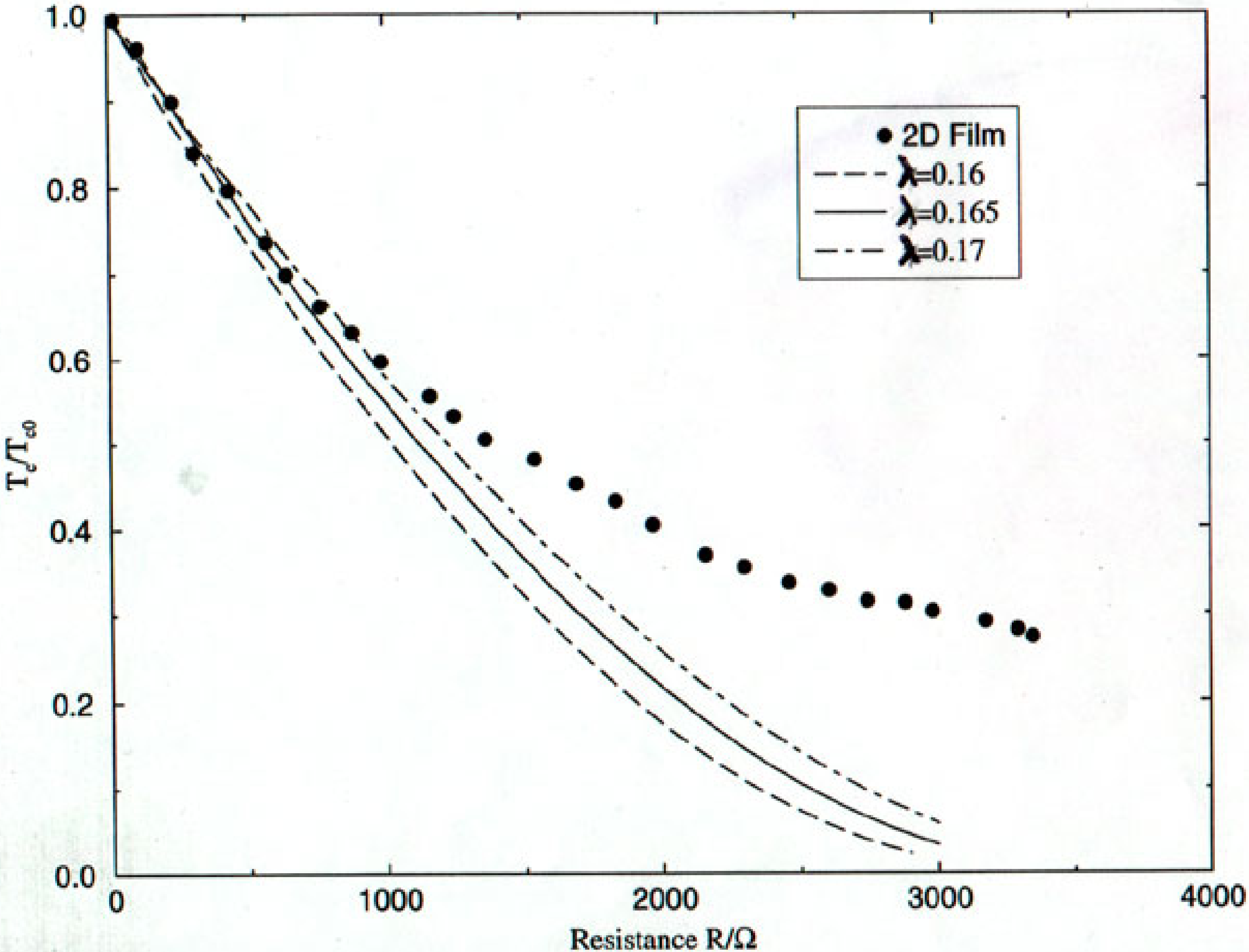
[Sharifi et al PRL 71 428 (1993)]



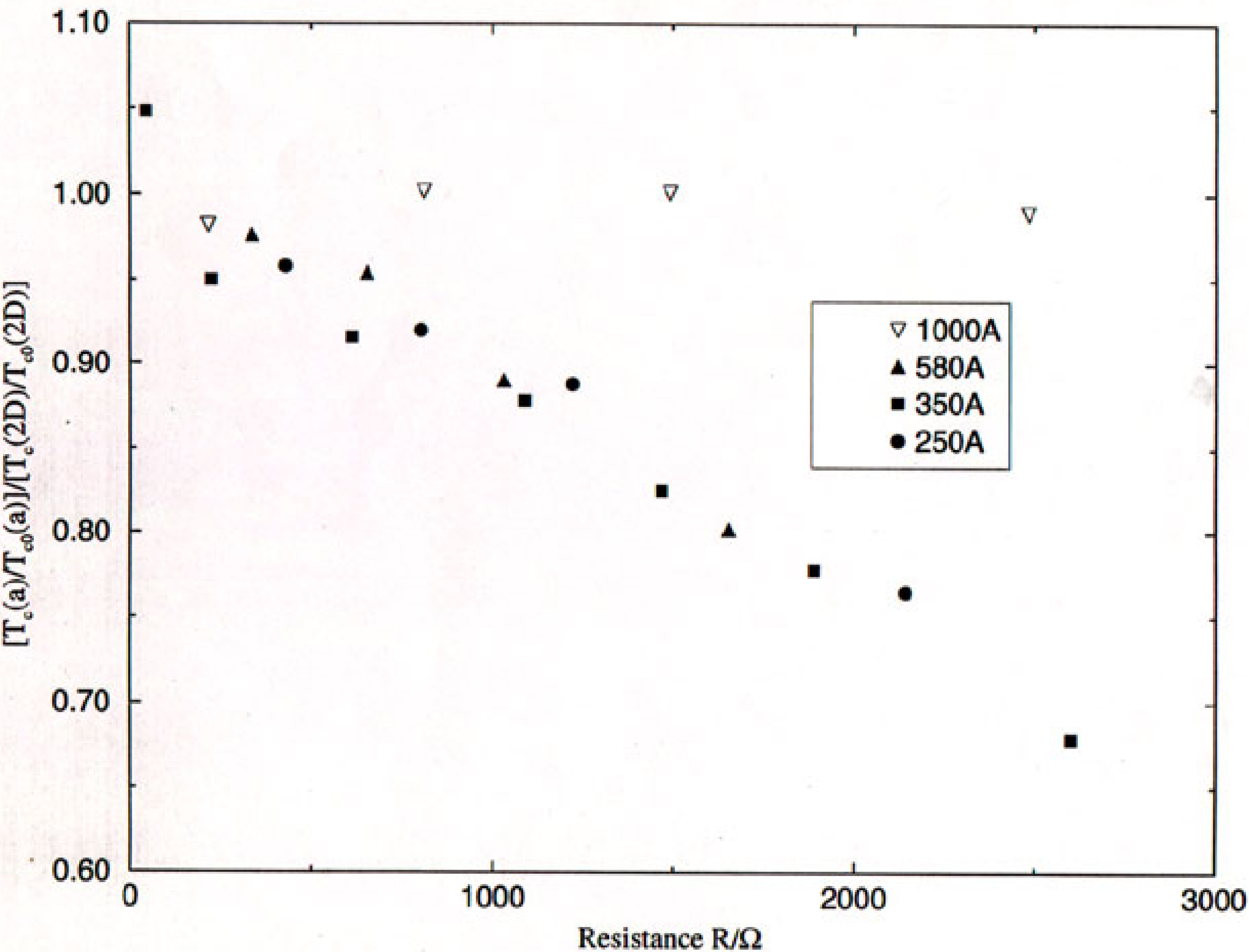
T_c vs Normal State $R_\square$ for 2D Pb Films + Wires [Xiong et al PRL 78 927 (1997)]



Xiong Data: Obtaining λ by Fitting 2D Data to Exact Diagonalization Results

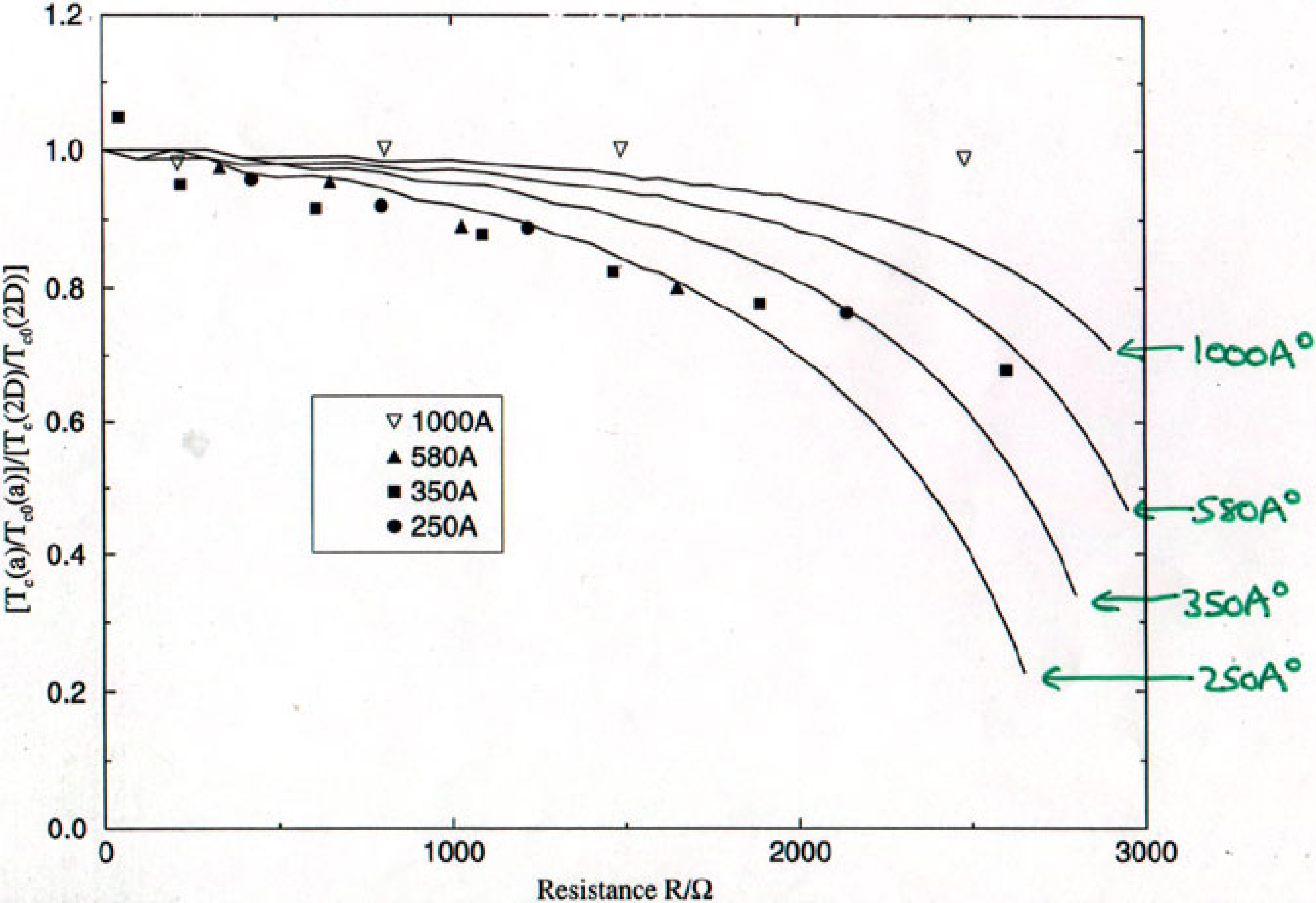


Xiong Data: Fractional T_c Suppression in Wires Relative to Films



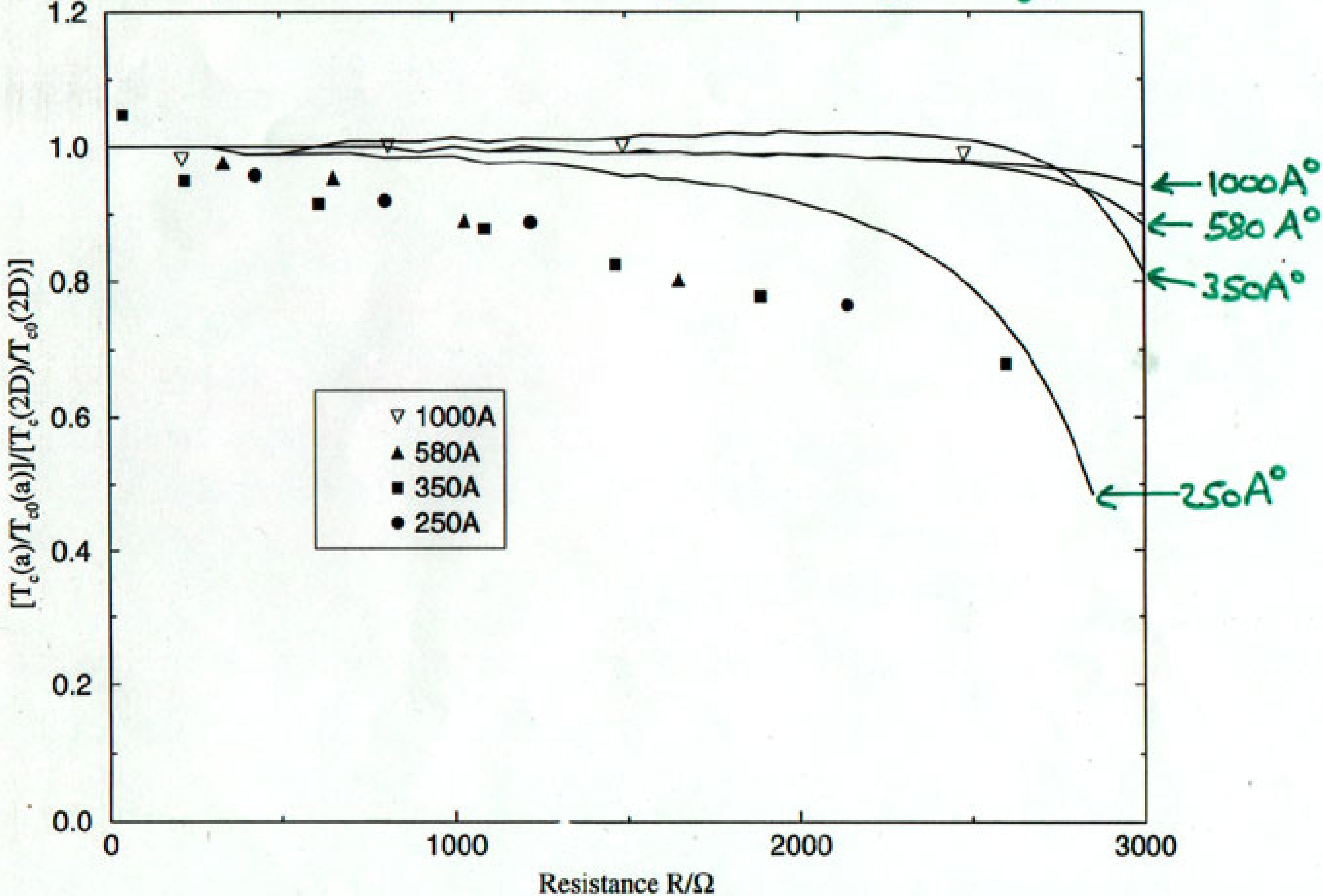
Theory vs Expt

zero gradient b.c. : $q_a = 0, \frac{\pi}{a}, \frac{2\pi}{a}, \frac{3\pi}{a} \ldots$



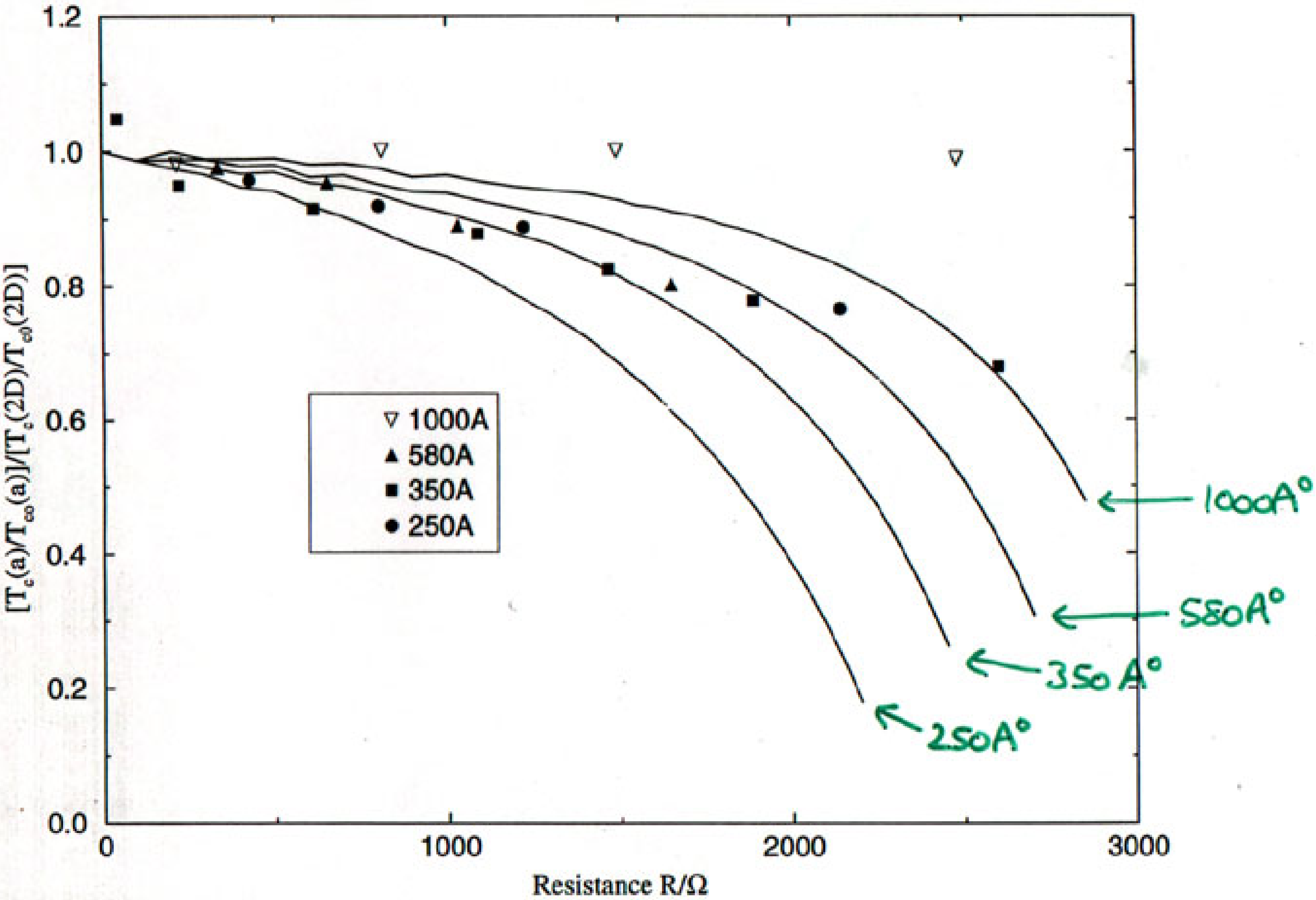
Theory vs Expt

periodic b.c. : $q_a = 0, \pm \frac{2\pi}{a}, \pm \frac{4\pi}{a} \ldots$



Theory vs Expt

Oreg's b.c. : $q_a = \pm 0, \pm\frac{2\pi}{a}, \pm\frac{4\pi}{a} \ldots$



Turning Off Coulomb Interaction

- Start with superconducting Ta film of $R_\square \sim 100\Omega$ which shows T_c suppression.

- Add thin oxide layer and thick Au layer: image charges induced in Au weaken effective Coulomb interaction.

- T_c increases showing suppression due to Coulomb interaction.

Spin-Orbit Scattering

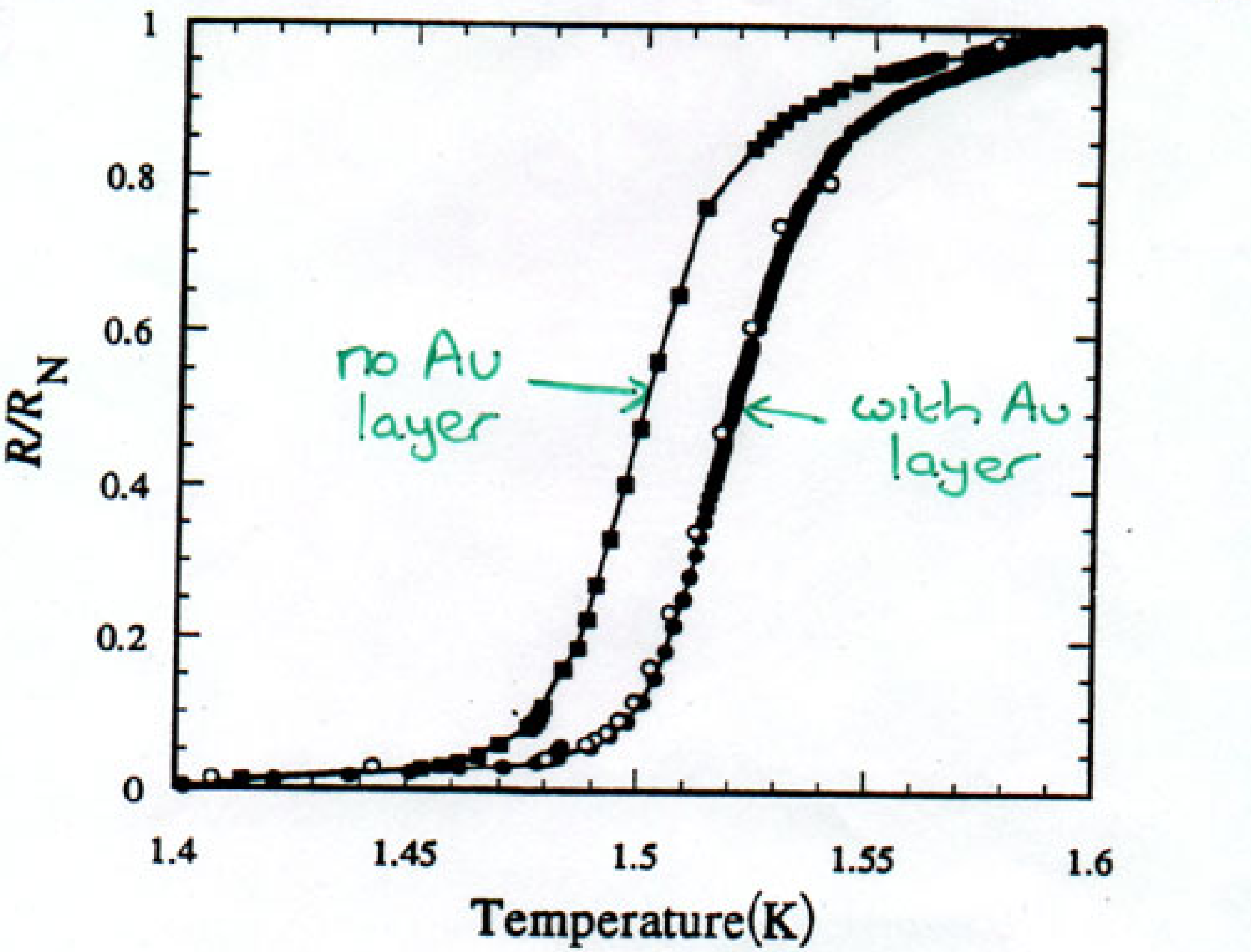
- Spin-orbit scattering leads to an antilocalization effect: $\sigma(T)$ increases as T decreases.

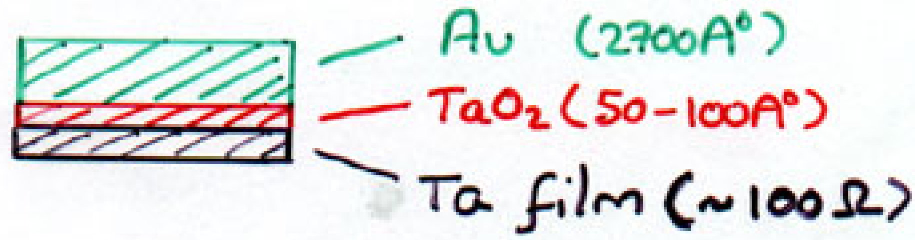
- Would this lead to turning off T_c suppression effects: ie $T_c \uparrow$ as $1/\tau_{so} \uparrow$?

- Seen in expt with granular Al + Bi.

Turning Off T_c Suppression in Ta Films by Turning off Coulomb Interaction

[Astrakharchik + Adkins PRB50 13622(1994)]





Effect of Spin-Orbit Scattering on T_c Suppression in Granular Al

[Miller et al PRL 61 2717 (1989)]