

The Suppression of Superconductivity by Disorder

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Thanks :

Beccy Handy (Birmingham)

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6 Lectures on Superconductivity in Disordered Metals

VA I : Green's functions, Matsubara and Cooper.

VA II : BCS-Gorkov, Ordinary + Spin-Flip Impurities + Gauge Invariance.

RAS I : Suppression of Superconductivity by Disorder: Perturbation Theory

RAS II : Beyond Perturbation Theory + Tuning Parameters

VA III : Strong Coupling

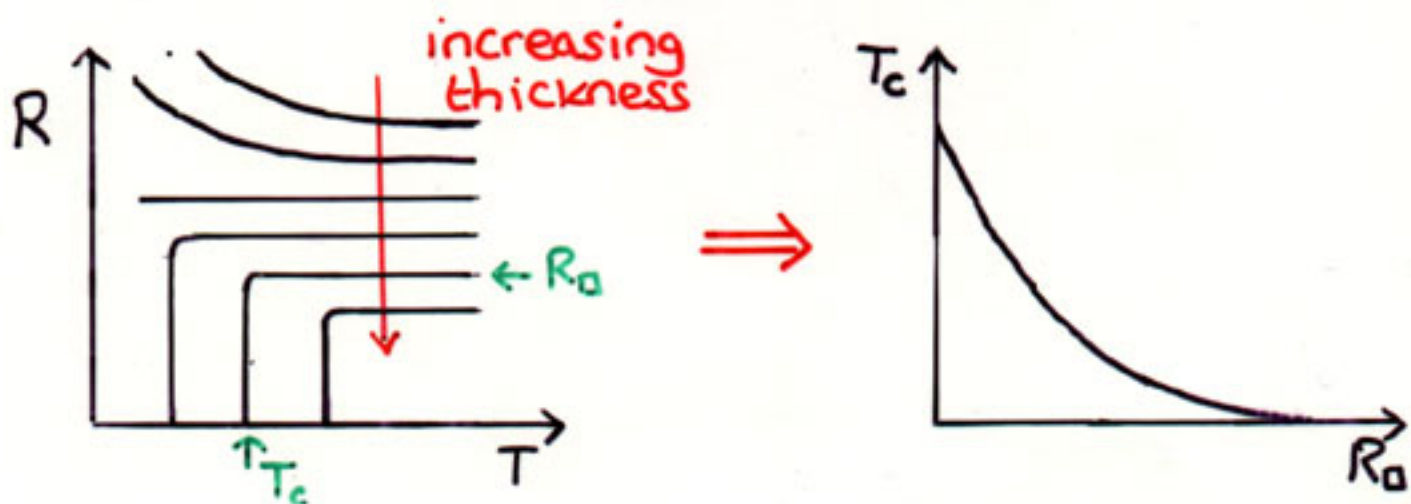
RAS III : Disordered Superconductors: Collective Modes + Strong Coupling

Lecture I

- Expt : superconductor-insulator transition
 T_c suppression $T_c(\lambda, \mu^*, R_0/R_0)$
- Anderson's theorem
- Mechanism of T_c suppression
- Disorder physics - weak localization
 - one-parameter scaling
 - interaction effects
 - superconducting T_c
- Calculating T_c from pair propagator
- Details of first-order perturbation theory - a cautionary note.

Superconductor-Insulator Transition

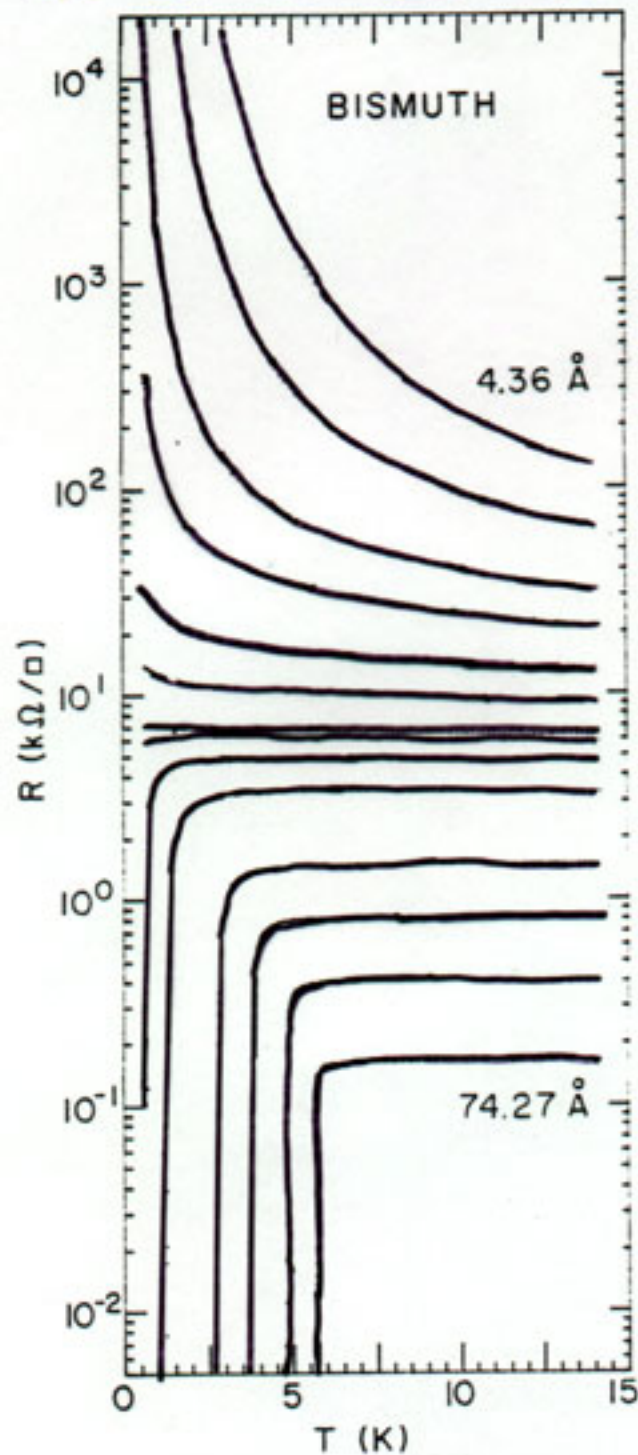
- System is thin metal film ($10-100\text{\AA}$) deposited on substrate by vapour deposition (e.g. Pb/Ge, Sn/Ge)
- As film thickness, $d \downarrow$, normal state resistance per square, $R_0 \uparrow$, and superconductivity is suppressed.



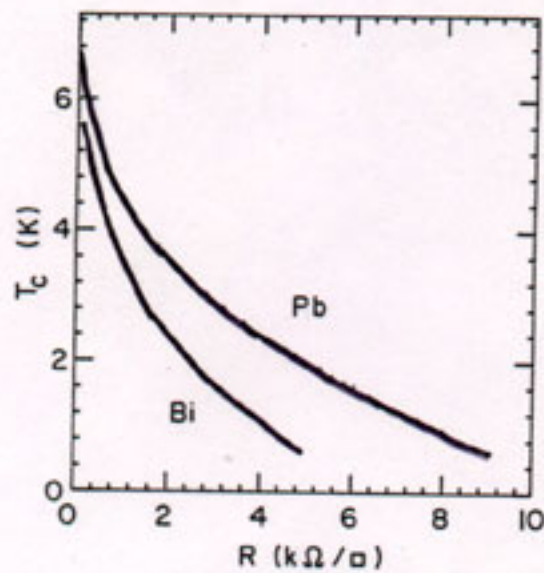
- Complete suppression of superconductivity occurs when R_0 is of order the quantum resistance $R_0 = h/4e^2 \approx 6.5\text{ k}\Omega$ [depends on sample]
- Can treat as extension of previous lectures
 $T_c(\lambda, \mu^*) \rightarrow T_c(\lambda, \mu^*, R_0/R_0)$
- Doesn't this violate Anderson's thm?

Suppression of Superconductivity by Disorder in Thin Film Pb + Bi

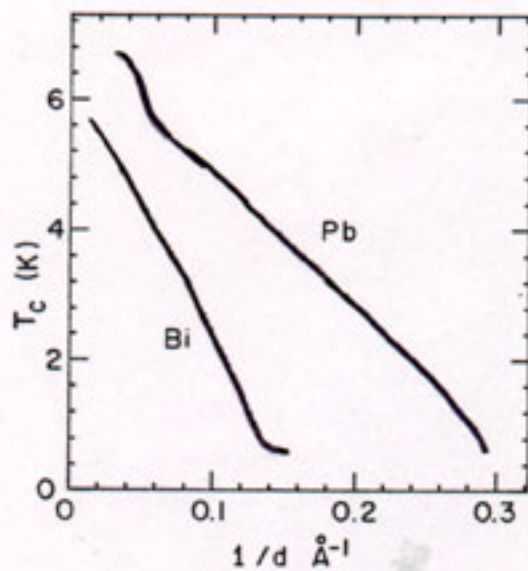
[Haviland et al PRL 62 2180 (1989)]



Suppression of T_c by Disorder in Thin Pb + Bi Films



T_c vs normal state resistance per square, R_0 , of films.



T_c vs inverse thickness, $1/d$, of films.

$\Rightarrow R_0$ correct measure of disorder.

What is Anderson's Thm?

- There are three important energy scales in a superconductor with impurities

$\frac{1}{\tau}$ - elastic scattering rate

T_c - transition temperature

ϵ_F - Fermi energy

$$\frac{1}{\tau} \ll T_c \ll \epsilon_F \Rightarrow \text{clean s.c.}$$

$$T_c \ll \frac{1}{\tau} \ll \epsilon_F \Rightarrow \text{dirty sc}$$

- Basic point of Anderson's thm:

$$\frac{1}{\tau} \sim T_c \Rightarrow T_c \text{ unaffected}$$

$$\frac{1}{\tau_s} \sim T_c \Rightarrow T_c \rightarrow 0$$

- Perturbative parameter for disorder effects we are looking at is $1/\epsilon_F \tau$: Anderson's thm says nothing about this.

- See why R_D/R_0 important parameter:

$$\frac{R_D}{R_0} = \frac{4e^2}{h} \cdot \frac{1}{2N(0)De^2} = \frac{2}{\epsilon_F \tau}$$

What is Physics of T_c Suppression?

- Diffusive motion of electrons leads to inefficient dynamical screening of Coulomb potential - **low- q singularity remains!**
- Discussion in Ambegaokar I gives in 3D

$$V_c(q, \omega) = \frac{V_c(q)}{\epsilon(q, \omega)} = \frac{4\pi e^2}{q^2} \frac{Dq^2 - i\omega}{D\chi_3^2 + Dq^2 - i\omega} \quad \chi_3^2 = 8\pi N(\omega)e^2$$
$$\simeq \underline{\underline{\frac{1}{2N(\omega)} \frac{Dq^2 - i\omega}{Dq^2}}}$$

- Consider BCS eqn for T_c :

$$\underline{T_c = 1.13 \omega_D \exp[-1/N(\omega)[\lambda - \mu^*]]}$$

Enhanced Coulomb repulsion causes $\mu^* \uparrow$ and $N(\omega) \downarrow$ (the latter because Landau quasiparticles short-lived near Fermi surface). Both effects **decrease T_c** .

- Nature of low- q singularity depends on dimension:

$$\epsilon(q, \omega) = 1 - V_c(q) \Pi(q, \omega) \Rightarrow \underline{\underline{\Pi(q, \omega) = -2N(\omega) \frac{Dq^2}{Dq^2 - i\omega}}}$$

$$\Rightarrow \underline{\underline{V_{2D}(q, \omega) = \frac{2\pi e^2}{q} \frac{Dq^2 - i\omega}{Dq\chi_2 + Dq^2 - i\omega}}} \quad \chi_2 = 4\pi N(\omega)e^2$$

Disorder - Screened Coulomb Interaction

- Electric current due to particle number and electric potential gradients

$$\underline{j} = -eD\nabla n - \sigma \nabla \phi$$

- Continuity eqn + Einstein relation gives

$$e \frac{\partial n}{\partial t} = -\nabla \cdot \underline{j} = eD \nabla^2 n + 2N(\omega)De^2 \nabla^2 \phi$$

- Poisson's eqn relates ϕ to total charge density

$$\nabla^2 \phi = -4\pi \rho_{\text{tot}} = -4\pi [en + \rho_{\text{ext}}]$$

$$\Rightarrow \phi(q, \omega) = \frac{4\pi}{q^2} \rho_{\text{tot}}(q, \omega) = \frac{1}{e^2} V_c(q) \rho_{\text{tot}}(q, \omega)$$

- Finally relate ϕ to ρ_{ext} not ρ_{tot} :

$$(Dq^2 - i\omega)(\rho_{\text{tot}} - \rho_{\text{ext}}) = -2N(\omega)Dq^2 V_c(q) \rho_{\text{tot}}$$

$$\Rightarrow \frac{\rho_{\text{ext}}}{\rho_{\text{tot}}} = \epsilon(q, \omega) = 1 + 2N(\omega) \frac{Dq^2}{Dq^2 - i\omega} V_c(q)$$

$$V_c(q, \omega) = \begin{cases} \frac{4\pi e^2}{q^2} \frac{Dq^2 - i\omega}{D\chi_3^2 + Dq^2 - i\omega} & (d=3) & \chi_3^2 = 8\pi N(\omega)e^2 \\ \frac{2\pi e^2}{q} \frac{Dq^2 - i\omega}{Dq\chi_2 + Dq^2 - i\omega} & (d=2) & \chi_2 = 4\pi N(\omega)e^2 \end{cases}$$

A Brief Recap of Diffusion

- Particle current due to gradient in particle number density:

$$\underline{j} = -D \underline{\nabla} n \quad \frac{\partial n}{\partial t} + \underline{\nabla} \underline{j} = 0$$

$$\Rightarrow \underline{\left(\frac{\partial}{\partial t} - D \nabla^2 \right) n(\underline{r}, t) = 0}$$

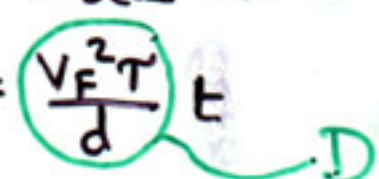
- Green function is solution with initial condition $n(\underline{r}, t=0) = \delta^d(\underline{r})$:

$$\left(\frac{\partial}{\partial t} - D \nabla^2 \right) n(\underline{r}, t) = \delta^d(\underline{r}) \delta(t)$$

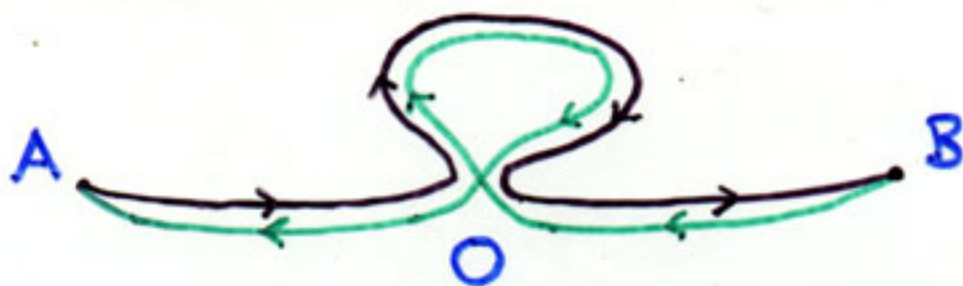
$$\Rightarrow (D q^2 - i\omega) n(q, \omega) = 1$$

$$\Rightarrow \underline{n(\underline{r}, t) = \frac{1}{(4\pi Dt)^{d/2}} \exp[-r^2/4Dt]}$$

- For electron scattering off impurities

$$\begin{aligned} \langle r^2(t) \rangle &= N_{\text{scat}} \langle r^2 \rangle = \frac{t}{\tau} \cdot \frac{\ell^2}{d} \\ &= \left(\frac{v_F^2 \tau}{d} \right) t \end{aligned}$$


Weak Localization



- Localization physics involves phase coherence between different scattering processes.
- Probability to get from A to B is given by Feynman sum over histories

$$p(A \rightarrow B) = \left| \sum_i A_i e^{i\phi_i} \right|^2 = \sum_{i,j} A_i A_j e^{i(\phi_i - \phi_j)}$$

- Simplest approximation assumes all paths phase incoherent [$\equiv$ Drude conductivity]

$$P_0(A \rightarrow B) = \sum_i A_i^2$$

- Consider pairs of self-intersecting paths shown in figure : get probability $(A+A)^2$ not $A^2 + A^2$ i.e. extra $2|A|^2$ probability to remain at O.

- Fractional suppression of conductivity is ratio of phase volume $V_F \lambda_F^{d-1} dt$ of loop with crosssectional area λ_F^{d-1} to volume $(Dt)^{d/2}$ of all diffusive paths

$$\frac{\Delta\sigma}{\sigma_0} = - \int_0^{\tau_\phi} \frac{dt V_F \lambda_F^{d-1}}{(Dt)^{d/2}} \quad \tau_\phi - \text{coherence time}$$

- At $T=0$ for cube of size L^d , upper cutoff in time is $t_0 = L^2/D$:

$$\frac{\Delta\sigma}{\sigma_0} = - \frac{V_F \lambda_F^{d-1}}{D} \int_{\ell^2}^{L^2} \frac{d(Dt)}{(Dt)^{d/2}} = - \frac{V_F \lambda_F^{d-1}}{D} \left[\frac{L^{2-d} - \ell^{2-d}}{2-d} \right]$$

- Since DoS at Fermi surface is given by

$$N(0) = \frac{dn}{d\varepsilon_F} = \frac{dk_F}{d\varepsilon_F} \frac{dn}{dk_F} = \frac{1}{v_F} \frac{d}{dk_F} (k_F^d) = \frac{1}{v_F \lambda_F^{d-1}}$$

relative size of quantum interference is

$$\frac{V_F \lambda_F^{d-1}}{DL^{d-2}} = \frac{1}{N(0)DL^{d-2}} = \frac{1}{(\sigma L^{d-2})/(e^2/h)} = \frac{1}{g}$$

ie inverse dimensionless conductance is the small parameter.

• Results in $d=1,2,3$ are

$$\delta\sigma(L) = \begin{cases} -\frac{e^2}{\pi^2\hbar} L & d=1 \\ -\frac{e^2}{\pi^2\hbar} \ln\left(\frac{L}{\ell}\right) & d=2 \\ \text{const} + \frac{e^2}{2\pi^2\hbar} \frac{1}{L} & d=3 \end{cases}$$

• At finite T upper cutoff is phase coherence time τ_ϕ . Since $1/\tau_\phi \sim T^p$ 2D localization correction is

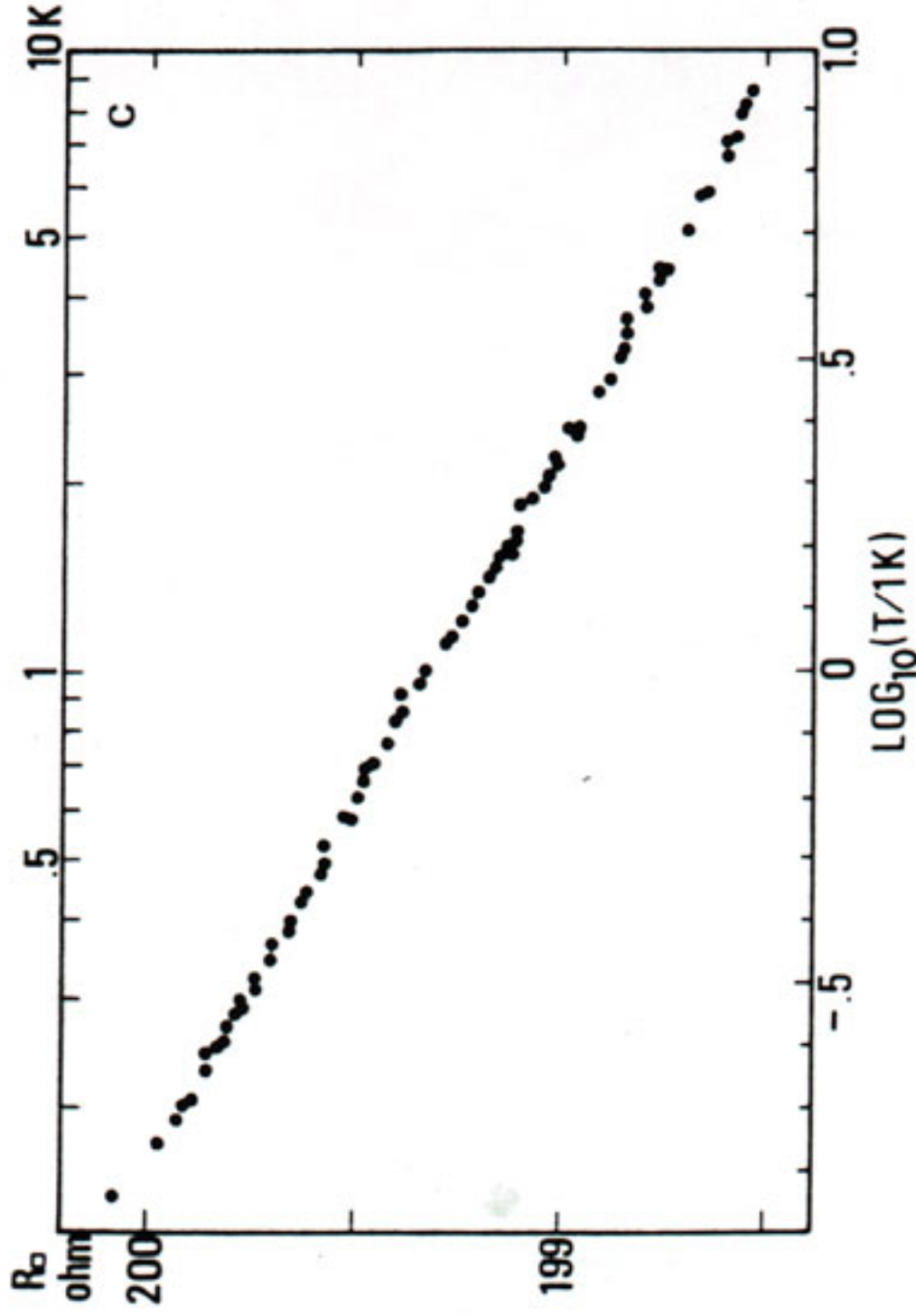
$$\underline{\underline{\delta\sigma_w(T) = -\frac{e^2}{2\pi^2\hbar} p \ln\left(\frac{T}{T_0}\right)}}$$

• Perpendicular magnetic fields destroy weak localization by scrambling phases of loops $\rightarrow$ negative magnetoresistance

• In 2D small parameter $1/g$ identical to $1/E_F\tau$ discussed earlier

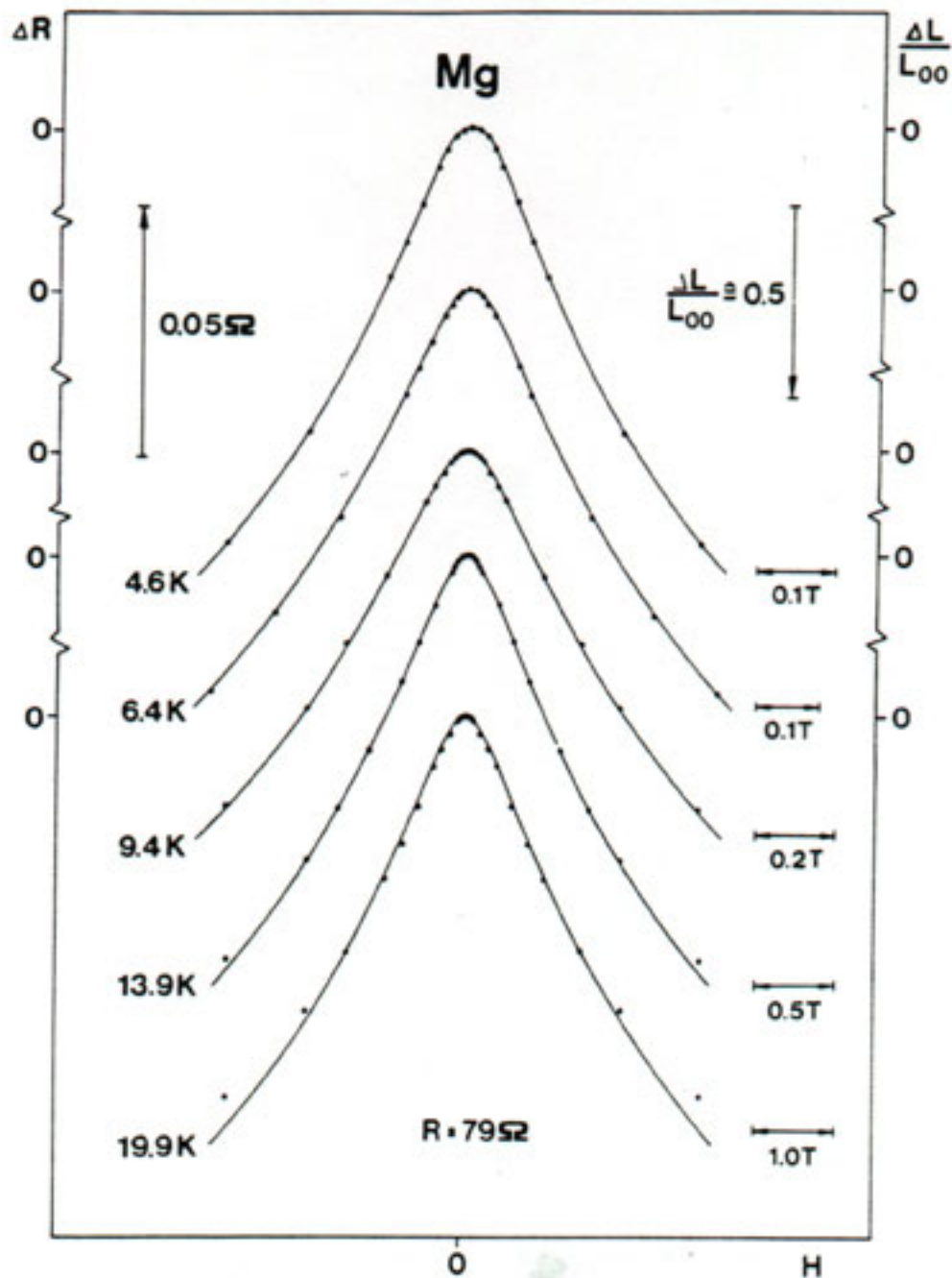
$$\underline{\underline{\frac{1}{g} = \frac{v_F \lambda_F}{D} = \frac{v_F}{k_F v_F^2 \tau} = \frac{1}{k_F \ell} \equiv \frac{1}{E_F \tau}}}$$

Temperature Dependence of Resistance in Coupled Fine Cu Particles



[Kobayashi et al J Phys Soc Jpn 49 1635 (1980)]

Negative Magnetoresistance
of Mg Film at Various Temperatures
[Bergmann PRB25 2937(1982)]

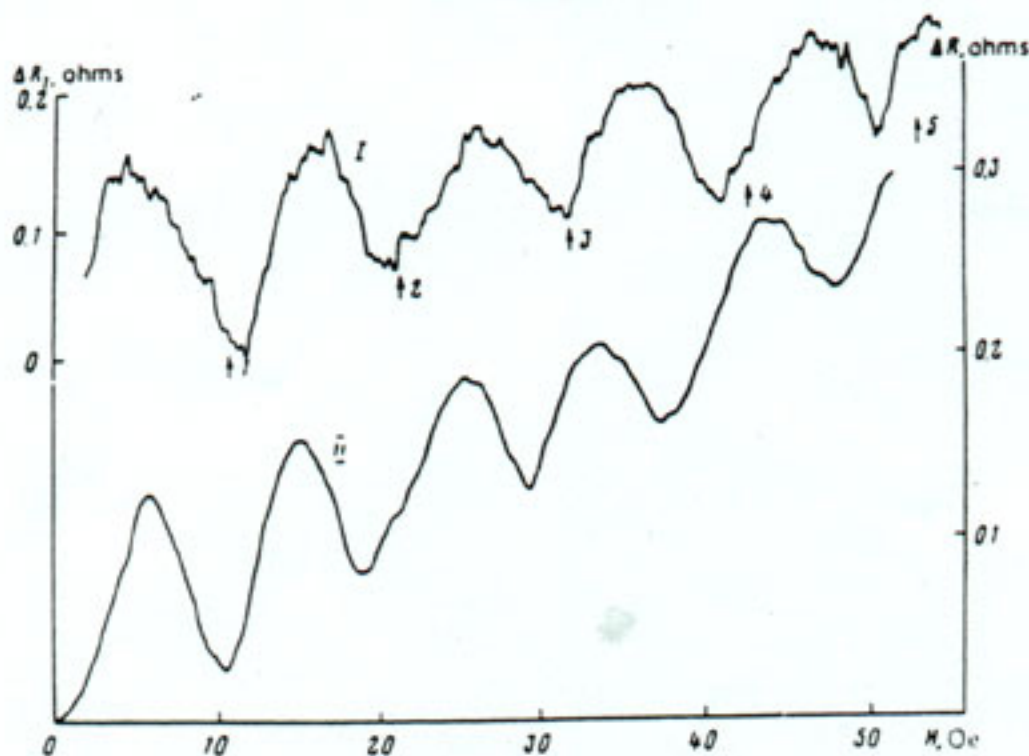
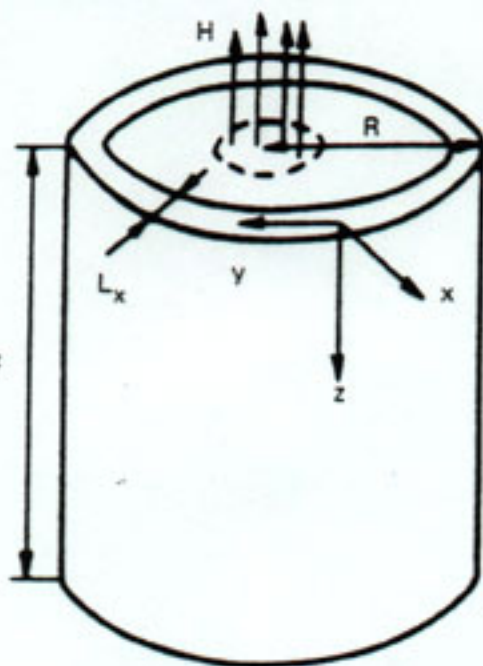


points = expt lines = theory

Oscillatory Magnetoresistance of a Disordered Mg Cylinder

[Sharvin + Sharvin JETP Lett 34 272 (1981)]

flux period
 $\phi = h/2e$
 [reversed paths
 each enclose
 flux]



Scaling Theory of Localization

[Abrahams et al PRL 1979]

- Quantum interference effects evolve continuously with scale L , their effective size being of order $1/g(L)$.
- If we change $L \rightarrow L + \delta L$ so that $g \rightarrow g + \delta g$ the change δg is a function only of $g(L)$:

$$\frac{1}{g(L)} \frac{\delta g}{\delta L} \rightarrow \underline{\underline{\frac{d \ln g(L)}{d \ln L} = \beta[g(L)]}}$$

- For $g \gg 1$ have good conductor:

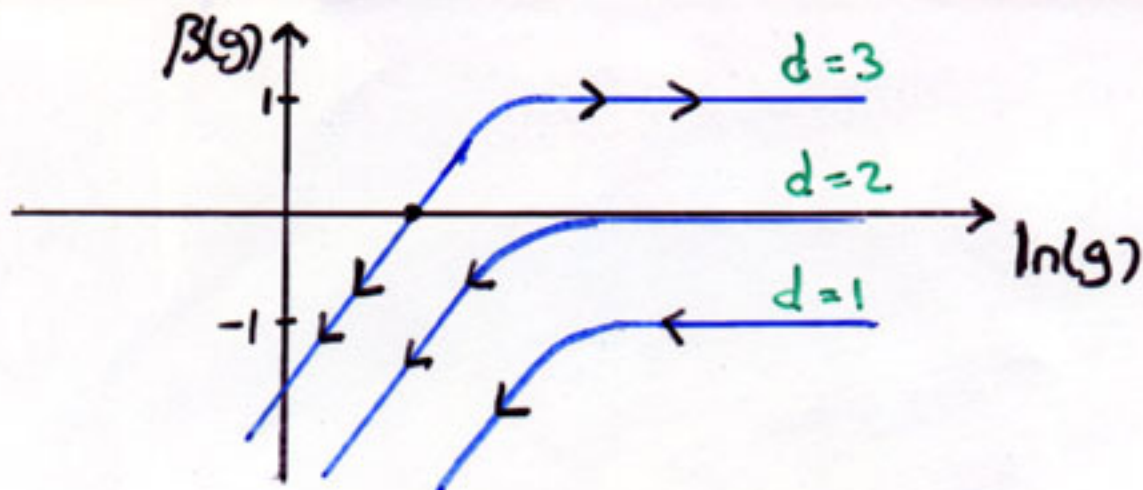
$$\underline{g(L) = \sigma L^{d-2}/e^2} \Rightarrow \underline{\beta(g) = d-2}$$

- For $g \ll 1$ have exponential localization:

$$\underline{g(L) = g_0 e^{-L/\xi}} \Rightarrow \underline{\beta(g) = \ln(g/g_0)}$$

- Weak localization corrections

$$\underline{g(L) = \frac{\sigma L^{d-2}}{e^2} - \frac{a}{2-d} \left[1 - \left(\frac{L}{L_0} \right)^{2-d} \right]} \Rightarrow \underline{\beta(g) = d-2 - a/g}$$



• In 1D and 2D see that system always scales to insulator as $L \rightarrow \infty$.

• Can estimate localization length ξ by assuming $\sigma(L=\xi) = 0$:

$$\sigma_{1D}(L) = \frac{ne^2\tau}{m} - \frac{e^2}{\pi\hbar} L \Rightarrow \xi \approx l \quad (1D)$$

$$\approx (\hbar^2 A) l \quad (\text{quasi-1D})$$

$$\sigma_{2D}(L) = \frac{ne^2\tau}{m} - \frac{e^2}{\pi^2\hbar} \ln\left(\frac{L}{l}\right) \Rightarrow \xi \approx l e^{\frac{\pi}{k_F} l} \quad (2D)$$

• In 3D scale to conducting or insulating behaviour if $g(l) > g^*$ or $g(l) < g^*$ respectively where $\beta(g^*) = 0$.

• For $g(l)$ close to g^* get scaling behaviour.

Suppression of DoS by Interaction

[Aronov Physica 126B 314 (1984)]

- Consider suppression of DoS at Fermi surface due to repulsive Coulomb interaction
- Two states of energy difference ω that are connected by Coulomb interaction will remain coherent for times $\tau < t < \hbar/\omega$
- The effective interaction will be enhanced from its bare value by the probability of two electrons being at the same point:

$$P_{\omega} = \int_{\tau}^{\hbar/\omega} dt \frac{V_F \lambda_F^{d-1}}{(Dt)^{d/2}}$$
$$= \frac{1}{N(\omega) D^{d/2}} \left[\frac{\tau^{1-d/2} - (\hbar/\omega)^{1-d/2}}{1-d/2} \right]$$

- For featureless potentials V in 2D+3D relative suppression of DoS proportional to enhanced effective potential:

$$\frac{\delta N(\epsilon)}{N(0)} = \begin{cases} -\frac{N(0)V}{N(0)D} \ln\left(\frac{1}{\epsilon\tau}\right) & 2D \\ \frac{N(0)V}{N(0)D} \left(\frac{\epsilon}{D}\right)^{1/2} & 3D \end{cases}$$

- Long-range behaviour in disorder screened Coulomb interaction enhances suppression:

$$\underline{\frac{\delta N(\epsilon)}{N(0)} = -\frac{1}{N(0)D} \ln\left(\frac{1}{\epsilon\tau}\right) \ln\left(\frac{D^2\chi^2\tau}{\epsilon}\right)}$$

- To see effect on T_c go back to BCS self-consistency eqn:

$$\begin{aligned} \frac{1}{\lambda} &= \int d\epsilon N(\epsilon) T \sum_{\omega} \frac{1}{\omega^2 + \epsilon^2} \\ &= \underbrace{N(0) \int d\epsilon T \sum_{\omega} \frac{1}{\omega^2 + \epsilon^2}}_{N(0) \ln\left(\frac{1.13 \omega_D}{T_c}\right)} + N(0) \int d\epsilon \frac{\delta N(\epsilon)}{N(0)} T \sum_{\omega} \frac{1}{\omega^2 + \epsilon^2} \end{aligned}$$

- To accuracy needed in second term:

$$T \sum_{\omega} \frac{1}{\omega^2 + \epsilon^2} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{\omega^2 + \epsilon^2} = \frac{1}{2\epsilon}$$

$$\Rightarrow \ln\left(\frac{T_c}{T_{c0}}\right) = \begin{cases} -\frac{R_0}{R_0} \ln^2\left(\frac{1}{2\pi T_c \tau}\right) \leftarrow 2D \\ -\frac{R_0}{R_0} \left\{ \frac{1}{3} \ln^3\left(\frac{1}{2\pi T_c \tau}\right) + \frac{1}{2} \ln^2\left(\frac{1}{2\pi T_c \tau}\right) \ln(D^2 \chi^2 \tau^2) \right\} \leftarrow 2D \text{ screened Coulomb} \\ -A(\rho/\rho_0)^{3/2} \leftarrow 3D \end{cases}$$

- Consider also effect from increase in Coulomb pseudopotential:

$$\ln\left(\frac{T_c}{1.13\omega_D}\right) = -\frac{1}{N(0)[\lambda - \mu^*]}$$

$$\Rightarrow \ln\left(\frac{T_c}{T_{c0}}\right) = \left\{ \frac{1}{N(0)[\lambda - \mu^*]} \right\}^2 \cdot -\frac{\delta\mu^*}{\mu^*}$$

$$\sim -\frac{R_0}{R_0} \ln^3\left(\frac{1}{2\pi T_c \tau}\right) \leftarrow 2D$$

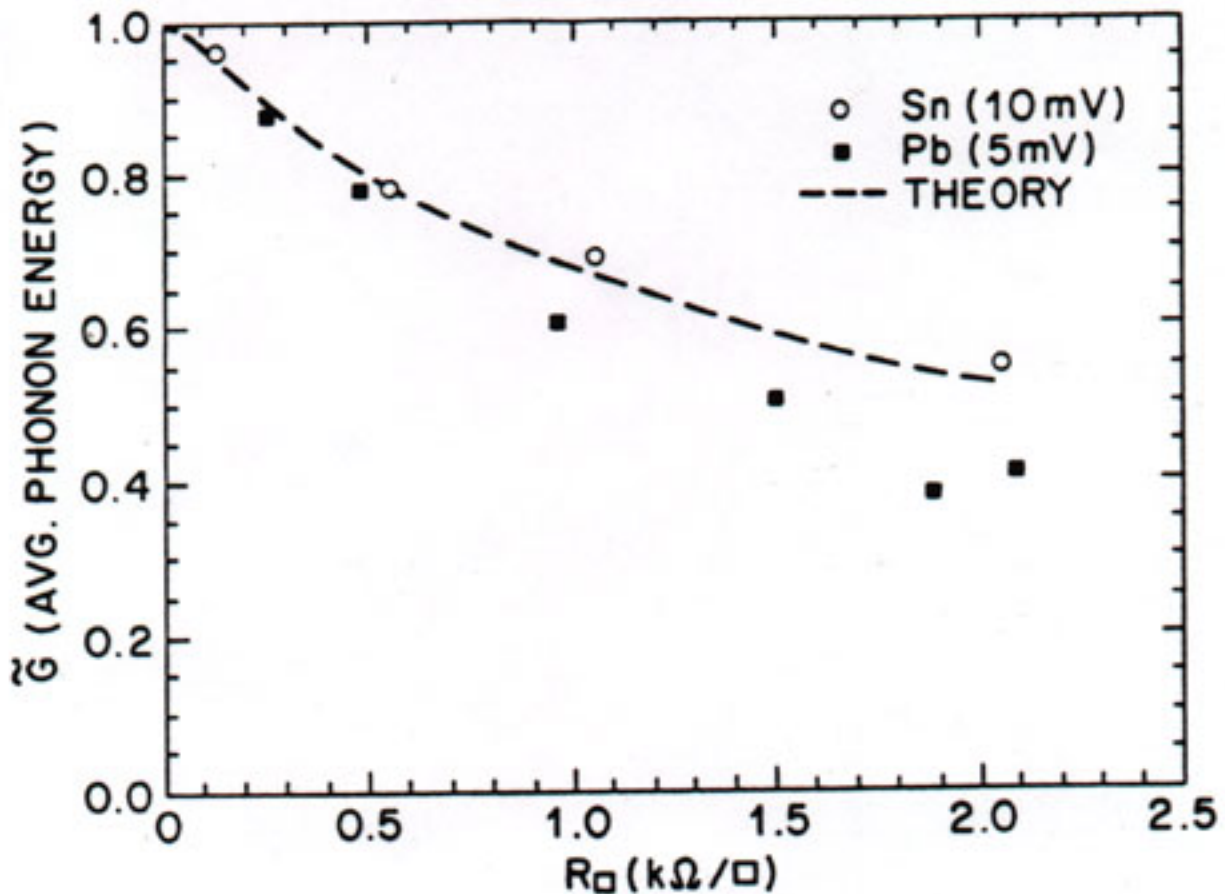
- Will need to consider self-energy and pseudopotential contributions, and details of long-range Coulomb
- Prediction for featureless potential turns out to be correct!! Why?

$$\ln\left(\frac{T_c}{T_{c0}}\right) = -\frac{R_0}{R_0} \left\{ \ln^3\left(\frac{1}{2\pi T_c \tau}\right) + \ln^2\left(\frac{1}{2\pi T_c \tau}\right) \right\}$$

↑
pseudopotl
↑
DoS

Suppression of One Electron DoS in Thin Pb + Sn Films

[Valles et al PRB40 6680 (1989)]



Plot of normalised DoS obtained from tunneling data vs resistance per square of film.

How We Calculate T_c

- Evaluate electron pair propagator and find temperature at which it diverges i.e. bounce electrons off each other till they stick

$$= \text{crossed line} + \text{crossed line with bubble} + \text{crossed line with two bubbles}$$

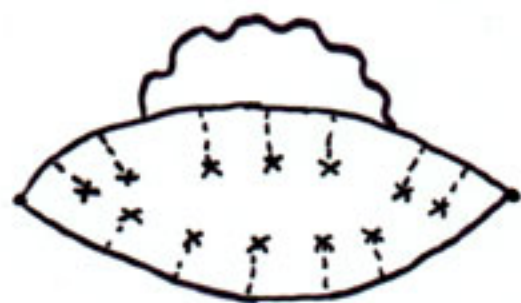
$$\Rightarrow \underline{L(q, i\eta) = [\lambda^{-1} - P(q, i\eta)]^{-1}} \quad L(0,0) = \infty \Rightarrow T = T_c$$

- Zeroth-order (mean field) contribution to pair polarisation bubble $P(q, i\eta)$ is

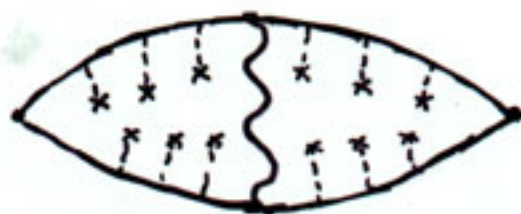
$$\Rightarrow \underline{L_0(q, i\eta)^{-1} = N(0) \left[\ln\left(\frac{T}{T_c}\right) + \psi\left(\frac{1}{2} + \frac{Dq^2 + |\eta|}{4\pi T}\right) - \psi\left(\frac{1}{2}\right) \right]}$$

P_0

- First-order perturbative contribution will come from disorder average of density of states and pseudopotential terms



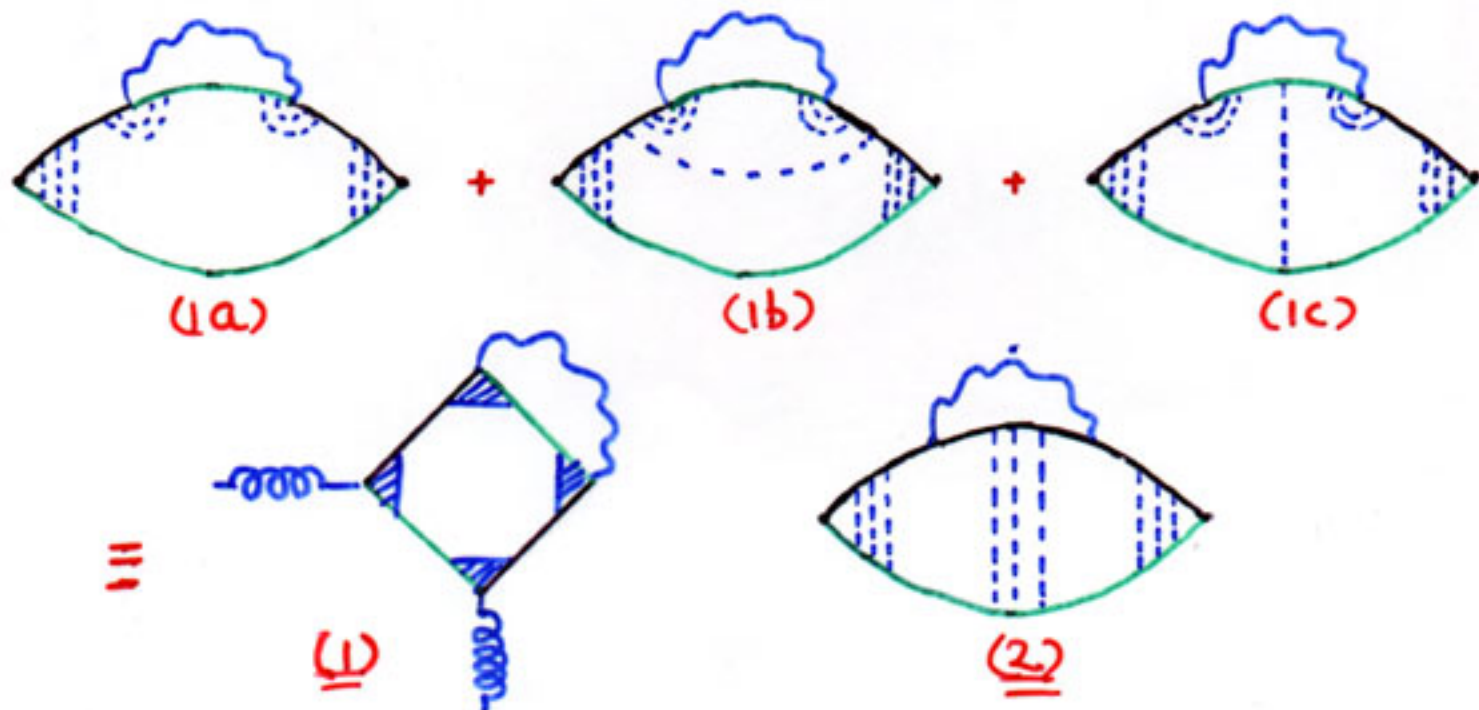
density of states



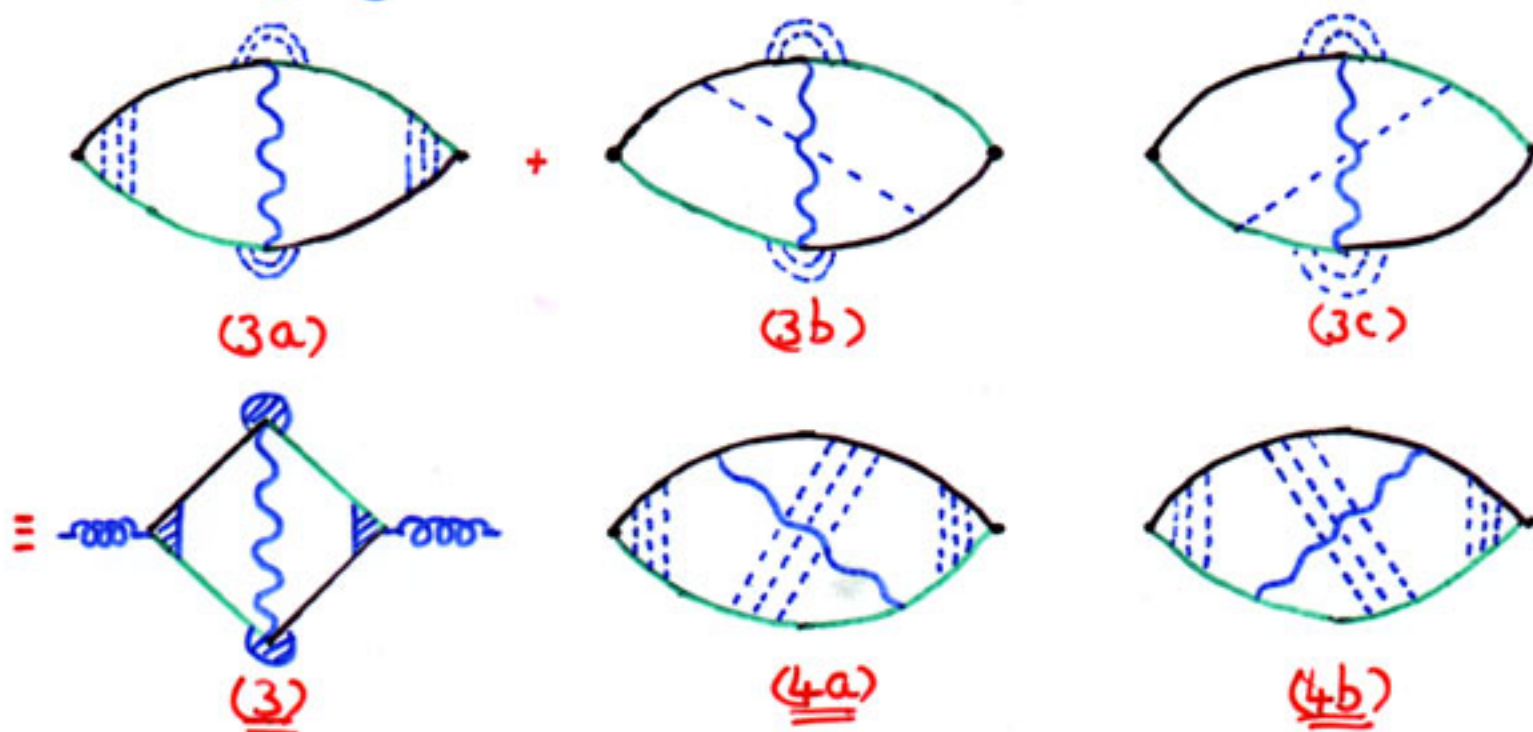
Coulomb pseudopot!

First-Order Perturbation Theory

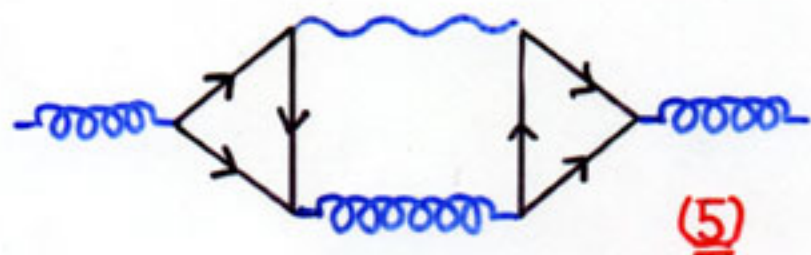
• Averaging DoS diagrams gives :



• Averaging pseudopotential diagrams gives



- Careful analysis shows existence of one more contribution:



- Understand as follows:
 - superconducting fluctuations reduce $N(0)$
 - above diagram is Coulomb suppression of this
- This diagram is very important as adding diagrams 1-5 gives cancellation of $1/q^2$ singularity of screened Coulomb p.t.
- We will show that the cancellation is due to gauge invariance.
- Upshot is that we can ignore diagram 5 and treat Coulomb as constant: $N(0)V = 1/2$.

Diagram 1



$$P_i = 2T \sum_{\omega} T \sum_{\Omega} \sum_q 2\pi N(0) \tau^4 [Dq^2 + 3|\omega| + |\omega + \Omega|] \\ \times \left(\frac{1}{2|\omega|\tau} \right)^2 \left(\frac{1}{[Dq^2 + |\omega| + |\omega + \Omega|]\tau} \right)^2 (-V) \Theta(-\omega(\omega + \Omega))$$

- For $\Omega > 0$ need $0 > \omega > -\Omega$
 $\Omega < 0$ need $0 < \omega < -\Omega$
 In either case $Dq^2 + |\omega| + |\omega + \Omega| = Dq^2 + |\Omega|$

$$P_i = -2\pi N(0)V T \sum_{\Omega > 0} T \sum_{\Omega > \omega > 0} \sum_q \left[\frac{1}{|\omega|^2 (Dq^2 + |\Omega|)} + \frac{2}{|\omega| (Dq^2 + |\Omega|)^2} \right]$$

- Do q -sum by noting in 2D:

$$\sum_q \equiv \int \frac{2\pi q dq}{(2\pi)^2} = \int \frac{d(Dq^2)}{4\pi D}$$

and cut off at $Dq^2 = 1/\tau$.

- Write Matsubara frequencies $\Omega = 2\pi T m$ and $\omega = 2\pi T(\ell + 1/2)$, cutting off Ω sum at $\Omega = 1/\tau$ to give:

$$P_i = - \frac{2\pi N(0) V T^2}{(4\pi D)(2\pi T)^2} \sum_{m=1}^M \sum_{\ell=0}^{m-1} \left[\frac{1}{(\ell + \frac{1}{2})^2} \ln\left(\frac{M}{m}\right) + \frac{2}{(\ell + \frac{1}{2})m} \right]$$

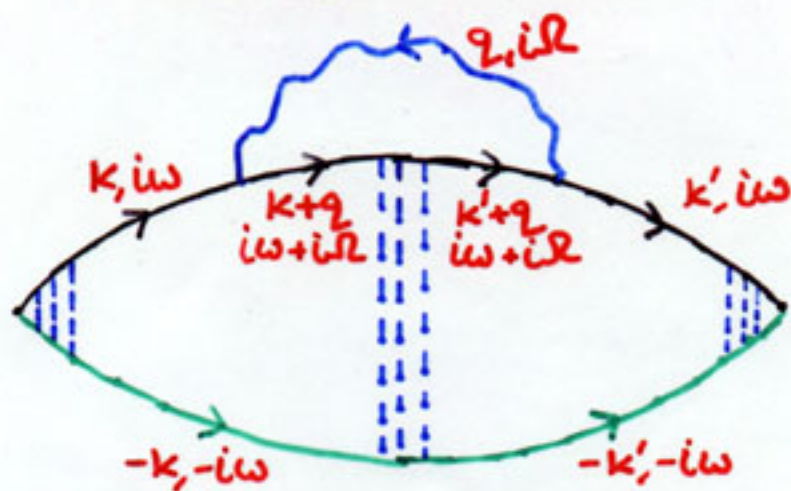
- Since $N(0)V = \frac{1}{2}$, doing ℓ -sum gives

$$\frac{P_i}{N(0)} = - \frac{1}{16\pi^2 N(0) D} \sum_{m=1}^M \left\{ [\psi'(\frac{1}{2}) - \psi'(\frac{1}{2} + m)] \ln\left(\frac{M}{m}\right) + [\psi(\frac{1}{2} + m) - \psi(\frac{1}{2})] \frac{2}{m} \right\} \quad M = \frac{1}{2\pi T \tau}$$

- Finally approximating sums by integrals

$$\boxed{\frac{P_i}{N(0)} = - \frac{3}{32\pi^2 N(0) D} \ln^2 M}$$

Diagram 2



$$P_2 = 2 T \sum_{\omega} T \sum_{\Omega} \sum_q \left(\sum_k G_R(k)^2 G_A(k) \right) \left(\sum_{k'} G_R(k')^2 G_A(k') \right) \\ \times \left(\frac{1}{2|\omega|\tau} \right)^2 \frac{1}{2\pi N(0) \tau^2 (Dq^2 + |\omega| + |\omega + \Omega|)} (-V) \theta(\omega(\omega + \Omega))$$

- k and k' sums give $(2\pi N(0) \tau^2 i)^2$
- Can swap relative sign of $\omega, \omega + \Omega$ to $\theta(-\omega(\omega + \Omega))$
 For $\Omega > 0$ need $0 > \omega > -\Omega$
 $\Omega < 0$ need $0 < \omega < -\Omega$
 In both cases $Dq^2 + |\omega| + |\omega + \Omega| = Dq^2 + |\Omega|$

$$\Rightarrow P_2 = +2\pi N(0) V T \sum_{\Omega > 0} T \sum_{\Omega > \omega > 0} \sum_q \frac{1}{|\omega|^2 (Dq^2 + |\Omega|)}$$

- Do q sum by noting in 2D:

$$\sum_q \equiv \int \frac{2\pi q dq}{(2\pi)^2} = \int \frac{d(Dq^2)}{4\pi D}$$

and cut off at $Dq^2 = 1/\tau$.

- Write Matsubara frequencies $\Omega = 2\pi T m$ and $\omega = 2\pi T (\ell + 1/2)$, cutting off Ω -sum at $\Omega = 1/\tau$ to give

$$P_2 = \frac{2\pi N(\omega) V T^2}{(4\pi D)(2\pi T)^2} \sum_{m=1}^M \sum_{\ell=0}^{m-1} \frac{1}{(\ell + \frac{1}{2})^2} \ln\left(\frac{M}{m}\right) \quad M = \frac{1}{2\pi T \tau}$$

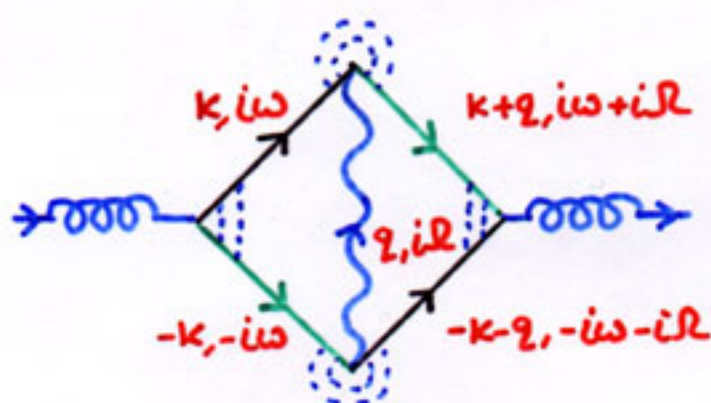
- Since $N(\omega) V = 1/2$, doing ℓ -sum gives

$$\frac{P_2}{N(\omega)} = \frac{1}{16\pi^2 N(\omega) D} \sum_{m=1}^M [\psi'(\frac{1}{2}) - \psi'(\frac{1}{2} + m)] \ln\left(\frac{M}{m}\right)$$

- Finally approximating sums by integrals

$$\boxed{\frac{P_2}{N(\omega)} = \frac{1}{32\pi^2 N(\omega) D} \ln^2(M)}$$

Diagram 3



Hikami box

$$P_3 = T \sum_{\omega} T \sum_{\Omega} \sum_{\mathbf{q}} 2\pi N(\omega) \tau^4 [2Dq^2 + 2|\omega| + 2|\omega + \Omega|]$$

$$\times \left(\frac{1}{2|\omega|\tau}\right) \left(\frac{1}{2|\omega + \Omega|\tau}\right) \frac{1}{(Dq^2 + |\omega| + |\omega + \Omega|)^2 \tau^2} (-V) \theta(-\omega(\omega + \Omega))$$

- For $\Omega > 0$ need $0 > \omega > -\Omega$ (so set $\omega \rightarrow -\omega$)
- $\Omega < 0$ need $0 < \omega < -\Omega$ (so set $\Omega \rightarrow -\Omega$)

In both cases $Dq^2 + |\omega| + |\omega + \Omega| = Dq^2 + |\Omega|$

$$\Rightarrow P_3 = -2\pi N(\omega) V T \sum_{\Omega > 0} T \sum_{\omega > 0} \sum_{\mathbf{q}} \frac{1}{\omega(\Omega - \omega)(Dq^2 + \Omega)}$$

- Do q -sum by noting in 2D:

$$\sum_{\mathbf{q}} \equiv \int \frac{2\pi q dq}{(2\pi)^2} = \int \frac{d(Dq^2)}{4\pi D}$$

and cut off at $Dq^2 = 1/\tau$.

- Write Matsubara frequencies $\Omega = 2\pi T m$ and $\omega = 2\pi T(\ell + 1/2)$, cutting off Ω sum at $\Omega = 1/\tau$:

$$\underline{P_3 = - \frac{2\pi N(0) V T^2}{(4\pi D)(2\pi T)^2} \sum_{m=1}^M \sum_{\ell=0}^{m-1} \frac{1}{(\ell + \frac{1}{2})(m - \ell - \frac{1}{2})} \ln\left(\frac{M}{m}\right)}$$

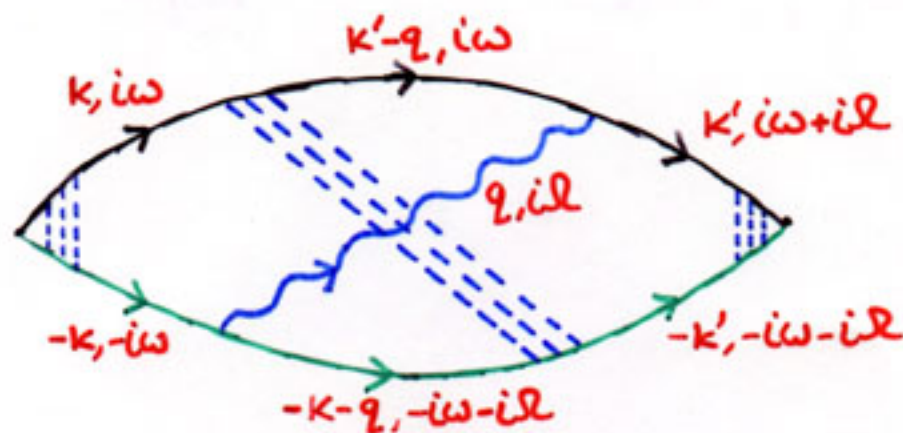
- Since $N(0)V = 1/2$ doing ℓ sum gives

$$\underline{\frac{P_3}{N(0)} = - \frac{1}{8\pi^2 N(0) D} \sum_{m=1}^M \frac{1}{m} [\psi(\frac{1}{2} + m) - \psi(\frac{1}{2})] \ln\left(\frac{M}{m}\right)}$$

- Approximating sums by integrals and noting that $\psi(\frac{1}{2} + m) \sim \ln m$ for large m gives

$$\boxed{\frac{P_3}{N(0)} = - \frac{1}{48\pi^2 N(0) D} \ln^3(M)}$$

Diagram 4



$$P_4 = 2T \sum_{\omega} T \sum_{\Omega} \sum_{\mathbf{q}} \left(\sum_{\mathbf{k}} G_R(\mathbf{k}) G_A(\mathbf{k})^2 \right) \left(\sum_{\mathbf{k}'} G_R(\mathbf{k}')^2 G_A(\mathbf{k}') \right) \\ \times \left(\frac{1}{2|\omega|\tau} \right) \left(\frac{1}{2|\omega+\Omega|\tau} \right) \frac{1}{2\pi N(0) T^2 (Dq^2 + |\omega| + |\omega+\Omega|)} (-V) \Theta(\omega(\omega+\Omega))$$

- $\mathbf{k}$ and $\mathbf{k}'$ sums give $(-2\pi N(0) T^2 i)(2\pi N(0) T^2 i)$.
- Can swap relative sign of $\omega, \omega+\Omega$ to $\Theta(-\omega(\omega+\Omega))$.
For $\Omega > 0$ need $0 > \omega > -\Omega$ (so set $\omega \rightarrow -\omega$)
 $\Omega < 0$ need $0 < \omega < -\Omega$
- In both cases $Dq^2 + |\omega| + |\omega+\Omega| = Dq^2 + |\Omega|$.

$$\Rightarrow P_4 = -2\pi N(0) V T \sum_{\Omega > 0} T \sum_{\omega > 0} \sum_{\mathbf{q}} \frac{1}{\omega(\Omega - \omega)(Dq^2 + \Omega)}$$

- Do q sum by noting in 2D:

$$\sum_q \equiv \int \frac{2\pi q dq}{(2\pi)^2} = \int \frac{d(Dq^2)}{4\pi D}$$

and cut off at $Dq^2 = 1/\tau$.

- Write Matsubara frequencies $\Omega = 2\pi Tm$ and $\omega = 2\pi T(\ell + 1/2)$, cutting off Ω sum at $\Omega = 1/\tau$:

$$\underline{P_4 = -\frac{2\pi N(\omega) V T^2}{(4\pi D)(2\pi T)^2} \sum_{m=1}^M \sum_{\ell=0}^{m-1} \frac{1}{(\ell + \frac{1}{2})(m - \ell - \frac{1}{2})} \ln\left(\frac{M}{m}\right)}$$

- Since $N(\omega)V = 1/2$ doing ℓ -sum gives

$$\underline{\frac{P_4}{N(\omega)} = -\frac{1}{8\pi^2 N(\omega) D} \sum_{m=1}^M \frac{1}{m} [\psi(\frac{1}{2} + m) - \psi(\frac{1}{2})] \ln\left(\frac{M}{m}\right)}$$

- Approximating sums by integrals and noting that $\psi(\frac{1}{2} + m) \sim \ln m$ for large m gives

$$\boxed{\frac{P_4}{N(\omega)} = -\frac{1}{48\pi^2 N(\omega) D} \ln^3(M)}$$

Summary of Perturbation Theory

- Sum of 4 diagrams gives:

$$\ln\left(\frac{T_c}{T_\infty}\right) = -t \sum_{m=1}^M \sum_{\ell=0}^{m-1} \frac{1}{m(\ell + \frac{1}{2})} \left[1 + 2\ln\left(\frac{M}{m}\right)\right]$$

where $M = \frac{1}{2\pi T_c \tau}$ and $t = \frac{1}{8\pi^2 N(0) D}$.

- Einstein's relation gives $\sigma = 2e^2 N(0) D$ so

$$t = \frac{1}{4\pi^2} \cdot \frac{e^2}{\hbar} \cdot \frac{1}{\sigma} = \frac{R_D}{R_0} \quad \text{where} \quad R_0 = \frac{4\pi^2 \hbar}{e^2} = 162 \text{ k}\Omega$$

- Approximating sums by integrals gives:

$$\ln\left(\frac{T_c}{T_\infty}\right) = -\frac{R_D}{R_0} \left[\frac{1}{3} \ln^3\left(\frac{1}{2\pi T_c \tau}\right) + \frac{1}{2} \ln^2\left(\frac{1}{2\pi T_c \tau}\right) \right]$$

- Need input parameter $\beta = \frac{1}{2\pi T_\infty \tau}$ in terms of which above eqn is

$$x = t \left[\frac{1}{3} (x + \beta)^3 + \frac{1}{2} (x + \beta)^2 \right] \quad x = \ln\left(\frac{T_{c0}}{T_c}\right)$$