Lecture III (V in A-S series) <u>Introduction to Strong Coupling</u>

1. General remarks and Model
2. Cooper instability and T_c; without and with impurities
3. Below T_c

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1. Developments by Migdal & Eliashberg very soon after BCS allowed one to understand deviations from 'Laws of Corresponding States' in materials like lead and mercury.

<u>Example</u> $\dfrac{2\Delta(0)}{T_c} = 2\pi e^{-\gamma} = 3.53$ (BCS)

~ 4.2 (Pb expt.) explained by theory

<u>Explanation</u> Stronger electron-phonon coupling leads to breakdown of quasi-particle approximation. Electron lifetimes become short in the sense $\tau(\omega_D).\omega_D \lesssim 1$ so that dynamics of phonon clouds surrounding electrons enters Cooper pair formation.

Aim of the last two lectures in the Ambegaokar-Smith series is to show how the problem of T_c suppression can be grafted on to these old considerations

<u>Model</u> Assume spherical Fermi surface in extended zone scheme.

$$H = \sum_{k\sigma} \varepsilon_k c^+_{k\sigma} c_{k\sigma} + \sum_{q\lambda} \omega_{q\lambda} (a^+_{q\lambda} a_{q\lambda} + \tfrac{1}{2})$$
$$+ \sum_{\substack{kk' \\ \sigma\lambda}} g_{kk'\lambda} c^+_{k\sigma} c_{k'\sigma} [a_{q\lambda} + a^+_{-q\lambda}]$$
$$+ \frac{1}{2} \sum_{\substack{kk'q \\ \sigma\sigma'}} V(q) c^+_{k\sigma} c^+_{k'+q\sigma'} c_{k'\sigma'} c_{k+q\sigma} \qquad (\text{III}.1)$$

<u>Phonon Green Function</u>

$$\mathcal{D}_\lambda(q,t) = -i \left\langle T \left(a_\lambda(q t) + a^+_\lambda(-q,t) \right) \left(a^+_\lambda(q 0) + a_\lambda(-q,0) \right) \right\rangle$$

Matsubara transform $\qquad (\text{III}.2)$

$$\mathcal{D}_\lambda(q, i\Omega_n) = \frac{2\omega_q}{(i\Omega_n)^2 - \omega_q^2} \qquad \Omega_n = \frac{2\pi n}{T}$$

$$n = 0, \pm 1, \pm 2 \ldots$$

Above ω_q are the renormalized phonons (including screening by electrons) and $g_{kk'\lambda}$ is the renormalized coupling constant.

Phonon effects will be expressed in terms of averages of $\mathcal{D}$ and g. Coulomb effects will be treated in the lowest relevant order

Fermi surface averages

$$\langle f(\vec{k})\rangle_k \equiv \int_{S_F} \frac{d^2k}{v_k} f(k) \Big/ \int\int_{S_F} \frac{d^2k}{v_k} = \frac{1}{N(0)} \int_{S_F} \frac{d^2k}{v_k} f(k)$$

Note that for weakly $|\vec{k}|$ dependent quantities

$$\sum_k = \int d\varepsilon_k \int_{S(\varepsilon)} \frac{d^2k}{v_k} \simeq \int d\varepsilon_k \int_{S_F} \frac{d^2k}{v_k} \qquad (\text{III}.3)$$

El-phonon information contained in the dimensionless function

$$\alpha^2 F(\omega) \equiv N(0) \left\langle \sum_\lambda |g_{kk'\lambda}|^2 \delta(\omega - \omega_{k-k',\lambda}) \right\rangle_{kk'} \qquad (\text{III}.4)$$

Note that at $T=0$ the electron lifetime due to phonon emission is

$$\frac{1}{\tau(\omega)} = 2\pi \int_0^\omega d\omega' \, \alpha^2 F(\omega') \qquad (\text{III}.5)$$

Strong coupling materials are precisely those for which $\tau(\omega_D)^{-1} \gtrsim \omega_D$ where ω_D is the phonon band-width.

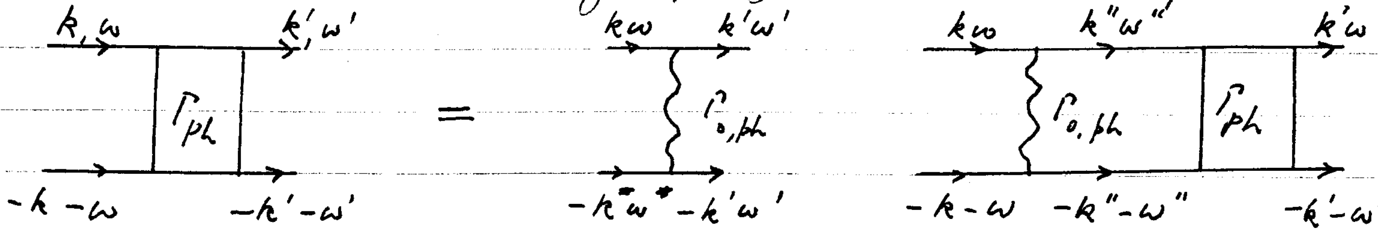
McMillan (c. 1965) showed how to extract $\alpha^2 F$ from the differential conductance of tunnel junctions.

For later use we define dimensionless interaction

$$\lambda(\omega-\omega') \equiv \int_0^\infty dz \, \alpha^2 F(z) \, 2z \Big/ (\omega-\omega')^2 + z^2 \qquad (\text{III}.6)$$

2. Mean field T_c neglecting Coulomb repulsion

As in previous lecture, look for instability in multiple scattering of zero momentum pairs



This denotes the integral equation

$$\Gamma_{ph}(kk'\omega\omega') = \sum_{\lambda} |g_{kk'\lambda}|^2 \mathcal{D}_\lambda(k-k', \omega-\omega')$$

$$- T \sum_{\substack{\omega'' k'' \\ \lambda}} |g_{kk''\lambda}|^2 \mathcal{D}_\lambda(k-k'', \omega-\omega'')\, G(k''\omega'')\, G(-k''-\omega'')\, \Gamma_{ph}(k''k'\omega''\omega') \qquad (\text{III}.7)$$

Let

$$\Gamma_{ph}(\omega\omega') \equiv \langle \Gamma_{ph}(kk'\omega\omega') \rangle_{kk'} \qquad (\text{III}.8)$$

After averaging (III.7) over k, k', k'' at the Fermi surface

$$N(0)\,\Gamma_{ph}(\omega\omega') = -\lambda(\omega\omega') + T N(0) \sum_{\omega''} \lambda(\omega-\omega'')\, R(\omega'')\, \Gamma_{ph}(\omega''\omega')$$

where $$R(\omega'') = \int d\epsilon'' \frac{1}{i\omega''-\epsilon''}\,\frac{1}{-i\omega''-\epsilon''} = \frac{\pi}{|\omega''|} \qquad (\text{III}.9)$$

The homogeneous part of (III.7) has a solution when

$$\Delta(\omega) = \pi T \sum_{\omega''} \lambda(\omega-\omega'') \frac{\Delta(\omega'')}{|\omega''|} \qquad (\text{III}.10)$$

Note $\lambda(\omega-\omega') \to$ const. within bandwidth reproduces BCS T_c equation [$N(0)$ absorbed in λ.]

Coulomb correction to T_C equation

Treat as weakly k-k' and ω-ω' dependent and write

$$N(0)\,\Gamma_{0,C} = \mu \qquad (\text{III}.11)$$

In simplest model $\mu = N(0)\left\langle \frac{4\pi e^2}{(k-k')^2+k_s^2}\right\rangle_{kk'}$ but this has no deep justification. Including this effect in the T_c equation yields

$$\Delta(\omega) = \pi T \sum_{\omega'} \left[\lambda(\omega-\omega') - \mu\right] \frac{\Delta(\omega')}{|\omega'|} \qquad (\text{III}.12)$$

Now λ is naturally cut-off at phonon frequency band width, but μ has a scale ε_F. Split up Δ according to

$$\Delta(\omega) = \Delta_\ell(\omega)\,\theta(\omega_c - \omega) + \Delta_h\,\theta(\omega - \omega_c) \qquad (\text{III}.13)$$

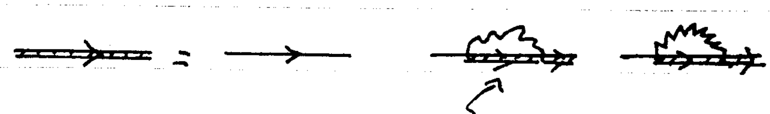
Substituting (III.13) into (III.12) confirms that Δ_h is independent of ω. Conclusion is that Δ_ℓ obeys

$$\Delta_\ell(\omega) = \pi T \sum_{\substack{\omega' \\ |\omega'|<\omega_c}} \left[\lambda(\omega-\omega') - \mu^*\right] \frac{\Delta_\ell(\omega')}{|\omega'|} \qquad (\text{III}.14)$$

with $\mu^* = \dfrac{\mu}{1+\mu \ln(\varepsilon_F/\omega_c)}$, and ω_c to be determined semi-empirically.

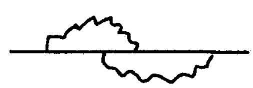
Self-energy Corrections for $T \geq T_c$

In general $\quad G^{-1}(k,i\omega) = G_0^{-1}(k,i\omega) - \Sigma(k,i\omega) \quad$ (III.15)

Might approximate by 

$\Sigma_{ph}(i\omega) \qquad \Sigma_c$

The approximation for Σ_{ph} can be justified,
that for Σ_c cannot

Migdal's theorem (i)  smaller than (ii) 

by a factor of order ω_D/ε_F, because in (ii) all electron intermediate states can be near the FS, thus avoiding large electron 'energy denominators', whereas this is not true for (i)

No similar theorem for Coulomb interaction. However, it is essentially a constant which can be absorbed into the chemical potential.

Ansatz $\quad \Sigma(i\omega) = i\omega(1 - Z(i\omega)) \Rightarrow G^{-1}(k,i\omega) = i\omega Z(i\omega) - \varepsilon_k$

(III.16)

Effect on the pair propagator eq. (III.9) is to replace $\pi/|\omega''|$ by

$$R(\omega'') = \frac{\pi}{|\omega''| Z(i\omega'')} \qquad \text{(III.17)}$$

So the homogeneous part of the equation now has a solution when

$$\boxed{Z(i\omega)\,\Delta(i\omega) = \pi T \sum_{|\omega''|<\omega_c} \left[\lambda(\omega-\omega'') - \mu^*\right] \frac{\Delta(i\omega'')}{|\omega''|}} \qquad (\text{III}.18)$$

Z can be constructed from the explicit form of Σ_{ph}:

$$\Sigma_{ph}(i\omega) = N(0)\,T \sum_{\omega';\lambda} \left\langle |g_{kk'\lambda}|^2 \mathcal{D}_\lambda(k-k',\omega-\omega') \right\rangle_{kk'} \int d\varepsilon' \, G(k'\omega') \qquad (\text{III}.19)$$

and

$$\int d\varepsilon' \, G(k'\omega') = \int d\varepsilon' \frac{1}{i\omega' Z(\omega') - \varepsilon'} = \int d\varepsilon' \frac{-i\omega' Z - \varepsilon'}{\omega'^2 Z^2 + \varepsilon^2}$$

$$\doteq -\pi i \operatorname{sgn}\omega' \qquad (\text{III}.20)$$

Thus $\displaystyle \Sigma_{ph}(\omega) = -\pi i T \sum_{\omega'} \lambda(\omega-\omega') \operatorname{sgn}\omega'$ $\qquad (\text{III}.21)$

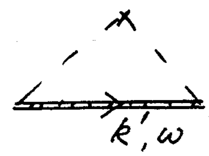
and Σ_c is a constant absorbed into the chemical potential. Using (III.16) in (III.21), (III.18) is completed by

$$\boxed{Z(i\omega) = 1 + \frac{\pi T}{|\omega|} \sum_{\omega'} \lambda(\omega-\omega') \operatorname{sgn}\omega \operatorname{sgn}\omega'} \qquad (\text{III}.22)$$

Can eliminate Z by multiplying (III.22) by Δ and subtracting (III.18).

Impurity corrections to T_c

Contribution to normal state electronic self-energy:

Σ_{imp} 

Make Ansatz $G(k, i\omega) = \dfrac{1}{i\omega \tilde{Z}(\omega) - \varepsilon_k}$ (III. 23)

It follows that

$$\Sigma_{imp}(i\omega) = \frac{1}{2\pi N(0)\tau} \sum_{k'} G(k, i\omega) = \frac{1}{2\pi\tau} \int d\varepsilon' \, \frac{i\omega \tilde{Z}(\omega) + \varepsilon'}{-\omega^2 \tilde{Z}^2(\omega) - \varepsilon'^2}$$

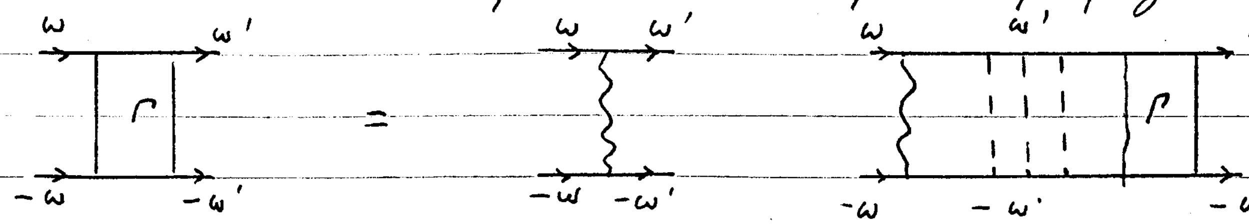
$$\doteq -\frac{i}{2\tau} \operatorname{sgn}\omega \qquad \text{(III. 24)}$$

It follows that

$$\tilde{Z}(\omega) = 1 + \frac{1}{2\tau|\omega|} + \frac{\pi T}{|\omega|} \sum_{\omega'} \lambda(\omega - \omega') \operatorname{sgn}(\omega) \times \operatorname{sgn}(\omega')$$

(III. 25)

Must also include impurities in the pair propagator



We see that the impurity ladder leads to the replacement

$$\frac{\pi N(0)}{\tilde{Z}(\omega'')|\omega''|} \longrightarrow \frac{\pi N(0)}{\tilde{Z}(\omega'')|\omega''|} \left[1 - \frac{1}{2\pi N(0)\tau} \, \frac{\pi N(0)}{\tilde{Z}(\omega'')|\omega''|} \right]^{-1}$$

$$= \frac{\pi N(0)}{|\omega''|\tilde{Z}(\omega'') - \frac{1}{2\tau}} = \frac{\pi N(0)}{|\omega''| Z(\omega'')} \qquad \text{(III. 26)}$$

Thus the impurity correction drop out of the pair propagator. The pair of equations (III.18) and (III.22) continue to determine T_c.

Anderson's theorem continues to hold at the mean field level in strong coupling theory.

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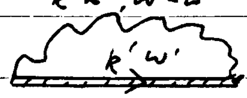
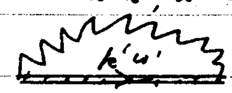
3. Strong Coupling Mean Field Theory below T_c

Calculations are in many ways easier because in the Nambu language there is only one type of self energy diagram

$k-k', \omega-\omega'$ $\qquad\qquad$ $k-k', \omega-\omega'$





$\Sigma_{ph}(k,\omega)$ $\qquad\qquad$ $\Sigma_c(k\omega)$

(III.27)

Assume self energy of form $\Sigma(k\omega) = i\omega[1-Z(\omega)] + \phi(\omega)\tau_1$

Then
$$G(k\omega) = \frac{i\omega Z(\omega) + \epsilon_k\tau_3 + \phi(\omega)\tau_1}{-\omega^2 Z^2(\omega) - \epsilon_k^2 - \phi^2(\omega)} \qquad (III.28)$$

The phonon self energy is given by

$$\Sigma_{ph}(\omega) = N(0)\,T\sum_{\omega'}\Big\langle \sum_\lambda |g_{kk'\lambda}|^2 \mathcal{D}_\lambda(k-k', \omega-\omega')\Big\rangle_{kk'} \int d\varepsilon''\, G(\varepsilon'', \omega')$$

$$= -\pi T\sum_{\omega'} \lambda(\omega-\omega')\,\frac{i\omega' Z(\omega') + \phi(\omega')\tau_1}{\sqrt{\omega'^2 Z^2(\omega') + \phi^2(\omega')}} \qquad (III.29)$$

For Σ_c replace $\lambda(\omega-\omega')$ by $-\mu$

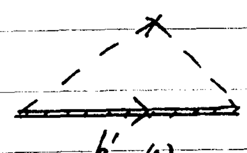
Writing $\Delta(\omega) = \phi(\omega)/Z(\omega)$ and equating coefficients of 1 and τ_1 in (II.27) and (III.29) yields

$$\boxed{\begin{aligned} Z(\omega) &= 1 + \frac{\pi T}{|\omega|} \sum_{\omega'} \left[\lambda(\omega-\omega') - \mu\right] \frac{\omega'}{\sqrt{\omega'^2+\Delta'^2}}\, \mathrm{sgn}(\omega) \\ Z(\omega)\Delta(\omega) &= \pi T \sum_{\omega'} \left[\lambda(\omega-\omega') - \mu\right] \frac{\Delta(\omega')}{\sqrt{\omega'^2+\Delta'^2}} \end{aligned}} \qquad \text{(III.30)}$$

In the eq. for $Z(\omega)$ the μ can be dropped because of oddness in ω' of the summand. Also $\mu \to \mu^*$ with upper cut-off ω_c works as before because $\Delta(\omega) \to 0$ for $\omega \sim \omega_c$. To lowest non-vanishing order in Δ we recover the equations (III.18) and (III.22) which determine T_c.

Impurity Effects below Tc

Need to include an extra self-energy diagram



Ansatz $\Sigma = i\omega(\tilde{Z}-1) + \tilde{\phi}\tau_1$

$$\Sigma_{imp}(\omega) = \frac{1}{2\pi N(0)\tau} \sum_{k'} \tau_3\, G(k'\omega)\, \tau_3 = \frac{1}{2\pi\tau} \int d\epsilon' \frac{i\omega\tilde{Z}(\omega) - \epsilon'\tau_3 - \tilde{\phi}\tau_1}{-\omega^2\tilde{Z}^2 - \epsilon'^2 - \tilde{\phi}^2}$$

$$= -\frac{1}{2\tau} \frac{i\omega\tilde{Z} - \tilde{\phi}\tau_1}{\sqrt{\omega^2\tilde{Z}^2 + \tilde{\phi}^2}} \qquad \text{(III.31)}$$

Define $\tilde{\Delta} \equiv \tilde{\phi}/\tilde{Z}$

The self consistency equations become

$$\tilde{Z}(\omega) = 1 + \frac{1}{2\tau\sqrt{\omega^2+\tilde{\Delta}^2(\omega)}} + \frac{\pi T}{\omega}\sum_{|\omega'|<\omega_c} \lambda(\omega-\omega')\frac{\omega'}{\sqrt{\omega'^2+\tilde{\Delta}(\omega')^2}}$$

$$\tilde{Z}(\omega)\tilde{\Delta}(\omega) = \frac{\tilde{\Delta}(\omega)}{2\tau\sqrt{\omega^2+\tilde{\Delta}^2(\omega)}} + \pi T\sum_{|\omega'|<\omega_c}\left[\lambda(\omega-\omega')-\mu^*\right]\frac{\tilde{\Delta}(\omega')}{\sqrt{\omega'^2+\tilde{\Delta}^2(\omega')}}$$

III.32

Note that $\tilde{Z}(\omega) - \frac{1}{2\tau\sqrt{\omega^2+\tilde{\Delta}^2(\omega)}}$ occurs in both these equations which reduce to (III.30) for the pure system, showing that $\tilde{\Delta}(\omega) = \Delta(\omega)$.

In short, not only T_c but the single particle density of states are unaffected, at the mean field level, by time-reversal invariant impurity scattering — including strong coupling.

Smith will show in his last lecture how the physics of localization destroys these results.

Overview of Mean field strong coupling theory

Successes Given $\alpha^2 F(\omega)$, μ^*, ω_c obtained from tunneling data, the theory has succeeded in explaining equilibrium [$H_c(T)$, $H_{c_2}(T)$, $2\Delta(0)/T_c$] and transport [heat conduction, ultrasonic attenuation] properties of lead and mercury

Failures Calculating T_c from 'first principles' or predicting T_c in new materials has not been a triumph.