Lecture II

1. Mean field theory — BCS, Gor'kov
2. Ordinary and spin flip impurities in MFT
3. Charge conservation and consequences.

---

1. Order parameter

As in Bose Einstein condensation superconductivity can be thought to manifest itself by the appearance of anomalous averages. The order parameter corresponding to pairing in a state of zero total momentum is

$$\psi_{\alpha\beta}(\vec{k}) = \langle c_\alpha(\vec{k})\, c_\beta(-\vec{k})\rangle = \begin{cases} \text{singlet} & (i\sigma_2)_{\alpha\beta}\, d(\vec{k}) \\ \text{triplet} & \left[i\,\vec{d}(\vec{k})\cdot\vec{\sigma}\,\sigma_2\right]_{\alpha\beta} \end{cases} \qquad (\text{II}.1)$$

with $d(-\vec{k}) = d(\vec{k})$ and $\vec{d}(-\vec{k}) = -\vec{d}(\vec{k})$, thus preserving the symmetry $\psi_{\alpha\beta}(\vec{k}) = -\psi_{\beta\alpha}(-\vec{k})$ for both spin states.

For simple metals the orbital state is usually s-wave i.e. $d(\vec{k}) \neq f(\vec{k})$.

Gorkov's Equations - Clean limit

Gorkov's Equations determine anomalous pairing averages self consistently. Hamiltonian on I-12 yields equation of motion for field operator in Heisenberg picture

$$i\frac{\partial\psi_\uparrow}{\partial t} = [\psi_\uparrow, H] = -\frac{\nabla^2}{2m}\psi_\uparrow - \tilde{\lambda}\psi_\downarrow^\dagger\psi_\downarrow\psi_\uparrow \quad \text{[All operators at } x,t] \quad (\text{II}.2)$$

This yields

$$\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m}\right) G_{\uparrow\uparrow}(xt, x'0) = \delta^3(x-x')\,\delta(t) + i\tilde{\lambda}\left\langle T\left[\left(\psi_\downarrow^\dagger\psi_\downarrow\psi_\uparrow\right)_{xt}\psi_\uparrow^\dagger(x'0)\right]\right\rangle \quad (\text{II}.3)$$

Introduce the Gorkov amplitudes

$$F(xt, x't') = -i\left\langle T\left[\psi_\uparrow(xt)\,\psi_\downarrow(x't')\right]\right\rangle$$
$$\bar{F}(xt, x't') = -i\left\langle T\left[\psi_\downarrow^\dagger(xt)\,\psi_\uparrow^\dagger(x't')\right]\right\rangle \quad (\text{II}.4)$$

Now $\bar{F}(xt, x't) \propto e^{2i\mu t}$ because two particles added to condensate. Tuck out of sight by measuring energies with respect to the chemical potential. Factorizing the right side of (II.3) and Fourier transforming

$$(i\omega_\ell - \epsilon_k)\, G(k, \omega_\ell) = 1 + \Delta\bar{F}(k, \omega_\ell) \quad (\text{II}.5)$$

where

$$\Delta \equiv \tilde{\lambda}\langle\psi_\uparrow(xt)\,\psi_\downarrow(xt)\rangle = \frac{i\tilde{\lambda}}{-i\beta}\sum_\omega\int\frac{d^3k}{(2\pi)^3}\,F(k,\omega_\ell) \quad (\text{II}.6)$$

To close the system of equations use the adjoint of (II.2) to obtain an equation for $\bar{F}$:

$$\left(-\frac{i\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu\right) \bar{F}(xt, x'0) = i\tilde{\lambda} \langle T[(\psi_\downarrow^\dagger \psi_\uparrow^\dagger \psi_\uparrow)_{x,t} \, \psi_\uparrow^\dagger(x',0)] \rangle \quad (II.7)$$

Keeping the term involving G on the right and Fourier transforming gives

$$(-i\omega_\ell - \epsilon_k) \bar{F}(k, \omega_\ell) = -\Delta^* G(k, \omega_\ell) \quad (II.8)$$

Solving (II.5) and (II.8) yields

$$G(k, \omega_\ell) = \frac{+i\omega_\ell + \epsilon_k}{-\omega_\ell^2 - E_k^2}, \quad \bar{F}(k, \omega_\ell) = \frac{\Delta^*}{-\omega_\ell^2 - E_k^2} \quad (II.9)$$

with $\quad E_k = +\sqrt{\epsilon_k^2 + |\Delta|^2} \quad (II.10)$

Since Δ^* is connected to $\bar{F}$ in the same way as Δ is to F in (II.6)

$$\Delta^* = +\frac{\tilde{\lambda}}{\beta} \Delta^* \sum_\omega \int' \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_\ell^2 + E_k^2} \quad (II.11)$$

Canceling out the Δ^* and doing the ω_ℓ sum one gets the famous BCS gap equation for $\Delta(T)$

$$1 = N(0)\tilde{\lambda} \int_{-\omega_D}^{\omega_D} d\epsilon \, \frac{\tanh \frac{1}{2}\beta E}{2E} \quad (II.12)$$

Clearly $\Delta \to 0$ at the T_c calculated in I.

To see that $|\Delta|$ is a gap in the single particle spectrum one calculates the spectral density to find

$$\frac{1}{2\pi} A(k,\omega) = u_k^2\, \delta(\omega - E_k) + v_k^2\, \delta(\omega + E_k) \qquad (\mathrm{I}.13)$$

with $\left\{ \begin{matrix} u_k^2 \\ v_k^2 \end{matrix} \right\} = \frac{1}{2}\left(1 \pm \frac{\varepsilon_k}{E_k}\right)$,

and notes that $E_k \geq \Delta$.

<u>Nambu space</u> To treat normal and anomalous propagators on the same footing it is convenient to consider a two component space

$$\Psi_i = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow^+ \end{pmatrix} \qquad \Psi_i^+ = \left(\psi_\uparrow^+ \ \psi_\downarrow \right) \qquad (\mathrm{I}.14)$$

The 2×2 Green function

$$\mathcal{G}_{ij}(xt, x'0) \equiv -i \left\langle T\left[\Psi_i(xt)\Psi_j^+(x'0)\right]\right\rangle$$

$$\xrightarrow[F.T.]{} \begin{pmatrix} G(k,\omega_\ell) & F(k,\omega_\ell) \\ \bar{F}(k,\omega_\ell) & -G(-k,-\omega_\ell) \end{pmatrix} \qquad (\mathrm{II}.15)$$

where the spin indices on F and $\bar{F}$ are as indicated in (II.4), and those on G can be ignored for the model being treated.

By working out the equations of motion for the 4 elements of $\mathcal{G}$ one finds

$$\mathcal{G}(k,\omega_\ell) = \frac{i\omega_\ell + \varepsilon_k\tau_3 + \Delta\tau_+ + \Delta^*\tau_-}{-\omega_\ell^2 - E_k^2} \qquad (\text{II}.16)$$

where $\tau_{1,2,3}$ are Pauli matrices in Nambu space and $\tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)$. Check against (II.9) and (II.15) by noting

$$\mathcal{G}_{11}(k,\omega_\ell) = G(k,\omega_\ell) \qquad \mathcal{G}_{21}(k,\omega_\ell) = \frac{\Delta^*}{-\omega_\ell^2 - E_k^2} = \bar{F}(k,\omega_\ell) \qquad (\text{II}.17)$$

The self consistency condition is

$$\Delta = -\tilde{\lambda} T \sum_\omega \int' \frac{d^3k}{(2\pi)^3} \mathcal{G}_{12}(k,\omega_\ell) \qquad (\text{II}.18)$$

Gorkov's Equations - Spin independent impurities

To write the impurity scattering potential (I.44) in the Nambu language note

$$\psi_\uparrow^\dagger\psi_\uparrow + \psi_\downarrow^\dagger\psi_\downarrow = :\left(\psi_\uparrow^\dagger\psi_\uparrow - \psi_\downarrow\psi_\downarrow^\dagger\right):$$

$$= :\Psi^\dagger\tau_3\Psi: \qquad (\text{II}.19)$$

where the : : remind one to put the adjoint field operators to the left (with a change of sign) when the relative order matters. Let $\bar{\Delta}$ be the average order parameter, which we choose to be real. Averaging over impurities in the manner of I gives

$$\tilde{\mathcal{G}}(k\omega_\ell) = \mathcal{G}(k\omega_\ell) + \mathcal{G}(k\omega_\ell)\,\Sigma(k\omega_\ell)\,\tilde{\mathcal{G}}(k\omega_\ell) \qquad (\text{II}.20)$$

with products implying a trace in Nambu space and

$$\Sigma(k\omega_\ell) = \frac{\Gamma}{2\pi N(0)}\int\frac{d^3k'}{(2\pi)^3}\;\tau_3\,\tilde{\mathcal{G}}(k\omega_\ell)\,\tau_3 \qquad (\text{II}.21)$$

These equations are solved self consistently by making the Ansatz

$$\Sigma(k\omega_\ell) = i(\omega_\ell - \tilde{\omega}_\ell) - (\bar{\Delta} - \tilde{\Delta})\tau_1 \qquad (\text{II}.22)$$

from which it follows that

$$\tilde{\mathcal{G}}(k\omega_\ell)^{-1} = i\tilde{\omega}_\ell - \epsilon_k\tau_3 - \tilde{\Delta}\tau_1 \qquad (\text{II}.23)$$

Eq (II.21) now requires that

$$i(\omega_\ell - \tilde{\omega}_\ell) = -\frac{\Gamma}{2\pi}\int d\epsilon' \;\; i\tilde{\omega}_\ell\,/\,(\epsilon'^2 + \tilde{\omega}_\ell^2 + \tilde{\Delta}^2)$$

$$\bar{\Delta} - \tilde{\Delta} = -\Gamma\int d\epsilon'\;\tilde{\Delta}\,/\,(\epsilon'^2 + \tilde{\omega}_\ell^2 + \tilde{\Delta}^2)$$

Then

$$\tilde{\omega}_\ell = \omega_\ell + \frac{\Gamma}{2}\frac{\tilde{\omega}_\ell}{\sqrt{\tilde{\omega}_\ell^{\,2}+\tilde{\Delta}^2}}$$

$$\tilde{\Delta} = \Delta + \frac{\Gamma}{2}\frac{\tilde{\Delta}}{\sqrt{\tilde{\omega}_\ell^{\,2}+\tilde{\Delta}^2}} \qquad (\text{II}.25)$$

with $\mathrm{Re}\sqrt{\tilde{\omega}_\ell^2+\tilde{\Delta}^2} > 0$. In the limit $\tilde{\Delta}\to 0$ we recover Eq. (I.45)

$$\tilde{\omega}_\ell = \omega_\ell + \frac{\Gamma}{2}\,\mathrm{sgn}\,\omega_\ell$$

Eq (II.25) have a simple solution. Let

$$u \equiv \frac{\tilde{\omega}_\ell}{\tilde{\Delta}} \qquad (\text{II}.26)$$

Then

$$\tilde{\omega}_\ell = u\tilde{\Delta} = \omega_\ell + \frac{\Gamma}{2}\frac{u}{\sqrt{u^2+1}}$$

$$\tilde{\Delta} = \bar{\Delta} + \frac{\Gamma}{2}\frac{1}{\sqrt{u^2+1}} \implies u = \frac{\omega_\ell}{\bar{\Delta}}$$

$$(\text{II}.27)$$

The equation determining $\bar{\Delta}$ [Eq (II.18)] is

$$\bar{\Delta} = \tilde{\lambda} T \sum_\omega \int' \frac{d^3k}{(2\pi)^3}\frac{\tilde{\Delta}}{\tilde{\omega}_\ell^{\,2}+\epsilon_k^2+\tilde{\Delta}^2} \qquad (\text{II}.28)$$

Because of (II.27), (II.28) implies that $\bar{\Delta}(T) = \Delta(T)$ as follows. Add and subtract the same quantity to get from (II.28)

$$\bar{\Delta} = \lambda T \sum_{\omega} \int' \frac{d^3k}{(2\pi)^3} \left[\frac{\tilde{\Delta}}{\tilde{\omega}_\ell^2 + \xi_k^2 + \tilde{\Delta}^2} - \frac{\bar{\Delta}}{\omega_\ell^2 + \xi_k^2 + \bar{\Delta}^2} \right]$$

$$+ \lambda T \sum_{\omega} \int' \frac{d^3k}{2\pi} \frac{\bar{\Delta}}{\omega_\ell^2 + \xi_k^2 + \bar{\Delta}^2} \qquad (\text{II}.29)$$

Because the first line is not logarithmically divergent we may do the integral over ξ_k whereupon we find for this term

$$\lambda T \sum_{\omega} N(0) \left[\frac{\pi \tilde{\Delta}}{\sqrt{\tilde{\omega}^2 + \tilde{\Delta}^2}} - \frac{\pi \bar{\Delta}}{\sqrt{\omega^2 + \bar{\Delta}^2}} \right] = 0 \quad (\text{II}.30)$$

using (II.27). Thus $\bar{\Delta}$ is determined by (II.11) implying $\bar{\Delta} = \Delta$, and recovering Anderson's "Theorem."

The back of a postcard argument for Anderson's "Theorem" is that one could imagine finding the eigenstates for non interacting electrons in the impurity potential (which would come in degenerate time reversed pairs of states) and then make a condensate from these pairs. Assuming that the density of states and the attraction is unaffected by interactions gives the same T_c as for the clean case.

Spin-Flip Scattering

The key point above is time-reversal invariance. When this is broken, the threshold temperature is reduced even at the mean field level. Suppose that the scattering potential is

$$\int \psi_\alpha^\dagger(x) \sum_{n=1}^{n_{imp}} \left[V(x-R_n)\delta_{\alpha\beta} + J(x-R_n)\vec{S}_n \cdot \vec{\sigma}_{\alpha\beta} \right] \psi_\beta(x) \quad (\text{II}.31)$$

where α, β refer to spin [and not Nambu] space and $\vec{\sigma}$ is the Pauli spin matrix vector. Assuming a random distribution of impurities and no average impurity spin $\langle \vec{S} \rangle$, Abrikosov & Gorkov worked out the effect on BCS superconductivity in the scattering approximation used here. Now there are two scattering rates

$$\Gamma_1 = 2\pi N(0)\, n_{imp} \langle |V|^2 \rangle \quad , \quad \Gamma_S = 2\pi N(0)\, n_{imp} \langle |J|^2 \rangle\, S(S+1) \quad (\text{II}.32)$$

The result for $\tilde{G}^{-1}$ is as given in (II.23) but now, again defining $u \equiv \tilde{\omega}_\ell / \tilde{\Delta}_\ell$, with

$$\frac{\omega_\ell}{\bar{\Delta}} = u\left(1 - \frac{\Gamma_s}{\bar{\Delta}} \frac{1}{\sqrt{1+u^2}}\right) \qquad (II.33)$$

The previous results are recovered when $\Gamma_s \to 0$.

Now the equation determining $\bar{\Delta}$ is

$$\bar{\Delta} = \pi N(0) T \tilde{\lambda} \sum_\omega{}' \frac{\tilde{\Delta}}{\sqrt{\tilde{\omega}_\ell^2 + \tilde{\Delta}^2}} = 2\pi N(0) T \tilde{\lambda} \sum_{\omega>0}{}' \frac{1}{\sqrt{u^2+1}} \qquad (II.34)$$

where we have done the ε_k integral and regularized the logarithmic divergence by cutting off the ω-sum. Very interestingly <u>$\bar{\Delta}$ is no longer the gap</u> in the single particle excitation spectrum

<u>Transition Temperature</u> Look for $\bar{\Delta} \to 0$ solution

$$u \Rightarrow \frac{\omega_\ell + \Gamma_s}{\bar{\Delta}} \quad \text{and } T_c \text{ is determined by}$$

$$1 = 2\pi N(0) \tilde{\lambda} T_c \sum_{\omega>0} \left[\frac{1}{\omega_\ell + \Gamma_s} - \frac{1}{\omega_\ell}\right] + N(0)\tilde{\lambda} \ln \frac{1.13\,\omega_D}{T_c}$$

$$= N(0)\tilde{\lambda} \ln \frac{1.13\,\omega_D}{T_c^0} \qquad (II.35)$$

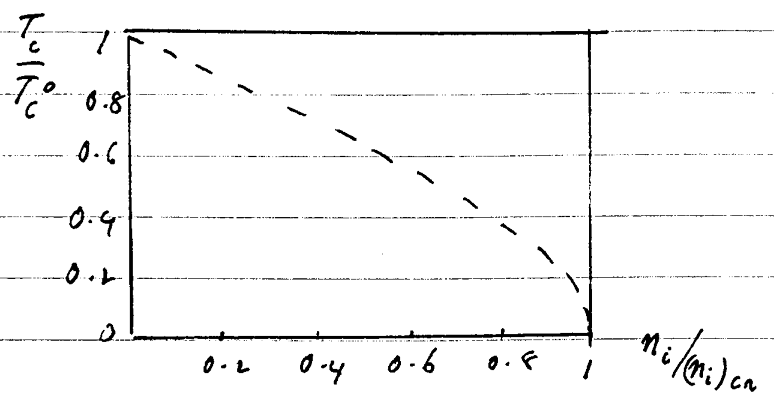
Putting it all together, we get

$$\ln \frac{T_c}{T_c^\circ} = -2 \sum_{l=0}^{\infty} \left[\frac{1}{2l+1} - \frac{1}{2l+1+\Gamma_s/\pi T_c} \right] \quad (\text{II}.36)$$

$$= \psi\left(\tfrac{1}{2}\right) - \psi\left(\tfrac{1}{2} + \frac{\Gamma_s}{2\pi T_c}\right)$$

where $\psi(z) = \frac{d \ln \Gamma(z)}{dz}$ is the digamma function.

[See ~~last~~ next transparency.]



$(n_i)_{cr}$ occurs when

$$\Gamma_s = \frac{\pi T_c^\circ}{2\gamma} = \frac{\Delta_p(0)}{2}$$

$$= \frac{T_c^\circ}{1.13}$$

Dirty Limit Critical Field

An equation identical in form to (II.36) is obtained from our dirty limit for ordinary impurities at finite pair momentum Eq (I.48)

$$\tilde{L}_0(q,0) = -\frac{1}{\tilde{\lambda}} \Rightarrow \ln \frac{T_c(q)}{T_c^\circ} = \psi\left(\tfrac{1}{2}\right) - \psi\left(\tfrac{1}{2} + \frac{Dq^2}{4\pi T_c(q)}\right) \quad (\text{II}.37)$$

In the dirty limit one can show that a penetrating field corresponds to replacing q^2 by $\underline{\underline{2eH_{c2}}}$.

This can be motivated by arguing that

$$\frac{q^2}{2m} \rightarrow \frac{1}{2m}\left(q - \frac{2e}{c}A\right)^2 \longrightarrow \frac{1}{2}\frac{2eH}{mc} \quad \text{for the}$$

lowest eigenstate in a constant magnetic field.

[Warning! There are subtleties here, carefully discussed in Kurkijarvi, Eilenberger and Ambegaokar PRB 5, 868 (1972) which, however, are unimportant in the dirty limit.]

Some properties of the Digamma function

$$\psi(z) = \frac{d \ln \Gamma(z)}{dz} \qquad \Gamma - \text{Gamma function}$$

$\Gamma(n+1) = n!$
for n - integer

$$\psi(1+z) = -\gamma + \sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+z}\right] \qquad \gamma = 0.577\ldots$$

[Euler's const]

$$\psi\left(\frac{1}{2}+\frac{x}{2}\right) - \psi\left(\frac{1}{2}\right) = 2\sum_{l=0}^{\infty}\left[\frac{1}{2l+1} - \frac{1}{2l+1+x}\right]$$

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2$$

For large x $\quad \psi\left(\frac{1}{2}+\frac{x}{2}\right) \longrightarrow \ln x + \frac{1}{6x^2}$

3. Charge conservation and consequences

To go beyond mean field theory we have to look into how the interactions are modified by fluctuations. As in the simple considerations in I, charge conservation plays an essential role.

One can start by rewriting the interactions in an instructive way

$$H_{int} = -\lambda \int d^3x\, \psi_\uparrow^+(x)\psi_\downarrow^+(x)\psi_\downarrow(x)\psi_\uparrow(x)$$

$$+\frac{1}{2}\int d^3x\, d^3x'\, \psi_\alpha^+(x)\psi_\beta^+(x')\frac{e^2}{|x-x'|}\psi_\beta(x')\psi_\alpha(x)$$

$$= \frac{1}{2}\int d^3x\, d^3x' : O_i(x)\, V^0_{ij}(x-x')\, O_j(x) : \qquad (II.38)$$

where $\quad O_i(x) = Tr\, \Psi^+(x)\, \tau_i\, \Psi(x) \qquad (II.39)$

and
$$V^0_{ij} = \begin{pmatrix} -\frac{\lambda}{2}\delta^3(x-x') & 0 & 0 \\ 0 & -\frac{\lambda}{2}\delta^3(x-x') & 0 \\ 0 & 0 & \frac{e^2}{|x-x'|} \end{pmatrix}$$

Ψ refers to the Nambu representation so that, e.g.

$$O_1(x) = (\psi_\uparrow^+\ \psi_\downarrow)\begin{pmatrix}\psi_\downarrow^+ \\ \psi_\uparrow\end{pmatrix} = \psi_\uparrow^+\psi_\downarrow^+ + \psi_\downarrow\psi_\uparrow \equiv O_\Delta$$

$$O_2(x) = (\psi_\uparrow^+ \psi_\downarrow) \begin{pmatrix} -i\psi_\downarrow^+ \\ i\psi_\uparrow \end{pmatrix} = -i\,\psi_\uparrow^+\psi_\downarrow^+ + i\,\psi_\downarrow\psi_\uparrow \equiv O_\phi$$

$$O_3(x) = (\psi_\uparrow^+ \psi_\downarrow) \begin{pmatrix} \psi_\uparrow \\ -\psi_\downarrow^+ \end{pmatrix} = \psi_\uparrow^+\psi_\uparrow - \psi_\downarrow\psi_\downarrow^+ \equiv O_\rho \qquad (II.40)$$

from the explicit form of the Pauli matrices τ. [Note: Colons in (II.38) refer to a normal product as in (II.19).]

The fluctuation modified interactions obey an integral equation of the form

$$V = V^\circ + V^\circ \Pi V \qquad (II.41)$$

If one were to neglect the phonon induced attraction, only density fluctuations would matter and the electronic dielectric function defined by

$$V(q,\Omega) = \frac{V^\circ(q,\cancel{\Omega})}{1 - V^\circ \Pi} \equiv \frac{V^\circ(q)}{\varepsilon(q,\Omega)} \qquad (II.42)$$

would modify the interaction. In the long wave length low frequency dirty limit we argued that

$$\varepsilon(q,\Omega) = 1 + k_s^2/[|\Omega|/\mathcal{D} + q^2].$$

Smith will discuss the matrix screening equation (II.41). I will close this lecture by showing that charge conservation requires certain identities connecting the different entries in the "proper polarization part" Π_{ij}.

The Hamiltonian (II.38) clearly commutes with charge density. Since only the kinetic energy does not, conservation of charge is guaranteed in an exact solution:

$$i\left[\frac{\partial \rho}{\partial t}\right] = [\rho, H] = -i\,\vec{\nabla}\cdot\vec{j} \qquad (\text{II}.43)$$

where $\rho(x) = :O_\rho(x):$ and $\vec{j} = \frac{\hbar}{2mi} :\Psi^\dagger \vec{\nabla} \tau_3 \Psi: + c.c.$

Smith will be discussing fluctuations in a "conserving approximation." However in calculating the Π-matrix one uses an approximation in which Δ is taken to be real. This is not a gauge invariant choice. In fact one is approximating (II.38) by

$$H_{int} \to (H_{int})_{MF} = \int d^3y\, \Delta\, O_\Delta(y) + \text{const} \qquad (\text{II}.44)$$

This means that within the calculation of Π

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = i\,[H_{int_{MF}}, \rho] = 2\Delta\, O_\phi \qquad (\text{II}.45)$$

A Ward Identity follows from this equation

$$\frac{\partial}{\partial t_2} \langle T[\Psi_i(x_1 t_1)\, \rho(x_2 t_2)\, \Psi_j^+(x_3 t_3)] \rangle =$$

$$\delta(\vec{x}_1 - \vec{x}_2)\,\delta(t_1 - t_2)\, i\,[\tau_3\, G(x_2 t_2, x_3 t_3)]_{ij}$$

$$- \delta(\vec{x}_2 - \vec{x}_3)\,\delta(t_2 - t_3)\, i\,[G(x_1 t_1, x_2 t_2)\tau_3]_{ij}$$

$$- T\langle \Psi_i\, \vec{\nabla}_2 \cdot \vec{j}(x_2 t_2)\, \Psi_j^+ \rangle + 2\Delta T\langle \Psi_i(x_1 t_1)\, O_\phi(x_2 t_2)\, \Psi_j(x_3 t_3) \rangle$$

$$(\text{II}.46)$$

Multiplying (II.46) by τ_3, taking the trace, Fourier transforming in space and imaginary time leads in the limit of zero wave vector to

$$\Omega\, \Pi_{\rho\rho}(0,\Omega) = 2\Delta\, \Pi_{\phi\rho}(0,\Omega) \qquad (\text{II}.47)$$

Also, multiplying by τ_2 and performing these same operations yields

$$\Omega\, \Pi_{\phi\rho}(0,\Omega) = -\frac{i}{\beta}\, Tr[\tau_1 G] + 2\Delta\, \Pi_{\phi\phi}$$

$$= 2\Delta\left[-\frac{1}{\tilde{\lambda}} + \Pi_{\phi\phi}\right] \qquad (\text{II}.48)$$

where the self consistency condition has been used.

These identities will be of importance in understanding long range forces and their effect on the singularities at small q in disordered superconductors.