6 Lectures on Superconductivity in Disordered Metals

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The aim of these lectures is to tie together early developments in the theory of superconductivity in simple metals with recent studies on the effect of disorder in destroying superconductivity — most effectively in films and wires.

The time span covered is roughly 40 years. Progress has occurred in fits and starts. And yet, in a manner typical of science in general and physics in particular, there is nothing quaint about the early work. Instead it is essential for understanding recent and current research.

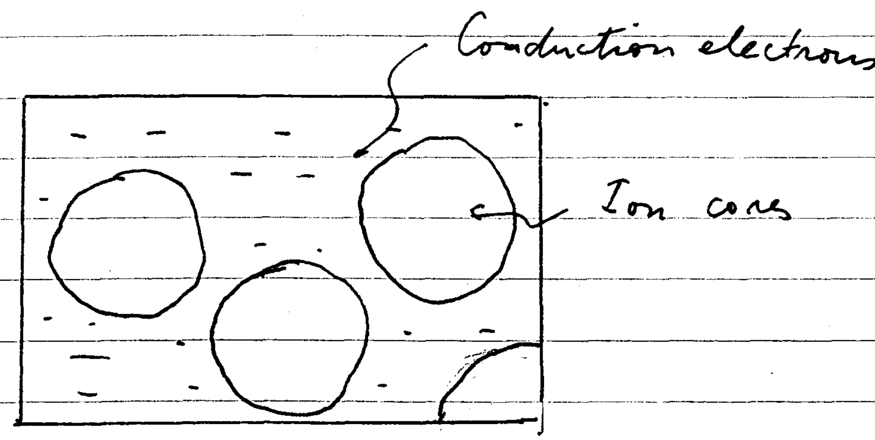
Lecture I

1. Elementary remarks about effective interaction between electrons in simple metals

2. Introduction to methods of solution

3. Calculation of superconducting transition temperature in clean and dirty systems in lowest order.

1. A Simple Metal



Model Basic ingredients

(1) Electrons in periodic potential + Coulomb repulsion
(2) Ions + Ion-Ion interaction ⇒ phonons
(3) Electron phonon interaction
(4) Impurities

Dielectric Function - Screening

Charge conservation essentially implicated.

Electronic system Suppose frequencies $\omega \ll \omega_{pl} = \sqrt{\frac{4\pi n e^2}{m}}$

[n - electron density, m - electron mass

At sufficiently long wavelengths: e - charge]

$\vec{j} = -eD\vec{\nabla}n - \sigma\vec{\nabla}\phi$ [D - diffusivity; σ - conductivity; ϕ electric potential]

$\simeq -eD\frac{\partial n}{\partial\mu}\vec{\nabla}\mu - \sigma\vec{\nabla}\phi$ (I.1) In equilibrium $\mu + e\phi = \text{const}$

$j = 0$

$\Rightarrow \sigma = e^2\frac{\partial n}{\partial\mu}D$ [Einstein]

Now $e\frac{\partial n}{\partial t} + \nabla . j = 0$

$\frac{\partial n}{\partial\mu} = 2N(0)$

density of states per spin

i.e $\frac{\partial n}{\partial t} = D\nabla^2 n + 2N(0)De\nabla^2\phi$ (I.2)

Now $\nabla^2\phi = -4\pi e n - 4\pi\rho_{ext} = -4\pi\rho_T$ (I.3)

Fourier transforming $(\vec{q}, \omega)$ gives

$(-i\omega + Dq^2)en = -4\pi . 2N(0)e^2 D\rho_T \equiv -k_s^2 D\rho_T$ (I.4)

Dielectric function defined by

$en(q\omega) = \left[\frac{1}{\varepsilon(q\omega)} - 1\right]\rho_{ext}(q\omega)$ (I.5)

Thus $\quad \varepsilon_{el}(q\omega) \cong 1 + \dfrac{k_s^2}{\frac{-i\omega}{D} + q^2} \quad$ for $q \ll k_F \quad \omega \ll \omega_{pl}$ (I.6)

$-Ze$ – ion charge

<u>Unscreened ions</u> $\quad q \to 0$ limit $\quad M$ – ion mass

$$\frac{\partial J_{ion}}{\partial t} = -\frac{n_{ion}(Ze)^2}{M}\nabla\phi \qquad n_{ion} \text{ – ion density}$$

Charge conservation gives $\quad \varepsilon_{ion} = 1 - \dfrac{\Omega_{pl}^2}{\omega^2} \quad$ (I.7)

$$\Omega_{pl}^2 = \frac{4\pi n_{ion}(Ze)^2}{M}$$

Thus the $q \ll k_F$, $\omega \ll \omega_{pl}$ dielectric function for the electron ion system is roughly

$$\varepsilon(q\omega) = 1 + \frac{k_s^2}{\frac{-i\omega}{D} + q^2} - \frac{\Omega_{pl}^2}{\omega^2} \qquad (I.8)$$

Coulomb forces thus screen longitudinal phonons and electron-electron interactions. If we ignore the $-i\omega/D$ correction the screened ~~Coulomb~~ interaction is between charges is roughly

$$\frac{4\pi e^2}{q^2\varepsilon(q\omega)} = \frac{4\pi e^2}{q^2+k_s^2}\left[1 + \frac{\Omega_{pl}^2/\omega^2}{q^2+k_s^2 - \Omega_{pl}^2/\omega^2}\right] \qquad (I.9)$$

$$= \frac{4\pi e^2}{q^2+k_s^2}\left[1 + \frac{\Omega_q^2}{\omega^2 - \Omega_q^2}\right] \qquad \Omega_q^2 = \frac{q^2\Omega_{pl}^2}{k_s^2+q^2}$$

Coulomb repulsion (screened) plus exchange of screened longitudinal phonons which is attractive at low frequencies. Very crude, but physically sound approximations.

The repulsion and attraction work on different band widths. The attraction is short range in space and has a frequency range of the order of ω_D – a phonon frequency. The repulsion is also short range but affects electrons with energies on the scale of the Fermi energy. Averaging over the Fermi surface (assumed spherical) one obtains

$$\left\langle \frac{4\pi e^2}{q^2+k^2} \right\rangle = \frac{\alpha^2}{2} \ln \frac{1+\alpha^2}{\alpha^2} \equiv \mu N(0) \,, \quad \alpha^2 \equiv \frac{k_s^2}{4k_F^2} \qquad (I.10)$$

To put this on the same band width as the attraction one introduces a pseudo potential

$$\mu^* = \frac{\mu}{1 + N(0)\mu \ln(\epsilon_F/\omega_D)} \qquad (I.11)$$

Within these approximations one reaches the BCS-Gorkov local effective interaction

$$-\frac{\tilde{\lambda}}{2} \int d^3x \, \tilde{\psi}_\sigma^+ \tilde{\psi}_{\sigma'}^+ \tilde{\psi}_{\sigma'} \tilde{\psi}_\sigma \quad \text{where } \tilde{\psi}_\sigma \equiv \sum_k{}' e^{ik\cdot x} c_{k\sigma} \qquad (I.12)$$

The prime means a sum over a shell of width ω_D.

$\tilde{\lambda} = \lambda - \mu^*$ the net effect of phonon induced attraction and Coulomb repulsion

2. Introduction to methods of Solution

Physical properties can often be related to equilibrium averages of Heisenberg representation operators. Describe equilibrium by Grand Canonical Ensemble.

$$\langle O \rangle = \frac{Tr\, e^{-\beta(H-\mu N)}\, O}{Tr\, e^{-\beta(H-\mu N)}} \quad (I.13)$$

O - Some observable
H - Hamiltonian
N - Number of conserved particles
β - Reciprocal Temperature
μ - Chemical potential

$$O(t) \equiv e^{iHt}\, O\, e^{-iHt} \quad (I.14)$$

$\hbar = 1$, $k_B = 1$

When one considers averages of products, say $A(t)B(t')$, the formal similarity between e^{-iHt} and $e^{-\beta H}$ has useful consequences.

Information about single particle or hole excitations is contained in the averages

$$G^>(xt, x't') \equiv -i \langle \psi(xt)\, \psi^+(x't') \rangle \quad (I.15)$$

$$G^<(xt, x't') \equiv +i \langle \psi^+(x't')\, \psi(xt) \rangle$$

where $\psi(xt) = \sum_r c_r(t)\, u_r(x)$ is the electron field operator $\{c_r, c^+_{r'}\} = \delta_{rr'}$ at equal times.

Because of time translational invariance, and the cyclic invariance of the Trace operation

$$G^{>}(x,x',t-t') = -e^{-\beta\mu}\, G^{<}(x,x',t-t'+i\beta) \qquad (I.16)$$

Check for a non interacting system with

$$H = \sum_r \varepsilon_r c_r^+ c_r \qquad \langle c_r^+ c_r \rangle = \frac{1}{e^{\beta(\varepsilon_r-\mu)}+1} \equiv f_r^-$$

$$\langle c_r c_r^+ \rangle = \frac{1}{e^{-\beta(\varepsilon_r-\mu)}+1} \equiv f_r^+$$

$$G^{\gtrless}(x,x',t) = \mp i \sum_r u_r(x) u_r^*(x') e^{-i\varepsilon_r(t\cancel{-t'})} f_r^{\pm} \qquad (I.17)$$

Checks because $\quad f_r^+ = e^{\beta(\varepsilon_r-\mu)} f_r^-$

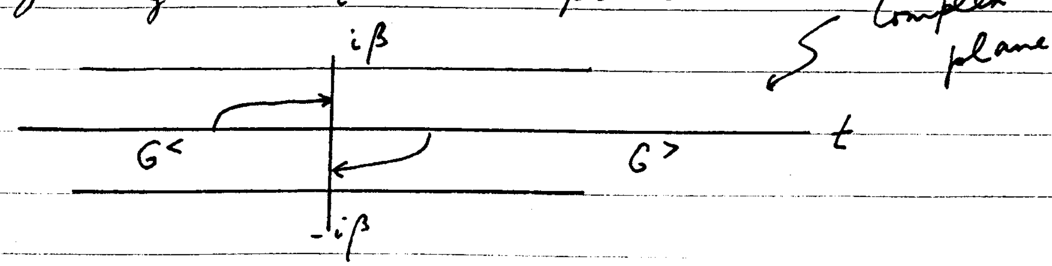
Ways to exploit this structure:

1. <u>Matsubara method</u>

Consider $\quad G(x,x',t) \equiv G^{>}(x,x',t)\,\theta(t) + G^{<}(x,x',t)\,\theta(-t)$

$$(I.18)$$

Can analytically extend to complex t



In region $Re\, t, t' = 0 \qquad -\beta < Im\, t, t' < 0$ antiperiodicity

$$G(t\,t') = \frac{i}{\beta} \sum_{\omega_\ell} G(i\omega_\ell)\, e^{+\omega_\ell (t-t')} \qquad \text{negative imaginary } t, t' \qquad (I.19)$$

$$i\omega_\ell = \mu + \frac{(2\ell+1)\pi i}{\beta}$$

2. Keldysh Redundant for equilibrium averages. Make 3 functions out of $G^>$ and $G^<$:

$$G_R \equiv (G^> - G^<)\,\theta(t) \qquad G_A \equiv -(G^> - G^<)\,\theta(-t)$$

$$G_K \equiv (G^> + G^<) \qquad (I.20)$$

R – Retarded
A – Advanced
K – Keldysh

In terms of these quantities calculations can be more easily done in real time.

Spectral Function

$$A(\omega) \equiv i\,(G^> - G^<)_\omega = i\,(G_R - G_A)_\omega \qquad (I.21)$$

For a non-interacting system [Note $f_r^+ + f_r^- = 1$]

$$A(x, x', \omega) = 2\pi \sum_r u_r(x)\, u_r^*(x')\, \delta(\omega - \varepsilon_r) \qquad (I.22)$$

Describes the spectrum of excitations

Quite generally

$$G^{\gtrless}(\omega) = \pm \frac{1}{i} A(\omega) \frac{1}{e^{\mp\beta(\omega-\mu)}+1} \tag{I.23}$$

As a result for equilibrium averages

$$G_K(\omega) = \frac{1}{i} A(\omega) \tanh \tfrac{1}{2}\beta(\omega-\mu) = \left[G_R(\omega) - G_A(\omega)\right] \tanh \tfrac{1}{2}\beta(\omega-\mu) \tag{I.24}$$

This displays the redundancy but shows that G_K carries information about the population of excitations.

Keldysh Matrix

$$\hat{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix} \tag{I.25}$$

Matrix products occur in a natural way in perturbation theory

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The Matsubara method will suffice for this course

Perturbation Theory

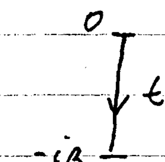
Suppose $H = H_0 + V$ (I.26). Wish to expand in V

For the Matsubara method, consider times in the Heisenberg picture to be pure imaginary and between 0 and $-i\beta$. Expansion in V based on the Dirac-Feynman method. Consider

$$U(\beta,0) \equiv e^{\beta H_0} e^{-\beta(H_0+V)} \qquad (I.27)$$

Clearly $\dfrac{\partial U}{\partial \beta} = - e^{\beta H_0} V e^{-\beta H_0} U$ and $U(0,0) = 1$

By iteration one finds



$$U(\beta,0) = 1 - i\int_0^{-i\beta} dt\, e^{iH_0 t} V e^{-iH_0 t} + \frac{(-i)^2}{2!}\int_0^{-i\beta} dt \int_0^{-i\beta} dt'\, T\left[V^I(t)\, V^I(t')\right] + \cdots$$

$$= T\left[\exp -i\int_0^{-i\beta} dt\, V^I(t)\right] \qquad (I.28)$$

where $V^I(t) = e^{iH_0 t} V e^{-iH_0 t}$ and T is Wick's operator ordering the V^I's from left to right according to the magnitudes of their negative imaginary times.

With the use of this operator the Green function introduced on I-5 is

$$G(xx', t-t') = -i \langle T[\psi(xt)\, \psi^{+}(x't')] \rangle \qquad (I.29)$$

where T is now required to include a sign change for every interchange of Fermion operators.

Higher order Green functions are defined by

$$G(x_1t_1, x_2t_2; x_3t_3, x_4t_4) \equiv (-i)^2 \langle T[\psi(x_1t_1)\psi(x_2t_2)\psi^{+}(x_3t_3)\psi^{+}(x_4t_4)] \rangle$$

etc $\qquad$ (I.30)

For non-interacting grand canonical averages, Wick's theorem states

$$G_0(1,2..n;1'2'..n') = \begin{vmatrix} G_0(11') & G_0(12') & & G_0(1n') \\ G_0(21') & G_0(22') & & G_0(2n') \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ G_0(n1') & \cdots & & G_0(nn') \end{vmatrix} \qquad (I.31)$$

The right hand side is the determinant of the square array.

Since perturbation theory converts averages in the interacting system to sums of averages in H_0 (which we take to have no interactions) according to

$$\langle T[O_1(t_1)\,O_2(t_2)\ldots O_n(t_n)]\rangle = \frac{\langle T[U(\beta 0)\,O_1^I(t_1)\,O_2^I(t_2)\ldots O_n^I(t_n)]\rangle_0}{\langle T[U(\beta 0)]\rangle_0} \qquad (I.32)$$

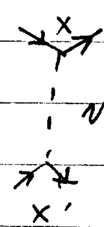
perturbation theory plus Wick's theorem allows for a systematic expansion in interactions.

<u>Feynman graphs</u>

Draw a picture for each term in the determinant.

$$x't' \longrightarrow xt$$

$$G_0(xt, x't')$$



$v(x,x')$ two body interaction

<u>Subtleties</u>

1. Linked Cluster Theorem

2. Counting. Many Wick contractions give the same integral

3 Sign. $(-1)^c$ c: number of closed fermion loops

3. Cooper Instability

The model Hamiltonian arrived at in Eq (I.12) is in second quantized language ($\lambda - \mu^* \to \tilde{\lambda}$)

$$H = -\int d^3x\, \psi_\sigma^+(x) \frac{\nabla^2}{2m} \psi_\sigma(x) - \tilde{\lambda} \int d^3x\, \tilde{\psi}_\uparrow^+(x)\, \tilde{\psi}_\downarrow^+(x)\, \tilde{\psi}_\downarrow(x)\, \tilde{\psi}_\uparrow(x) \qquad (I.33)$$

where $\tilde{\psi}_\sigma(x) = \frac{1}{\sqrt{\Omega}} {\sum_k}' e^{i\vec{k}\cdot\vec{x}} C_{\vec{k},\sigma}$ and the prime means a sum restricted to a shell around the FS.

Consider the propagation of a pair of particles in this system, i.e. contemplate

$$L(x_1 t_1, x_2 t_2) \equiv -i \langle T[\psi_\uparrow(x_1 t_1)\, \psi_\downarrow(x_1 t_1)\, \psi_\downarrow^+(x_2 t_2)\, \psi_\uparrow^+(x_2 t_2)] \rangle \qquad (I.34)$$

Expand in powers of λ $\qquad L = L_0 + \cdots$

In the Matsubara scheme after Fourier transformation in $x_1 - x_2$ and imaginary $t_1 - t_2$, L and L_0 are functions of q and $i\Omega_n = 2\mu + \frac{2\pi n i}{\beta}$ $\quad n = 0, \pm 1 \ldots$ integers

[Even instead of odd, because pair]

Using Wick's Theorem

$$L_0(12) = i\, G_{0\uparrow}(12)\, G_{0\downarrow}(12) \qquad \text{so}$$

$$L_0(q, \Omega) = \frac{i}{-i\beta} \sum_{\omega} \int \frac{d^3p}{(2\pi)^3}\, G_0(q-p, \Omega-\omega)\, G_0(p, \omega) \qquad (I.35)$$

where

$$G_0(p,\omega_\ell) = \int_0^{-i\beta} dt\, e^{-\omega_\ell t}\, G_0(pt) = \int_0^{-i\beta} dt\, e^{-\omega_\ell t}\, G_0^{>}(pt)$$

$$= -i \int_0^{-i\beta} dt\, e^{-\omega_\ell t}\, e^{-i\epsilon_p t}\, \frac{1}{e^{-\beta(\epsilon_p-\mu)}+1}$$

$$= \frac{-i}{-\omega_\ell - i\epsilon_p}\left[-e^{-\beta(\epsilon_p-\mu)} - 1\right]\frac{1}{e^{-\beta(\epsilon_p-\mu)}+1}$$

$$= \frac{1}{i\omega_\ell - \epsilon_p} \qquad \text{using } e^{-i\beta\cdot\frac{(2\ell+1)\pi}{\beta}} = -1 \qquad (I.36)$$

Thus

$$L_0(q,\Omega) = -\frac{1}{\beta}\sum_{\omega}\int \frac{d^3p}{(2\pi)^3\left[i(\Omega-\omega_\ell)-\epsilon_{q-p}\right]\left[i\omega_\ell-\epsilon_p\right]} \qquad (I.37)$$

Both $\vec{p}$ and $\vec{q}-\vec{p}$ are restricted to a shell about the FS. Maximum phase space for $q=0$.

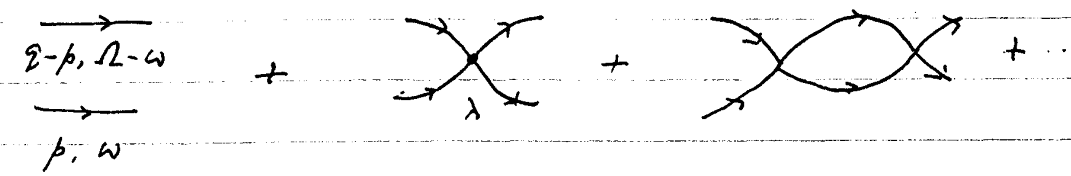
$$L_0(0,0) = -\frac{1}{\beta}N(0)\sum_{\omega}\int_{-\omega_D}^{\omega_D} d\epsilon\, \frac{1}{\omega_\ell^2+\epsilon^2} \qquad (I.38)$$

$\omega_\ell = \frac{(2\ell+1)\pi}{\beta}$, ϵ measured from the FS. To logarithmic accuracy in $\omega_D\beta$

$$L_0(0,0) = -N(0)\ln 1.13\,\omega_D\beta \qquad 1.13 = \frac{2}{\pi}e^{\gamma}$$

[illegible]

Now consider multiple scattering of injected pair. Feynman graphs $\xrightarrow[p\omega]{}$ for G $\bullet$ for λ

$L =$ 

$$L = L_0 - \tilde{\lambda} L_0^2 + \cdots = L_0 [1 - \tilde{\lambda} L] \qquad (I.40)$$

Sign important

i.e
$$L = \frac{L_0}{1 + \tilde{\lambda} L_0} \qquad (I.41)$$

Perturbation theory goes badly wrong at $L_0 = -\frac{1}{\tilde{\lambda}}$

$-L_0$ increases monotonically as β increases or T decreases. Breakdown of perturbation theory occurs at

$$-N(0) \ln \frac{1.13\, \omega_D}{T_c} = -\frac{1}{\tilde{\lambda}}$$

$$T_c = 1.13\, \omega_D\, e^{-\frac{1}{N(0)(\lambda - \mu^*)}} \qquad (I.42)$$

reintroducing μ^*.

One can show that for $T < T_c$ the perturbation calculation gives

$$L_R(t) \propto e^{+\alpha t}\,\theta(t) \qquad (I.43)$$

So T_c signals an instability towards ~~the~~ reconstructed states.

Impurity Scattering Consider randomly distributed spinless impurities:

$$V_{imp} = \int d^3r\ \psi^\dagger_\sigma(x) \sum_{n=1}^{n_{imp}} V(x - R_n)\,\psi_\sigma(x) \qquad (I.44)$$

Standard treatment, which is argued to be leading order in the small parameter $(k_F l)^{-1}$ with $l = v_F \tau$ the mean free path, calculates correction to the electron propagator according to

$$\underset{\tilde G_0(p\omega)}{\Longrightarrow} = \underset{G_0(p\omega)}{\longrightarrow} \qquad \underset{G_0}{\rightarrow}\ \underset{\Sigma}{\times}\ \underset{\tilde G_0}{\Longrightarrow},$$

yielding

$$\tilde G_0(p\omega) = \frac{1}{i\tilde\omega - \epsilon} \qquad \tilde\omega = \omega + \frac{1}{2\tau}\,\mathrm{sgn}\,\omega$$

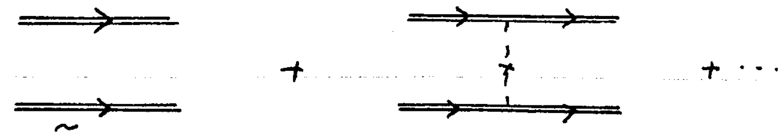
$$\frac{1}{\tau} \equiv \Gamma = 2\pi N(0)\, n_{imp} \langle |V|^2 \rangle \qquad (I.45)$$

This gives the spectral weight function a width $\frac{\Gamma}{2}$.

Cooper Instability including impurities (lowest order)

Pair propagator including impurity scattering

$\tilde{L}_0 =$



$\tilde{L}_{00}$

$$\tilde{L}_0 = -\frac{1}{\beta}\sum_{\omega}{}' \frac{\int \frac{d^3p}{(2\pi)^3}\, \tilde{G}_0(q-p, \Omega-\omega)\, \tilde{G}_0(p,\omega)}{1 - \frac{1}{2\pi N(0)\tau}\int \frac{d^3p}{(2\pi)^3}\, \tilde{G}_0(q-p,\Omega-\omega)\,\tilde{G}_0(p,\omega)} \qquad (I.46)$$

Convenient to transfer the cut-off to the ω-sum.
Numerator of the fraction is

$$N(0)\int_{-\infty}^{\infty} d\epsilon \int_{-1}^{1}\frac{du}{2}\, \frac{1}{i(\widetilde{\Omega-\omega}) - \epsilon + q v_F u}\; \frac{1}{i\tilde{\omega} - \epsilon}$$

$$= \theta[\omega(\omega-\Omega)]\, \frac{2\pi N(0)}{\Gamma}\int_{-1}^{1}\frac{du}{2}\, \frac{1}{\left[1 + \frac{|2\omega-\Omega|}{\Gamma} + \frac{i q v_F u}{\Gamma}\right]}$$

"Dirty limit"

$$\xrightarrow{\Gamma \gg \Omega,\, q v_F,\, T} \quad \theta[\omega(\omega-\Omega)]\, \frac{2\pi N(0)}{\Gamma}\left[1 - \frac{|2\omega-\Omega|}{\Gamma} - \frac{Dq^2}{\Gamma} + \cdots\right]$$

where $D \equiv \frac{v_F^2 \tau}{3}$. Then $\qquad (I.47)$

$$\tilde{L}_0 = -4\pi N(0)\, T\left\{\sum_{\omega>0}\left[\frac{1}{Dq^2 + 2\omega + |\Omega|} - \frac{1}{2\omega}\right] + \sum_{\omega>0}{}' \frac{1}{2\omega}\right\} \qquad (I.48)$$

Note that $\tilde{L}_0(0,0)$ is unchanged by impurity scattering to this approximation.

Thus the instability at $\tilde{L} = -\frac{1}{\tilde{\lambda}}$ gives the same T_c to this approximation as for the clean model

$$T_c = 1.13\, \omega_D\, e^{-\frac{1}{N(0)(\lambda - \mu^*)}} \qquad (I.49)$$

Lack of correction to T_c is called Anderson's "Theorem." However, questionable assumptions have been made. In particular, the change in the screening properties of electrons [contained in μ^*] has been ignored. We already saw that diffusing electrons screen less effectively than ballistic ones.

The connection between the Cooper instability and the onset of superconductivity is not, on the face of it, obvious. It is a historical fact that Cooper's slightly less general calculation was the key for the inventors (BCS) to guess a paired ground state and associated excitations which explained the zero resistance and the expulsion of a small magnetic field (Meissner effect) as well as other dissipative properties below T_c. The essential idea of the paired state is an order parameter → Lecture II.