

Three lectures on Homogenization of Mechanical Metamaterials @ Boulder Summer School 2026

Lecture Plan

- 1) parallelogram hole kirigami — mechanisms
- 2) parallelogram hole kirigami — mechanism gradients
- 3) origami / kagome / wrinkled shells

For references & further reading see Boulder Summer School website.

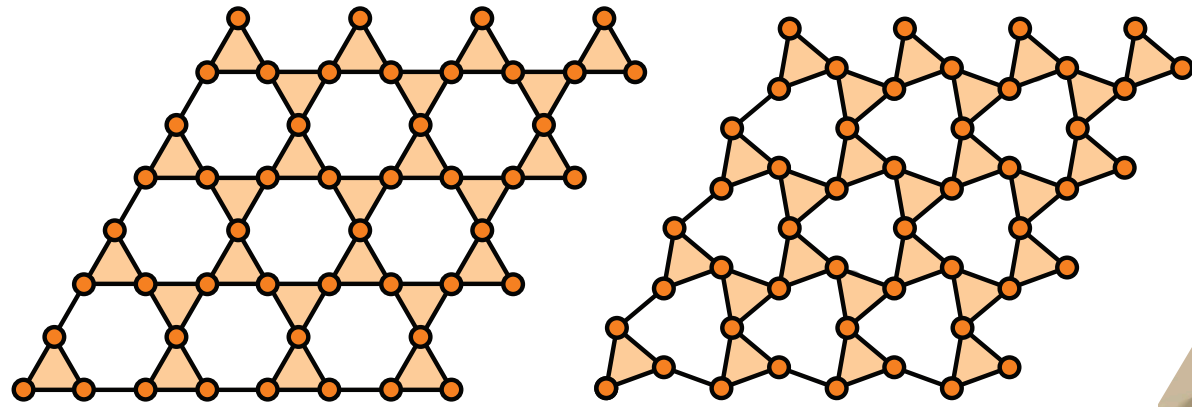
Thanks to my collaborators!

Parts 1+2: Paul Plucinsky, Paolo Celli, Yue Zheng, Hu Xu

Part 3: Joey Paulsen, Eleni Katifori, Yousra Timounay, Desislava Todorova

Mechanical Metamaterials

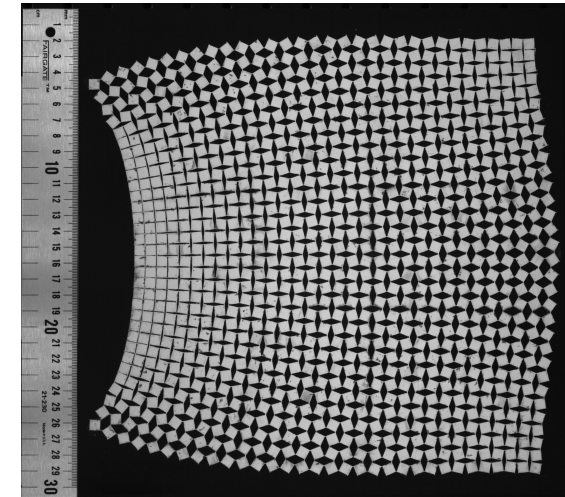
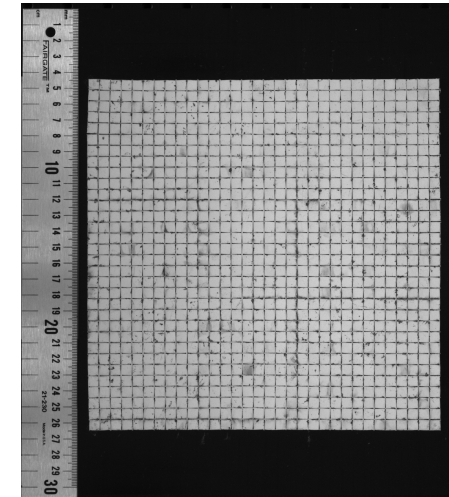
What is a mechanical metamaterial?



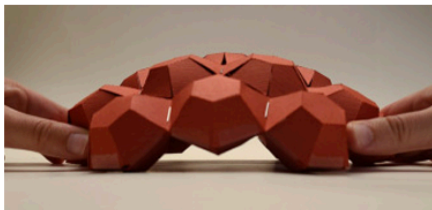
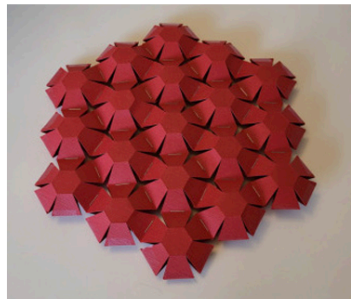
Lubensky, Kane, Mao, Souslov, Sun
Reports on Progress in Physics '15



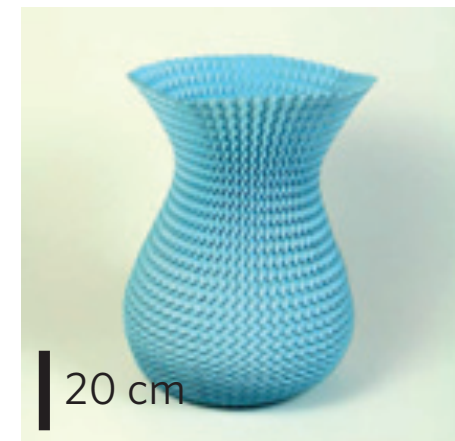
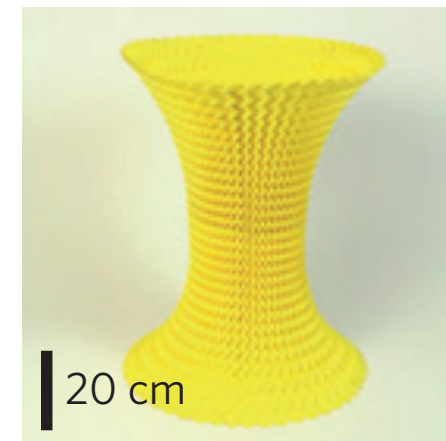
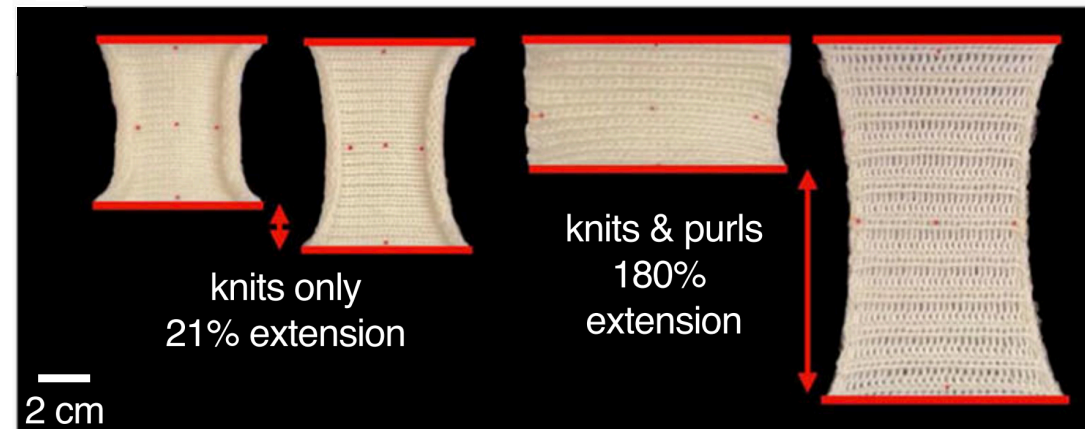
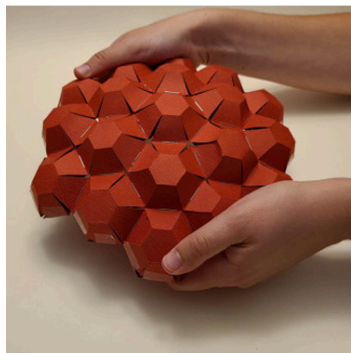
Niu, Dion, Kamien
PNAS '25



Zheng, Niloy, Celli, Tobasco, Plucinsky
Physical Review Letters '22



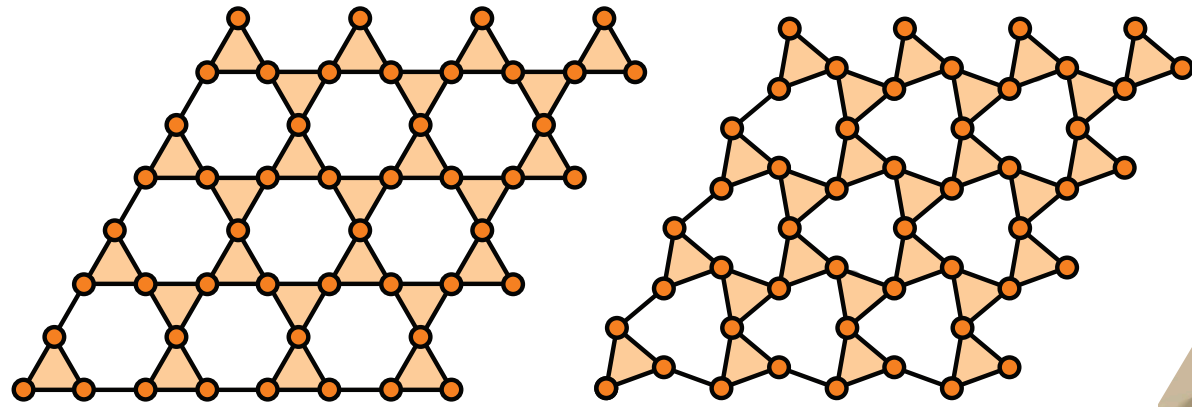
Singal, Dimitriyev, Gonzalez, Cachine, Quinn, Matsumoto
Nature Communications '24



Dudte, Vouga, Tachi, Mahadevan
Nature Materials '16

Redoutey, Roy, Filipov
International Journal of
Solids and Structures '21

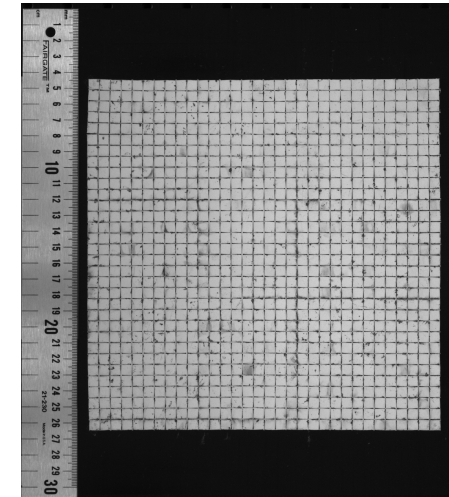
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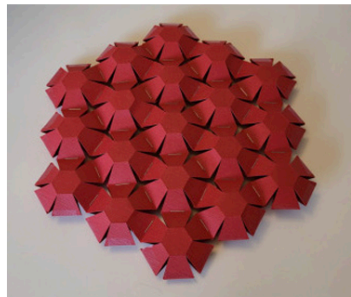
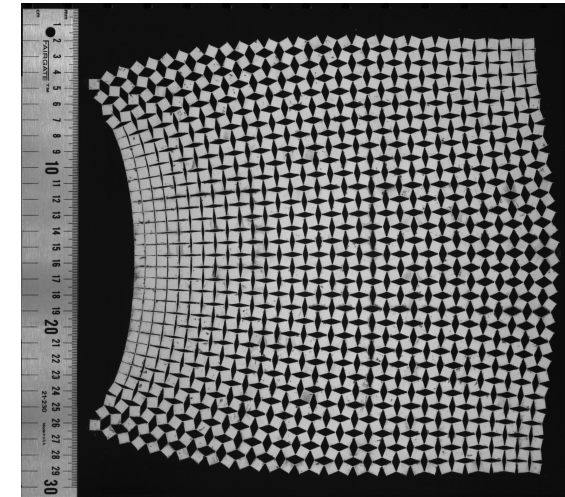
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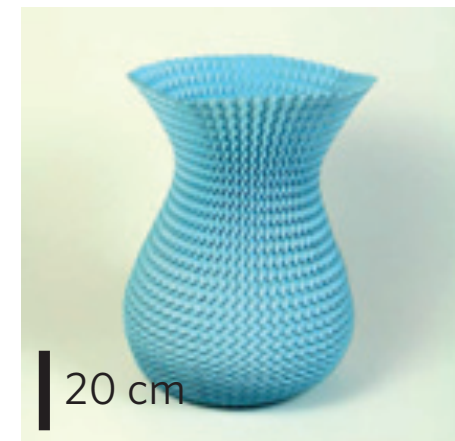
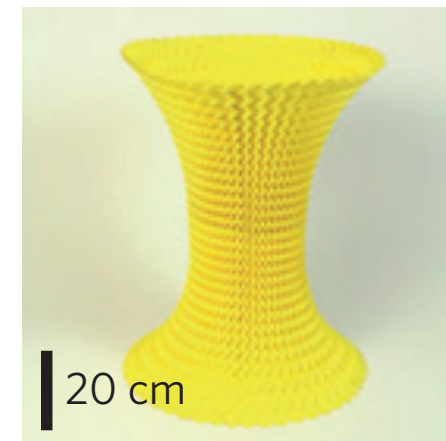
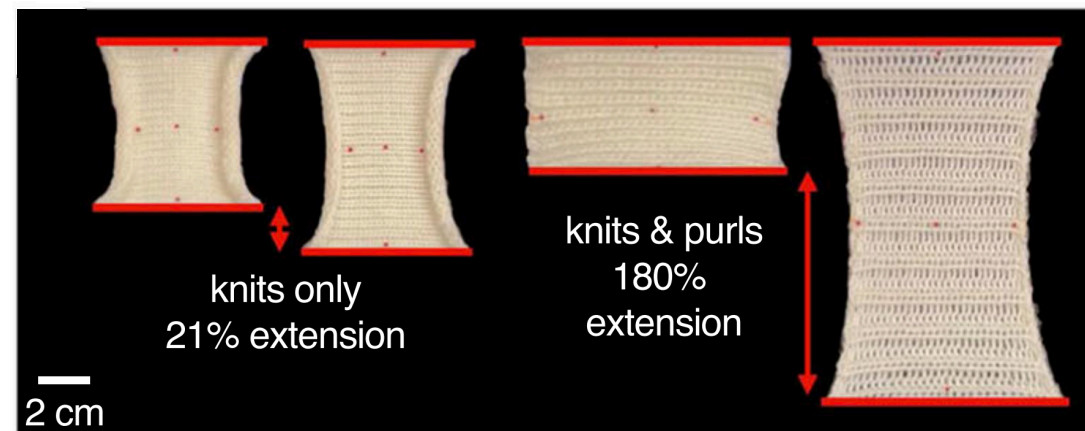
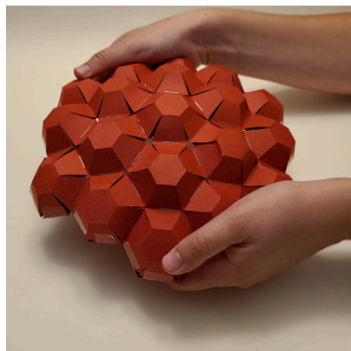
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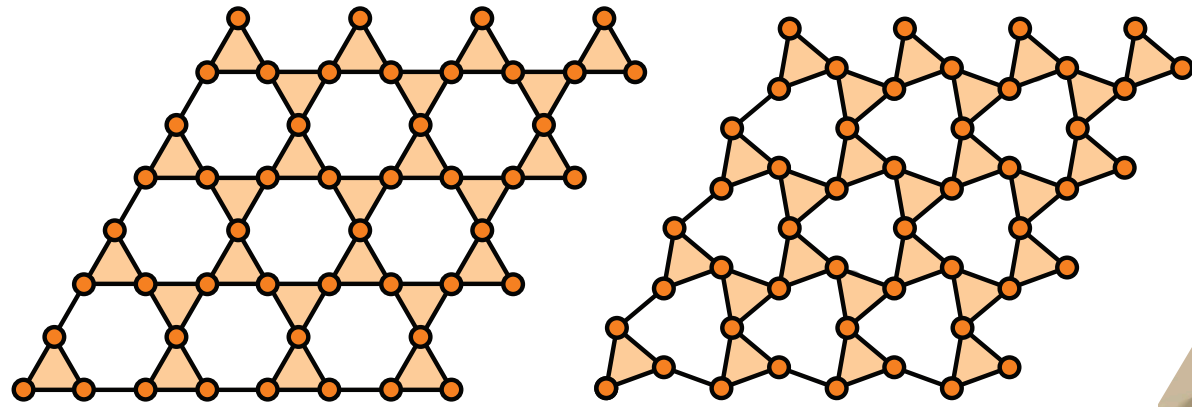


Dudte, Vouga, Tachi, Mahadevan
Nature Materials '16

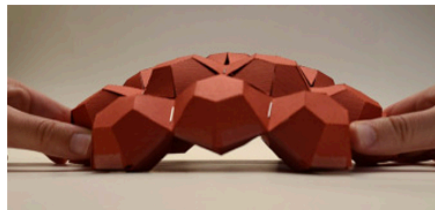
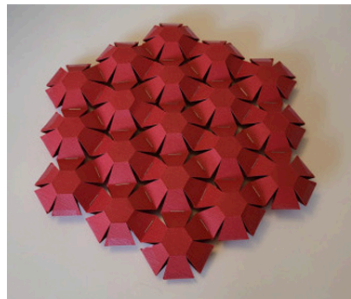
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Solids and Structures '21

*A many-body system with elastic interactions
and unusual bulk properties (e.g., negative Poisson's ratio)*

What is a mechanical metamaterial?



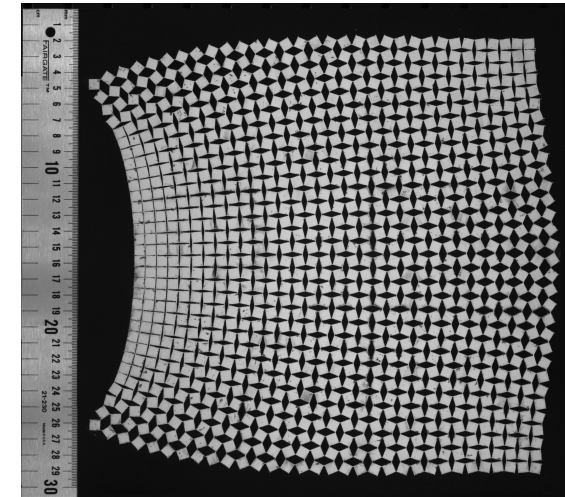
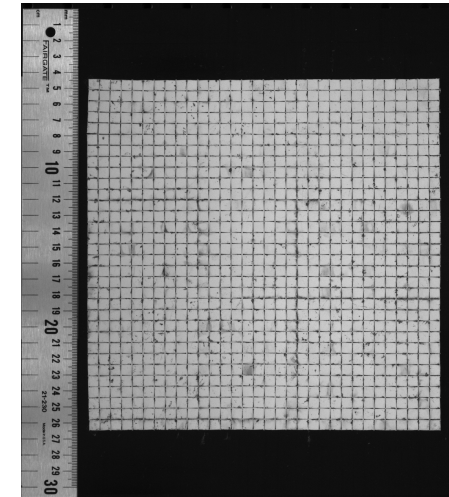
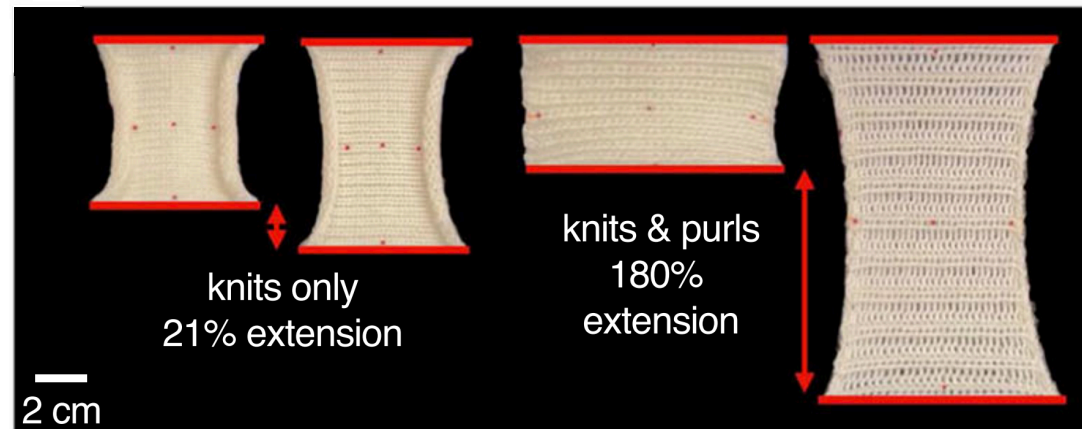
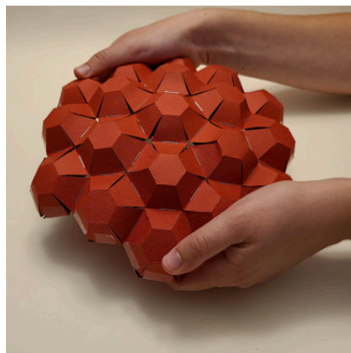
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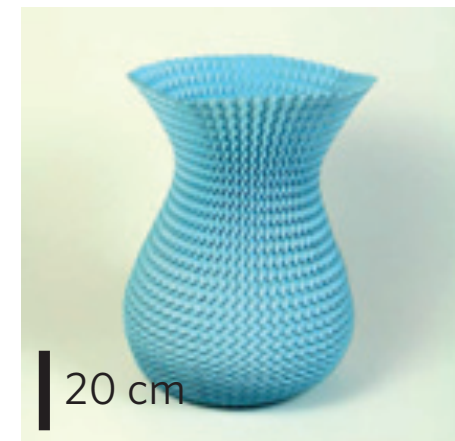
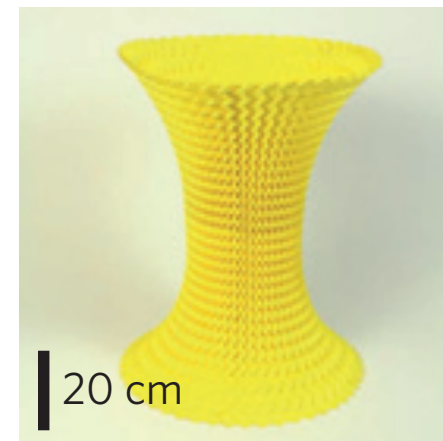
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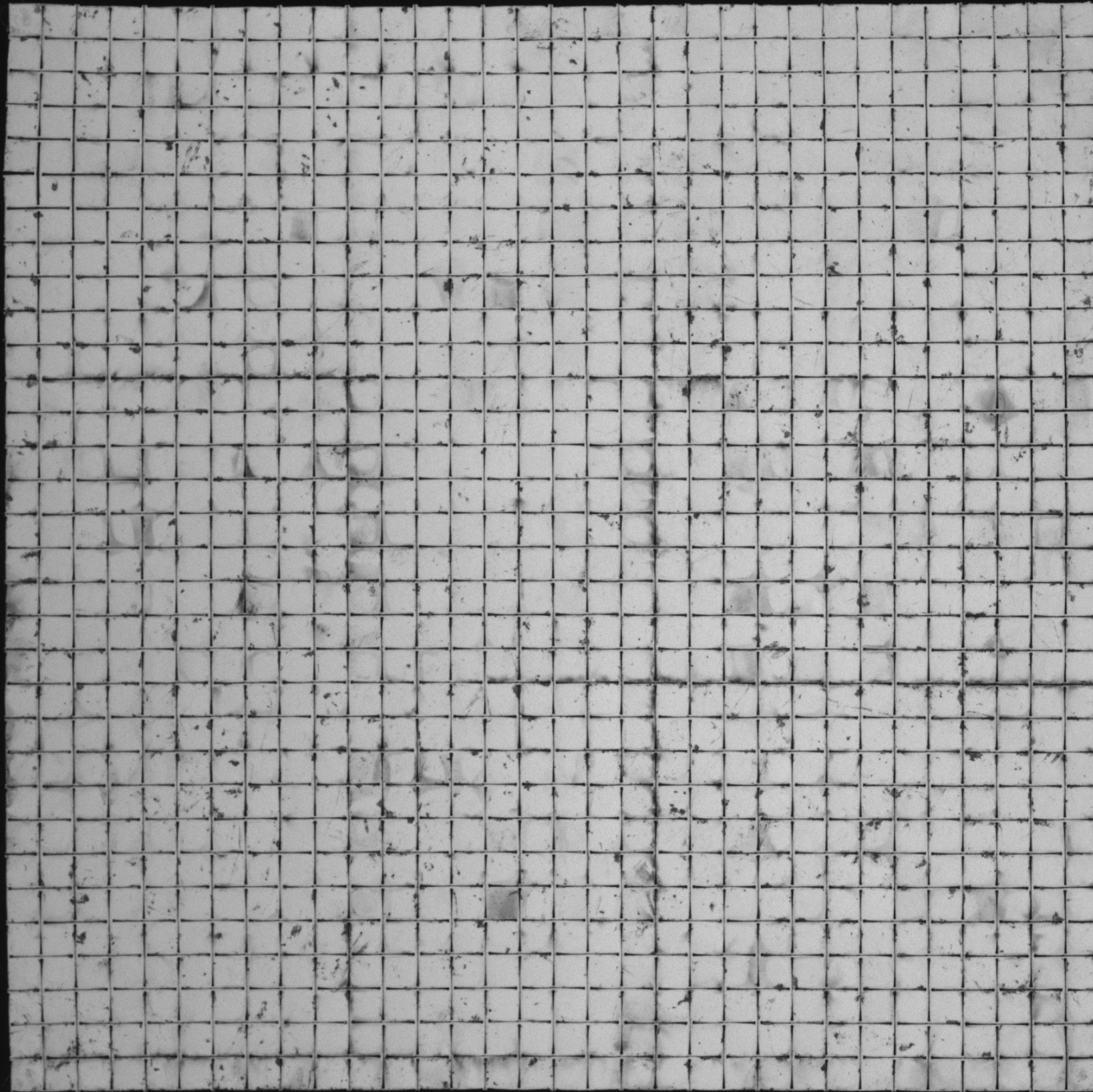
Dudte, Vouga, Tachi, Mahadevan
Nature Materials '16

*A many-body system with elastic interactions
and unusual bulk properties (e.g., negative Poisson's ratio)*

*Important point: some bulk properties turn out to be
dictated by **geometry** rather than **material***

**Pulling a
rubber
kirigami sheet**

from Paolo Celli's lab @ Stony Brook



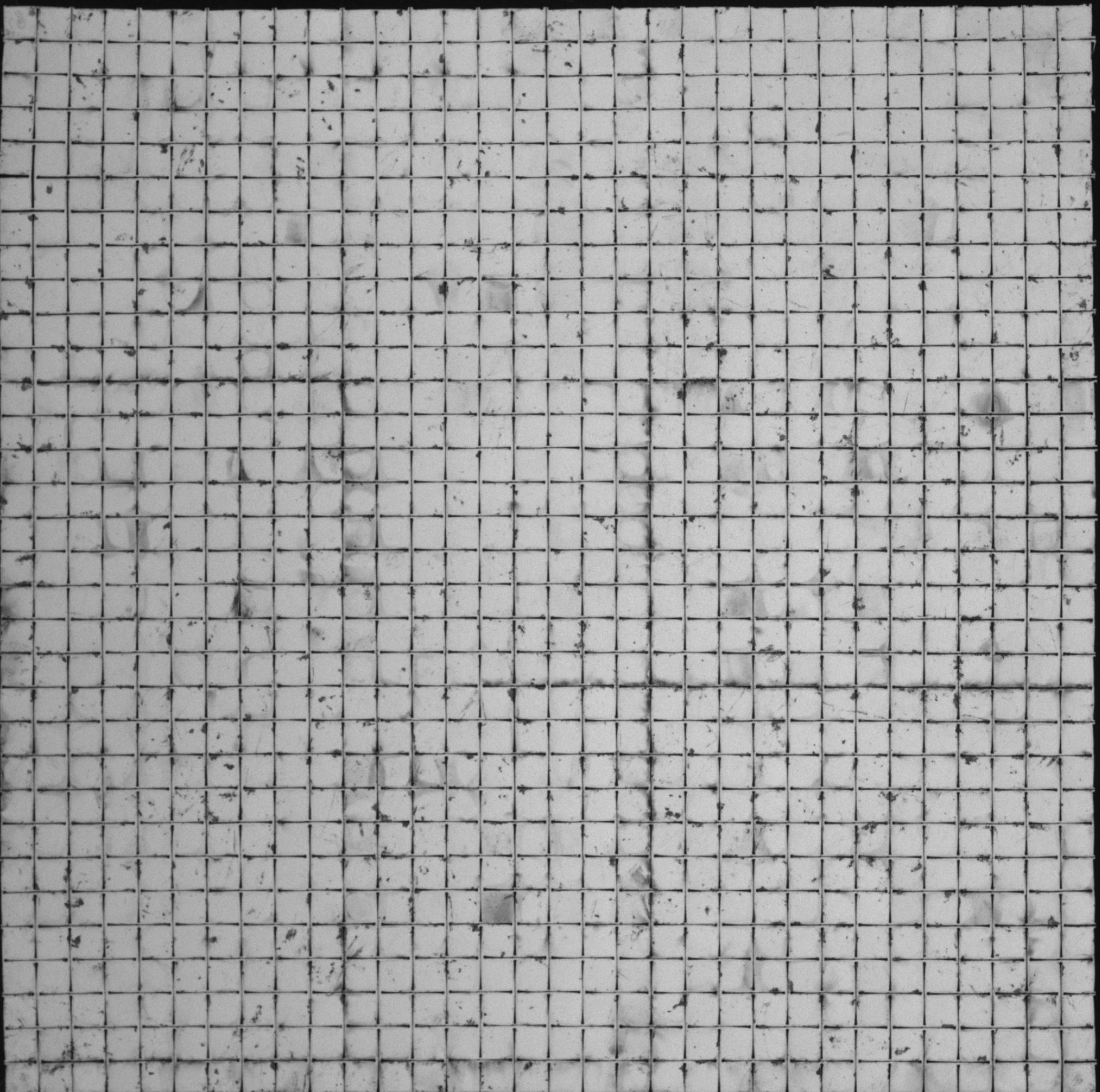
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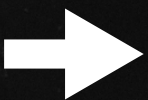
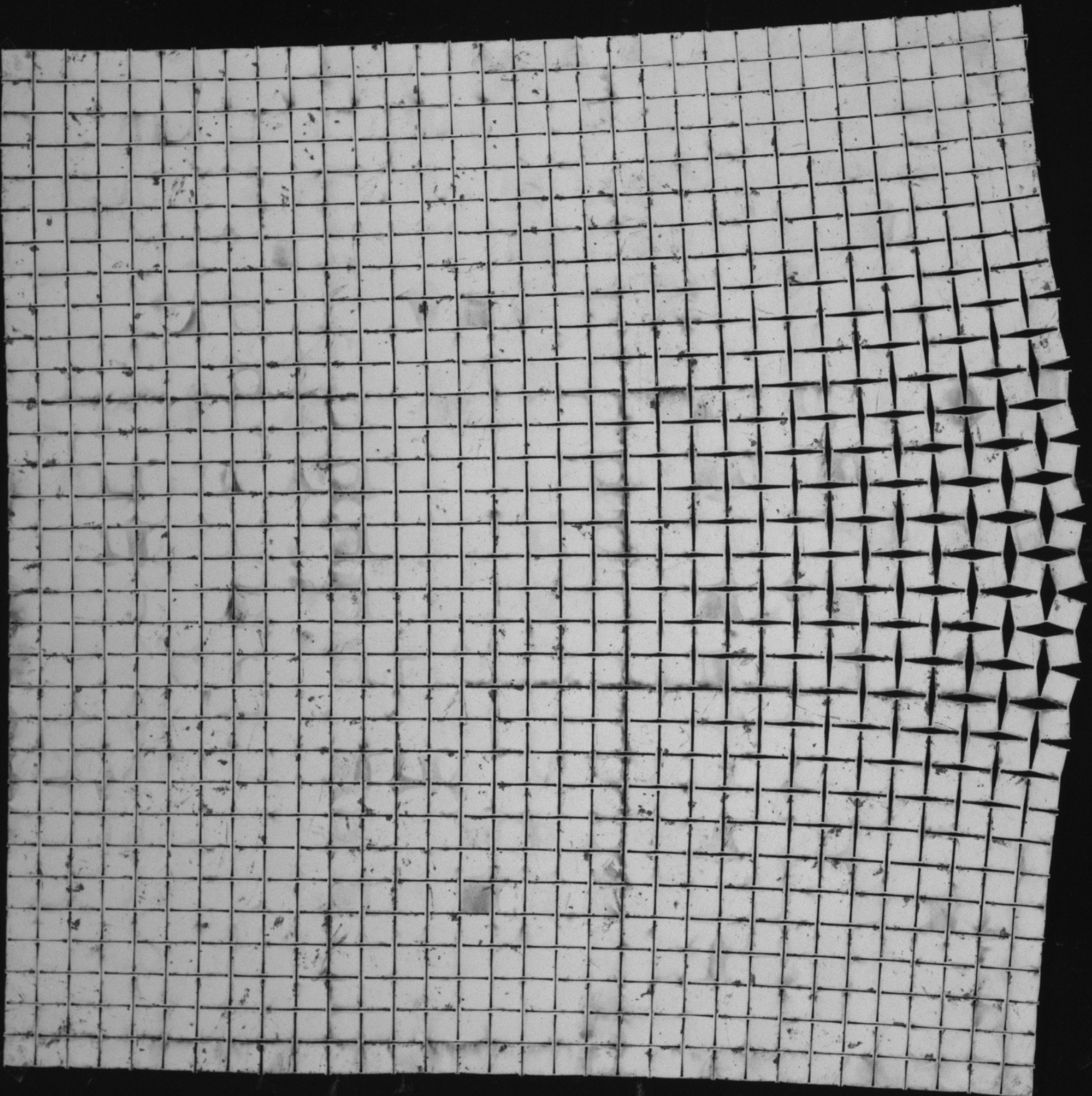
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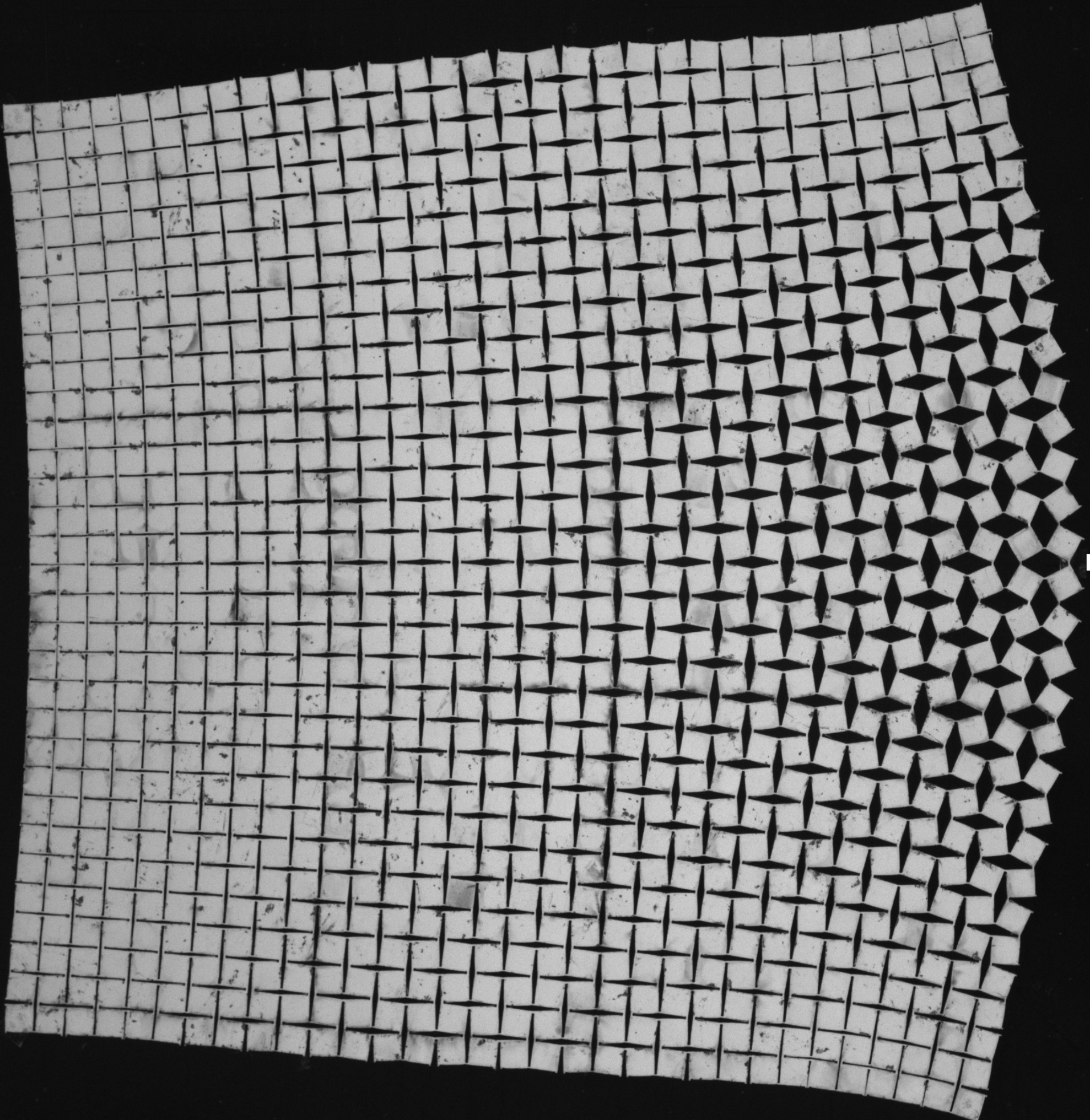
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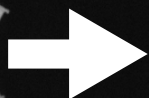
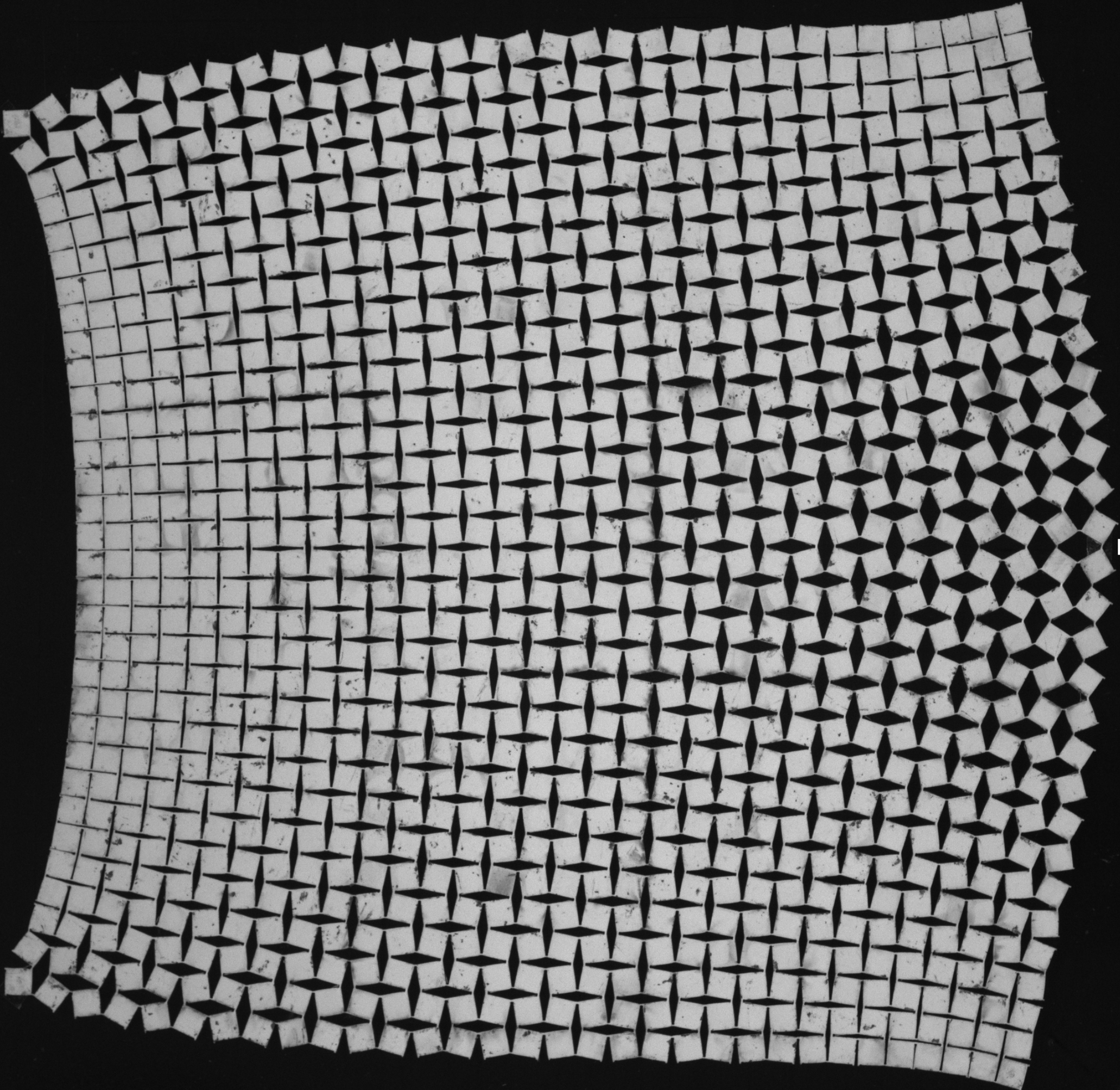
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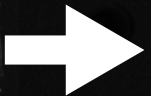
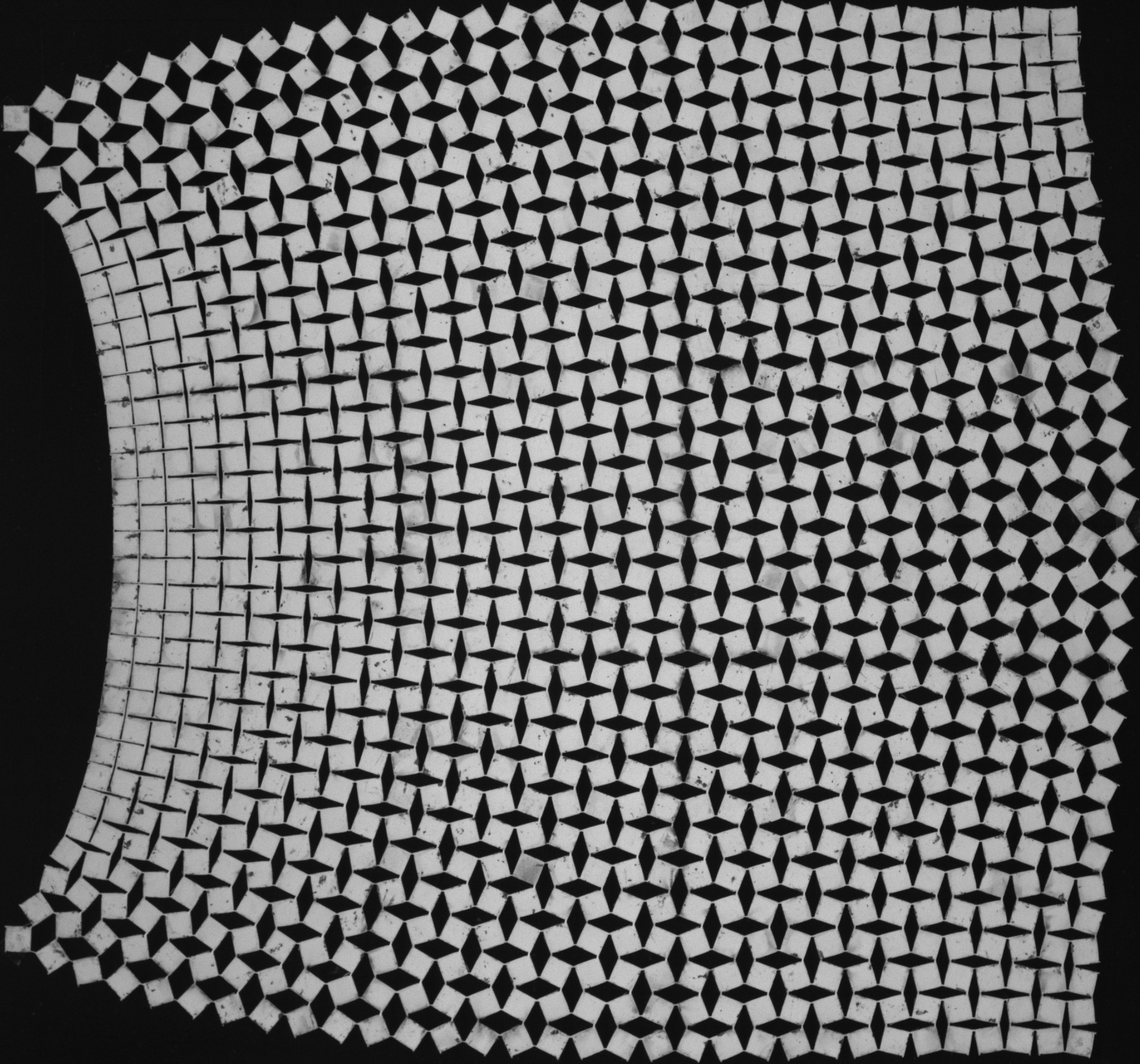
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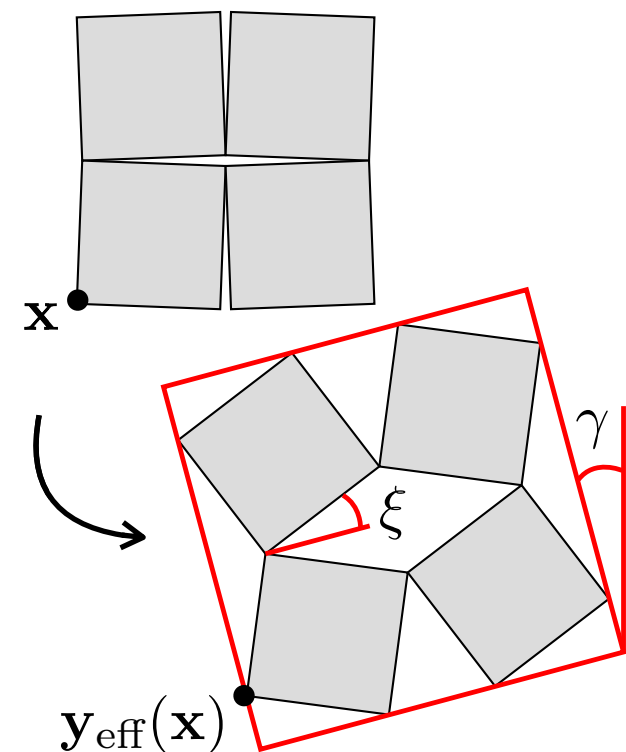
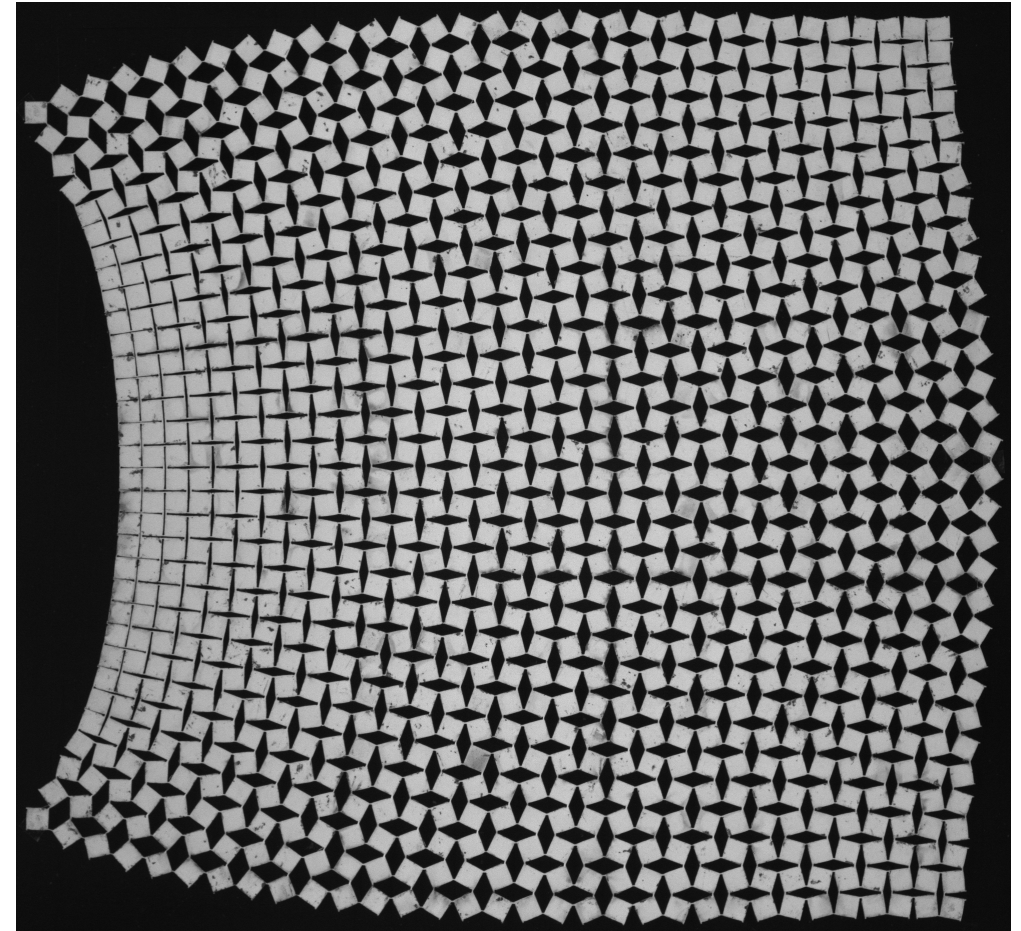
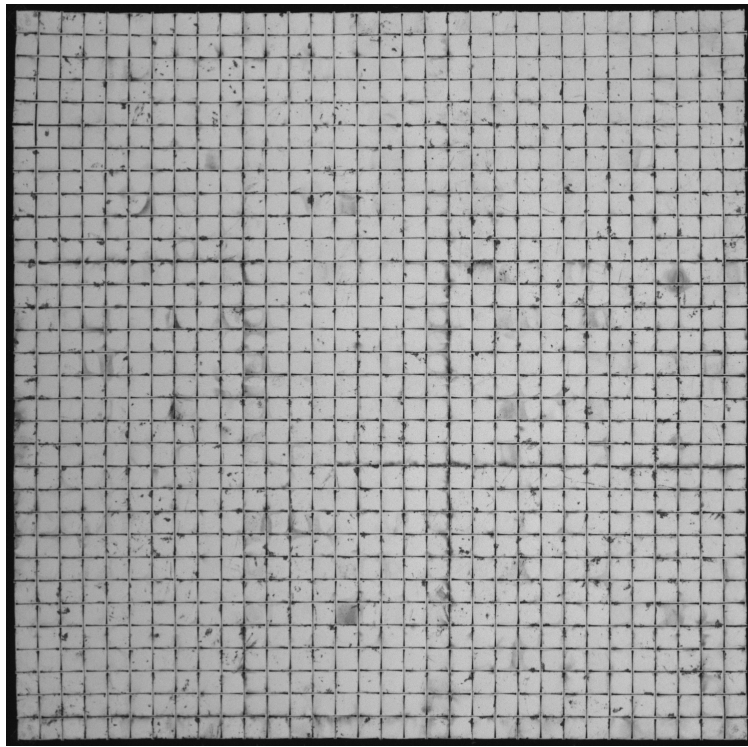
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21-230

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General soft modes look locally mechanistic

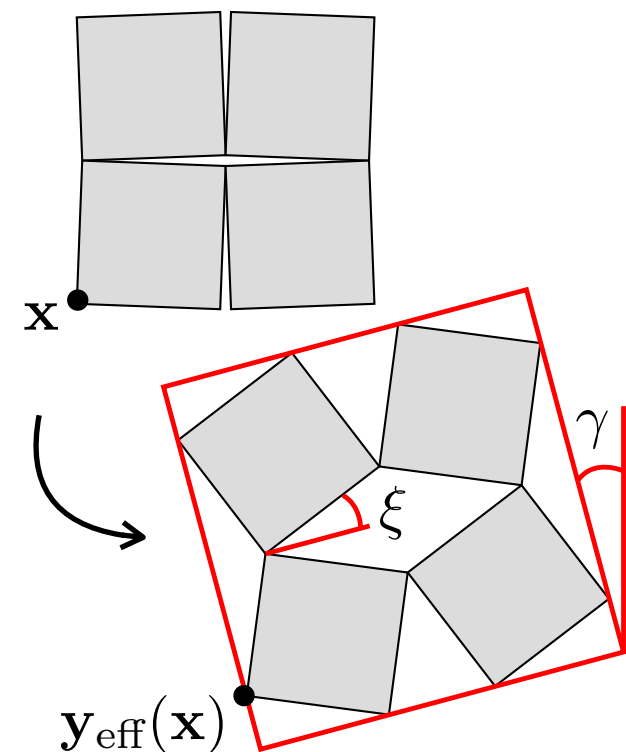
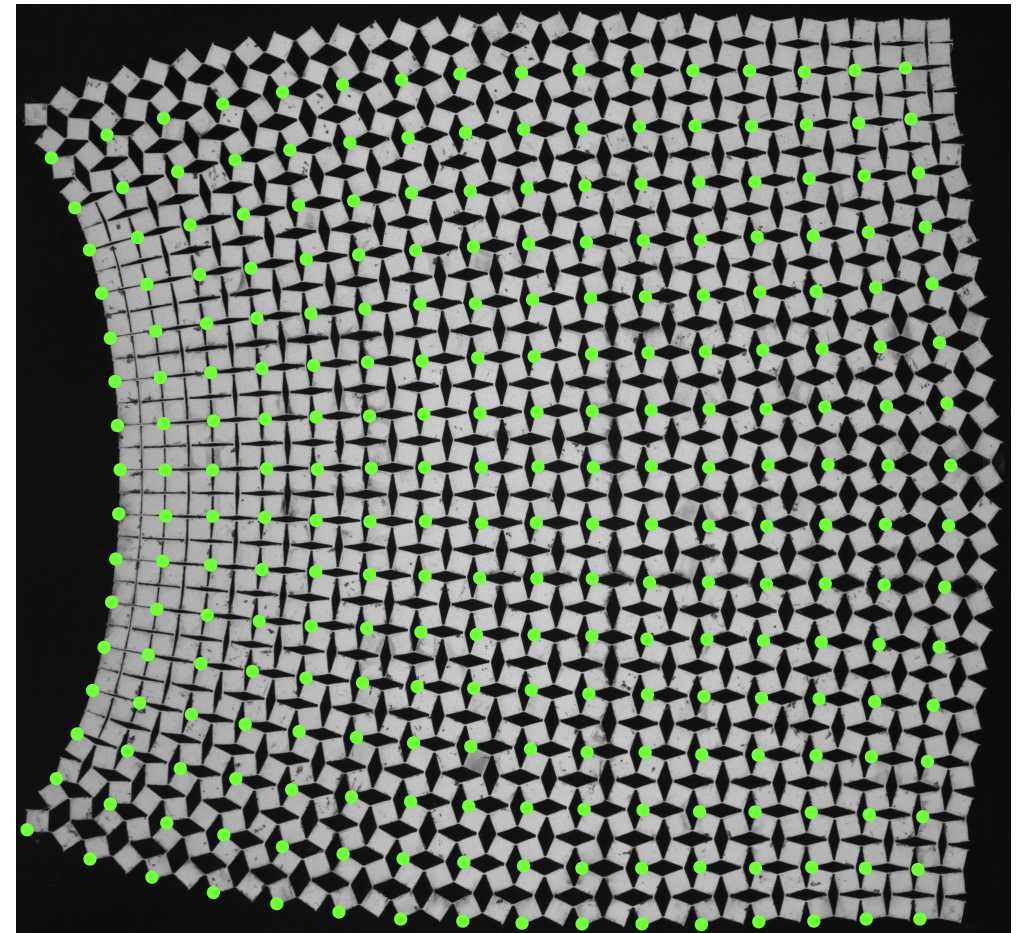
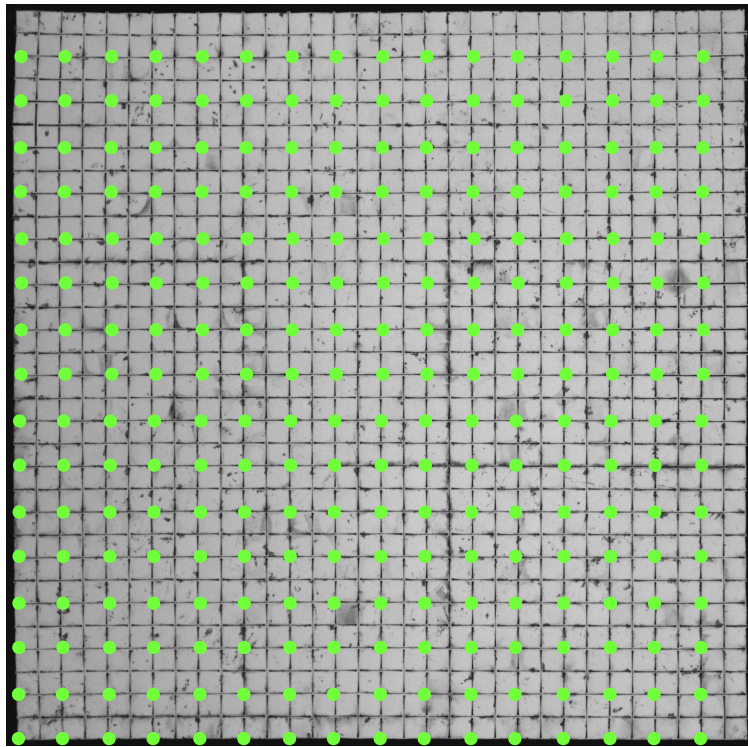


$\gamma(\mathbf{x}) = \text{c.c.w. rotation of cell at } \mathbf{x}$

$\xi(\mathbf{x}) = \text{counter-rotation of panels at } \mathbf{x}$

$\mathbf{y}_{\text{eff}}(\mathbf{x}) - \mathbf{x} = \text{translation of cell at } \mathbf{x}$

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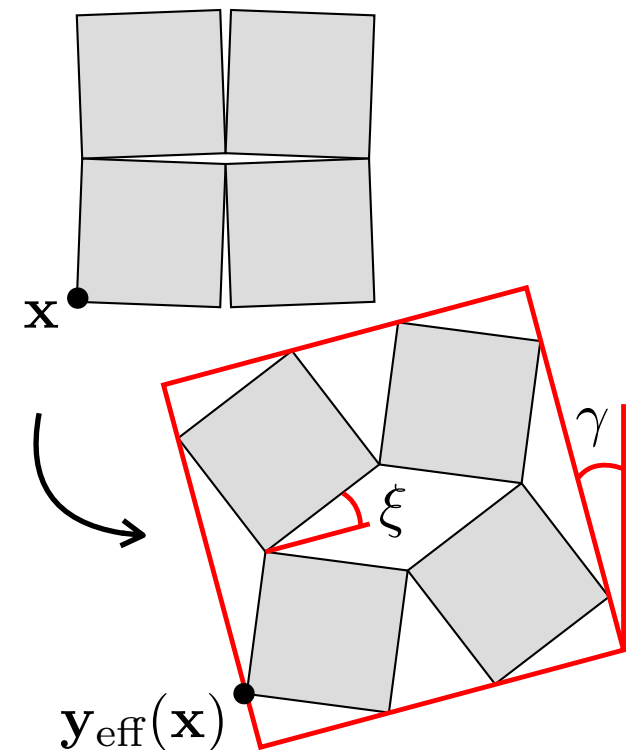
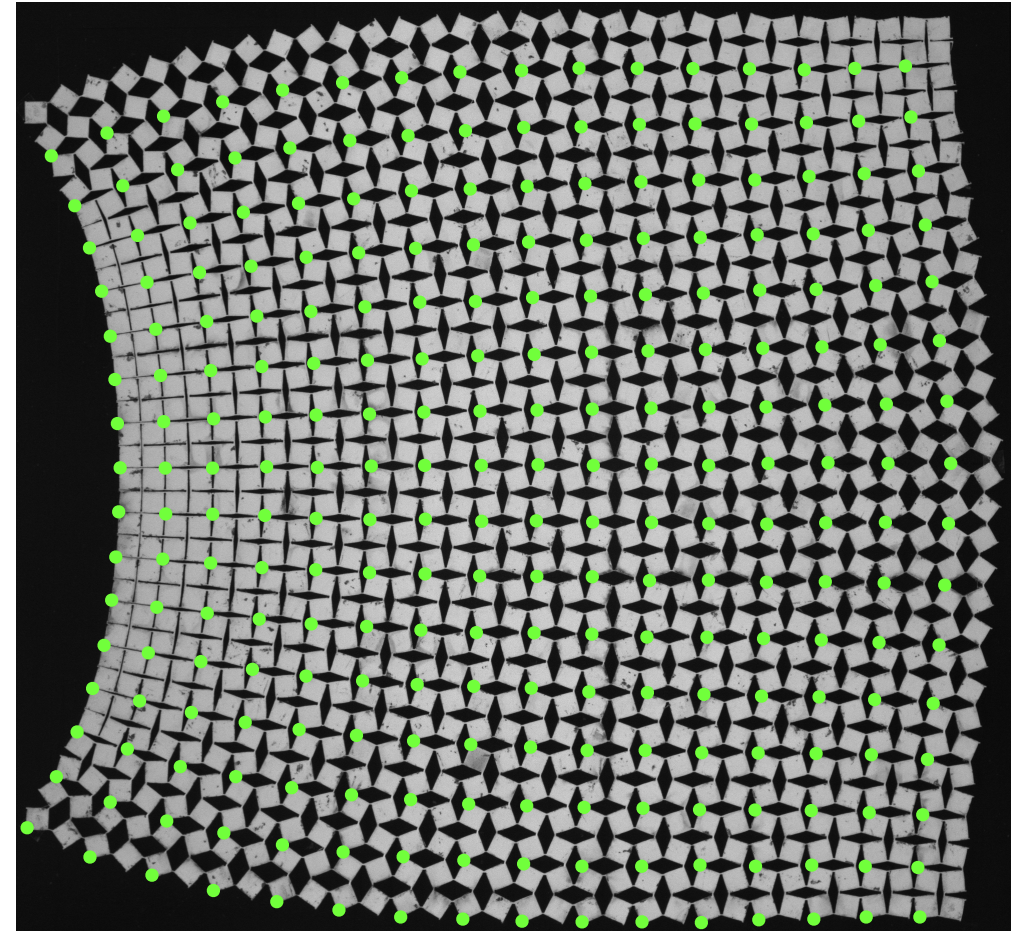
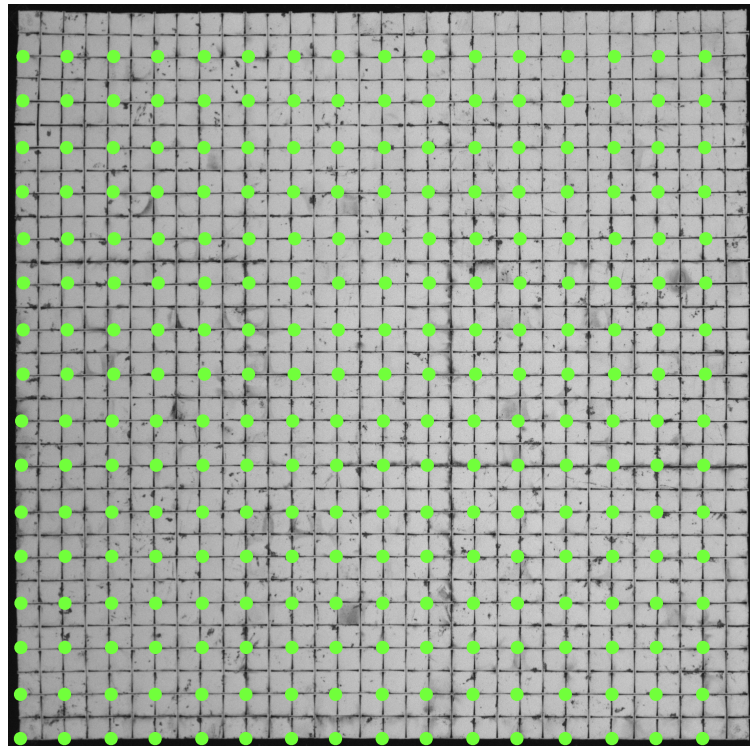


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General soft modes look locally mechanistic



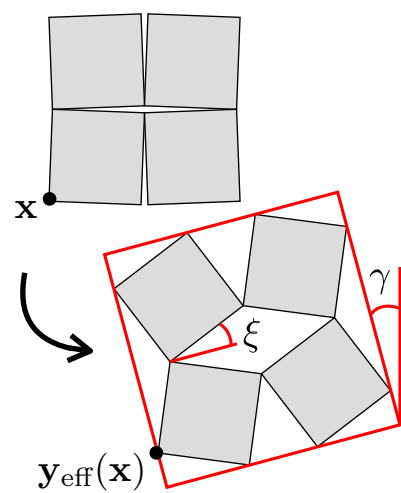
Homogenization rule:

$$\nabla \mathbf{y}_{\text{eff}}(\mathbf{x}) = \mathbf{R}(\gamma(\mathbf{x})) \mathbf{A}(\xi(\mathbf{x}))$$

$\mathbf{A}(\xi) = \text{shape change tensor}$

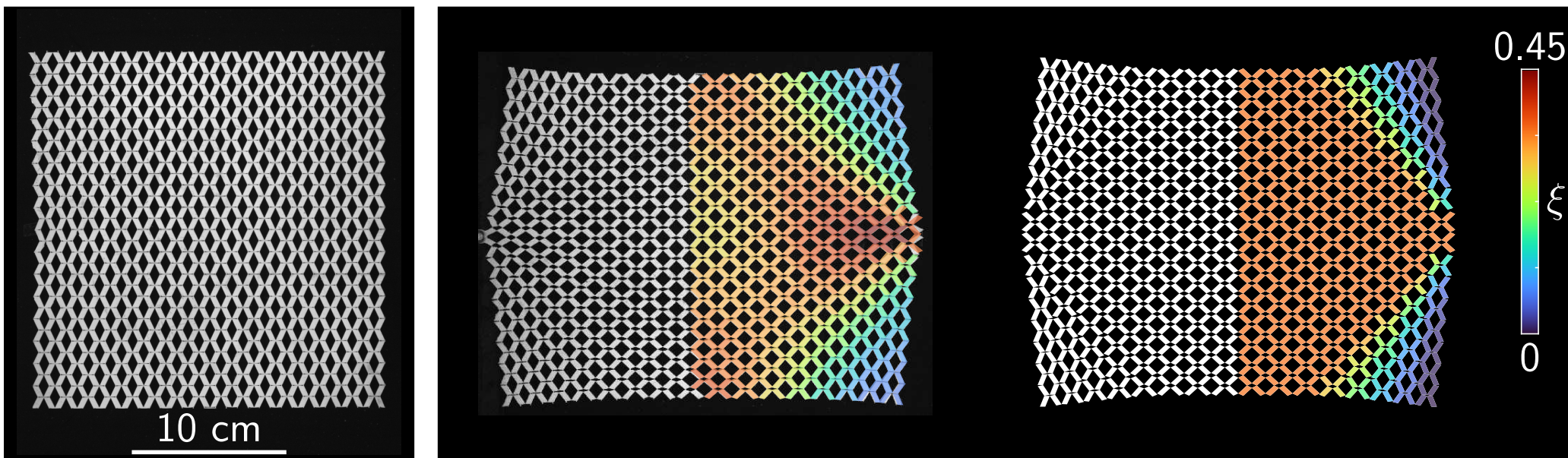
Experiments vs. PDE solutions

$$\nabla \mathbf{y}_{\text{eff}}(\mathbf{x}) = \mathbf{R}(\gamma(\mathbf{x})) \mathbf{A}(\xi(\mathbf{x})) \implies \nabla \gamma(\mathbf{x}) = \mathbf{\Gamma}(\xi(\mathbf{x})) \nabla \xi(\mathbf{x})$$



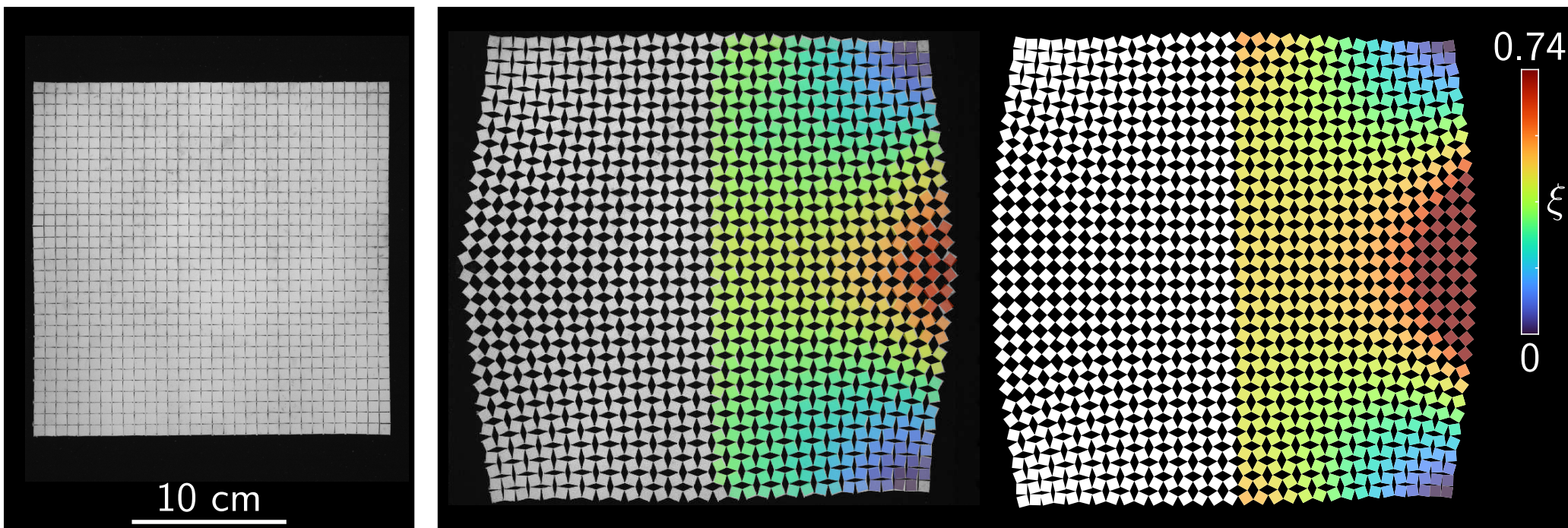
hyperbolic
(characteristic lines)

$$\mathbf{A}(\xi) = \begin{pmatrix} \lambda_1(\xi) & 0 \\ 0 & \lambda_2(\xi) \end{pmatrix}$$



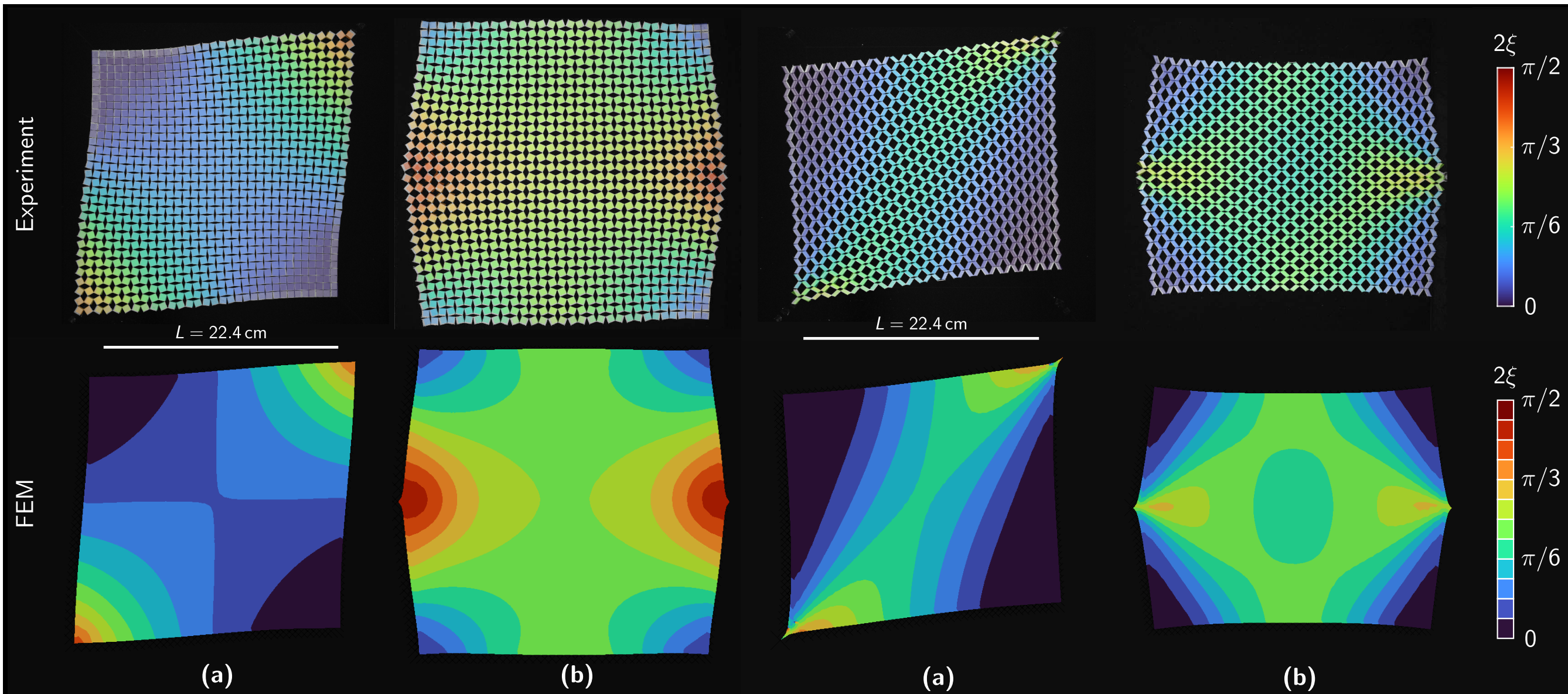
elliptic
(max. principle)

$$\mathbf{A}(\xi) = \lambda(\xi) \mathbf{I}$$



In general: hyperbolic = pos. Poisson's ratio / elliptic = neg. Poisson's ratio

FEM sims of mechanism gradient theory

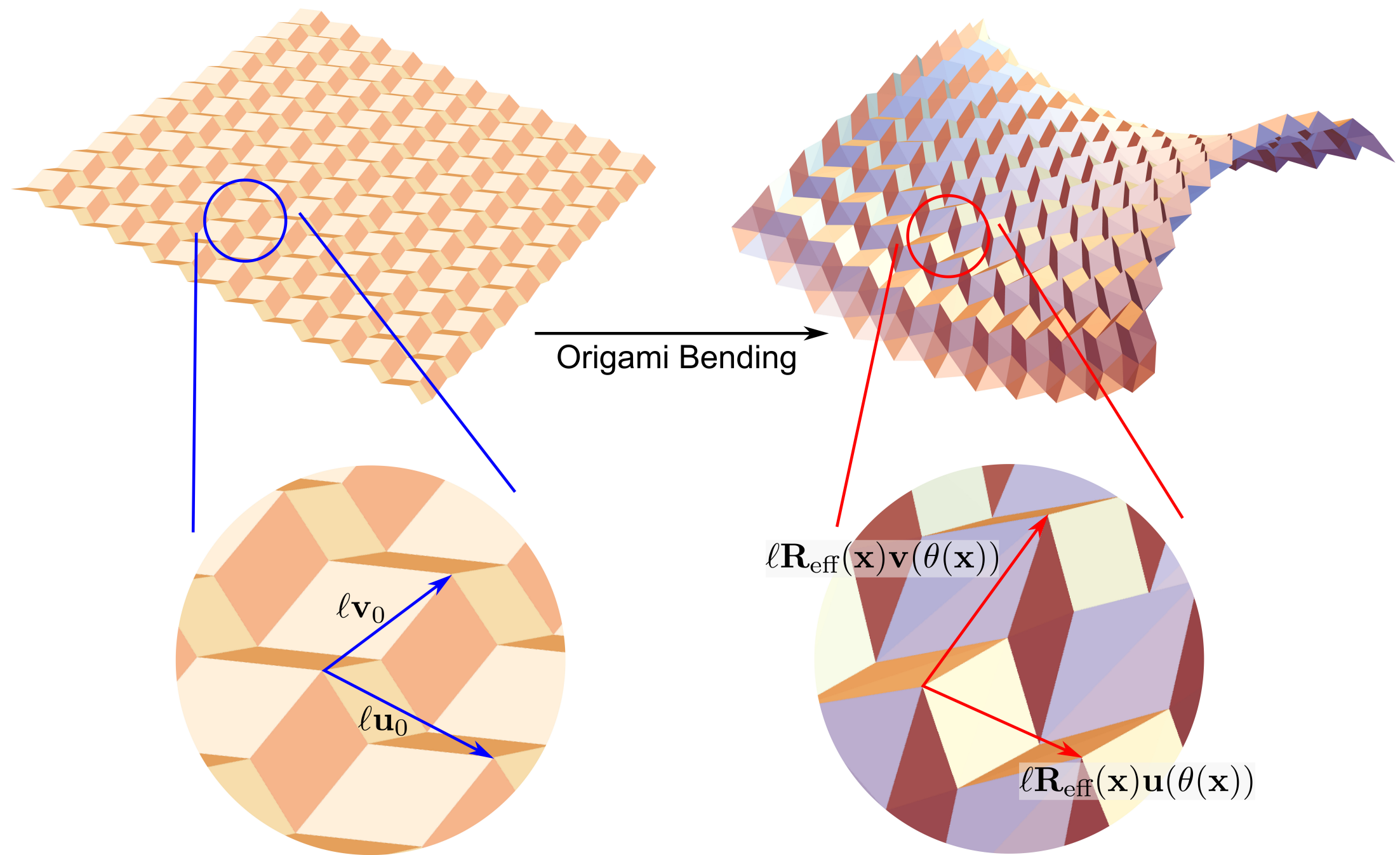


$$E = \int_{\Omega} c_0 d^2 (\nabla \mathbf{y}_{\text{eff}}(\mathbf{x}), SO(2) \mathbf{A}(\xi(\mathbf{x}))) + c_1 |\xi(\mathbf{x})|^2 + c_2 |\nabla \xi(\mathbf{x})|^2 d\mathbf{x}$$

enforces PDE
hinge bending
mech. gradient

$$c_0 \sim \frac{\mu}{\log(\ell/\delta)} \quad c_1 \sim \mu \frac{\delta^2}{\ell^2} \quad c_2 \sim \mu \frac{\ell^2}{\log(\ell/\delta)} \quad \mu = \text{shear modulus}$$

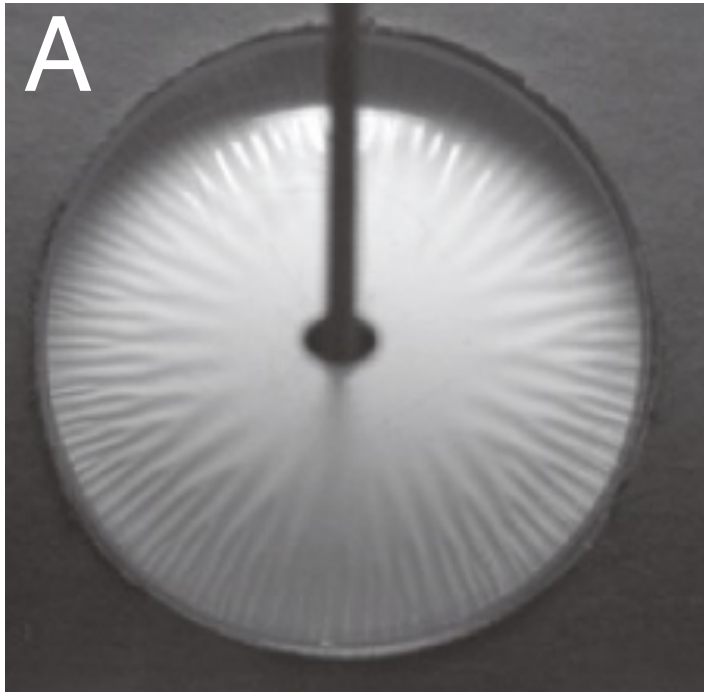
Parallelogram Origami



Wrinkle patterns

Curvature-Driven Wrinkling

Mismatch in lengths + confinement -> patterns



Hure, Roman
& Bico
PRL '12

Davidovitch, Sun
& Grason
PNAS '19

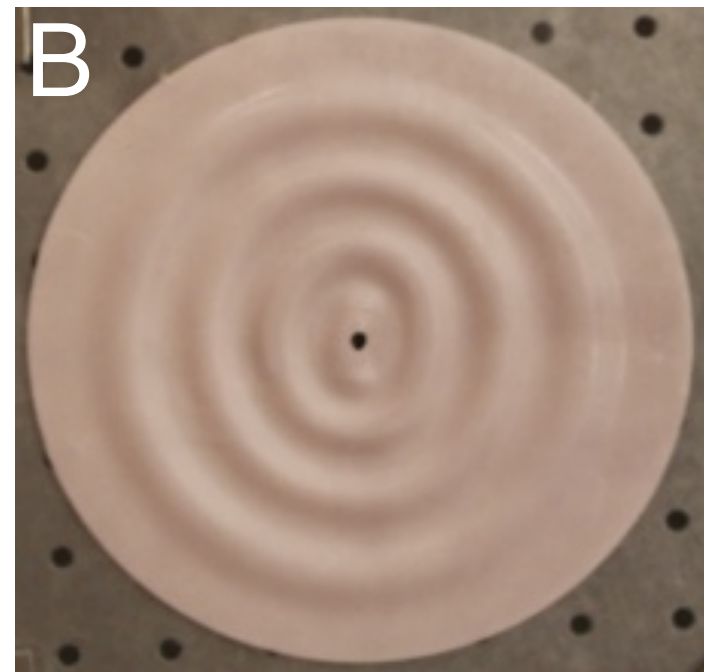
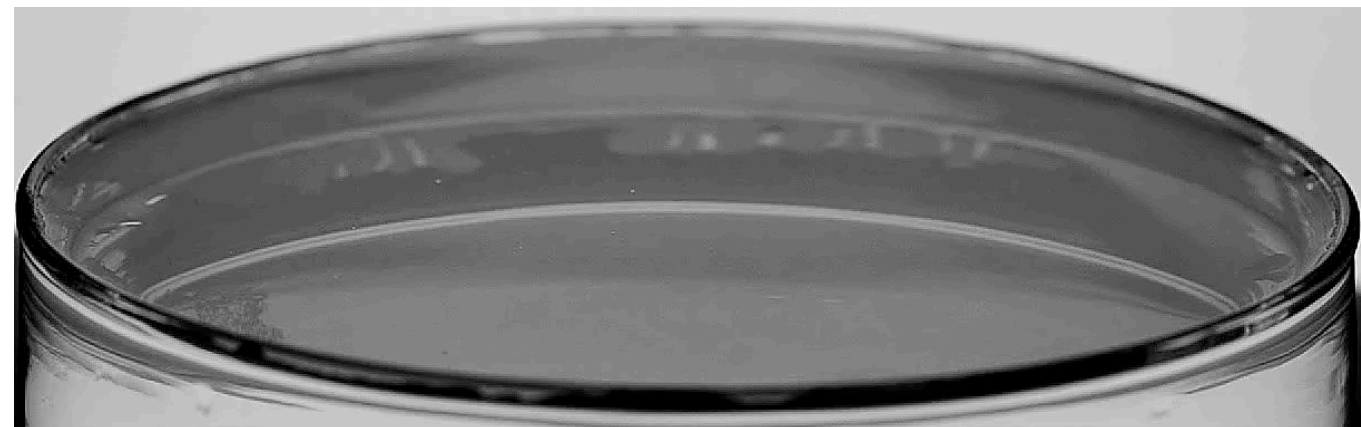


image credit to
Sharon



video credit to Octavio Albarrán (arXiv:1806.03718)

Wrinkle patterns depend on initial shape

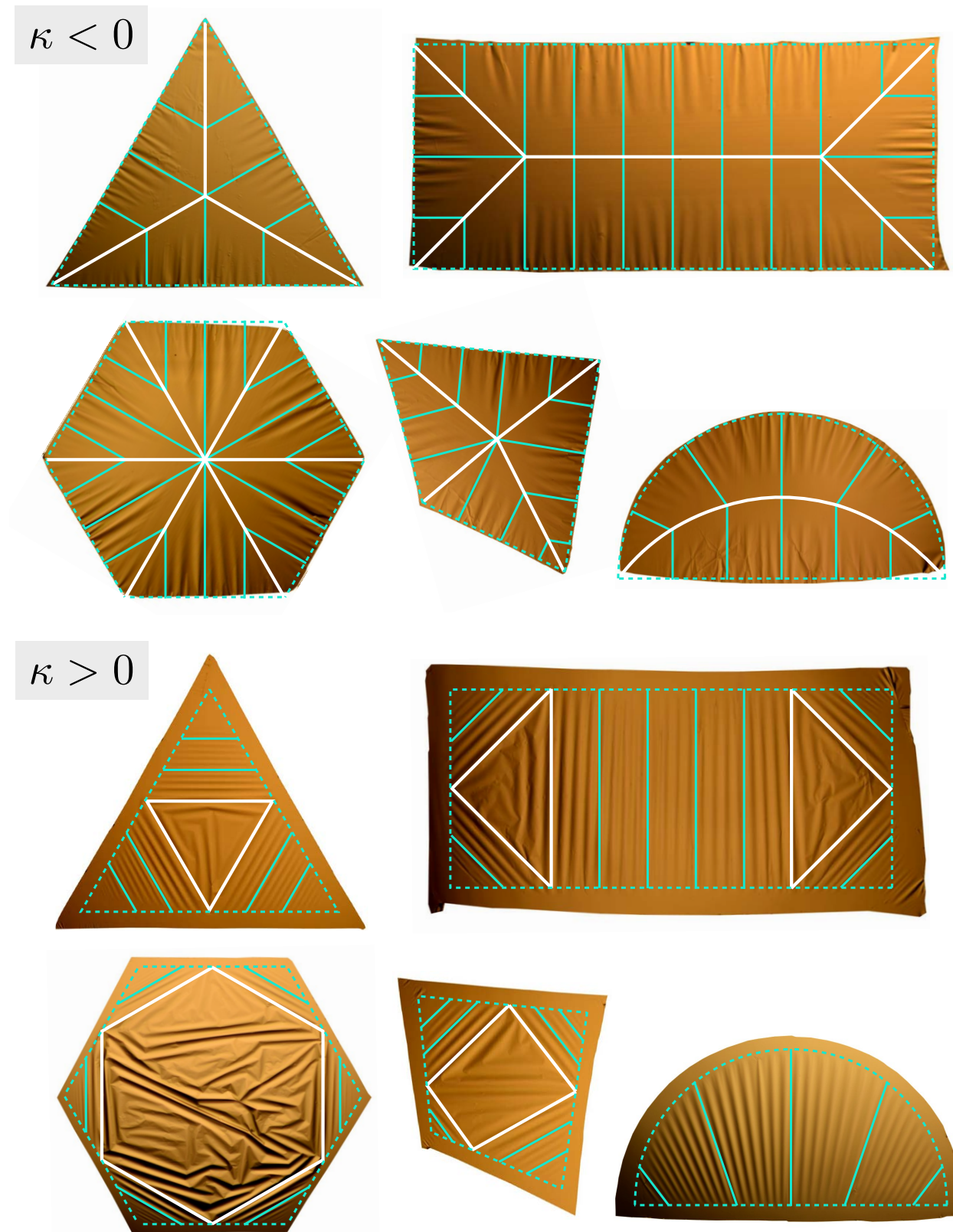
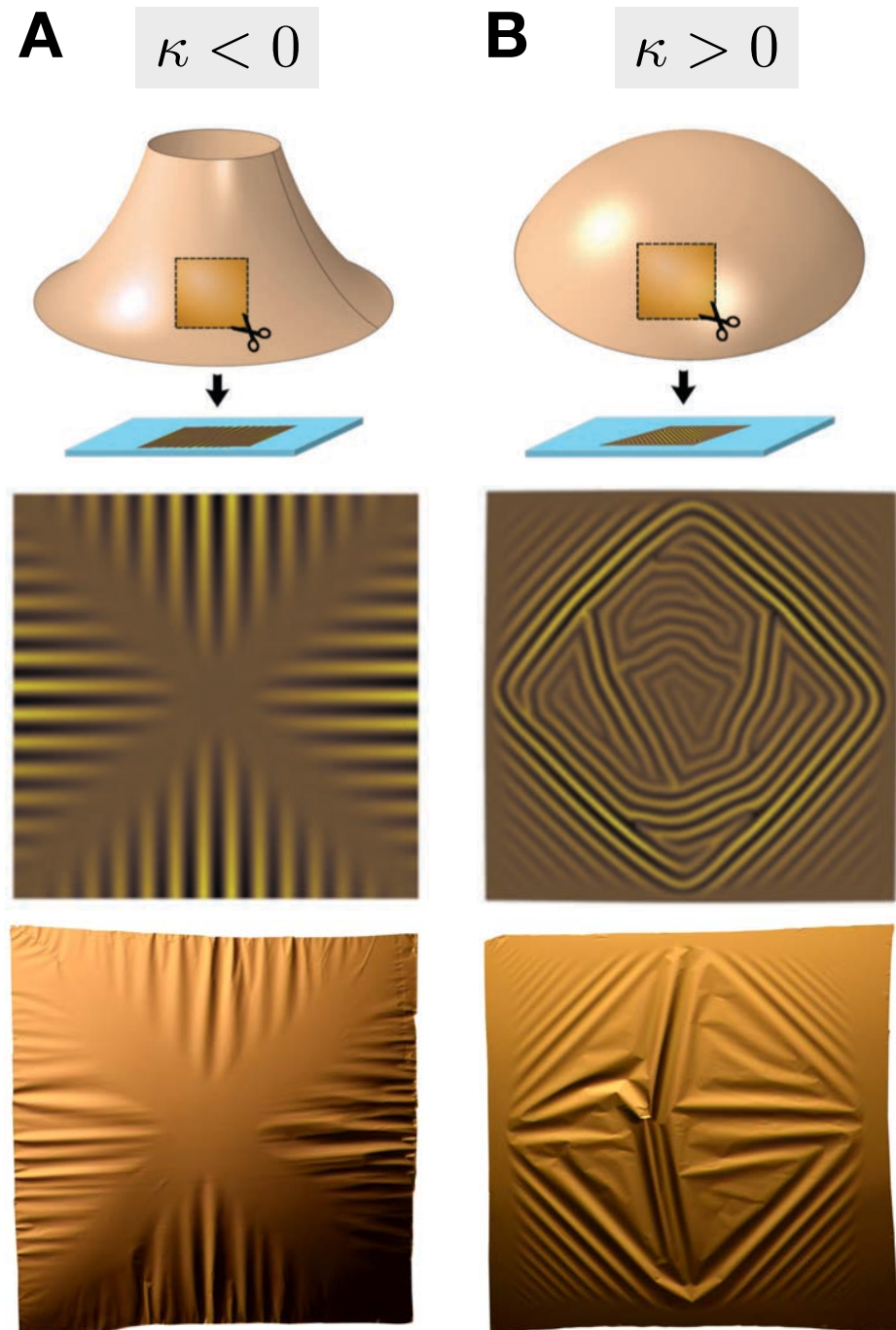


Fig 1. Thin floating shells, cut out of a sphere or a saddle

Wrinkle patterns depend on initial shape

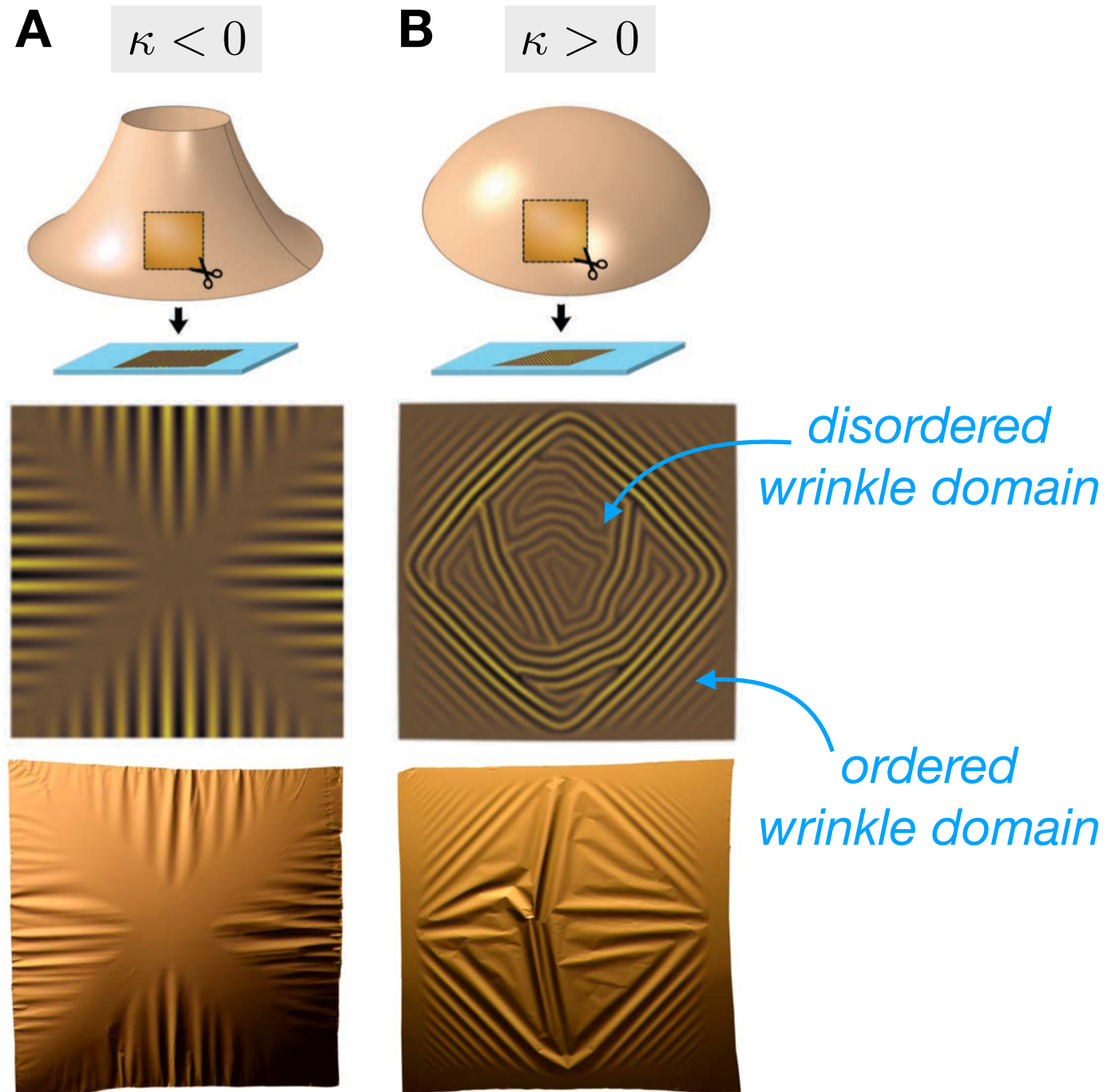
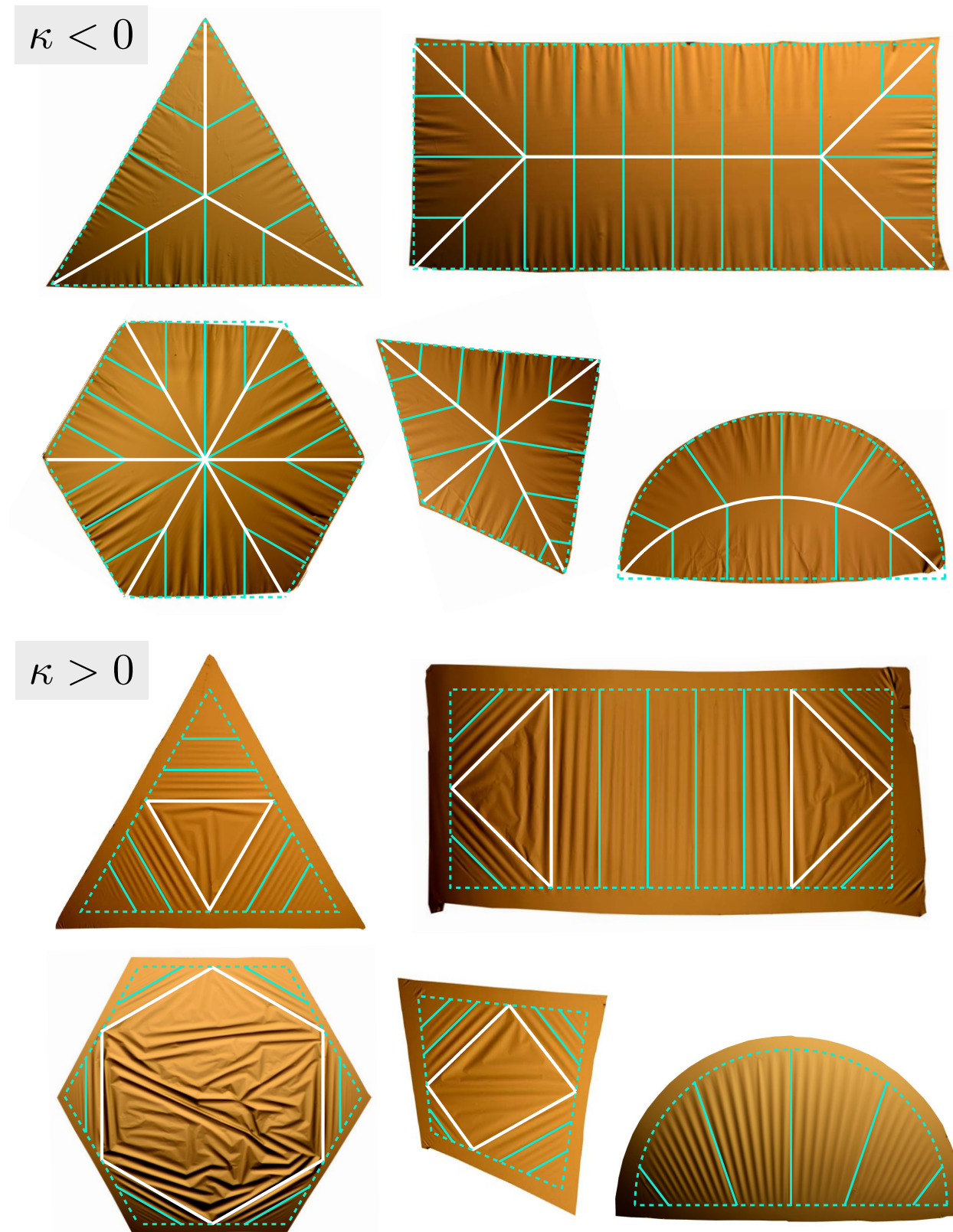
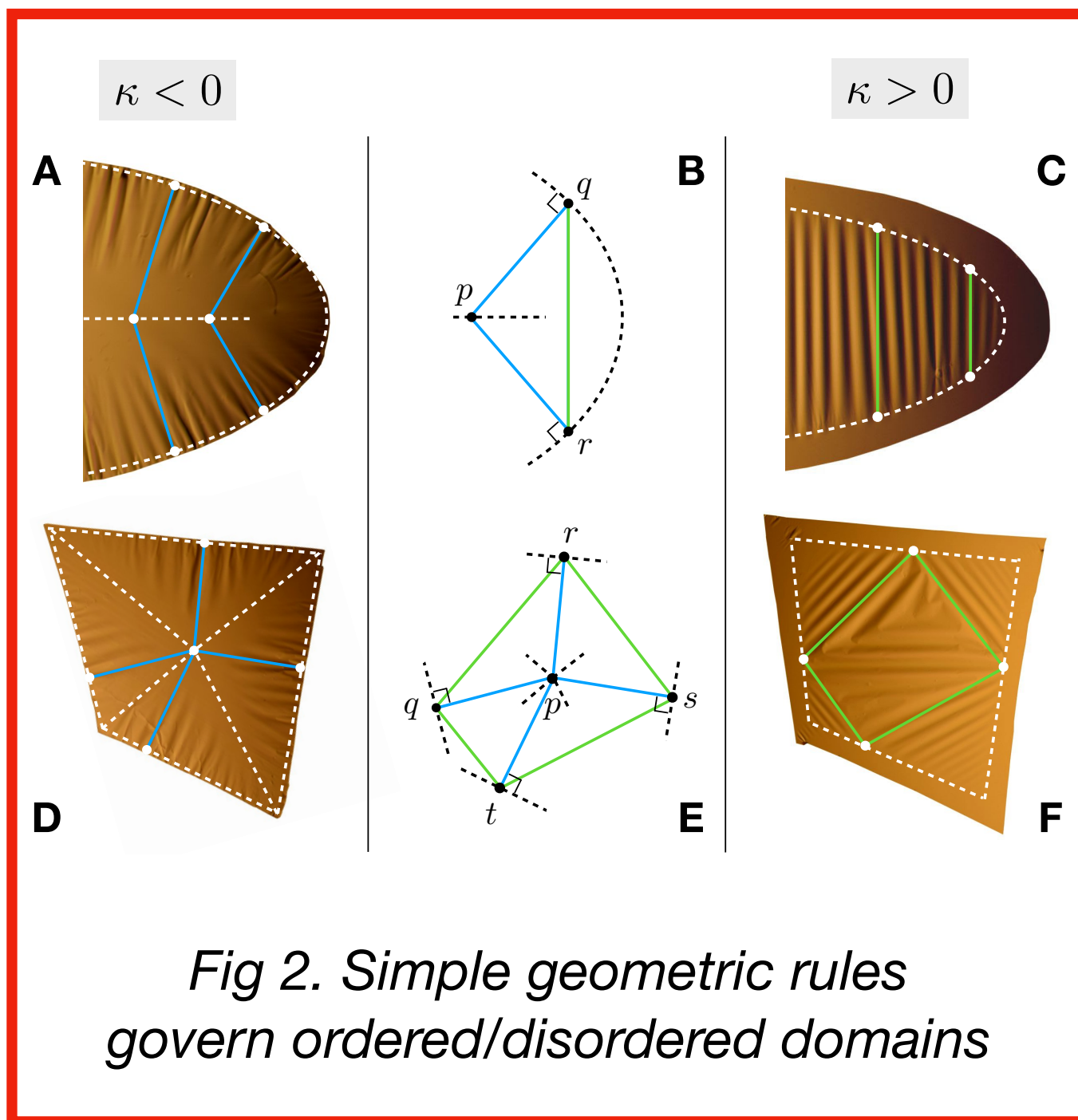


Fig 1. Thin floating shells, cut out of a sphere or a saddle

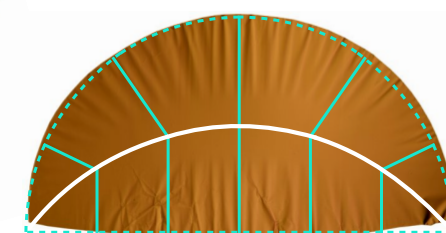
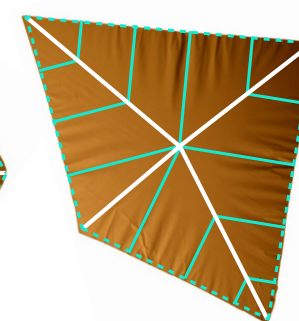
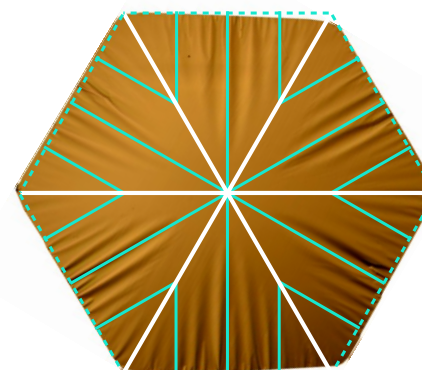
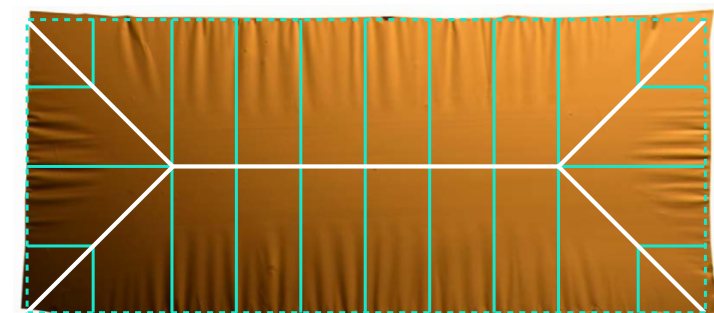
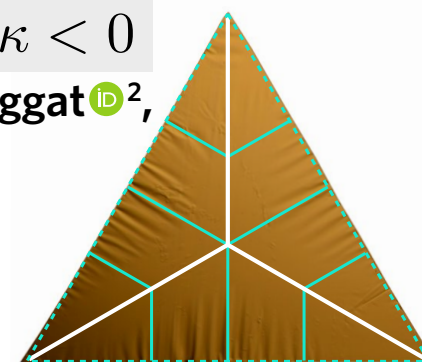


Exact solutions for the wrinkle patterns of confined elastic shells

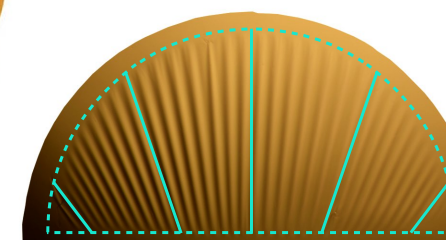
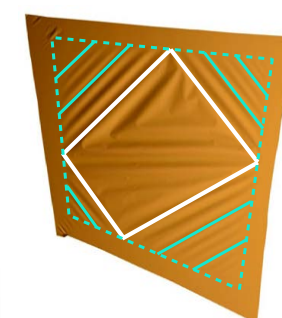
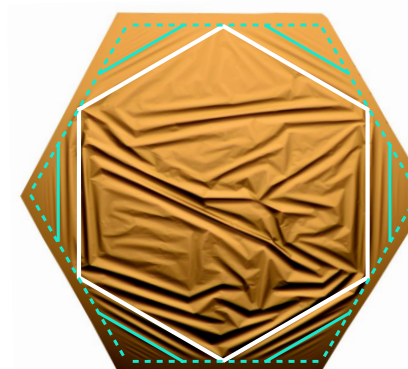
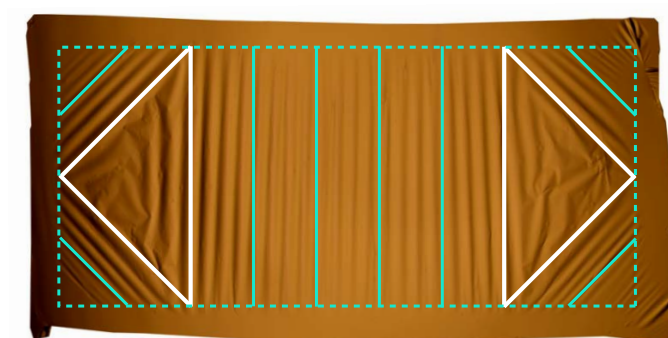
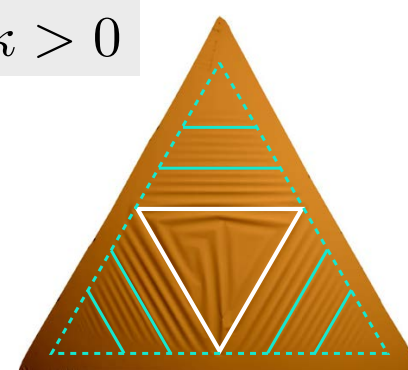
Ian Tobasco¹, Yousra Timounay^{2,3}, Desislava Todorova⁴, Graham C. Leggat², Joseph D. Paulsen^{2,3} and Eleni Katifori⁴



$\kappa < 0$



$\kappa > 0$



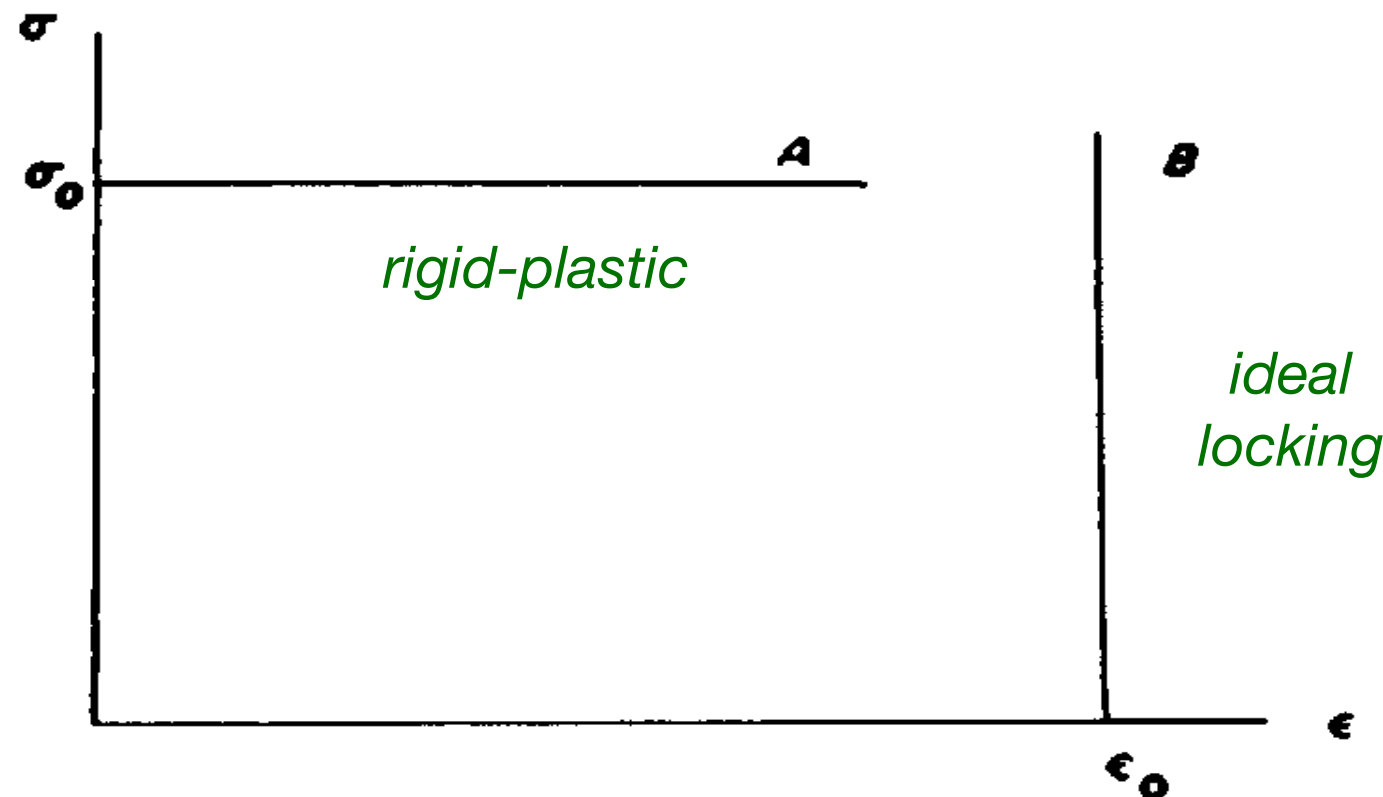
On Ideal Locking Materials

WILLIAM PRAGER, *Brown University, Providence, R. I.*

I. INTRODUCTION

Stress analyses in engineering are most frequently based on the linear theory of elasticity, though deviations from the linear stress-strain law are observed for practically all engineering materials.

Depending on the character of these deviations, materials may be classified as *soft* or *hard*. For a soft material, the stress increment required to produce a specified strain increment diminishes with progressing deformation; for a hard material, on the other hand, this stress increment grows as the deformation proceeds.



B

$\kappa > 0$

Locking stress Lagrange multiplier, convex Airy potential



$$\begin{cases} \varepsilon_{\text{eff}} = e(u_{\text{eff}}) - m \\ \nabla \cdot \sigma_L = 0 \\ \sigma_L : \varepsilon_{\text{eff}} = 0 \\ \varepsilon_{\text{eff}} \leq 0, \quad \sigma_L \geq 0 \end{cases} \quad \begin{array}{l} + \text{"b.c.s"} \\ \sigma_L = I \\ \text{outside shell} \end{array}$$

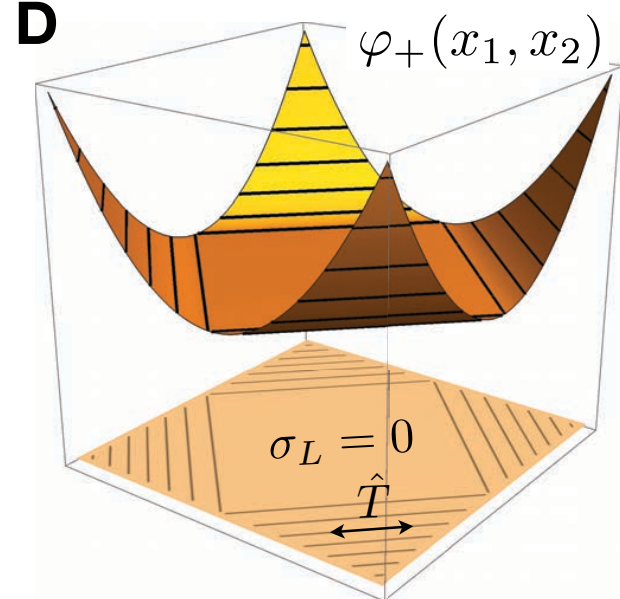
*equilibrium of the
infinitesimally wrinkled shell*

A

$\kappa < 0$



D

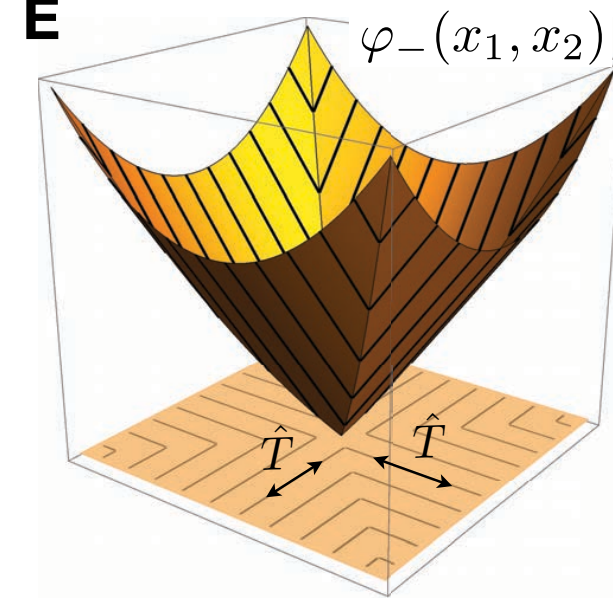


$$\sigma_L = \begin{pmatrix} \partial_{22}\varphi & -\partial_{12}\varphi \\ -\partial_{12}\varphi & \partial_{11}\varphi \end{pmatrix} \quad \text{locking stress}$$

$$\max_{\varphi(x)} \int \varphi \kappa$$

$$\varphi(x) \text{ is convex, and} \\ = \frac{1}{2}|x|^2 \text{ outside shell}$$

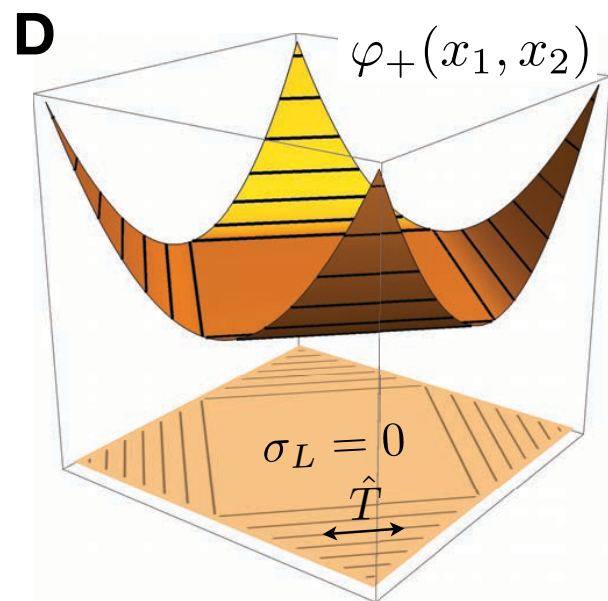
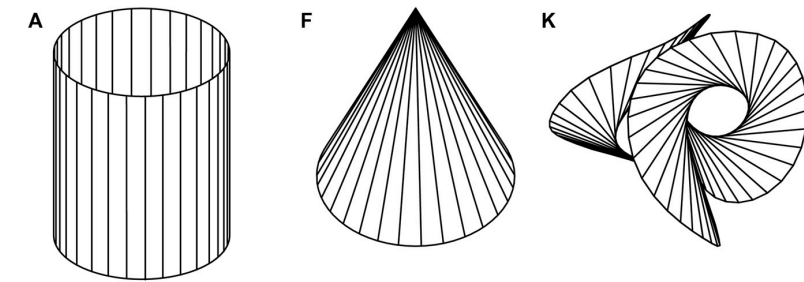
E



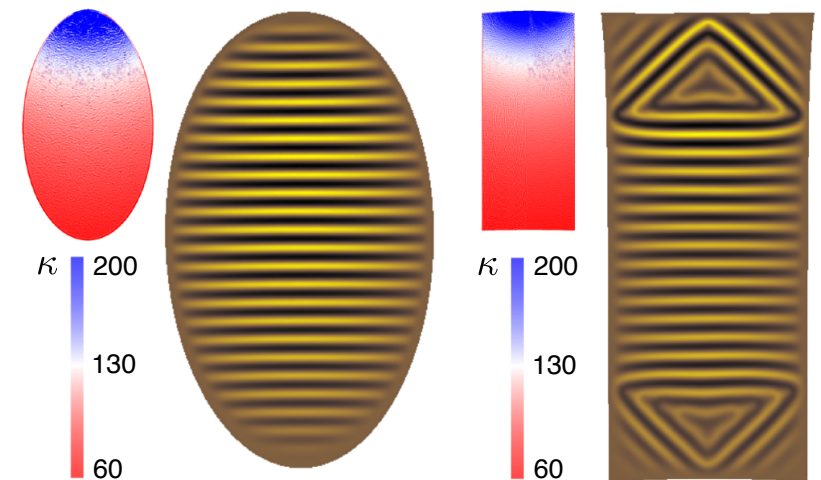
Consequences of the locking stress theory

$$\max_{\varphi(x)} \int \varphi \kappa$$

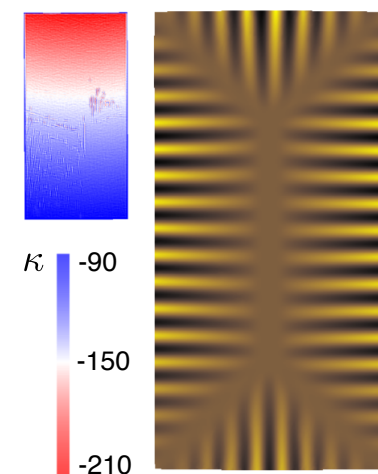
Wrinkles = projected ruling lines of the convex Airy potential



A $\kappa > 0$

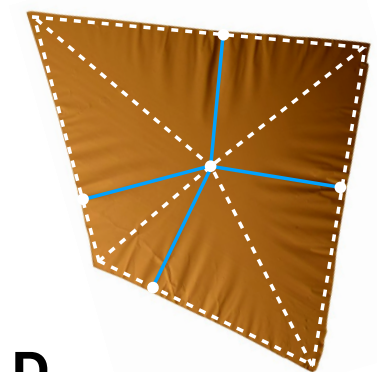
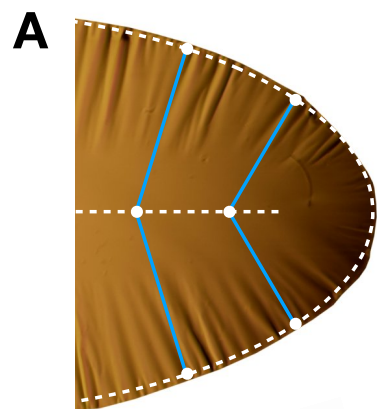


B $\kappa < 0$

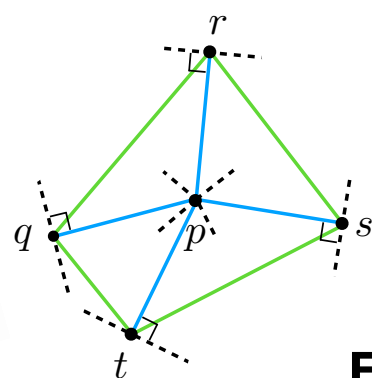
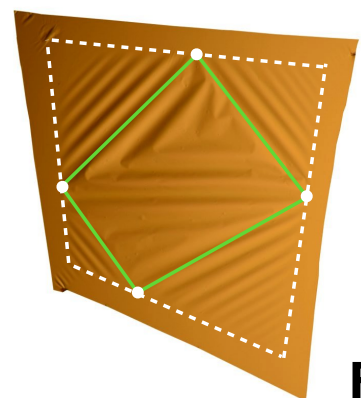
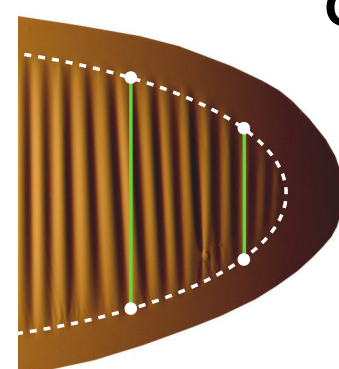
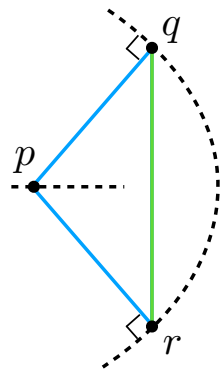


Their layout depends only on the sign of the Gaussian curvature

$\kappa < 0$



$\kappa > 0$



Wrinkle domains of oppositely curved shells come in reciprocal pairs