Elastic and "Plastic" Depinning in Driven Systems.

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Work done in collaboration with: Alan Middleton, Syracuse
Thomas Prellberg,Clausthal.

Phys. Rev. Lett. **85**, 1104 (2000).

Outline

- **Motivation**
  - Good understanding of the depinning transition of **elastic** media as a dynamical critical phenomenon (see talks by D.S. Fisher).
  - Many experimental systems are plagued with history dependence of the response and exhibit a crossover from elastic to "plastic" depinning as the parameters are varied (see talks by S. Bhattacharya and E. Andrei).
  - Need for a simple theoretical model to describe such systems.
- **The Model**
  - First ingredient: the generic coarse-grained model for an elastic medium driven over quenched disorder described by Daniel Fisher.
  - Second ingredient: the phenomenology of viscoelasticity described in one of Dave Weitz's lectures$\rightarrow$ frequency-dependent shear modulus and shear viscosity.
  - A model for a driven **viscoelastic** medium where elastic couplings to displacement are replaced by nonlocal (in time) couplings to local velocities.
- **Mean-Field Solution**
  - Transition from continuous to first order hysteretic depinning as a MF critical point.
  - Stick-slip-type instabilities for pinning potentials with cusps.
- **Relation to other systems and models**
  - The Random Field Ising Model of hystresis in magnets discussed by Karin Dahmen.
  - Models of crack propagation in hetrogeneous solids.
- **Work in progress**
  - Study of the model in finite dimensions by RG methods and numerical simulations.

Rain down dirty windshields

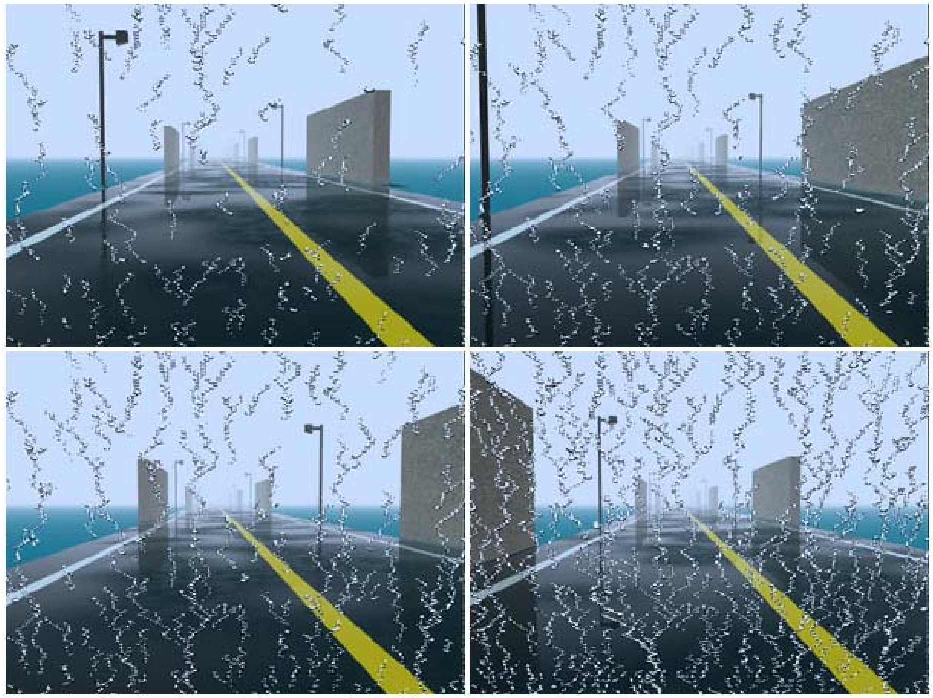
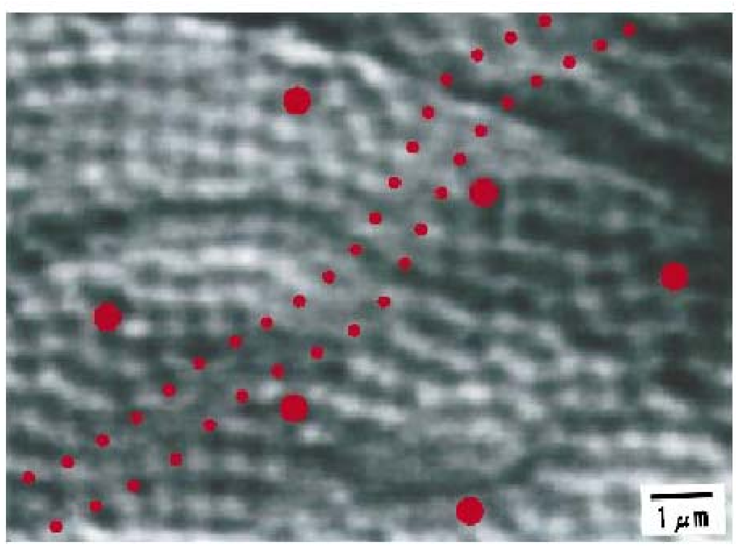


Figure 7: Animation of water droplets on a windshield.

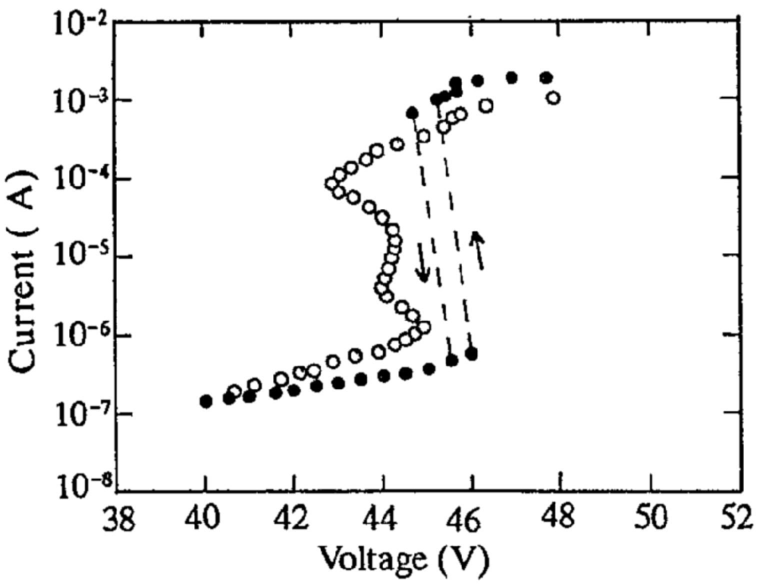
K. Kaneda, et al., JVCA **10** 15 (1999)

Vortices in superconductors



A. Tonomura, Micron **30**, 479 (1999) (*Nb* films)

From the lectures of S. Bhattacharya we learned that, while theoretical models of the depinning of driven elastic media yield a critical scaling of the velocity $v \sim (F - F_c)^\beta$, with $\beta < 1$ in both 2d and 3d, experiments in CDWs sometime show critical scaling of the IV characteristic with either $\beta < 1$ or $\beta > 1$, sometime show history dependendence and hysteresis with "switching" of the type shown below:



A. Maeda et al., 1990 ($K_{0.3}MoO_3$)

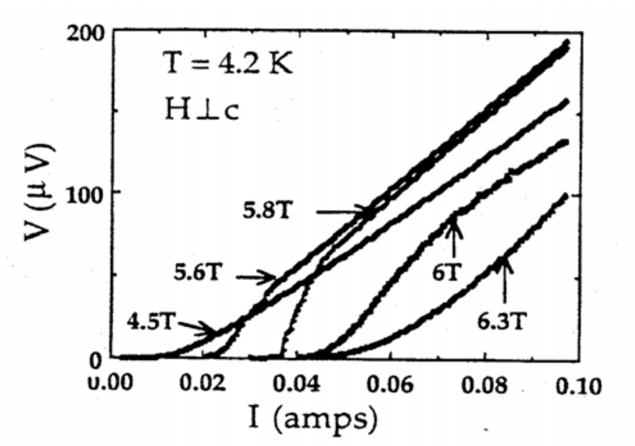
- **• constant Voltage (∼ driving force)**
- **◦ constant CDW current (∼ CDW velocity)**

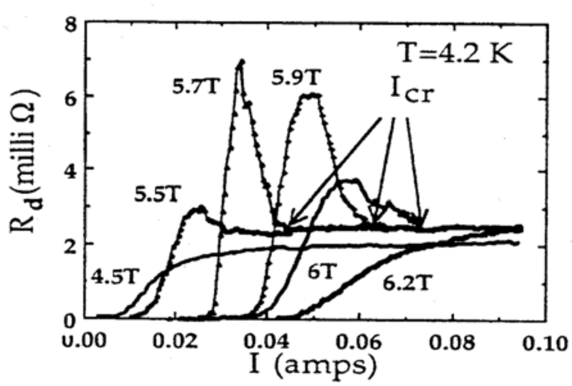
History dependence of response generally associated with tearing of the CDW due to phase slips

⇒ breakdown of the elastic model

From the lectures of Bhattacharya and Andrei: history dependence and plasticity in the response of current-driven vortex lattices in type-II superconductors.

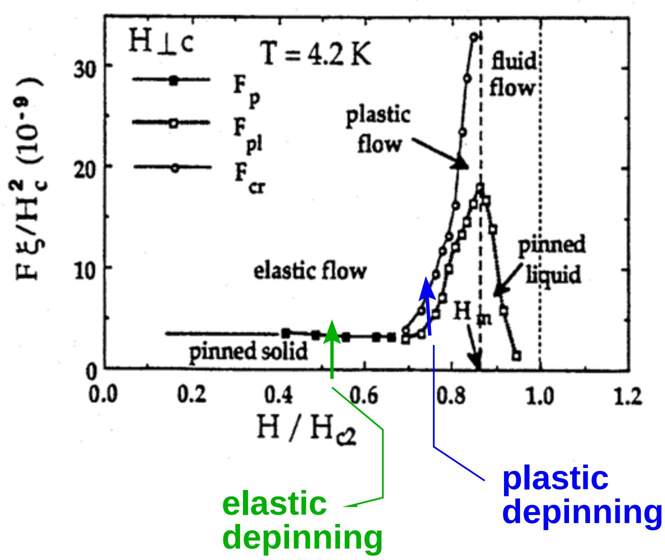
Evolution of shape of IV with applied field H:





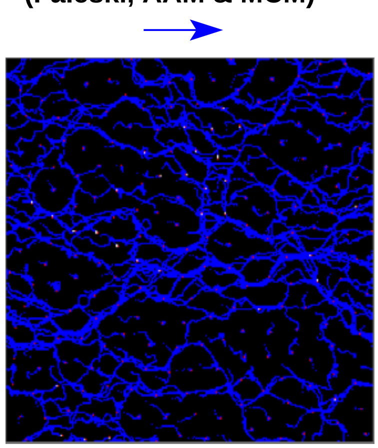
crossover from elastic to plastic depinning with decreasing interaction strength:

Bhattacharya & Higgins, 1993 ($NbSE_2$)



2d vortex simulations:

(Faleski, AAM & MCM)



Just above F_c vortex motion is confined to a network of channels.

General Continuum Model

of driven system, as introduced by Daniel Fisher

$$\gamma\partial u(\mathbf{r},t) = F_{\mathrm{ext}} + \int_{\mathbf{r}',t'} J(\mathbf{r}-\mathbf{r}',t-t') \qquad \left[u(\mathbf{r}',t') - u(\mathbf{r},t)\right]$$
$$+F_p(\mathbf{r},u(\mathbf{r},t)) \qquad (1)$$

Two classes:

- **Monotonic Models** $\Rightarrow J(\mathbf{r},t) \geq 0$
  "no passing rule" (AAM 1992, scalar field)

  - **uniqueness of sliding state**
  - **no hysteresis**

  $J(\mathbf{r},t) = \mu\delta(t)\nabla^2\delta(\mathbf{r}) \Rightarrow$ **conventional short-range elasticity.**

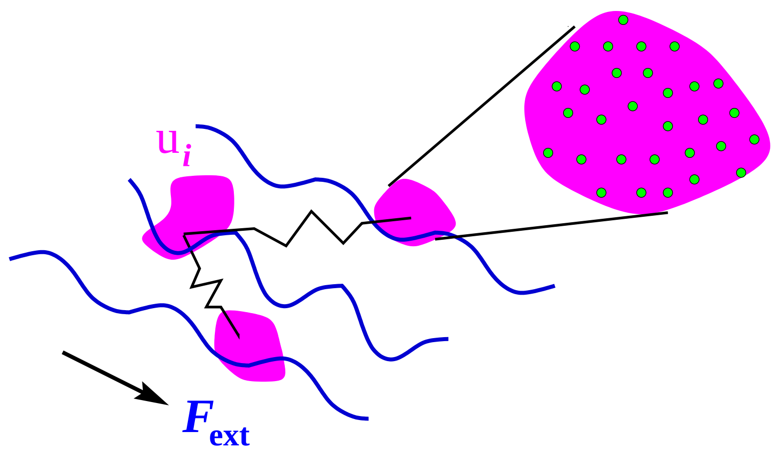
- **Nonmonotonic Models** $\Rightarrow J(\mathbf{r},t)$ can be < 0
  possibility of coexistence of pinned and sliding states and first order phase transitions.

Generic Model of Driven Elastic Medium

Generic coarse-grained model of an overdamped elastic medium driven through disorder:

local deformations (CDW phase) $\rightarrow \mathbf{u}(\vec{r}_i) = \mathbf{u_i}$

$\boxed{d = 1}$



$$\gamma_0 \dot{\mathbf{u}}_\mathbf{i} = \mu \sum_{\text{nbrs } \mathbf{j}} (\mathbf{u_j} - \mathbf{u_i}) + \mathbf{F}_{\text{ext}} - \frac{\partial \mathbf{V_i}(\mathbf{u_i})}{\partial \mathbf{u_i}}$$

elastic coupling	external drive	periodic pinning force

The pinning potential $V_i(\mathbf{u_i})$ at site i is periodic in $\mathbf{u_i}$, random in i, e.g.: $V_i(\mathbf{u_i}) = \mathbf{h_i} \cos(\mathbf{u_i} - \beta_\mathbf{i})$
with h_i and β_i random variables.

$\boxed{\textbf{Mean motion: } v(F) = \langle \dot{\mathbf{u}}_\mathbf{i} \rangle}$

Model for: CDWs, vortex lattice, driven interfaces, . . .

Properties of elastic depinning

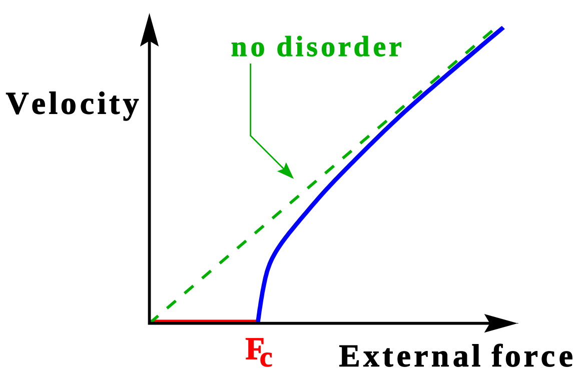
(see talks by D.S. Fisher)

- **Continuous transition** at a threshold F_C from **pinned** $(v = 0)$ to **sliding** $(v \neq 0)$ state.

  $\Rightarrow$**Universal critical behavior:**

  $$v \sim (F_{\rm ext} - F_c)^{\beta},$$
  $$\beta = 1 - (4-d)/6$$

  (Fisher (1985), Middleton, Narayan & Fisher, Natterman)



- **No hysteresis**: for a given $F_{\rm ext}(t)$, **unique** long-time solution (AAM, 1992).

- **Avalanches** when $F_{\rm ext}$ is slowly increased below F_c with power-law size distribution. Infinite avalanche at F_c. (related to Bak-Tang-Wiesenfeld sandpile models, self-organized criticality, Gutenberg-Richter distribution of earthquakes)

Phenomenology of Viscoelasticity

(see lectures by Dave Weitz)

When a frequency-dependent shear stress (of frequency ω) is applied to a viscous liquid, the liquid will respond elastically if $\omega >> 1/\tau$ and will flow if $\omega << 1/\tau$, with τ the characteristic structural relaxation time discussed by Jorge Kurchan.

- **Elastic response:**
  elastic stress= $\sigma_{xy}^{\mathrm{el}} = \mu\gamma$ $\qquad$ γ = strain, μ = shear modulus $\qquad$ $\gamma = \dfrac{\partial u_x}{\partial y}$
- **Viscous Flow:**
  viscous stress= $\sigma_{xy}^{\mathrm{vis}} = \eta\dot{\gamma}$ $\qquad$ $\dot{\gamma}$ = strain rate, η = shear viscosity $\qquad$ $\dot{\gamma} = \dfrac{\partial v_x}{\partial y}$

Here u_x denotes the elastic displacement field and v_x the flow velocity.

To describe viscoelasticity we introduce frequency-dependent moduli and viscosities (which are indeed probed in experiments)':

$$\sigma_{xy}^{vis-el}(\omega) = \tilde{\mu}(\omega)\gamma(\omega) = \tilde{\eta}(\omega)\dot{\gamma}(\omega)$$

or in the time domain:

$$\sigma_{xy}^{vis-el}(t) = \int_0^t dt' C(t-t') \frac{\partial v_x(t')}{\partial y}$$

It can be shown by a microscopic derivation of the above equation that $C(t)$ is the stress tensor autocorrelation function that describes the relaxation of local stresses in the system. It is related to the shear viscosity via a Green-Kubo formula (see, e.g., Hansen & McDonald, *Theory of Simple Liquids*, Academic Press; or Boon & Yip, *Molecular Hydrodynamics*, McGraw-Hill, 1980), which is this case simply reads

$$\tilde{\eta}(\omega) = \int_0^\infty dt e^{i\omega t} C(t)$$

Note that more in general the stress tensor autocorrelation function introduces both temporal and spatial nonlocaties in the viscous response. It has the general properties:

$$\int_0^\infty dt C(t) = \eta$$

$$C(t=0) = G_\infty$$

Finally the viscoelastic force is given by

$$F_x^{vis-el}(t) = \partial_y \sigma_{xy}^{vis-el} = \int_0^t dt' C(t-t') \partial_y^2 v_x(t').$$

Maxwell Model:

The Maxwell model assumes that $C(t)$ is simply an exponential, i.e., $C(t) = G_\infty e^{-t/\tau}$, corresponding to:

$$G(\omega) = G_\infty \frac{i\omega\tau}{1+i\omega\tau}$$

$$\tilde{\eta}(\omega) = \frac{\eta}{1+i\omega\tau} \qquad\qquad G_\infty \tau = \eta_0$$

We will use this form below. This approximation simplifies things considerably, but is not essential. The viscous force has the following limiting forms:

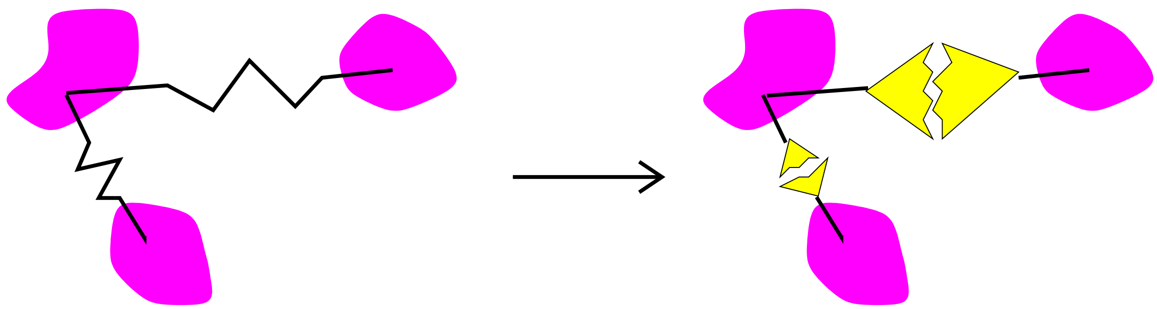
$$F_x^{vis-el}(t) \approx G_\infty \partial_y^2 u_x(t) \qquad \text{for} \qquad \begin{array}{l} \eta_0 \to \infty \\ \tau \to \infty \\ G_\infty = \eta_0/\tau = \text{constant} \end{array}$$

$$F_x^{vis-el}(t) \approx \eta_0 \partial_y^2 v_x(t) \qquad \text{for} \qquad \begin{array}{ll} \eta_0 & \text{finite} \\ \tau = 0 & \end{array}$$

A Model for Plastic Depinning

MCM, A.A. Middleton & Thomas Prellberg, PRL85, 1104 (2000)

Replace elastic couplings by viscoelastic couplings:



$$\mu(\mathbf{u_j} - \mathbf{u_i}) \to \int_{-\infty}^{\mathbf{t}} \mathbf{C}(\mathbf{t}-\mathbf{s})\,[\dot{\mathbf{u}}_\mathbf{j}(\mathbf{s}) - \dot{\mathbf{u}}_\mathbf{i}(\mathbf{s})]\,\mathbf{ds}$$

with $C(t)$ time-correlation function of local stress, e.g.

$$\boxed{C(t) = \mu e^{-t/\tau}}$$

$C(t=0) = \mu$ **(elastic modulus)**

$\int_0^\infty C(t)dt = \eta$ **(viscosity)**

$$\boxed{\eta = \mu\tau}$$

- **Nonlocal couplings to velocity generated generically upon coarse-graining (cf. derivation of hydrodynamics.)**
- **The model exhibits both elastic restoring forces and viscous forces on different length and time scales.**

Equations of motion:

$$\dot{\mathbf{u}}_\mathbf{i} = \sum_{\text{nbrs }\mathbf{j}} \int_{-\infty}^{\mathbf{t}} \mathbf{C}(\mathbf{t}-\mathbf{s})\,[\dot{\mathbf{u}}_\mathbf{j}(\mathbf{s}) - \dot{\mathbf{u}}_\mathbf{i}(\mathbf{s})]\,\mathbf{ds} + \mathbf{F}_{\text{ext}} - \frac{\partial \mathbf{V_i}(\mathbf{u_i})}{\partial \mathbf{u_i}}$$

Viscoelastic Model

With the Maxwell form of $C(t) = \mu e^{-t/\tau}$, the integro-differential equation of motion can be transformed into a second order differential equation:

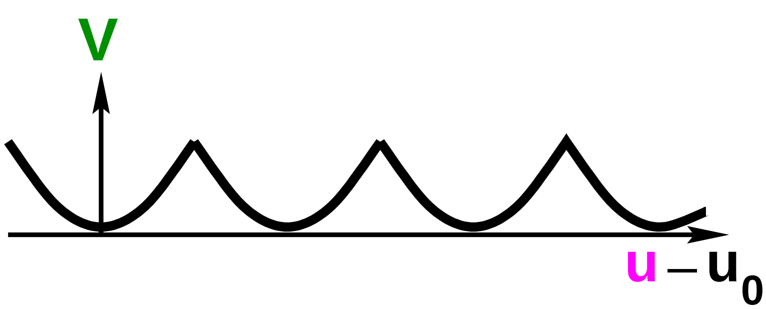
$$\boxed{\tau \ddot{\mathbf{u}}_{\mathbf{i}} + \left[\mathbf{1} + \eta + \tau \frac{\partial^2 \mathbf{V}_{\mathbf{i}}(\mathbf{u}_{\mathbf{i}})}{\partial {\mathbf{u}_{\mathbf{i}}}^{\mathbf{2}}}\right] \dot{\mathbf{u}}_{\mathbf{i}} = \eta \sum_{\text{nbrs } \mathbf{j}} (\dot{\mathbf{u}}_{\mathbf{j}} - \dot{\mathbf{u}}_{\mathbf{i}}) + \mathbf{F}_{\text{ext}} - \frac{\partial \mathbf{V}_{\mathbf{i}}(\mathbf{u}_{\mathbf{i}})}{\partial \mathbf{u}_{\mathbf{i}}}}$$

Pinning potential:

periodic, piecewise quadratic (scallops):

$V(\phi) = \frac{h}{2}(\phi^2 - \phi + \frac{1}{4})$ **distribution** $\rho(h)$ **of** h

$\rightarrow$ **linear pinning force**



Parameters:

- η **interaction strength**
- τ **duration of velocity coupling**
- $\rho(h)$ **disorder distribution of width** h_0

Below $\rho(h) = \frac{1}{h_0} e^{-h/h_0}$

Mean-Field Solution

MFT $(d = \infty)$**:** $\quad \frac{1}{N}\sum_{\text{nbrs } j} C(t)\dot{\mathbf{u}}_{\mathbf{j}} \to \mathbf{C}(\mathbf{t})\overline{\mathbf{v}}$

$$\overline{v} = \frac{1}{N}\sum_i \dot{\mathbf{u}}_{\mathbf{i}} = \langle \dot{\mathbf{u}}_{\mathbf{i}} \rangle$$

Mean Field Equation:

$$\tau \ddot{\mathbf{u}}_{\mathbf{i}} + \gamma(\eta, \tau, \mathbf{h}; \mathbf{u}_{\mathbf{i}})\dot{\mathbf{u}}_{\mathbf{i}} = \mathbf{F} + \eta\overline{\mathbf{v}} - \frac{\partial \mathbf{V}_{\mathbf{i}}(\mathbf{u}_{\mathbf{i}})}{\partial \mathbf{u}_{\mathbf{i}}}$$

with

$$\gamma(\eta, \tau, h; \mathbf{u}_{\mathbf{i}}) = \mathbf{1} + \eta + \tau \frac{\partial^{\mathbf{2}} \mathbf{V}_{\mathbf{i}}(\mathbf{u}_{\mathbf{i}})}{\partial \mathbf{u}_{\mathbf{i}}{}^{\mathbf{2}}}$$

For scalloped potential MFT can be solved exactly *under the assumption of steady* $\overline{v}$**.**

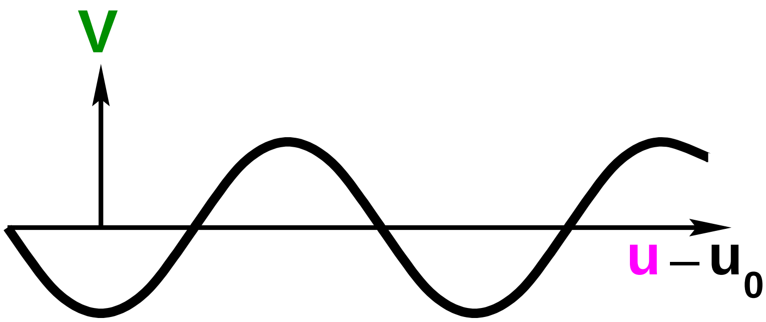
- **Solve** 2^{nd} **order linear DE.**
- **Find** $T_i(\eta\overline{v} + F_{\text{ext}}, h_i, \eta, \tau)$**.**
- **Impose self-consistency** $\left\langle T_i^{-1} \right\rangle = \overline{v}$**.**

Which Pinning Potential?

Mean-field Theory: Solved for

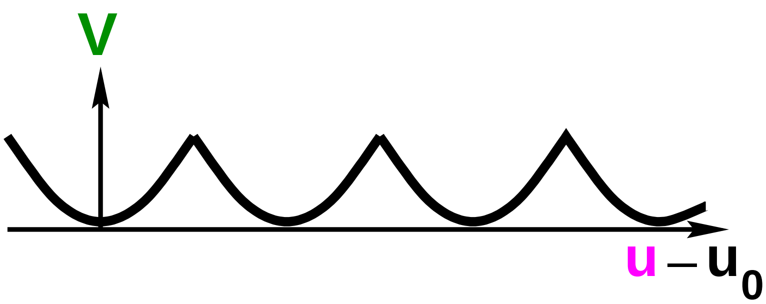
$$V_i = h_i \sin[2\pi(\mathbf{u_i} - \mathbf{u_{i,0}})]$$

(Fisher, 1985)



$\beta = 3/2$

For piecewise quadratic potential (scallops):



$\beta = 1$

Which is more generic?

$d \neq \infty$**:**

The effective potential for $\mathbf{u_i}$ changes rapidly when a neighbor "hops".

$\Rightarrow$ proper MFT given by scalloped potential.

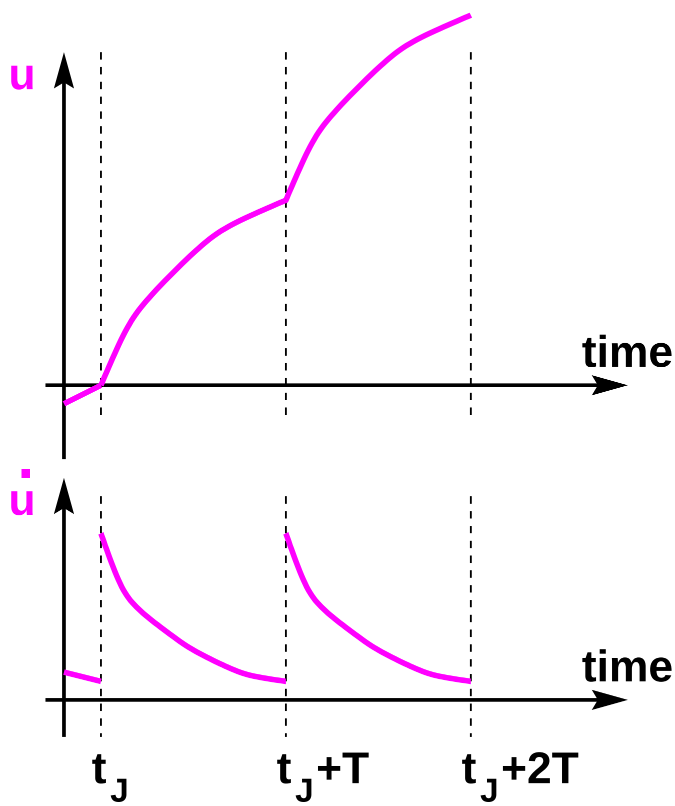
(supported by functional renormalization arguments)

$$\boxed{\tau = 0}$$

$$(1+\eta)\dot{\mathbf{u}}_{\mathbf{i}} = \mathbf{F}_{\text{ext}} + \eta\overline{\mathbf{v}} + \frac{\mathbf{h}_{\mathbf{i}}}{\mathbf{2}} - \mathbf{h}_{\mathbf{i}}\mathbf{u}_{\mathbf{i}}$$

cf. equation for an overdamped single particle:

- **effective friction:** $1+\eta$
- **effective driving force:** $F_{\text{ext}} + \eta\overline{v}$ – **depends on particle dynamics.**



Viscoelastic particle:

$$\overline{v} = \int_0^\infty dh \frac{h}{1+\eta} \frac{\rho(h)}{\ln\left(\frac{F_{\text{ext}}+\eta\overline{v}+h/2}{F_{\text{ext}}+\eta\overline{v}-h/2}\right)}$$

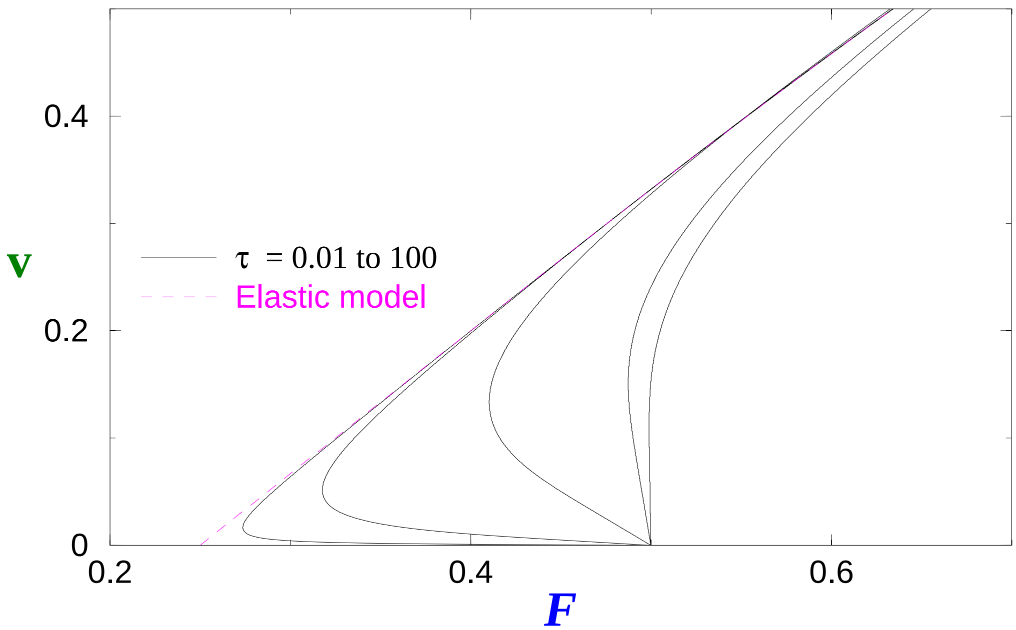
Single particle:

$$\overline{v} = \int_0^\infty dh \frac{h\rho(h)}{\ln\left(\frac{F_{\text{ext}}+h/2}{F_{\text{ext}}-h/2}\right)}$$

Solution for fixed h

$$\rho(h) = \delta(h-1)$$

$\mu = 1, \qquad \tau = 0.01 \to 100$



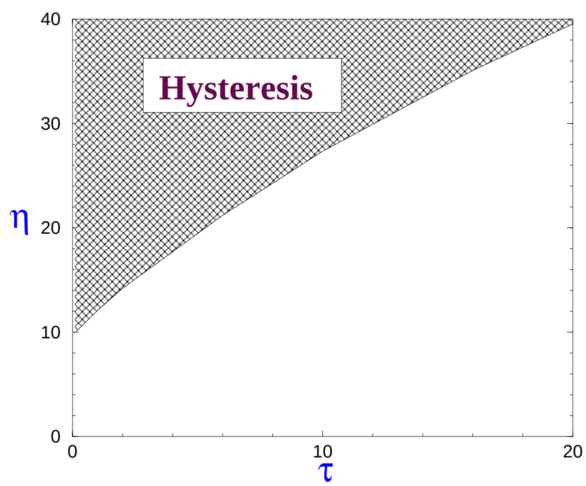
For fixed h, nonuniform initial conditions must be chosen to reflect the disorder.

- **Elastic model:** $\overline{v} = F_{\text{ext}} - F_c, \quad F_c = 0.25$
- **Single particle:** $\overline{v} = \left[\ln\left(\frac{F_{\text{ext}}+1/2}{F_{\text{ext}}-1/2} \right) \right]^{-1}$

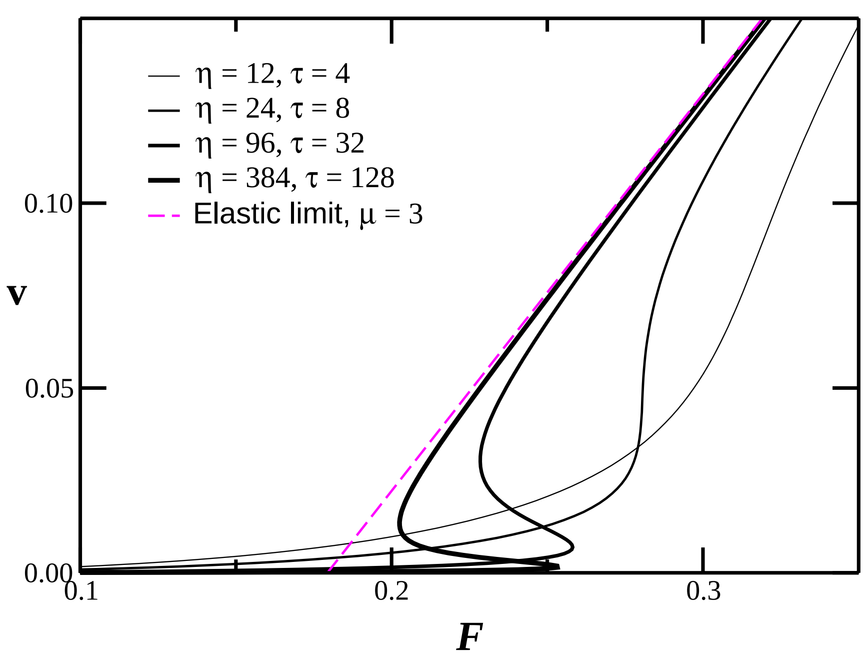
Analytical MF Solution

There is a critical line $\eta_c(\tau)$ separating single-valued solution (continuous depinning) from multi-valued solutions (hysteretic depinning).

$$\rho(h) = \frac{1}{h_0}\exp(-h/h_0)$$



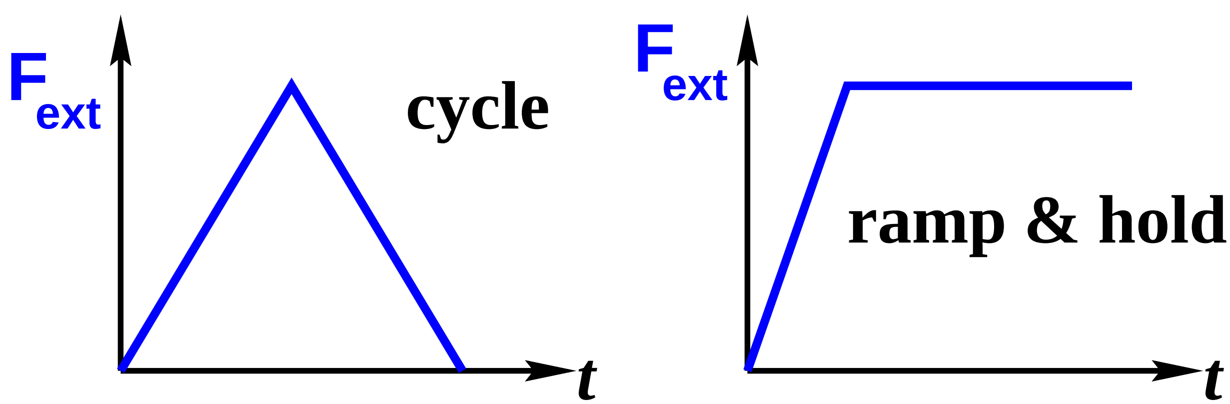
$\mu = 3.0$**:**



Simulation of MF Model

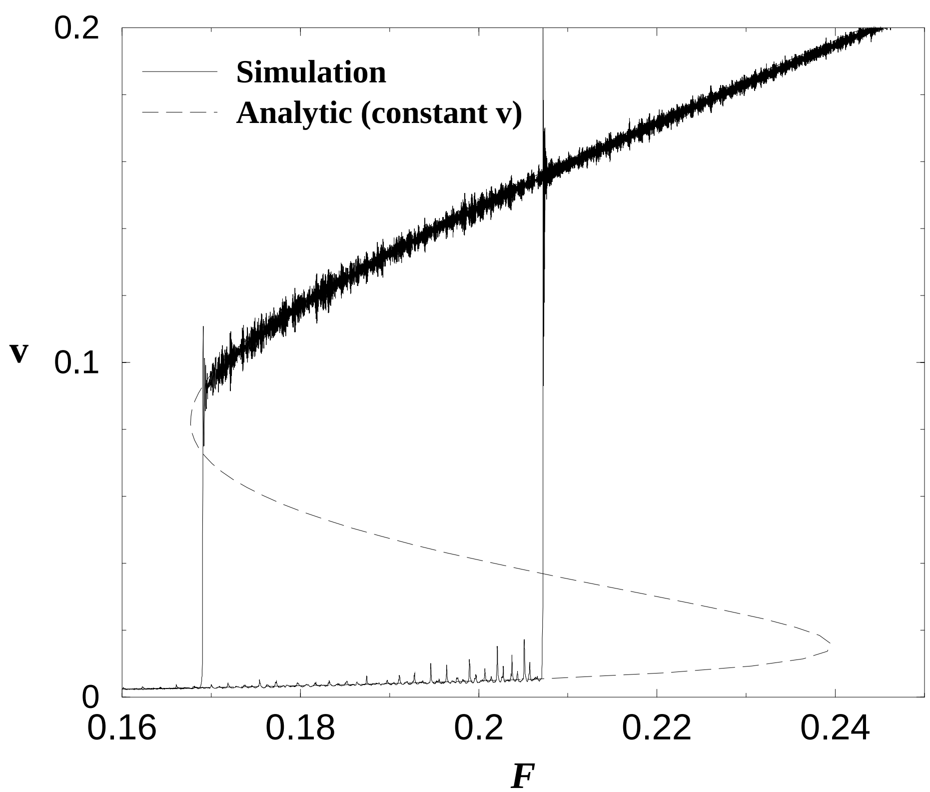
- Check stability of constant $\overline{v}$ solution.
- Constant $F_{\rm ext}$ or constant $\overline{v}$ drive.
- RK integration
- "Events" are "crossings" where $\mathbf{u_i}$ crosses from one scallop to another.

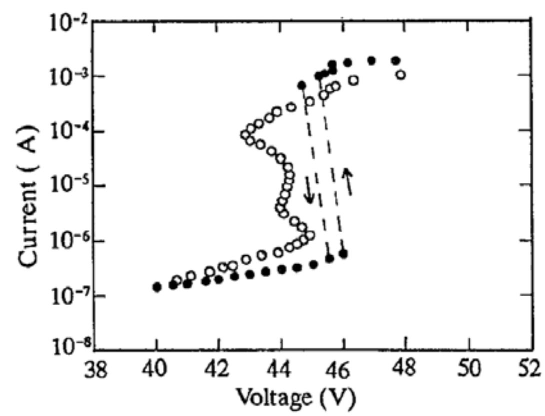
Record $\overline{v}(t)$, event times for $F(t)$ histories.



Numerical Simulation of MF Model

$\eta = 32.0$, $\tau = 0.8$, $N = 16384$:



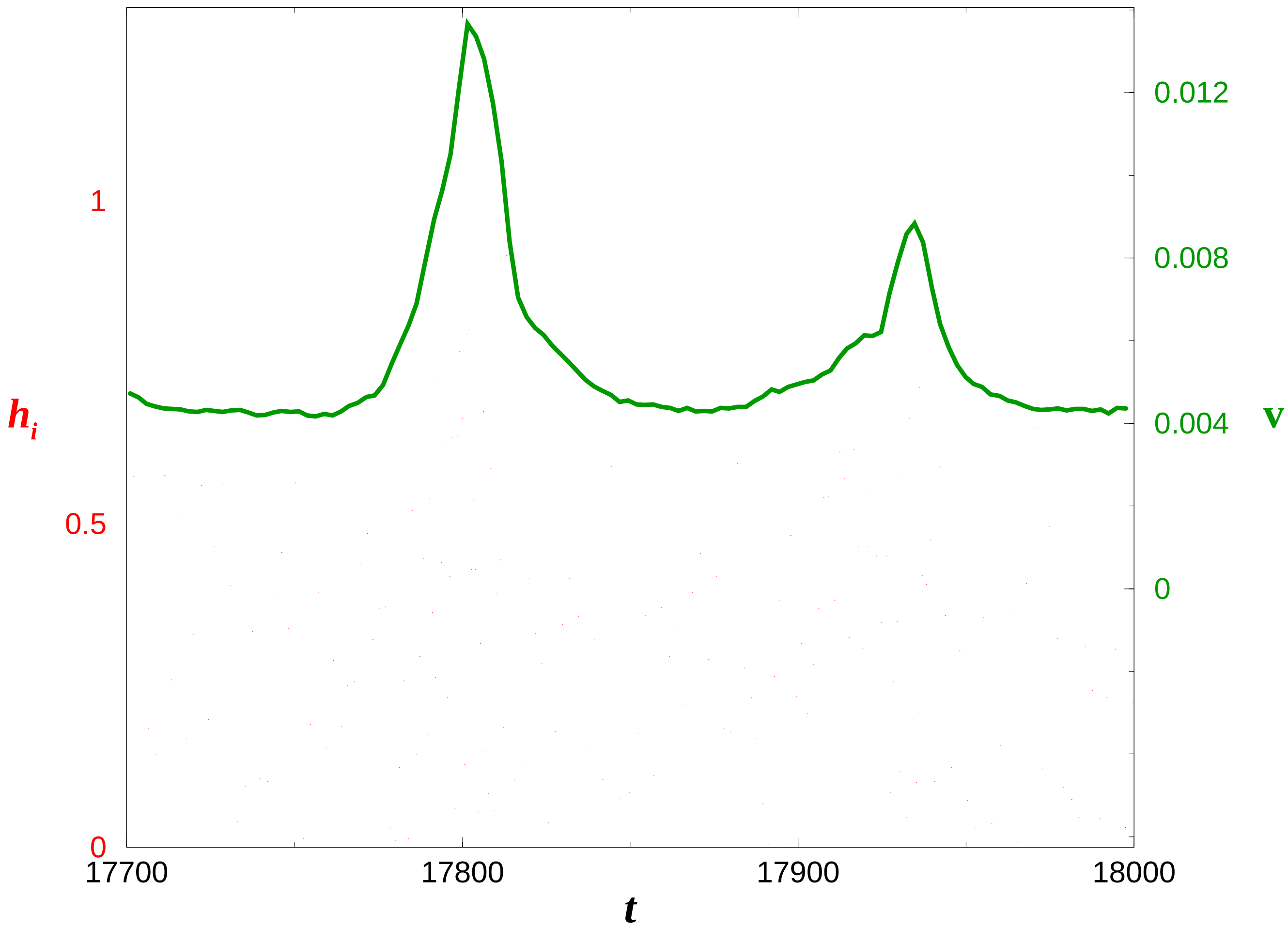


Comparison of MFT simulation with analytic results:

- **Noise in upper branch due to finite system size (decreases as $N^{-1/2}$).**
- **In the lower branch of the hysteresis cycle the mean field velocity occasionally spikes due to macroscopic events. These fluctuations saturate at large N and are due to stick-slip type instability.**
- **Premature switching from lower to upper branch.**

The Stick-Slip Instability

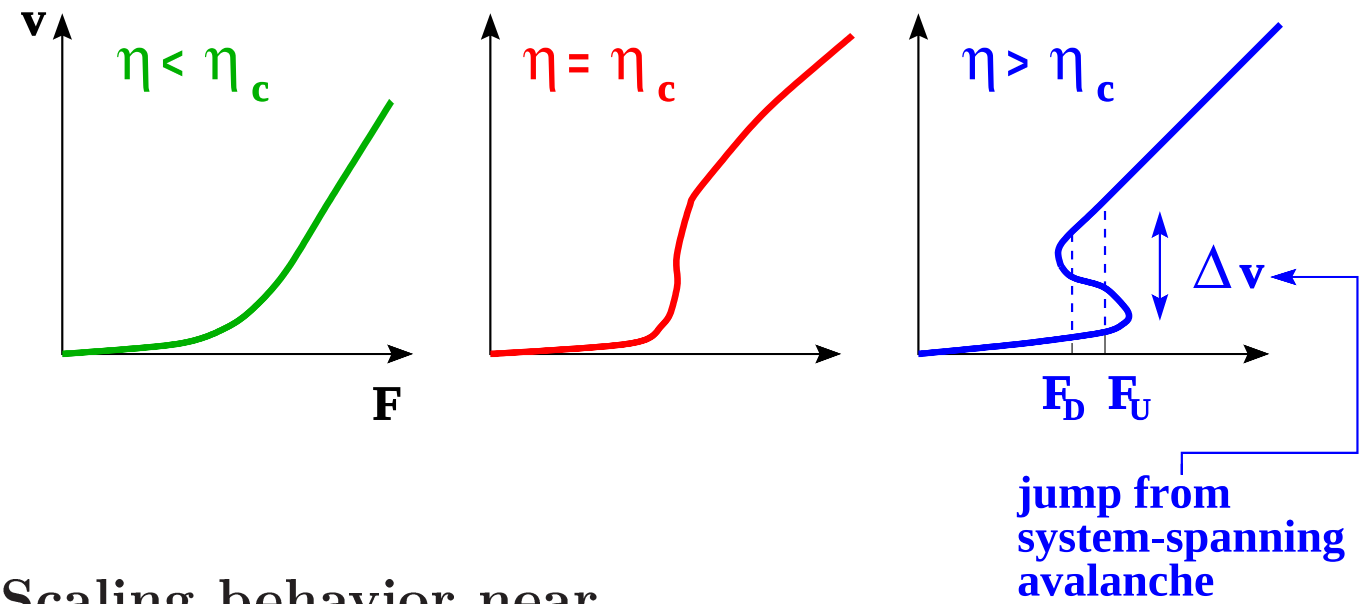
$\eta = 32.0,\ \tau = 0.0,\ N = 16384$:



Dots are time of crossing events where a $\mathbf{u_i}$ with pinning h_i crosses from one scallop to another.

Maximum pinning force is h; maximum drive from external and other u_i's is $\eta\overline{v} + F_{\rm ext}$.

(This instability is absent for a smooth pinning potential.)

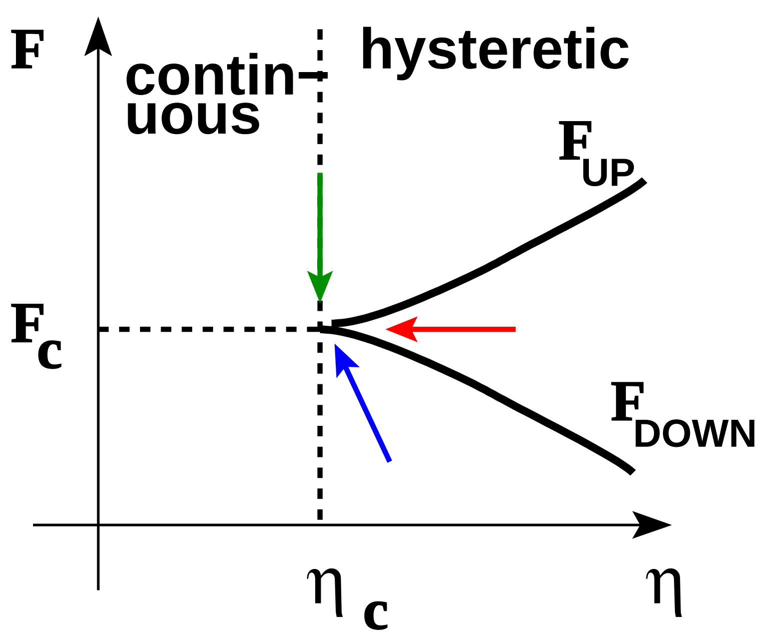


Scaling behavior near critical point: $(\tau = 0)$ η_c, F_c

$\Delta v \sim (F - F_c)^{1/\delta}$ $\delta_{MF} = 3$

$\Delta v \sim (\eta - \eta_c)^{\delta}$ $\beta_{MF} = 1/2$

$\Delta v \sim (\eta - \eta_c)^{\beta}$ cf. RFIM (Dahmen, Sethna, ...)



RFIM of Hysteresis in Magnets

Dahmen & Sethna, 1996

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} S_i S_j - \sum_i \big(H + f_i) S_i$$

random fields f_i with distribution $\rho(f)$.

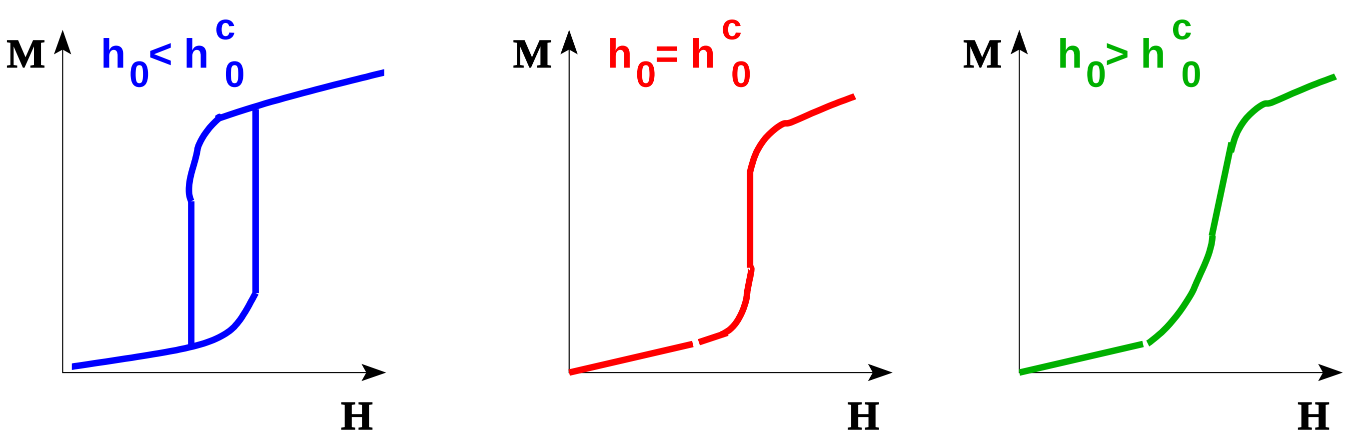
Dynamics:

$$\frac{1}{\Gamma_0} \partial_t S_i = \sum_{\langle ij \rangle} J_{ij} S_j - H - f_i$$

driving force $F \leftrightarrow$ applied field H

mean velocity $v \leftrightarrow$ Magnetization M

local velocities $v_i \leftrightarrow$ spins S_i

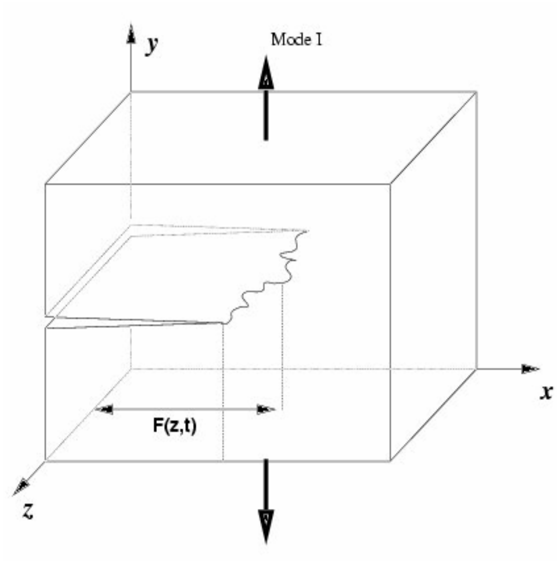


Work in progress:

RG calculation to derive a $6 - \epsilon$ expansion of the critical exponents about their MF values.

Expect same exponents as RFIM.

Crack Propagation in Brittle Materials



load ~ driving force

From:
S. Ramanathan & D.S. Fisher
(1997)

- **Elastic (long-range) coupling between points on the crack front.**
- **Stress overshoot: inertial stress transfer in which the motion of one segment creates a transient stress (*additional* to the elastic stress) on other segments.**

J. Schwarz & D.S. Fisher (cond-mat/0012246):

$$\sigma_i(t) = \sum_{\langle ij \rangle} \left[h_j(t) - h_i(t)\right] + M \sum_j \left[h_j(t) - h_j(t-1)\right]$$

From continuous ($M < M_c$)to discontinuous hysteretic depinning ($M > M_c$).

Summary and Outlook

- **Model of a driven disordered system that exhibits a transition from continuous to hysteretic depinning with disorder and driving force as tuning parameters.**

- **In MF the transition is a critical point in the universality class of the noneq. RFIM.**

- **Work in progress: preliminary RG calculations indicate that the critical point survives in finite dimensions.**

- **Connection with model of crack propagation and earthquakes.**

  (J. Schwarz & D.S. Fisher, cond-mat/0012246).