Lecture Plan

1) what is a mechanical metamatil ?

planar kirigami

mechanisms

homogenized desription ⇝ PDEs

auxeticity & PDE type

mechanism gradient theory

kagome lattice

mechanisms (there are many)

homogenized bulk energy &

cell problem

bounds on effective energy

2) parallelogram origami

skipped for time
see ref. online

mechanisms

homogenized description via

Cartan's method of moving frames

mechanism gradient theory

3) wrinkled shells as mech. metamatils

shallow shell theory

effective description of wrinkles

locking stress & convex Airy potentials

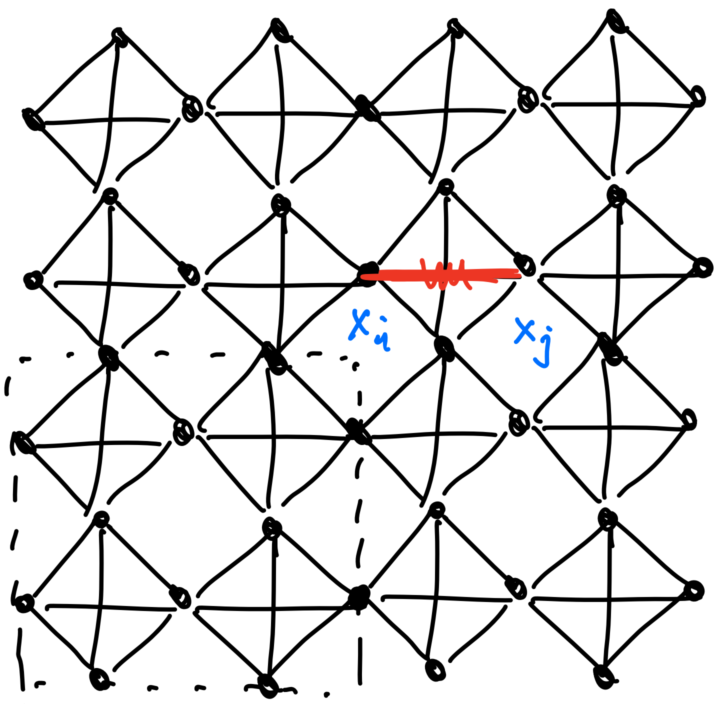
sol'n formulas for wrinkle patterns

Part 1. Homogenization of Kirigami

What is a mechanical metamaterial?

2 viewpoints:
- underconstrained spring lattices (i)
- collections of elastic bodies with elastic interactions (ii)

In (i), we have a finite portion of a periodic graph $G = (V, E)$ in $\mathbb{R}^d$ $(d = 2, 3)$



Σ unit cell

V = vertices = $\{x_i\}_{i=1}^N$

E = edges = $\{i \sim j\}$

= Hookean springs

deformation

$y : x_i \longmapsto y_i, \quad i = 1, \dots, N$

$$\text{Energy} = E(y) = \sum_{i \sim j} \tfrac{1}{2} k_{ij} \, \varepsilon_{ij}^2$$

$$\varepsilon_{ij} = \text{strain of } ij^{th} \text{ spring}$$

$$= \frac{1}{2} \frac{|y_i - y_j|^2 - |x_i - x_j|^2}{|x_i - x_j|^2}.$$

A <u>mechanism</u> y is a length preserving map $G \to \mathbb{R}^2$ that is <u>not</u> a rigid motion, i.e.,

$$\begin{cases} \varepsilon_{ij}(y) = 0 & \forall \ i \sim j \\ y(x) \neq Rx + c & \text{for } R \in SO(d) \\ & \quad c \in \mathbb{R}^d \end{cases}$$

Equivalently, y is a zero-energy map

$$E(y) = 0$$

and <u>not</u> a rigid body motion.

We are interested in mechanisms that cause "bulk shape change", meaning they are of the form

$$y(x) = F_{eff}\, x + \varphi(x)$$

where φ is periodic on the lattice.

For large lattices $y \approx F_{eff}\, x$. (zoom out)

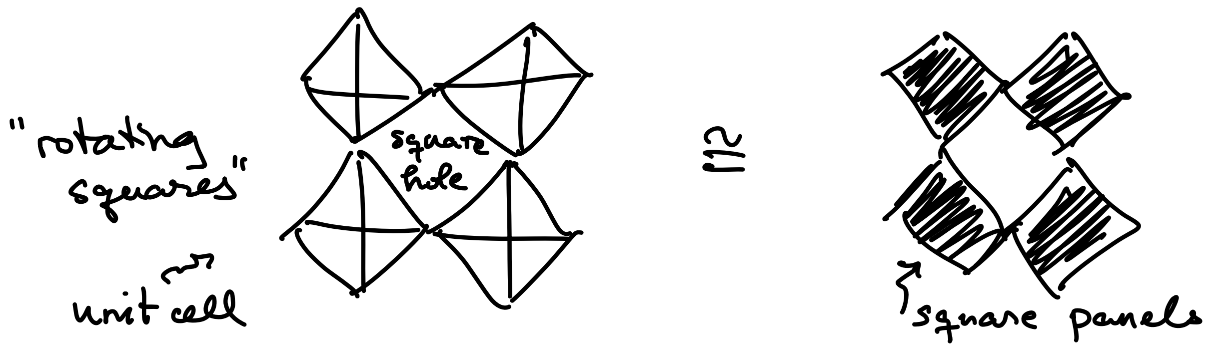
The shape change is associated with the <u>effective deformation</u>

$$y_{eff}(x) = F_{eff}\, x$$

which we think of as a deformation of $\mathbb{R}^d$.

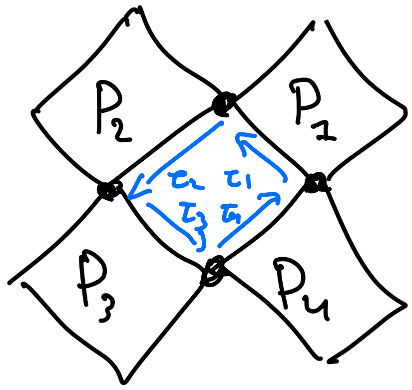
Main example for this lecture:

parallelogram hole kirigami in $\mathbb{R}^2$



General Def'n: 4 quad panels, 4 parallelogram holes in unit cell

It's useful to use viewpoint (ii) & make rigid collections of springs into rigid bodies, called "panels".

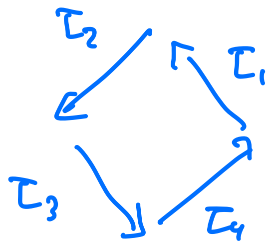


panels P_i, $i = 1, \dots, 4$ rigid

$$y_i(x) = R_i x + c_i, \quad i = 1, \dots, 4$$

$$R_i \in SO(2), \quad c_i \in \mathbb{R}^2$$

Is there a sol'n? Yes: focus on the hole



$$\sum_{i=1}^{4} \tau_i = 0$$

$$\hookrightarrow \sum_{i=1}^{4} R_i \tau_i = 0 \qquad \text{"loop condition"}$$

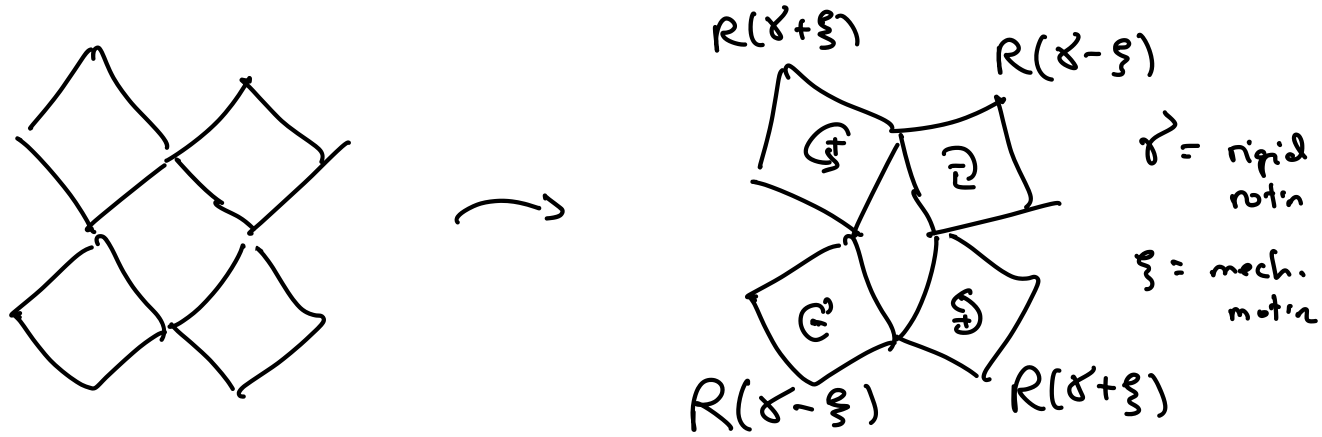
$\tau_3 = -\tau_1$ & $\tau_4 = -\tau_2$

$$\Rightarrow \quad (R_1 - R_3)\,\tau_1 + (R_2 - R_4)\,\tau_2 = 0$$

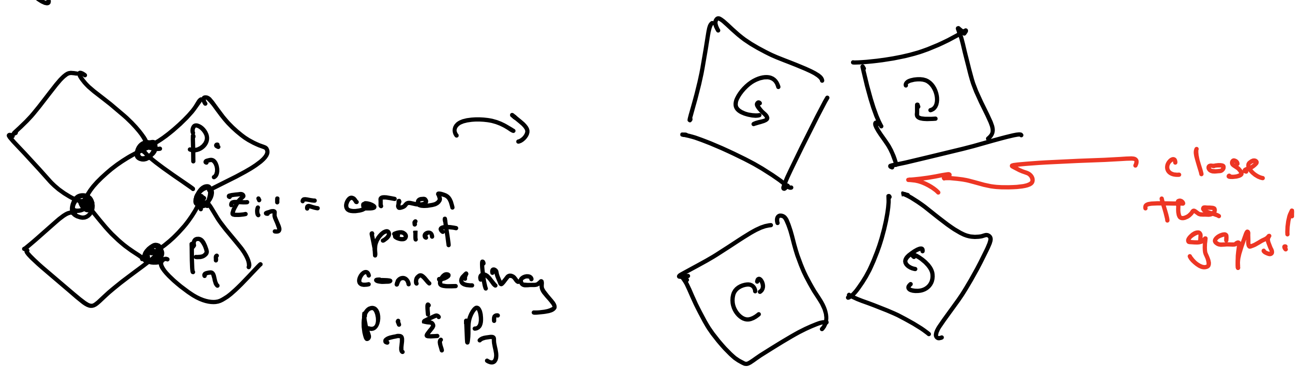
$$\Rightarrow \quad R_1 = R_3 \quad \& \quad R_2 = R_4$$

<u>Note</u>. this works if the holes are parallelograms, hence the name "parallelogram hole kirigami".

So the situation is this: $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



One can now find the translations c_i by enforcing a "no gap" condition:



$$g_{ij} = ij^{th} \text{ gap} = R_i z_{ij} + c_i - (R_j z_{ij} + c_j)$$

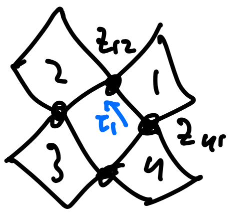
$$= (R_i - R_j) z_{ij} + c_i - c_j$$

$$g_{ij} = 0 \ \forall ij \iff \begin{cases} (R_1 - R_2) z_{12} + c_1 - c_2 = 0 \\ (R_2 - R_3) z_{23} + c_2 - c_3 = 0 \\ (R_3 - R_4) z_{34} + c_3 - c_4 = 0 \\ (R_4 - R_1) z_{41} + c_4 - c_1 = 0 \end{cases}$$

4 vectors $c_i \in \mathbb{R}^2$ unknown & 4 $\mathbb{R}^2$-val. constraints

but adding all eqns gives

$$\begin{aligned} 0 &= (R_1 - R_2) z_{12} + (R_2 - R_3) z_{23} \\ &\quad + (R_3 - R_4) z_{34} + (R_4 - R_1) z_{41} \\ &= R_1 (\overbrace{z_{12} - z_{41}}^{\tau_1}) + R_2 (\overbrace{z_{23} - z_{12}}^{\tau_2}) \\ &\quad + R_3 (\underbrace{z_{34} - z_{23}}_{\tau_3}) + R_4 (\underbrace{z_{41} - z_{34}}_{\tau}) \\ &= \sum_{i=1}^{4} R_i \tau_i \end{aligned}$$

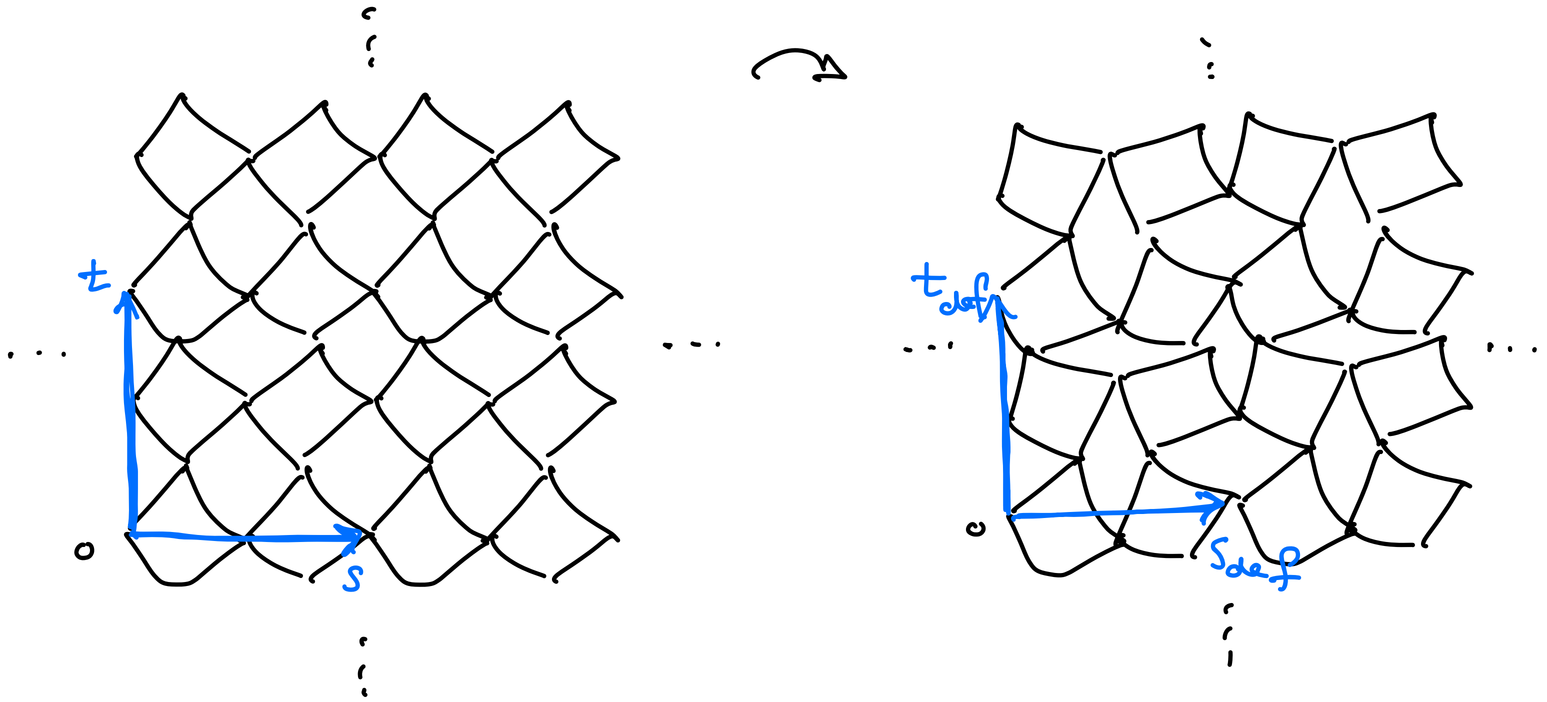


automatically zero by loop cond !

this is the only redundancy, e.g., we can choose $c_1 = 0$ & get

$$\begin{cases} c_2 = (R_1 - R_2) z_{12} \\ c_3 = (R_1 - R_2) z_{12} + (R_2 - R_3) z_{23} \\ c_4 = (R_1 - R_4) z_{41} \end{cases}$$

What is F_{eff} ?



$\{s, t\}$ lattice vectors of ref. config. $\longrightarrow$ $\{s_{def}, t_{def}\}$ lattice vectors of def. config by a mech.

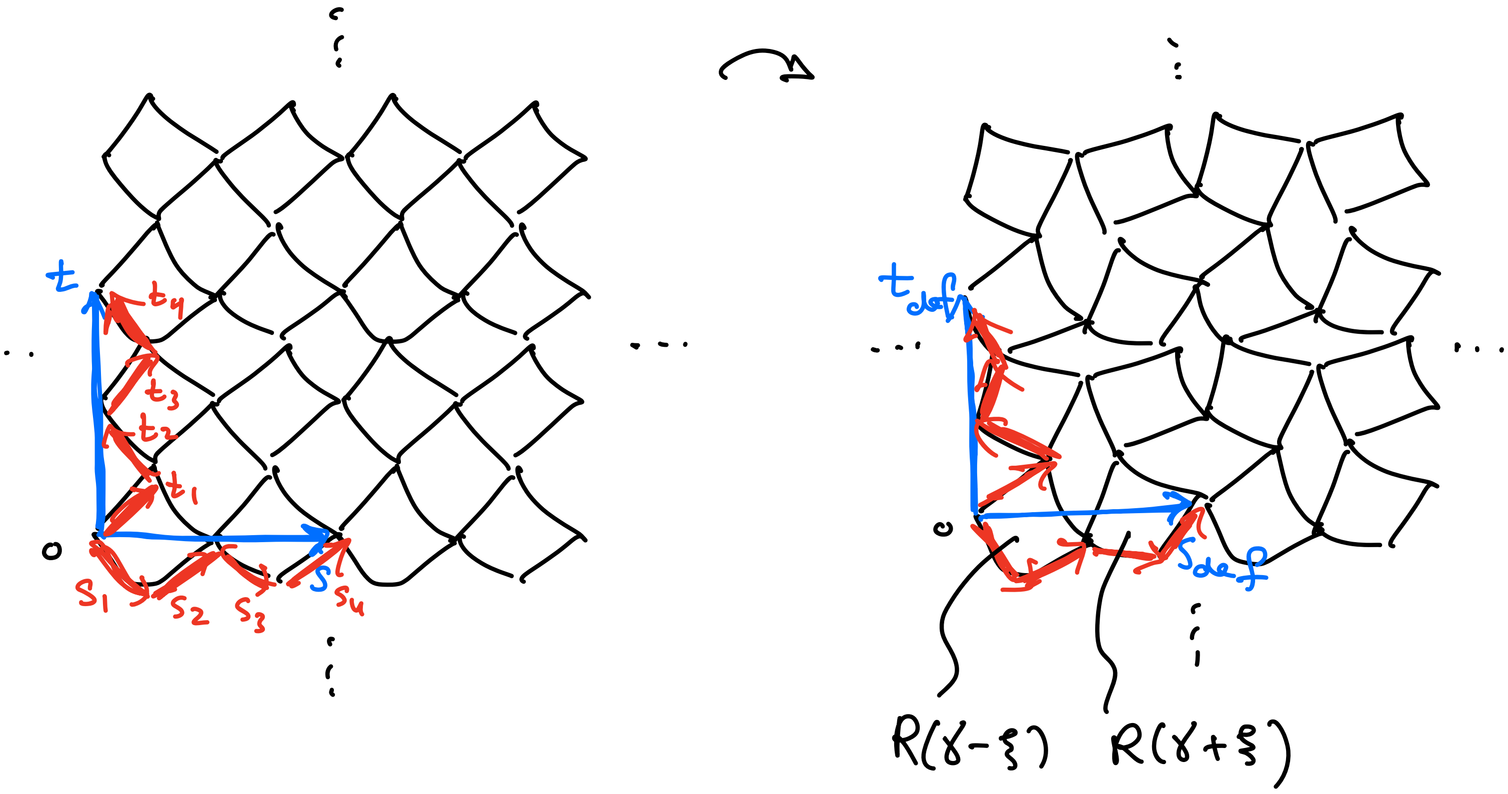
$$y(x) = F_{eff}\, x + \underbrace{\varphi(x)}_{\text{per.}}$$

$$\begin{cases} y(s) - y(0) = F_{eff}\, s + \underbrace{\cancel{\varphi(s) - \varphi(0)}}_{0} \\ y(t) - y(0) = F_{eff}\, t + \underbrace{\cancel{\varphi(t) - \varphi(0)}}_{0} \end{cases}$$

by periodicity

$$\Rightarrow \begin{cases} y(s) - y(0) = F_{eff}\, s \\ y(t) - y(0) = F_{eff}\, t \end{cases}$$

We have formulas for γ. Another way:



$$\begin{cases} s_{def} = R(\gamma-\xi)(s_1+s_2) + R(\gamma+\xi)(s_3+s_4) \\ t_{def} = R(\gamma-\xi)(t_1+t_4) + R(\gamma+\xi)(t_2+t_3) \end{cases}$$

2D rotations commute, $R(\gamma \pm \xi) = R(\gamma)R(\xi) = R(\xi)R(\gamma)$

$$\Rightarrow \begin{cases} s_{def} = R(\gamma)\Big[\underbrace{R(-\xi)(s_1+s_2) + R(\xi)(s_3+s_4)}_{:= A(\xi)s}\Big] \\ t_{def} = R(\gamma)\Big[\underbrace{R(-\xi)(t_1+t_4) + R(\xi)(t_2+t_3)}_{:= A(\xi)t}\Big] \end{cases}$$

this defines

$A(\xi)$ = "shape change tensor"

for parallelogram hole

kirigami

Conclusion: for parallelogram hole kirigami,

$$\boxed{\begin{aligned} F_{\text{eff}} &= R(\gamma)\, A(\xi) \\ \gamma &= \text{rigid rot'n} \\ \xi &= \text{mech. mot'n} \end{aligned}}$$

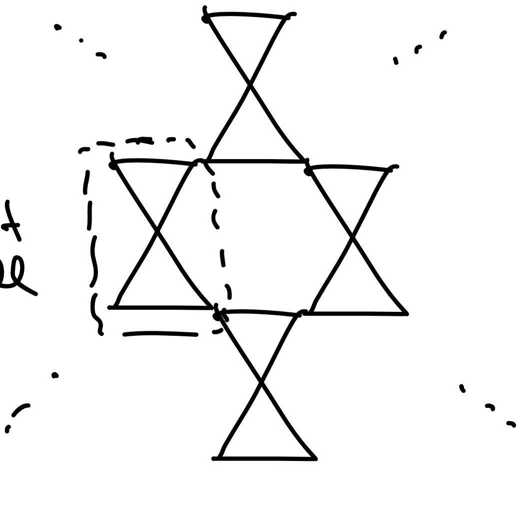
one can solve for γ, which is like finding c_i (exercise)

Ex. rotating squares has shape change tensor $A(\xi) = \lambda(\xi)\, I$ conformal

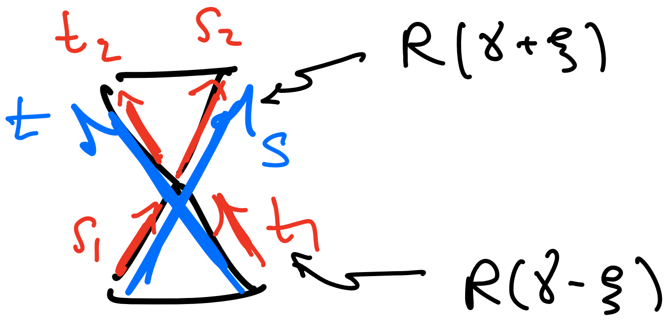
almost clear from the diagrams ... easy to check (exercise) $\lrcorner$

Ex. kagome lattice

unit cell



equilateral triangles



$$s = s_1 + s_2 \qquad R\left(\tfrac{\pi}{3}\right) s = t$$
$$t = t_1 + t_2$$

$$\begin{cases} F_{eff}\, s = R(\gamma-\xi)\, s_1 + R(\gamma+\xi)\, s_2 \\ F_{eff}\, t = R(\gamma-\xi)\, t_1 + R(\gamma+\xi)\, t_2 \end{cases}$$

$$F_{eff}\, s = R(\gamma)\left[R(-\xi)\,\frac{s}{2} + R(\xi)\,\frac{s}{2} \right]$$

$$= R(\gamma) \begin{pmatrix} \cos(-\xi)+\cos\xi & -\cancelto{0}{(\sin(-\xi)+\sin(\xi))} \\ \cancelto{0}{\sin(-\xi)+\sin\xi} & \cos(-\xi)+\cos\xi \end{pmatrix} \frac{s}{2}$$

$$= R(\gamma)\, \cos(\xi)\, s$$

& $$F_{eff}\, t = R(\gamma)\, \cos(\xi)\, t$$

$$\Rightarrow \quad \boxed{F_{eff} = R(\gamma)\, \cos(\xi)\, I} \qquad \underline{\text{conformal}}$$

<u>BUT</u> there are periodic mechanisms of kagome that are not described using one bowtie unit cell. More in Lect 2.

Q. Are there other shape change tensors $A(\xi)$?

A. Yes! Using parallelogram holes we can achieve many others. They can be classified by auxeticity, more later.

Q. How well does the effective description

$$y_{eff}(x) = F_{eff}\, x$$

capture the response of actual, physical mech. metamaterials?

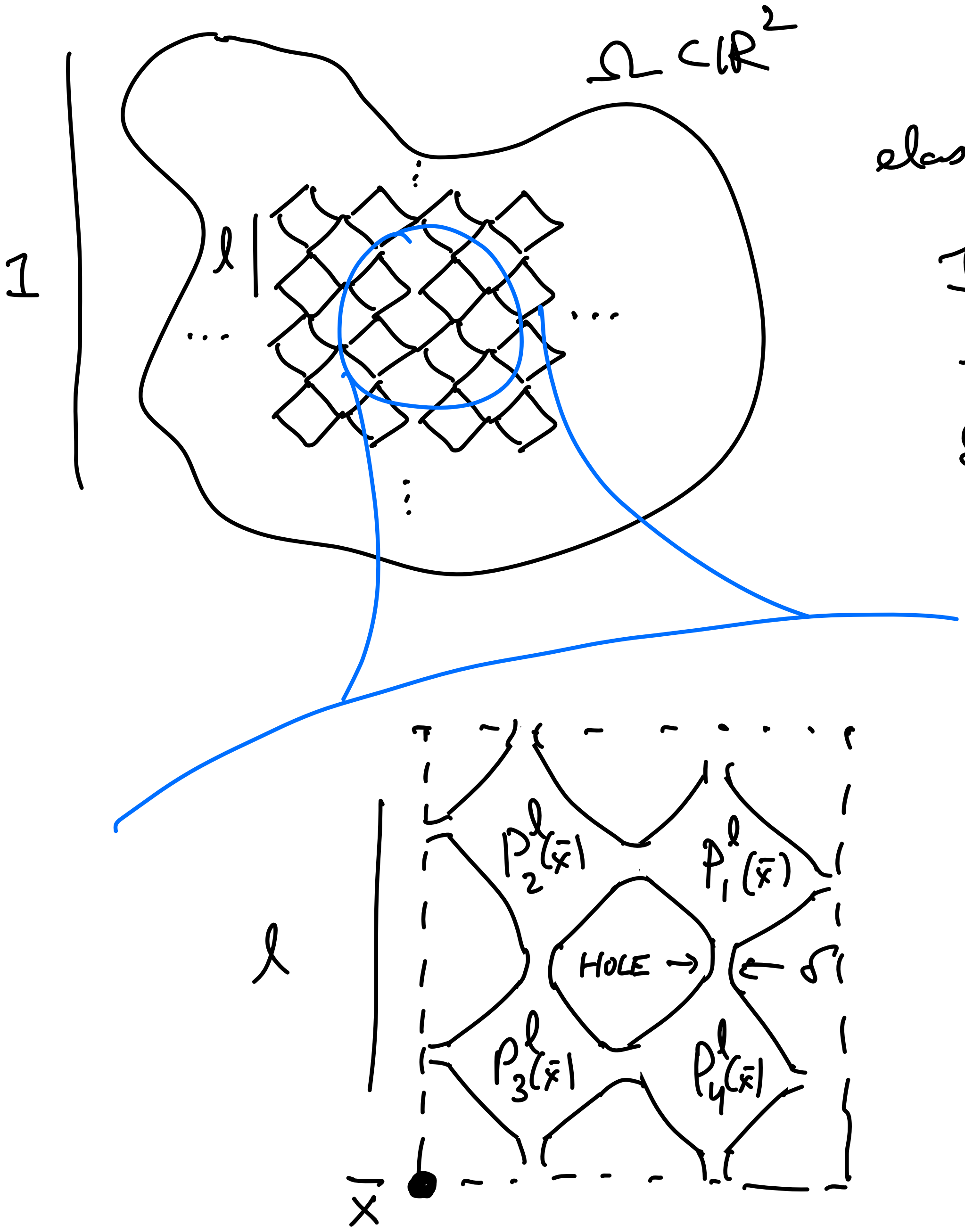
A. Very badly. [see PRL '22 Zheng et al for experiments showing the typical response]

What is going on here?

$\boxed{\text{KEY IDEA:}}$ mechanism gradients allow the system to relax its energy given b.c. or loads

To understand this we need a model for the energy of a "locally mechanistic" response.

Parallelogram kirigami, revisited.



elastic sheet in $\mathbb{R}^2$

1 = sheet size

ℓ = unit cell size $\ll 1$

δ = hinge size $\ll \ell$

$P_i^\ell(\bar x)$ = ith panel of $\bar x$th cell at scale ℓ

$\bar x \in \ell\mathbb{Z}^2$

(gen. by s,t)

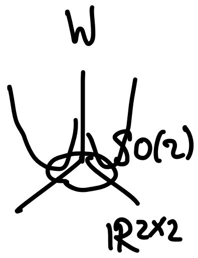
Really, the deformation

$$y : \underbrace{\Omega_{\delta,\ell}}_{\text{perforated kirigami sheet ref. config.}} \longrightarrow \mathbb{R}^2$$

$$\Omega_{\delta,\ell} = \Big(\underbrace{\bigcup_{\bar{x} \in \ell \mathbb{Z}^2 \cap \Omega} \bigcup_{i=1}^{4} P_i^\ell(\bar{x})}_{\text{scaled panels}} \Big)$$

$$\cup \quad \text{hinge regions of size } \delta$$

$$E(y) = \int_{\Omega_{\delta,\ell}} W(\nabla y(x))\, dA(x)$$



W = hyperelastic energy density

$(W \geq 0,\ W(F) = 0 \Leftrightarrow F \in SO(2))$

To get a handle on this we recognize it is a good idea to say the cell at $\bar{x}$ deforms by a perturbation of a mechanism:

$$y : x \in P_i^\ell(\bar{x}) \longmapsto y_{\text{eff}}(\bar{x}) + R_i(\bar{x})(\underbrace{x - \bar{x}}_{\sim \ell}) + \ell\, c_i(\bar{x}) + O(\ell^2)$$

$i = 1, \dots, 4$

$$R_i(\bar{x}) = R(\gamma(\bar{x}) \pm \xi(\bar{x})) \qquad \begin{matrix} + & i \text{ even} \\ - & i \text{ odd} \end{matrix}$$

Let's treat y_{eff}, γ, ξ as smooth & determine the strain energy of this ansatz.

$$\varepsilon = \frac{1}{2}\left(\nabla y^T \nabla y - I\right) = \text{strain}$$

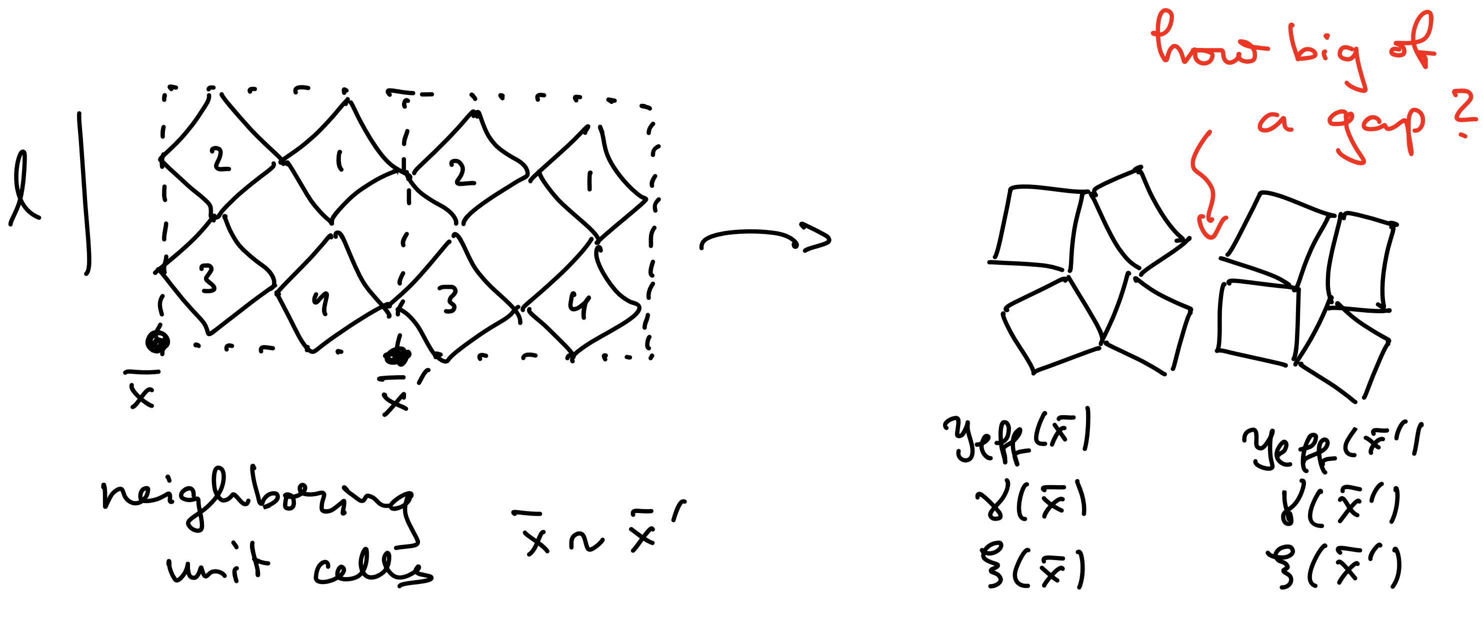
on panel $P_i^{\ell}(\bar{x})$,

$$\nabla y = R_i(\bar{x}) + O(\ell)$$

$$\begin{aligned}\nabla y^T \nabla y &= \left(R_i(\bar{x}) + O(\ell)\right)^T \left(R_i(\bar{x}) + O(\ell)\right) \\ &= R_i(\bar{x})^T R_i(\bar{x}) + O(\ell) \\ &= I + O(\ell)\end{aligned}$$

$$\therefore \quad \varepsilon = O(\ell)$$

Really? Something is fishy here...

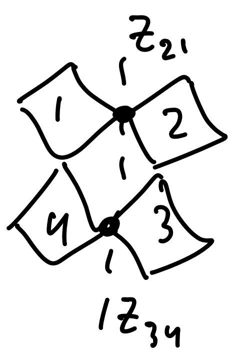


neighboring unit cells $\bar{x} \sim \bar{x}'$

If the gaps between cells are not $O(\ell^2)$ we have a problem.

Compute the gaps: say $\bar{x}' - \bar{x} = \ell s$

$$g_{34} = y_{\mathrm{eff}}(\bar{x}') + R_3(\bar{x}')(z_{34} - \bar{x}') + \ell c_3(\bar{x}')$$



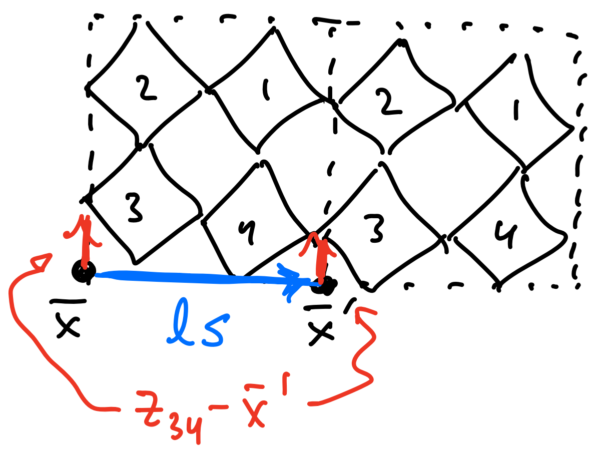
$$- \left(y_{\mathrm{eff}}(\bar{x}) + R_4(\bar{x})(z_{34} - \bar{x}) + \ell c_4(\bar{x}) \right) + O(\ell^2)$$

$$= y_{\mathrm{eff}}(\bar{x}') - y_{\mathrm{eff}}(\bar{x})$$

$$+ R_3(\bar{x}')(z_{34} - \bar{x}') + \ell c_3(\bar{x}')$$

$$- \left(R_4(\bar{x})(z_{34} - \bar{x}) + \ell c_4(\bar{x}) \right) + O(\ell^2)$$

Note a perfect mechanism has <u>no</u> gaps: by defn of F_{eff},



$$F_{\mathrm{eff}}(\bar{x})\,\ell s = R_4(\bar{x})(z_{34} - \bar{x}) + \ell c_4(\bar{x}) - \left(R_3(\bar{x})(\underbrace{z_{34} - \bar{x}'}_{\text{see diagram}}) + \ell c_3(\bar{x}) \right)$$

$$\therefore \quad g_{34} = y_{\mathrm{eff}}(\bar{x}') - y_{\mathrm{eff}}(\bar{x})$$

$$+ R_3(\bar{x}')\,(z_{34}-\bar{x}') + l\,c_3(\bar{x}')$$

$$- F_{eff}(\bar{x})\, l s$$

$$- R_3(\bar{x})\,(z_{34}-\bar{x}') + l\,c_3(\bar{x})$$

$$+ O(l^2)$$

$$= y_{eff}(\bar{x}') - y_{eff}(\bar{x}) - F_{eff}(\bar{x})\, l s$$

$$+ \left[R_3(\bar{x}') - R_3(\bar{x})\right](z_{34}-\bar{x}')$$

$$+ l\left(c_3(\bar{x}') - c_3(\bar{x})\right) + O(l^2)$$

Recall

$$F_{eff}(\bar{x}) = R(\gamma(\bar{x}))\, A(\xi(\bar{x}))$$

$$R_3(\bar{x}) = R(\gamma(\bar{x}) - \xi(\bar{x})) \qquad (-\text{ odd})$$

$$c_3(\bar{x}) = \text{a smooth func. of } \gamma, \xi$$

We can Taylor expand : $\bar{x}' - \bar{x} = l s$

$$y_{eff}(\bar{x}') - y_{eff}(\bar{x}) = \nabla y_{eff}(\bar{x})\, l s + O(l^2)$$

etc.

$$\Rightarrow \quad g_{34} = \left[\nabla y_{eff}(\bar{x}) - R(\gamma(\bar{x}))A(\xi(\bar{x}))\right] l s$$

$$+ O(l^2)$$

To make $g_{34} = O(\ell^2)$ must sat.

$$\left[\nabla y_{eff}(\bar{x}) - R(\gamma(\bar{x}))A(\xi(\bar{x}))\right] s = 0 .$$

Likewise, from $\bar{x}' - \bar{x} = \ell t$ we get

$$\text{gap} = O(\ell^2)$$
$$\iff \left[\nabla y_{eff}(\bar{x}) - R(\gamma(\bar{x}))A(\xi(\bar{x}))\right] t = 0 .$$

Hence we get panel strains $|\varepsilon| \sim \ell$
iff intercell gaps $\sim \ell^2$
iff $\nabla y_{eff}(\bar{x}) = R(\gamma(\bar{x}))A(\xi(\bar{x}))$.

Replacing $\bar{x} \in \ell\mathbb{Z}^2$ with $x \in \Omega$ we have derived the homogenized PDE description of parallelogram hole kirigami :

$y_{eff} : \Omega \to \mathbb{R}^2$ solves $\boxed{\nabla y_{eff}(x) = R(\gamma(x))A(\xi(x))}$

From here, we can consider

i) PDE analysis of kirigami

ii) mechanism gradient energies to select PDE sol'n using minimization w/ b.c.

We'll touch on both.

First, PDE analysis of 2D shape changing mat'ls. Starting point is

$$\nabla y_{eff}(x) = \underbrace{R(\gamma(x))}_{\substack{\text{rot'n} \\ \in SO(2)}} \underbrace{A(\xi(x))}_{\substack{\text{shape tensor,} \\ \text{not necc. sym.}}}$$

We can eliminate y_{eff}:

$$0 = \nabla \times \nabla y_{eff} = \nabla \times \left(R(\gamma) A(\xi) \right)$$

$$= \partial_1 \left(R(\gamma) A(\xi) e_2 \right) - \partial_2 \left(R(\gamma) A(\xi) e_1 \right)$$

$$R'(\theta) = W R(\theta) = R(\theta) W \qquad \text{(exercise)}$$

$$= \left[W R(\gamma)\, \partial_1 \gamma\, A(\xi) + R(\gamma)\, A'(\xi)\, \partial_1 \xi \right] e_2$$

$$- \left[W R(\gamma)\, \partial_2 \gamma\, A(\xi) + R(\gamma)\, A'(\xi)\, \partial_2 \xi \right] e_1$$

Rewrite in terms of $\nabla\gamma = \partial_1 \gamma e_1 + \partial_2 \gamma e_2$
$\nabla\xi = \partial_1 \xi e_1 + \partial_2 \xi e_2$

$$0 = W R(\gamma) A(\xi) \left(\partial_1 \gamma e_2 - \partial_2 \gamma e_1 \right)$$
$$+ R(\gamma) A'(\xi) \left(\partial_1 \xi e_2 - \partial_2 \xi e_1 \right)$$

$$= W R(\gamma) A(\xi) W \nabla\gamma$$
$$+ R(\gamma) A'(\xi) W \nabla\xi$$

$$\therefore \boxed{\begin{aligned} \nabla\gamma &= \Gamma(\xi)\, \nabla\xi \\ \text{w/ } \Gamma(\xi) &= \frac{A^T(\xi)\, A'(\xi)\, W}{\det A(\xi)} \end{aligned}}$$

<u>exercise</u>. show this using

$$\operatorname{cof} A = -WAW \qquad \text{in 2D}$$
$$= \det A \; A^{-T}$$

<u>Ex</u>. conformal shape change

$$A(\xi) = \lambda(\xi)\, I, \quad \lambda = e^{\alpha}$$

$$\Gamma(\xi) = \alpha'(\xi)\, W$$

$$\nabla\gamma = \alpha'(\xi)\, W \nabla\xi$$

$$\begin{cases} \partial_1\gamma = -\alpha'(\xi)\,\partial_2\xi \\ \partial_2\gamma = \alpha'(\xi)\,\partial_1\xi \end{cases}$$

$$\begin{cases} \partial_1\gamma = -\partial_2\,\alpha(\xi) \\ \partial_2\gamma = \partial_1\,\alpha(\xi) \end{cases}$$

Cauchy - Riemann eqns in $(\gamma, \alpha(\xi))$.

$$\Delta\gamma = 0 \quad \&\quad \Delta\alpha(\xi) = 0$$

$$\alpha = \log\lambda \quad \therefore \quad \underbrace{\Delta\log\lambda(\xi) = 0}$$

$K_{Gauss} = 0$ for conformal metric $(\lambda(\xi))^2(dx_1^2 + dx_2^2)$

Indeed,

$$\begin{aligned} g &= \nabla y^T \nabla y \\ &= (R(\gamma)A(\xi))^T R(\gamma)A(\xi) \\ &= A^T(\xi)\, A(\xi) \end{aligned}$$

& here $\quad A(\xi) = \lambda(\xi)\, I$

$$\Rightarrow \quad g = \lambda^2(\xi)\, I \qquad \underset{\equiv}{\text{conformal}}$$

$\lrcorner$

So then, generally,

$$g = A^T A(\xi)$$

$$K_{Gauss} = 0 \rightsquigarrow 2^{nd} \text{ order PDE in } \xi$$

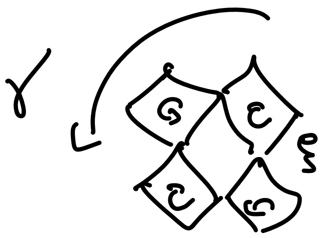
Here, it comes from direct manipulation:

$$\nabla \gamma = \Gamma(\xi) \nabla \xi$$

$$0 = \nabla \times \nabla \gamma = \nabla \times (\Gamma(\xi) \nabla \xi)$$

$$\boxed{0 = \nabla \cdot (W \Gamma(\xi) \nabla \xi)}$$

2^{nd} order nonlinear PDE for ξ = mech. angle field



PDEs have a "type": elliptic, hyperbolic, parabolic

defined via linearization about a sol'n; only highest order derivs. matter

Here, $\xi \to \xi + \delta\xi$

$$0 = \nabla\cdot\left(W\,\Gamma(\xi+\delta\xi)(\nabla\xi+\nabla\delta\xi)\right)$$

$$= \nabla\cdot\left[W\left(\Gamma(\xi)+\Gamma'(\xi)\delta\xi\right)(\nabla\xi+\nabla\delta\xi)\right]$$

$$= \cancel{\nabla\cdot\left[W\,\Gamma(\xi)\nabla\xi\right]} \rightarrow 0 \text{ by assumption}$$

$$+ \underbrace{\nabla\cdot\left[W\,\Gamma(\xi)\,\nabla\delta\xi\right]}_{} + \cdots$$

terms that are lower order in ∇ or $O((\delta\xi)^2)$

$$= \sum_{ij}\left(W\,\Gamma(\xi)\right)_{ij}\partial_{ij}\xi + \cdots$$

$$\text{principal part} = \sum_{ij}\mathrm{sym}\left(W\,\Gamma(\xi)\right)_{ij}\partial_{ij}\xi$$

$\det\ \mathrm{sym}(W\,\Gamma(\xi)) > 0 \iff$ elliptic

like $\partial_{11}+\partial_{22}$, Laplace

$= 0 \iff$ parabolic

like ∂_{11}

$< 0 \iff$ hyperbolic

like $\partial_{11}-\partial_{22}$, wave

How is this related to design?

Look at "effective Poisson's ratio":

$\nu = -\dfrac{\varepsilon_2}{\varepsilon_1}$, ε_i are principal strains gotten by linearization

Suffices to linearize around a mechanism $\gamma = 0$, $\xi = \xi_0 = \text{const.}$

$y(x) = A(\xi_0)\, x + u\,(A(\xi_0)\, x)$, $\gamma \to 0 + \delta\gamma$, $\xi \to \xi_0 + \delta\xi$

solve $\nabla y = R(\gamma)\, A(\xi)$

to linear order in $u, \delta\gamma, \delta\xi$

$$\nabla y = A(\xi_0) + \nabla u\,(A(\xi_0)x)\, A(\xi_0)$$

$$R(\gamma) = I + \cancel{R'(\gamma)} \, W \, \delta\gamma + O((\delta\gamma)^2)$$

$$A(\xi) = A(\xi_0) + A'(\xi_0)\,\delta\xi + O((\delta\xi)^2)$$

$$R(\gamma)\, A(\xi) = A(\xi_0) + W A(\xi_0)\,\delta\gamma + A'(\xi_0)\,\delta\xi + \dots$$

$$\therefore \quad \nabla u\,(A(\xi_0)x) = W\,\delta\gamma + A'(\xi_0)\, A^{-1}(\xi_0)\,\delta\xi$$

after linearization.

$$\varepsilon_{lin} = \operatorname{sym} \nabla u = \operatorname{sym}\left(A'(\xi_0) A^{-1}(\xi_0)\right) \delta\xi$$

$$\varepsilon_{lin} = \varepsilon_1\, v_1 \otimes v_1 + \varepsilon_2\, v_2 \otimes v_2$$

$\{v_1, v_2\}$ right handed ONB of eigenvects.

$\varepsilon_1, \varepsilon_2$ principal strains of linear part u.

$$\nu = -\frac{\varepsilon_2}{\varepsilon_1}$$

$$\Rightarrow \quad \operatorname{sign} \nu = -\operatorname{sign} \det \operatorname{sym}\left(A' A^{-1}(\xi_0)\right).$$

auxetic $\iff$ $\nu < 0$ $\iff$ $\det \operatorname{sym} A'A^{-1} > 0$

<u>exercise</u>: check if $\det A > 0$ then

$$\operatorname{sign} \det \operatorname{sym} A'A^{-1} = \operatorname{sign} \det W\Gamma$$

where $\Gamma = \dfrac{A^T A' W}{\det A}$.

<u>Hint</u>. show corresponding evals are of opp. sign (a stronger result)

<u>Conclusion</u>: parallelogram hole kirigami is

$$\begin{cases} \text{auxetic } (\nu < 0) \iff \text{elliptic} \\ \text{classical } (\nu > 0) \iff \text{hyperbolic} \end{cases}$$

<u>Ex</u>. rhombic holes produce diagonal $A(\xi) = \begin{pmatrix} \mu_1(\xi) & 0 \\ 0 & \mu_2(\xi) \end{pmatrix}$

Then, $W\Gamma = W \dfrac{A A' W}{\det A} = -\dfrac{1}{\mu_1 \mu_2} \begin{pmatrix} \mu_1 \mu_1' & \\ & \mu_2 \mu_2' \end{pmatrix}$

$$\Rightarrow \det \operatorname{sym} W\Gamma = \frac{\mu_1' \mu_2'}{\mu_1 \mu_2}$$

$$A'A^{-1} = \begin{pmatrix} \frac{\mu_1'}{\mu_1} & 0 \\ 0 & \frac{\mu_2'}{\mu_2} \end{pmatrix}$$

$$\Rightarrow \det \operatorname{sym} A'A^{-1} = \frac{\mu_1' \mu_2'}{\mu_1 \mu_2}$$

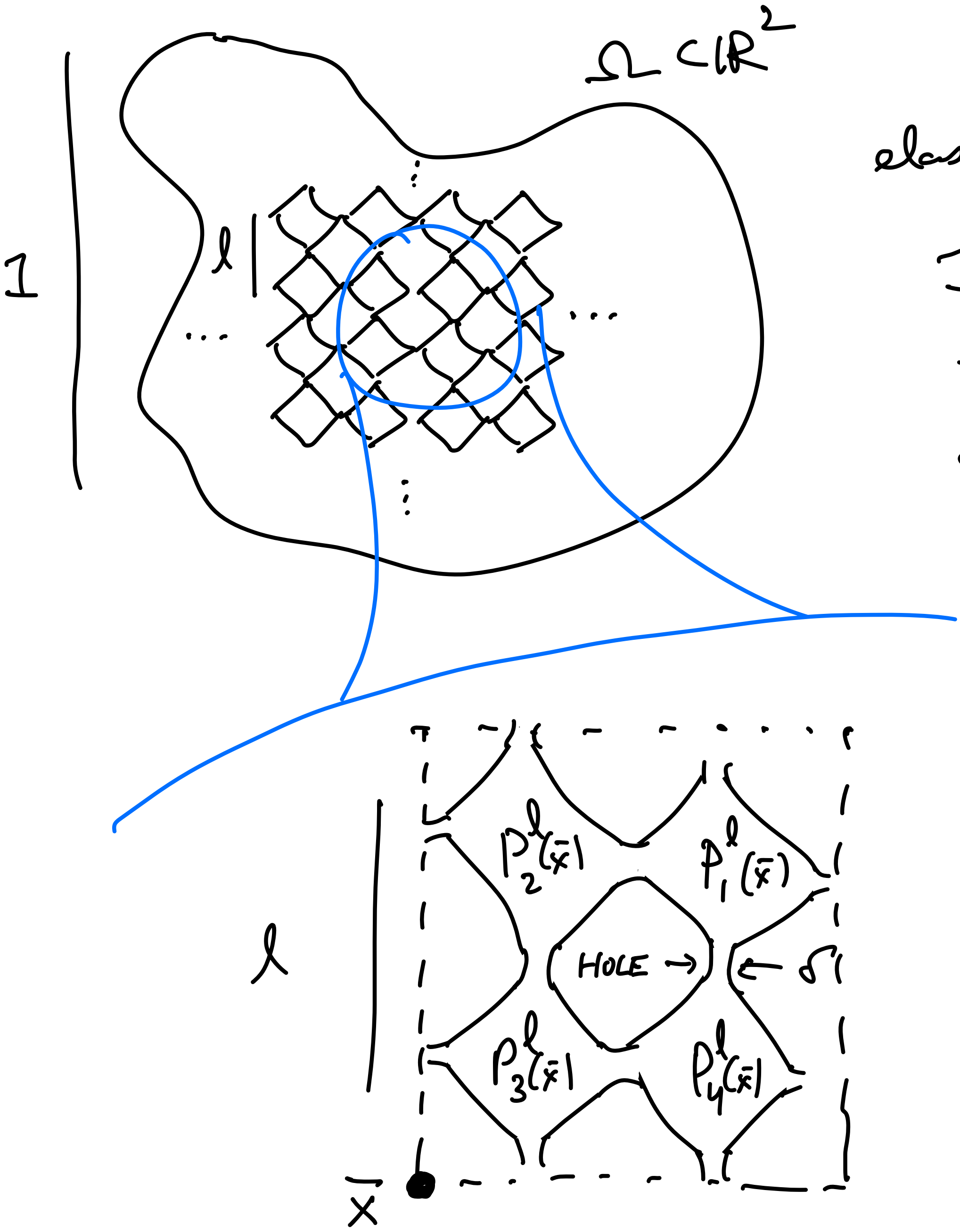
same!

(it is consistent)

Poisson's ratio is $\nu = -\frac{\varepsilon_2}{\varepsilon_1} = -\frac{\mu_2'/\mu_2}{\mu_1'/\mu_1}$.

[see examples from PRL '22 Zhang et al]

Final topic is mechanism gradient energies.



elastic sheet in $\mathbb{R}^2$

1 = sheet size

ℓ = unit cell size $\ll 1$

δ = hinge size $\ll \ell$

$P_i^\ell(\bar{x})$ = i^{th} panel of $\bar{x}^{\text{th}}$ cell at scale ℓ

$\bar{x} \in \ell\mathbb{Z}^2$

(gen. by s, t)

How can we select a PDE sol'n using b.c.?

$$E(y) = \int_{\Omega_{\delta,\ell}} W(\nabla y)\, dA$$

principal of virtual work : $\min\limits_{\substack{y \\ \text{b.c.}}} \; E(y) -$ work of loads

Now we think of parameterizing y via the locally mech. ansatz

$$y(x) = y_{eff}(\bar{x}) + R_i(\bar{x})(x-\bar{x}) + \ell c_i(\bar{x}) + O(\ell^2)$$

$$\bar{x} \in \ell \mathbb{Z}^2, \quad i = 1, \dots, 4 .$$

Recall gaps between cells $\sim |\nabla y_{eff} - R(\gamma)A(\xi)|\,\ell + O(|(\nabla\gamma, \nabla\xi)|\,\ell^2)$

So panel strain $\sim |\nabla y_{eff} - R(\gamma)A(\xi)| + O(|(\nabla\gamma, \nabla\xi)|\,\ell)$

This suggests a model $\quad (\nabla\gamma = \Gamma(\xi)\nabla\xi)$

$$E(y_{eff}, \gamma, \xi) = \int_{\Omega} W(\nabla y_{eff} A^{-1}(\xi)) + \ell^2 Q(\xi, \nabla\xi)\, dA$$

standard hyperelas. W
$W(F) = 0 \iff F \in SO(2)$
$W \geq 0$ in general

quadratic form in $\nabla\xi$ with coeffs. param. by ξ

Thought expt :

this is what we see in real expts. ($\nu > 0$ drawn)

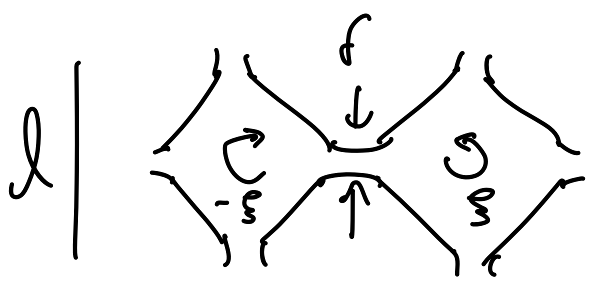
min E here $\Rightarrow$ mechanisms, i.e.,

$$\nabla \xi = 0$$

so $\xi = \text{const}$.

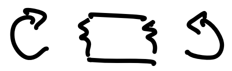
this has $E = 0$.

What's missing?



$\delta \ll l \ll 1$

there is energy in the necks!



$$E_{neck} \sim N(\xi)\, A_{neck}$$

say $A_{neck} \sim \delta^2$ then

summing up $\Rightarrow$ total $E_{necks} \sim \left(\frac{\delta}{l}\right)^2 \int_\Omega N(\xi(x))\, dA(x)$.

Altogether,

$$E(y_{eff}, \delta, \xi)$$
$$= \int_\Omega W(\nabla y_{eff} A^{-1}(\xi)) + \ell^2 Q(\xi, \nabla \xi) + \frac{\delta^2}{\ell^2} N(\xi) \, dA$$

Rmks. i) A better estimate on the mechanism gradient term includes a logarithmic correction coming from the fact that the linear strain of a wedge with point force at its tip $\sim \frac{1}{r}$.

ii) hinge bending dominates mech. gradient effects if

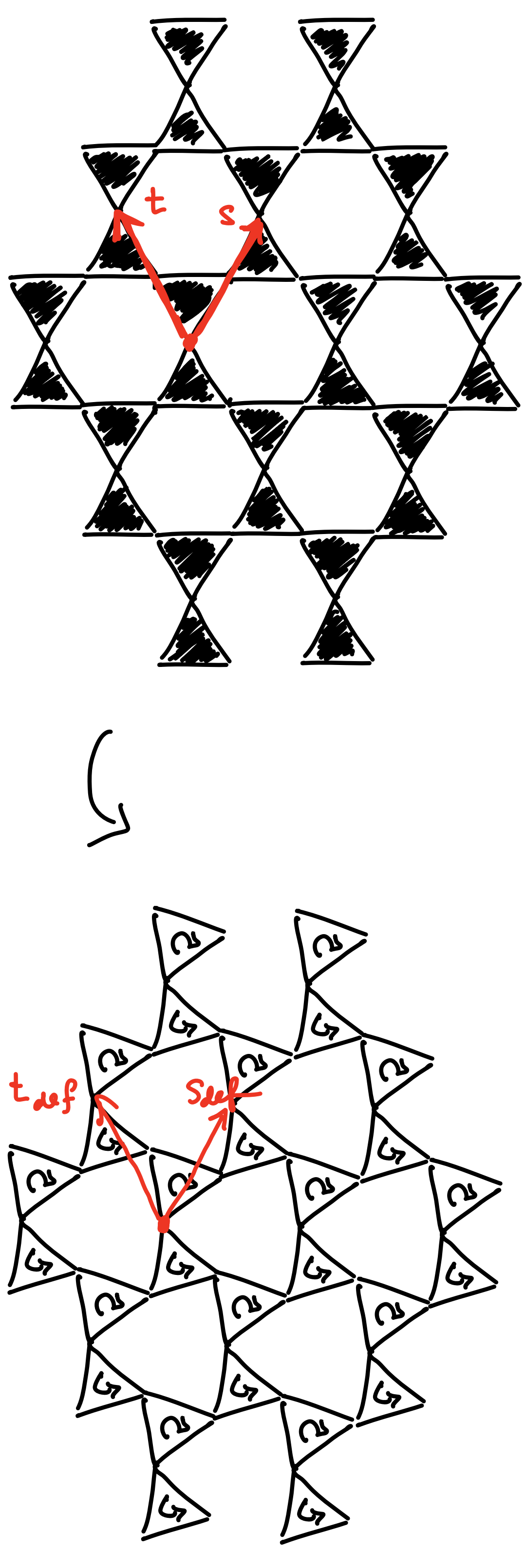
$$\frac{\delta^2}{\ell^2} \gg \ell^2$$

i.e. $\delta \gg \ell^2$ (non-dim by system size)

The hinges don't have to be that large to matter.

[see examples from PRSA '23 Zhang et al]

Kagome Lattice.



1-periodic mech.

lattice of equilateral triangles & hexagonal holes

Recall we showed for 1-periodic mechanisms

$$y(x) = F_{eff}\, x + \underbrace{\varphi(x)}_{1\text{-periodic}}$$

that

$$F_{eff} = R(\gamma)\, \lambda(\xi)\, I$$

i.e., the shape change tensor

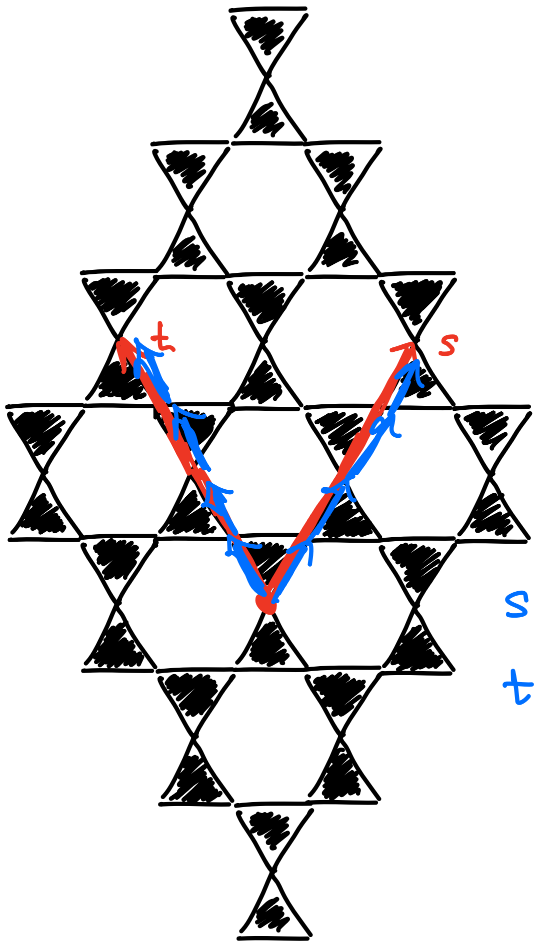
$$A(\xi) = \lambda(\xi)\, I \,.$$

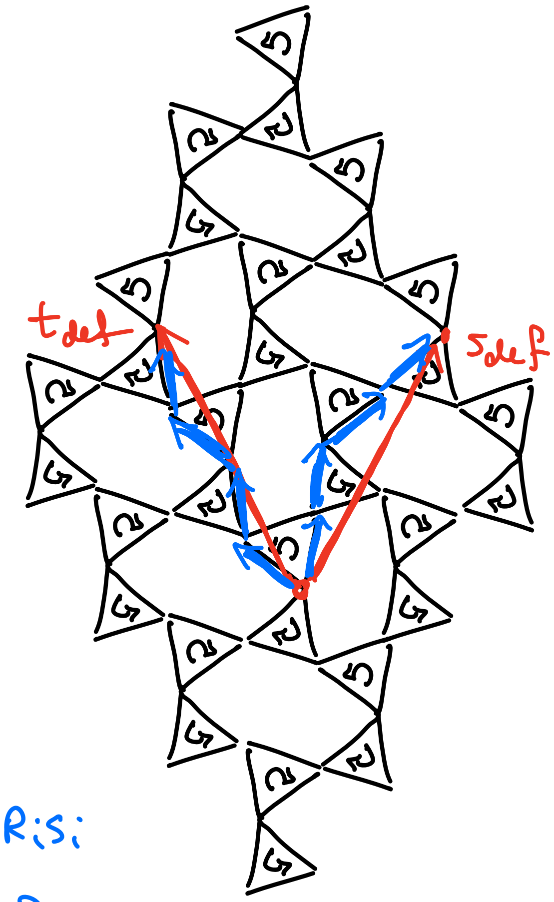
conformal

<u>BUT</u> there are many other mechanisms!

Are they all conformal?

Let's look at a 2-periodic mechanism.





$$s = \sum_{i=1}^{4} s_i$$

$$t = \sum_{i=1}^{4} t_i$$

$$s_{def} = \sum_{i=1}^{4} R_i s_i$$

$$t_{def} = \sum_{i=1}^{4} Q_i t_i$$

Looks conformal... how to prove it?

Recall

$$y(x) = F_{eff}\, x + \underbrace{\varphi(x)}_{2\text{-per.}}$$

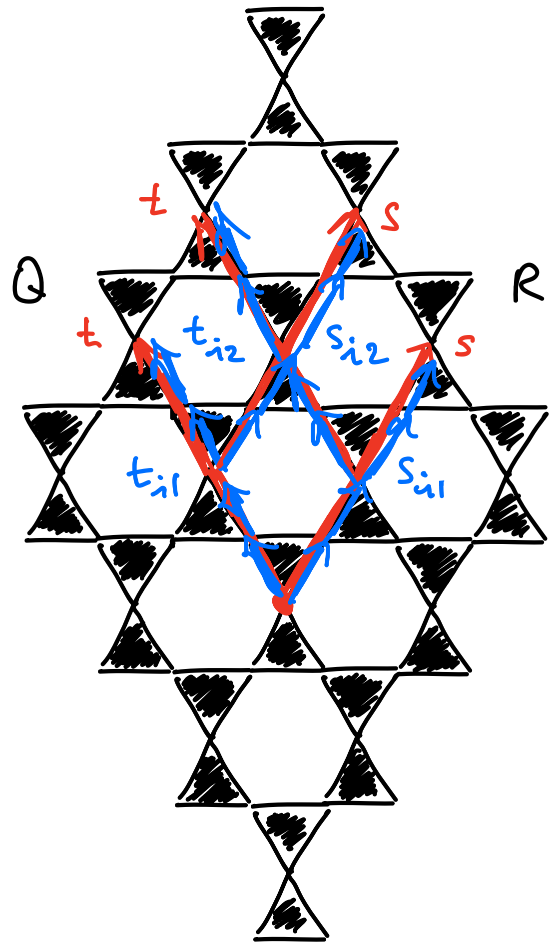
2 per. means $\varphi(x') = \varphi(x)$ if $x' - x = ns + mt$
for $n, m \in \mathbb{Z}$
w/ s, t as above

$$\Rightarrow \begin{cases} s_{def} = y(x+s) - y(x) = F_{eff}\, s \\ t_{def} = y(x+t) - y(x) = F_{eff}\, t \end{cases}$$

$$\Rightarrow \begin{cases} F_{eff}\, s = \sum_{i=1}^{4} R_i s_i \\ F_{eff}\, t = \sum_{i=1}^{4} Q_i t_i \end{cases}$$

but the rotins $\{R_i\}_{i=1}^4$ & $\{Q_i\}_{i=1}^4$ are not obviously related...

IDEA: use other choices of s, t.



2 ways to write s, t

$$\begin{cases} s = \sum_i s_{i1} = \sum_i s_{i2} \\ t = \sum_i t_{i1} = \sum_i t_{i2} \end{cases}$$

$$\begin{cases} s_{def} = \sum_i R_{i1} s_{i1} = \sum_i R_{i2} s_{i2} \\ t_{def} = \sum_i Q_{i1} t_{i1} = \sum_i Q_{i2} t_{i2} \end{cases}$$

The rotations $R_{i1}, R_{i2}, Q_{i1}, Q_{i2}$ are not all distinct.

e.g., $R_{11} = Q_{11}$

$R_{21} = Q_{42}$ by 2-per.

$R_{31} = Q_{12}$

$\vdots$

Of course, $R(\frac{\pi}{3})\, s = t$.

What about $R(\frac{\pi}{3})\, s_{def}$?

$$R(\tfrac{\pi}{3})\, s_{def} = R(\tfrac{\pi}{3})\, \frac{1}{2} \sum_{i,j} R_{ij}\, s_{ij}$$

$$= \frac{1}{2} \sum_{i,j} R_{ij}\, R(\tfrac{\pi}{3}) s_{ij}$$

$$\overset{!}{=} \frac{1}{2} \sum_{\alpha,\beta} Q_{\alpha\beta}\, t_{\alpha\beta} \qquad \text{by the diagram}$$

by pairing $(i,j) \leftrightarrow (\alpha,\beta)$

s.t. $R_{ij} = Q_{\alpha\beta}$

automatic $\rightsquigarrow$ $R(\tfrac{\pi}{3}) s_{ij} = t_{\alpha\beta}$

b/c $s_{ij} = \frac{s}{4}$

$t_{ij} = \frac{t}{4}$

<u>Note</u>: each triangle is used once.

$$\therefore \quad R(\tfrac{\pi}{3})\, s_{def} = t_{def}$$

We've proved

$$R(\tfrac{\pi}{3})\, F_{eff}\, s = F_{eff}\, R(\tfrac{\pi}{3})\, s$$

A similar arg. shows same for t, hence

$$R(\tfrac{\pi}{3})\, F_{eff} = F_{eff}\, R(\tfrac{\pi}{3}).$$

$R(\theta) = \cos\theta \, I + \sin\theta \, W \quad , \quad W = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\Rightarrow \qquad W F_{eff} = F_{eff} W$$

$$\Rightarrow \qquad R F_{eff} = F_{eff} R \qquad \forall R \in SO(2)$$

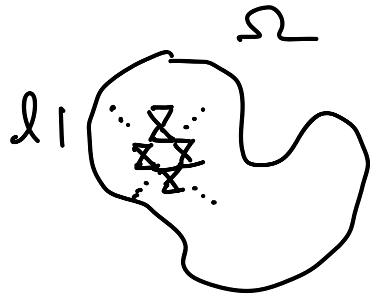
$$\therefore \qquad F_{eff} = R \lambda I \qquad \text{conformal}.$$

Same argument applies for all periods.

<u>Conclusion</u>. All periodic mech. of kagome are conformal.

To adress arbitrary mechanisms & soft modes one needs the "cell formula" from

Li + Kohn J. Elasticity '25



Roughly, their theorem says

$$E \longrightarrow \int_\Omega W(\nabla y_{eff}(x)) \, dA(x)$$

as lattice spacing $l \to 0$, where

$$E(y) = \sum_{\substack{i\sim j \\ \text{in } \Omega \\ \text{on scaled} \\ \text{lattice}}} \tfrac{1}{2} k_{ij} (\varepsilon_{ij}(y))^2 \, \underbrace{\ell^2}_{\text{area element}}$$

$$W(F) = \min_{\substack{\text{unit cell} \\ Q}} \; \min_{\substack{\varphi \\ Q\text{-per.}}} \; \frac{E_Q(x \mapsto Fx + \varphi(x))}{A(Q)}$$

$$E_Q(y) = \sum_{\substack{i\sim j \\ \text{in } Q}} \tfrac{1}{2} k_{ij} (\varepsilon_{ij}(y))^2$$

$A(Q) =$ area of Q (unscaled)

Note. the min is NOT necessarily achieved, i.e., one may need to take period of $Q \to \infty$.

Cor. $E(y_\ell) \to 0 \Rightarrow W(\nabla y_{eff}(x)) = 0 \;\; \forall x$

$$\Rightarrow \nabla y_{eff}(x) = R(x)\, \lambda(x)\, I$$

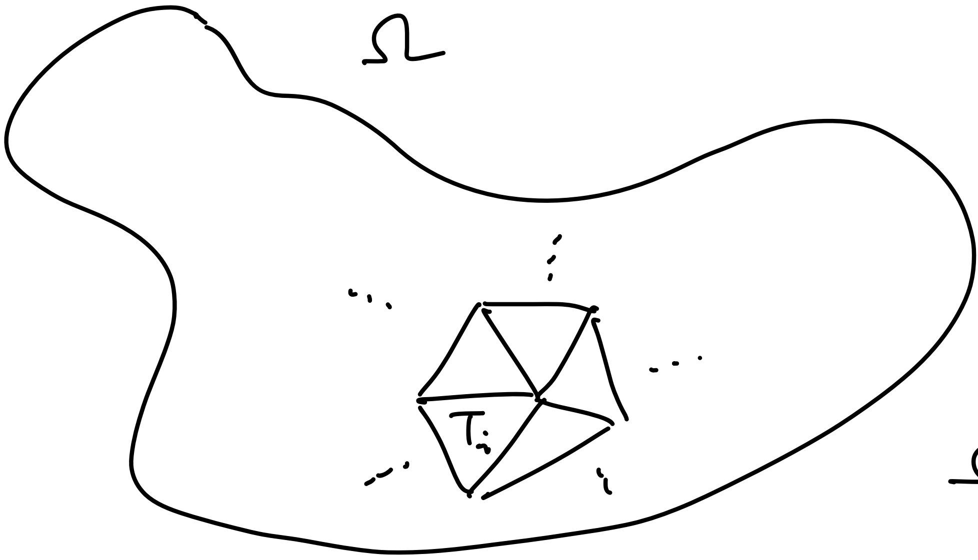
some $R : \Omega \to SO(2)$
$\lambda : \Omega \to \mathbb{R}$

$\therefore$ soft modes of kagome are conformal.

Pf. quantify the previous arg. to show that at any period, $E \approx 0 \Rightarrow F \approx$ conformal. Check that this doesn't degenerate as period $\to \infty$.

See Li + Kohn arXiv '25. $\square$

Picture of the proof of the cell formula:



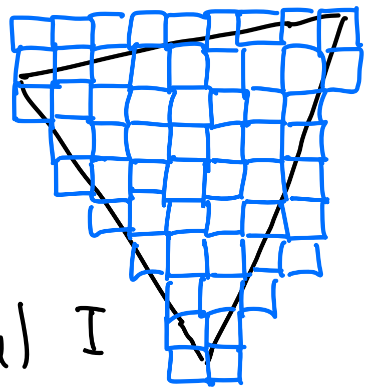
$\xrightarrow{y_{\text{eff}}} \mathbb{R}^2$

$$\Omega = \bigcup_i T_i$$

① $y_{\text{eff}} \approx$ piecewise linear map

$x \longmapsto F_i x + c_i$ on triangle T_i

like in FEM

② 

$\ell \cdot \begin{pmatrix}\text{optimal} \\ \text{period}\end{pmatrix}$ I

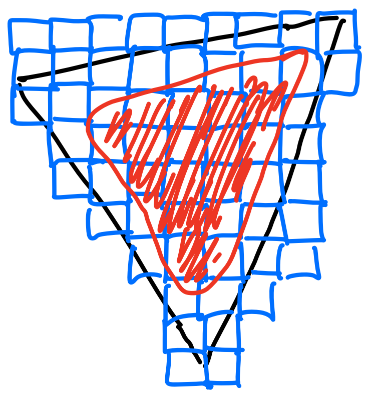
$$T_i \subset \bigcup_j Q_j$$

cover T_i with (nearly) optimal squares Q_j w.r.t. cell problem for F_i

in Q_j, $\quad x \longmapsto F_i x + c_i + \ell \underbrace{\varphi_i\left(\frac{x}{\ell}\right)}_{\text{optimal}}$

$$E \text{ in } Q_j = W(F_i)\, A(Q_j)$$

(3) cut off ansatz to match affine b.c. at ∂T_i



keep φ_i in red portion

truncate smoothly to 0 in b.l. nearby ∂T_i

this produces an ansatz y on T_i with

$$E(y) \text{ in } T_i \approx W(F_i)\,A(T_i)$$

& $\quad y(x) = F_i x + c_i \quad$ at ∂T_i

$\therefore \quad y$ is def'd on Ω.

(∂T_i & ∂T_j match if $i \sim j$)

(4) add up to get

$$E(y) = \sum_i E(y) \text{ in } T_i$$

$$\approx \sum_i W(F_i)\,A(T_i)$$

$$\approx \int_\Omega W(\nabla y_{eff})\,dA .$$

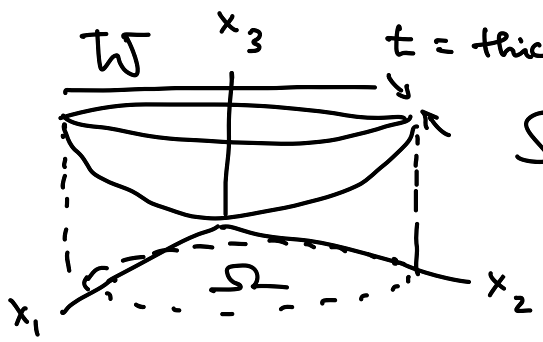
$\square$

Can another ansatz do better? Not in the limit $\ell \to 0$. This requires a different proof, based on lower bounds rather than constructions. For full details see Li + Kohn J. Elasticity '25.

Part 2. Wrinkled Shells.

Press a $\underbrace{\text{thin elastic shell}}_{\text{"curved surface"}}$ flat. What happens?

It wrinkles!



$$S = \{ (x_1, x_2, p(x_1,x_2)) : (x_1,x_2) \in \Omega \}$$
$$= \text{reference shell}$$

$$h = \frac{t}{W} \ll 1 \qquad (\sim 10^{-4} \text{ to } 10^{-6})$$

Experiments on ultrathin ($\sim 10^2$ nm) polymer shells use water to do the confinement.

$$E = E_{elastic} + E_{substrate} \longrightarrow \min$$

deformation $y : \underset{\underset{\Omega}{\cong}}{S} \longrightarrow \mathbb{R}^3$

$$(x_1,x_2) \longmapsto \big(x_1+u_1(x_1,x_2),\ x_2+u_2(x_1,x_2),\ w(x_1,x_2)\big)$$

$u = (u_1,u_2)$ = in-plane displacement

w = out-of-plane height ($w-p$ = displ.)

$$\varepsilon = \tfrac12\left(\nabla y^T\nabla y - g_S\right)$$

$e_{ij}(u) = \frac12(\partial_i u_j + \partial_j u_i)$

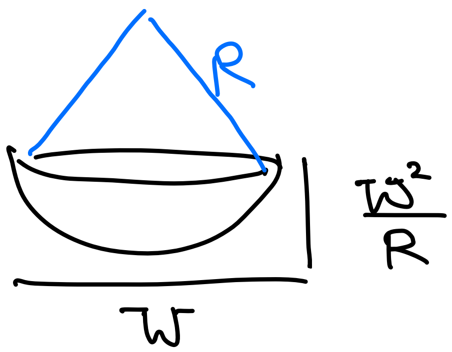
$$= \tfrac12\Big[\cancel{I} + e(u) + \nabla u^T\nabla u + \tfrac12\nabla w\otimes\nabla w - \big(\cancel{I} + \nabla p\otimes\nabla p\big)\Big]$$

<u>exercise</u>: show the metric of S is

$$g_S = I + \nabla p\otimes\nabla p$$

i.e., $(g_S)_{ij} = \delta_{ij} + \partial_i p\,\partial_j p \qquad i,j\in\{1,2\}$

The shell is <u>shallow</u> if $g_S \approx I$, i.e.,



e.g. $p(x_1,x_2) = \frac{1}{R}\,\frac{x_1^2+x_2^2}{2}$

$$\nabla p = \frac{(x_1,x_2)}{R}$$

so $|\nabla p| \sim \frac{W}{R} \ll 1$ shallow

a.k.a. "small slopes"

FvK approx.: $\varepsilon = \underbrace{e(u) + \tfrac12\nabla w\otimes\nabla w - \tfrac12\nabla p\otimes\nabla p}_{:=\,\varepsilon_{vK}} + \cancel{O(|\nabla u|^2)}$ ignore!

why? expect

$$|\nabla u| \sim |\nabla w|^2 \sim |\nabla p|^2 \sim \left(\frac{W}{R}\right)^2 \ll 1$$

$$\Rightarrow \quad |\nabla u|^2 \sim \left(\frac{W}{R}\right)^4 \lll 1 \quad \text{so drop it.}$$

$$E_{elas} = \int_\Omega \underbrace{\frac{Y}{2}|\varepsilon_{vK}|^2}_{\text{stretching}} + \underbrace{\frac{B}{2}|\nabla\nabla w - \nabla\nabla p|^2}_{\text{bending}} \, dx$$

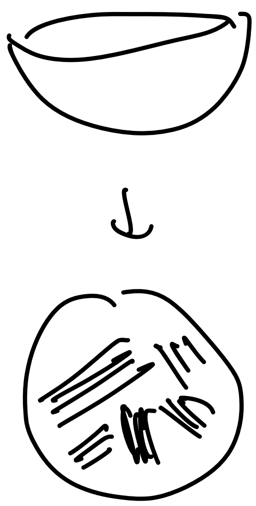
ASSUME: isotropic elasticity (def. of 1.1)

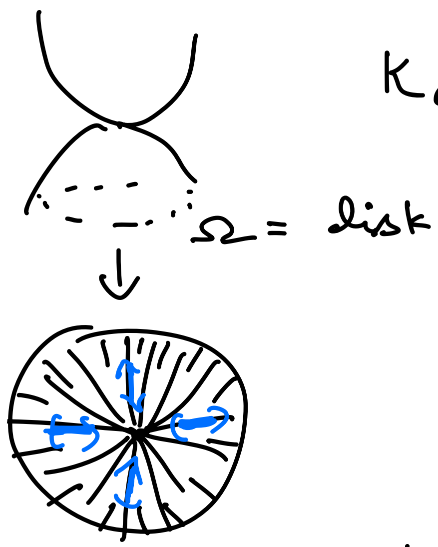
$$E_{sub} = \int_\Omega \frac{K}{2} w^2 \, dx$$

Problem. Expts. show the shell wrinkles.

Spherical cap $\qquad$ Saddle disk

$K_G > 0$ $\qquad$ $K_G < 0$

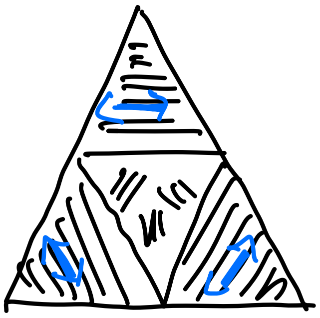




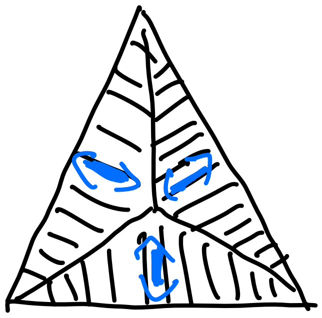
random wrinkles $\qquad$ ordered wrinkles, $\hat{e}_r \otimes \hat{e}_r$

Some more examples (see Tobasco et al Nat Phys '22 for many more + sims)

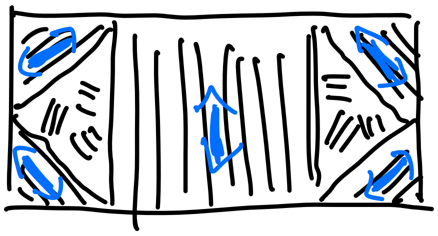
$K_G > 0$ $K_G < 0$



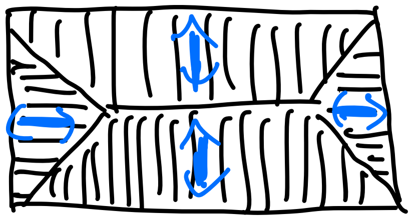
Ω = triangle



$\hat{T} \otimes \hat{T}$



Ω = rectangle



What's going on here?

$\boxed{\text{IDEA}}$ view highly wrinkled shells as "metamaterials"

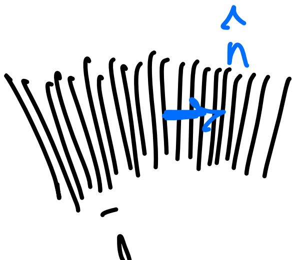
(i.e., use concept of homogenization)

<u>Goal</u>: define ε_{eff}, σ_{eff} & write a mechanical theory to predict the wrinkle patterns.

Step 1. What is ε_{eff} ?

$$\varepsilon_{vK} = e(u) + \tfrac{1}{2}\nabla w \otimes \nabla w - \tfrac{1}{2}\nabla p \otimes \nabla p \approx 0$$

e.g., $\quad w = a(x)\,\ell\,\sin\left(\dfrac{x\cdot\hat{n}(x)}{\ell}\right)$



$$\nabla w = a\cos\left(\frac{x\cdot\hat{n}}{\ell}\right)\hat{n} + O(\ell)$$

$$\nabla w \otimes \nabla w = a^2\cos^2\left(\frac{x\cdot\hat{n}}{\ell}\right)\hat{n}\otimes\hat{n} + O(\ell)$$

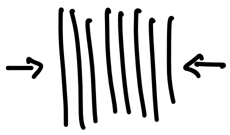
a & $\hat{n}$ are slowly varying w.r.t. ℓ

Key. How to choose u ?

$$u = u_{eff}(x) + u_{wr}\left(x, \frac{x}{\ell}\right)$$

Need to select u_{wr}, a, $\hat{n}$ to solve $\varepsilon_{vK} \approx 0$, i.e.,

$$\underbrace{e(u_{wr}) + \tfrac{1}{2}\nabla w\otimes\nabla w}_{\text{oscillatory}} + \underbrace{e(u_{eff}) - \tfrac{1}{2}\nabla p\otimes\nabla p}_{\text{slowly varying}} \approx 0$$



compression drives wrinkling

It's possible $\iff$

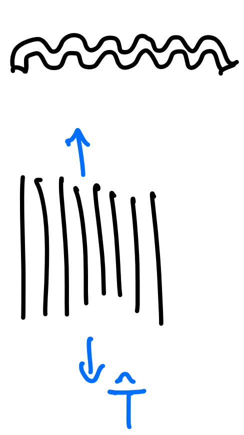
$$\boxed{\varepsilon_{eff} := e(u_{eff}) - \tfrac{1}{2}\nabla p\otimes\nabla p \leq 0}$$

Infinitesimally wrinkled shells cannot extend lengths.

(Nonlinearly, $g_{eff} \leq g_S$.)

<u>Step 2</u>. What is σ_{eff}?

What's preventing ε_{eff} from being > 0?

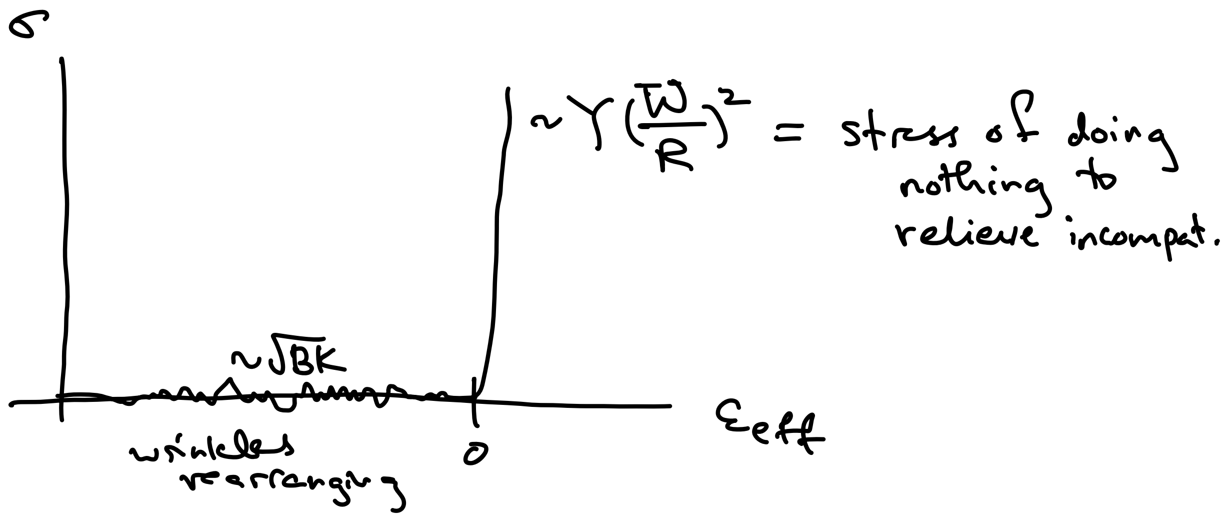


<u>A</u>. $\sigma \sim \sqrt{BK}$ in these wrinkle patterns, so there is a small amount of stress.

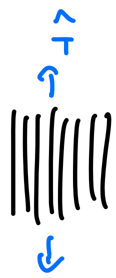
[see B. Davidovitch, Y. Sun, & G. Grason PNAS '19]

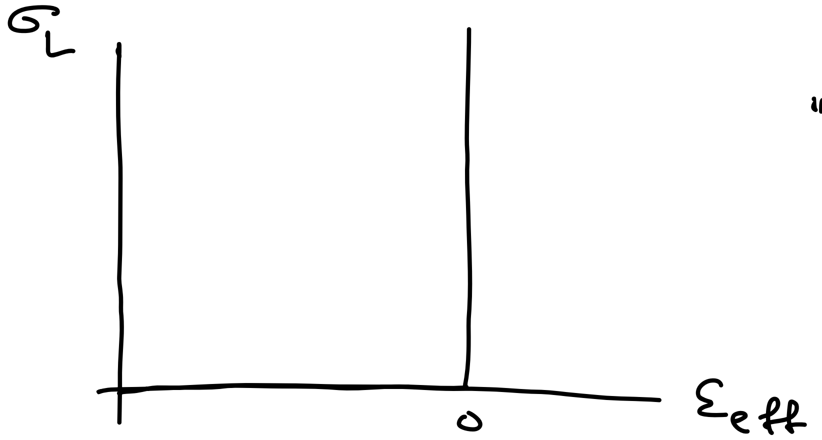
Stretching more $\parallel \hat{T}$ would give

$$E \sim E_{stretch} \gg E_{wrinkling}$$



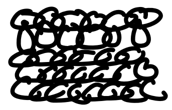
It makes sense to introduce a Lagrange multiplier σ_L to enforce the constraint of no stretching, i.e., no effective extension in the limit.





Prager's "ideal locking material"

like chain mail



Tensorially,

$$\boxed{\nabla \cdot \sigma_L = 0 \;, \quad \sigma_L \geq 0 \;, \quad \langle \sigma_L , \varepsilon_{eff} \rangle = 0}$$

E.g., $\quad \sigma_L = \lambda \, \hat{T} \otimes \hat{T} \;, \quad \lambda \geq 0$

$$\Rightarrow \quad \lambda \, (\varepsilon_{eff})_{\hat{T}\hat{T}} = 0$$

if $\lambda > 0$ then $(\varepsilon_{eff})_{\hat{T}\hat{T}} = 0$

i.e., wrinkles are $\parallel$ to $\hat{T}$.

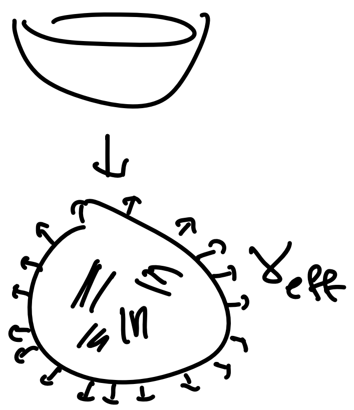
Aha! It's a way of solving for the wrinkle dir'n.

<u>Bonus</u> : $\sigma_L = 0$ permits any $\varepsilon_{eff} \leq 0$, so can have disordered patterns

<u>Step 3</u>. What b.c. to use ?

It is not obvious, since the model ($\min E$) has no tension at ∂S. But, there turns out to be an "effective" surface tension $\gamma_{eff} = 2\sqrt{BK}$!

$$\Rightarrow \quad \sigma_L \hat{n} = \hat{n} \quad \text{at } \partial\Omega .$$



σ_L models $\underset{\approx}{\text{dir'n}}$ $\perp$ wrinkling

so it is dimensionless

This derives from a homogenized formula for the energy of the wrinkles ($E_{bend} + E_{sub}$):

$$E_{wrinkles} = 2\sqrt{BK}\,\Delta A(u_{eff}) + \text{h.o.t.}$$

$$\text{w/} \quad \Delta A(u_{eff}) = A(S) - A(S_{eff})$$

and $$\frac{\delta \Delta A}{\delta u_{eff}} = 0 \quad \Rightarrow \quad \sigma_L \hat{n} = \hat{n} \text{ at } \partial\Omega ,$$

formally.

Conclusion. We can predict wrinkle dir'n by solving

$$
\begin{cases}
\varepsilon_{eff} = e(u_{eff}) - \frac{1}{2}\nabla p \otimes \nabla p \leq 0 & \text{in } \Omega \\
\nabla \cdot \sigma_L = 0 \quad , \quad \sigma_L \geq 0 & \text{in } \Omega \\
\langle \sigma_L , \varepsilon_{eff} \rangle = 0 & \text{in } \Omega \\
\sigma_L \hat{n} = \hat{n} & \text{at } \partial\Omega
\end{cases}
$$

Amazingly, this can be solved <u>exactly</u> if Ω is simply connected & K_G is of one sign (either $K_G(x) \geq 0 \ \forall x$ or $\leq 0 \ \forall x$).

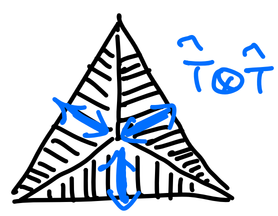
<u>Key</u>. $\sigma_L = \nabla^\perp \nabla^\perp \varphi \qquad \varphi =$ convex Airy potential

For $K_G \leq 0$, $\varphi_- = \frac{1}{2}|x|^2 - \frac{1}{2} d^2_{\partial\Omega}(x)$

distance to $\partial\Omega$

$$\Rightarrow \nabla \varphi_- = x - d_{\partial\Omega}(x) \nabla d_{\partial\Omega}(x) = P_{\partial\Omega}(x)$$

$$\Rightarrow \nabla\nabla\varphi_- = \nabla P_{\partial\Omega} = \lambda \hat{T}^\perp \otimes \hat{T}^\perp$$

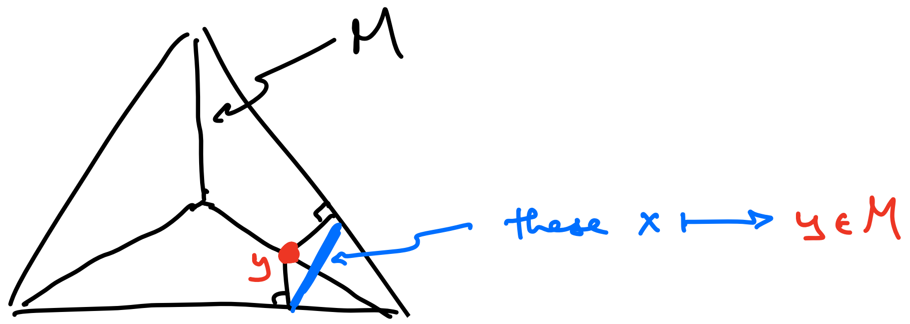


where $\hat{T} \cdot \nabla P_{\partial\Omega} = 0$

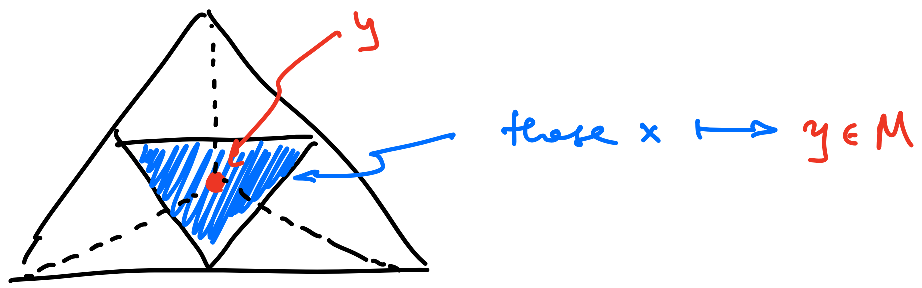
i.e. $\hat{T} \parallel$ level sets of $P_{\partial\Omega}$ for $K_G \leq 0$.

For $K_G \geq 0$, $\quad \varphi_+ \overset{!}{=} \varphi_-^{*}$ = Legendre transform of φ_-

$\Rightarrow \quad \nabla \varphi_+ = P_M \qquad M$ = medial axis of Ω

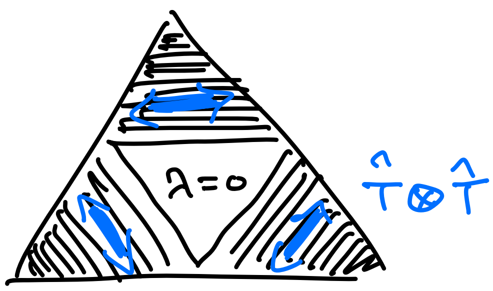


Def'n of P_M



$$\Rightarrow \quad \nabla\nabla \varphi_+ = \nabla P_M = \lambda\, \hat{T}^{\perp} \otimes \hat{T}^{\perp}$$

where $\hat{T} \cdot \nabla P_M = 0$ & $\lambda = 0$ if $\nabla P_M = 0$



See Tobasco et al Nat Phys '22 for the proof of these formulas & expts. + sims.