

Geometric Elasticity in Soft Matter Physics Boulder Summer School

Michael Moshe

Racah Institute of Physics

The Hebrew University of Jerusalem

Edmond J. Safra Campus, Jerusalem 9190401, Israel

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Contents

1. First Lecture	3
1.1. Motivation: Why Geometry	3
1.2. Mathematical Background: Differential Geometry of Surfaces	5
2. Second Lecture	14
2.1. Summary of previous lecture	14
2.2. Dimension reduction	14
2.3. Programming Shape	15
2.4. Geometric elasticity: zero codimension	18
2.5. Curvature-induced stress and the Elastic Munchausen Effect	24
3. Third Lecture: Geometric Charges and Generalized Defects	26
3.1. The cone and the disclination	26
3.2. Classification problem on a punctured disk	27
3.3. Complex-coordinate interpretation	29
3.4. The impossible cone	30
3.5. Local changes of reference metric generate quadrupoles	32
3.6. Physical consequences	32
4. Fourth Lecture: Anomalous Elasticity and Mechanical Screening	34
4.1. Motivation: Mechanical Screening in Holey Metamaterials	34
4.2. Continuum theory of mechanical screening	37
A. Extra Material on Thin Sheets	47
A.1. Non-Euclidean Plates: Thin Sheets and Dimension Reduction	47
A.2. Curvature potentials for non-Euclidean plates	48

1. First Lecture

Learning Objectives

- Why Geometry? Motivating systems and phenomena
- Geometry of surfaces
- Shape Selection and Inverse Design

1.1. Motivation: Why Geometry

One family of questions that motivates geometric elasticity is pattern formation and shape selection in nature. Nature exhibits many patterns, but in this lecture we focus on emergent shapes in thin sheets: plant leaves and petals, cell membranes and self-assembled molecular layers, and synthetic examples such as a torn plastic bag whose edge develops a fractal-like wavy pattern. By the end of this lecture, we will view these shapes as the outcome of a simple

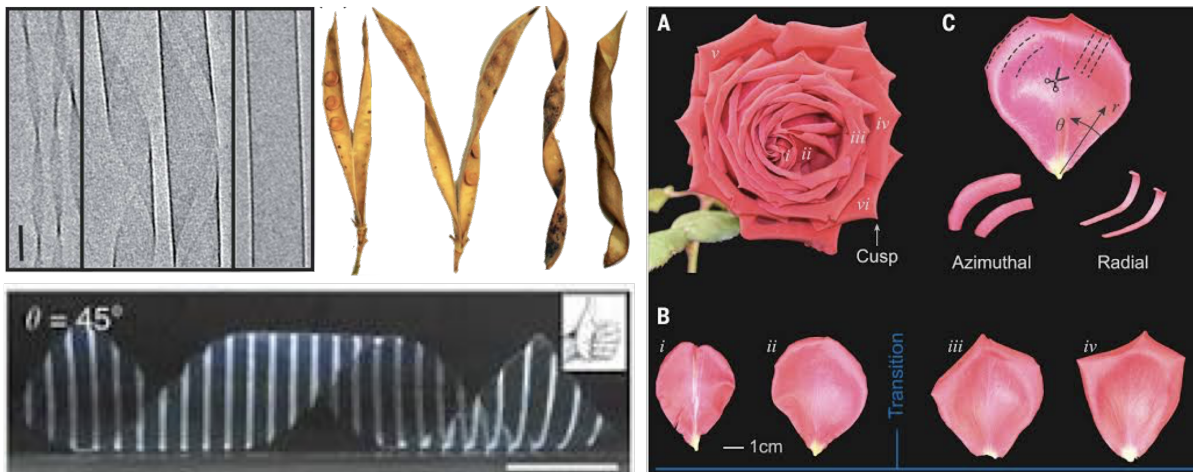


Figure 1.1.: Morphological patterns in natural and synthetic thin sheets: self-assembly, Bauhinia seed pods, responsive gels, and petals.

geometric conflict: the sheet has an intrinsic geometry that cannot be realized perfectly in the surrounding space, so the observed shape is the result of an energetic compromise.

To demonstrate the geometric perspective, consider a bay leaf. As it grows, its edge becomes wavy. This waviness is not simply a preferred three-dimensional shape: if the leaf is sliced into narrow strips, the strips can flatten while revealing a preferred intrinsic geometry with nonzero curvature. The shape is therefore emergent, arising from the way growth changes

local distances within the sheet. To understand such phenomena, we first need a language for quantifying shape. Geometry provides this language: it is the kinematics of surfaces and, more generally, the kinematic framework underlying continuum mechanics.



Figure 1.2.: Comparing narrow strips cut from the edges of flat and wavy leaves (top) and flattened between glass plates (bottom) reveals the difference between their intrinsic geometries. Strips cut from the flat leaf (left) form the expected sequence of arcs, whose radii increase away from the center. In contrast, the outer strips cut from the wavy leaf (right) form smaller arcs than the inner strips. The two leaves were collected from the same bay-leaf bush. Adapted from E. Sharon et al., *Leaves, Flowers and Garbage Bags: Making Waves* (2004).

With the geometric perspective you would be able to provide an explanation for the following:

- When a plastic bag is torn the edge buckles into a wavy, fractal-like pattern.
- When a properly designed three-dimensional printed plastic sheet is placed in hot water, it changes shape according to information programmed during printing. Heating activates a prescribed intrinsic geometry, and the sheet responds by adopting the corresponding three-dimensional form.
- When a strip of cellophane is cut and held in the hand, it can bend and flip back and forth as if it were alive.

Demonstration: Deformed Elastic Sheet and Torn Plastic Bag

Consider a plastic bag marked with a Cartesian array of points indicating Lagrangian coordinates. Distances between nearby material points are measured by a Euclidean reference metric. An elastic deformation changes these distances, which are measured by the actual metric. The strain measures the difference between the reference and actual distances. Tearing induces irreversible changes in the rest distances, which can lead to complex shapes.

But is that all? In the coming lectures, I hope to convince you that the geometric approach does more than explain physical phenomena: it also sheds new light on classical concepts, such as defects, and can predict and reveal new physics.

Before we start, a brief comment on notation is useful. I will often write vectors and tensors using components with upper indices for vectors and lower indices for covectors. A useful motivation for distinguishing vectors from covectors already appears in elementary Newtonian mechanics. A virtual displacement δx is a vector, while a force naturally acts on this displacement to produce the scalar virtual work

$$\delta W = F(\delta x) = F_i \delta x^i.$$

Thus force is intrinsically a covector: a linear map that takes a displacement vector and returns a number. In Cartesian Euclidean coordinates this distinction is often hidden because the metric identifies vectors and covectors, but in curvilinear coordinates or on a curved configuration space the distinction becomes essential. Similarly, stresses, strains, fluxes, and response coefficients are tensors whose meaning is clearest when they are viewed as multilinear maps. This motivates the language of vectors, covectors, tensors, dual spaces, and the Einstein summation convention used below.

1.2. Mathematical Background: Differential Geometry of Surfaces

1.2.1. Notations and definitions

An important property of the geometric approach is that it is naturally formulated intrinsically in a Lagrangian coordinate system. We describe a thin sheet as a manifold M with (Lagrangian) coordinate system $(x^1, x^2) = (u, v)$. This manifold can be embedded in $\mathbb{R}^2$ or $\mathbb{R}^3$.

A **configuration** is a map into either the plane or three-dimensional space:

$$\phi : M \rightarrow \mathbb{R}^2 \quad \text{or} \quad \phi : M \rightarrow \mathbb{R}^3.$$

Explicitly, a planar configuration is $\phi = \{X(u, v), Y(u, v)\}$, whereas a configuration embedded in $\mathbb{R}^3$ is $\phi = \{X(u, v), Y(u, v), Z(u, v)\}$.

Consider a curve $\gamma(s) = (u(s), v(s))$, and its image $\phi(\gamma(s))$. The vector $\dot{\gamma}$ is mapped to $\nabla\phi \cdot \dot{\gamma}$. For example, the tangent vector $(1, 0)$ is mapped to $\partial_u\phi$. Therefore, more generally, the tangent vectors are obtained by differentiating the configuration:

$$e_\alpha = \partial_\alpha\phi.$$

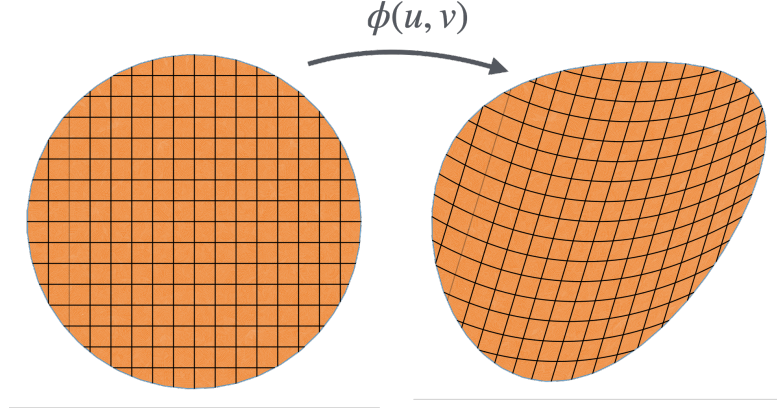


Figure 1.3.: Lagrangian coordinates label material points and move with the body during deformation.

Note that the tangent vectors are generally not normalized. For an embedding in $\mathbb{R}^3$, the corresponding unit normal is

$$n = \frac{e_1 \times e_2}{|e_1 \times e_2|}.$$

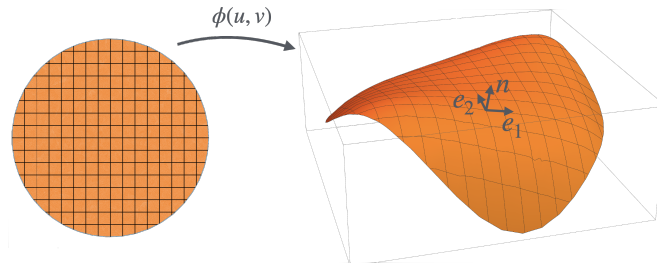


Figure 1.4.: Local surface basis vectors e_1 , e_2 , and the unit normal n .

The intrinsic and extrinsic geometry of the surface is encoded in the values and spatial variations of the triad (e_1, e_2, n) .

1.2.2. Distances: The Metric, or First Fundamental Form

Our goal is to define distances and angles on the deformed configuration. Lengths and angles are naturally expressed through an inner product. We now show that the inner product **on the manifold** can be inherited from the embedding space. Consider two tangent vectors at a given point. Then

$$V \cdot U = V^\alpha U^\beta e_\alpha \cdot e_\beta$$

Upon defining

$$g_{\alpha\beta} = e_\alpha \cdot e_\beta ,$$

we get

$$V \cdot U = V^\alpha U^\beta g_{\alpha\beta}.$$

1. First Lecture

As an inner product, the metric tensor g measures distances and angles.

Example 1.1: The metric tensor in the plane, the cone, and the sphere

$$\phi_{\text{polar}}(r, \theta) = \{r \cos \theta, r \sin \theta\}$$

$$\phi_{\text{cone}}(r, \varphi) = \{r \sin \theta_0 \cos \varphi, r \sin \theta_0 \sin \varphi, r \cos \theta_0\}$$

$$\phi_{\text{sphere}}(\theta, \varphi) = \{R \sin \theta \cos \varphi, R \sin \theta \sin \varphi, R \cos \theta\}$$

At this point, we can already see why the metric tensor is naturally suited to describing elastic materials. The rest metric is commonly denoted by $\bar{g}$, while the actual metric is denoted by g . The strain

$$u = \frac{1}{2}(g - \bar{g})$$

measures the deviation of the actual distances between material elements from their rest values.

Remark: The Dual Space and Linear Transformations revisited

In the motivation section, we briefly discussed the distinction between covectors and vectors. With the notion of the metric tensor in hand, this distinction takes another form. In general, it is useful and convenient to think of vectors and tensors as multilinear maps. The dual space T_p^*M is the space of linear maps from tangent vectors to scalars. The metric provides a natural map from a vector $V = V^\alpha e_\alpha$ to a covector $V^\flat$ by taking an inner product with another vector:

$$V^\flat(U) = V \cdot U = g_{\alpha\beta} V^\alpha U^\beta.$$

Thus the components of the associated covector are

$$V_\beta = g_{\alpha\beta} V^\alpha.$$

In this sense, the metric lowers an index. Conversely, the inverse metric $g^{\alpha\beta}$, defined by $g^{\alpha\mu} g_{\mu\beta} = \delta_\beta^\alpha$, raises an index:

$$V^\alpha = g^{\alpha\beta} V_\beta.$$

The same convention applies to any tensor: contracting an index with $g_{\alpha\beta}$ lowers it, while contracting with $g^{\alpha\beta}$ raises it. For example,

$$T^\alpha{}_\beta = g^{\alpha\mu} T_{\mu\beta}, \quad T_{\alpha\beta} = g_{\alpha\mu} T^\mu{}_\beta.$$

1.2.3. A note on tensors

We have good physical intuition for vectors, but there are several complementary ways to understand them. The multilinear-map viewpoint extends naturally to higher-rank tensors and is therefore worth discussing. A velocity vector, for example, can be understood through its

1. First Lecture

projections: given a unit direction, the projection of the velocity gives the rate of motion in that direction. Similar interpretations apply to forces, fluxes, and other vector quantities.

The stress tensor is also naturally understood in this language. As a linear map, it takes the direction of a material element and returns the force acting on that element. As a bilinear map, it returns the component of that force in one direction acting on an element oriented in another direction.

This viewpoint also clarifies equilibrium. The condition of vanishing total force leads, in a simple coordinate form, to an equation such as $\partial_\alpha \sigma^{\alpha\beta} = 0$. This is not yet the fully general form, because it omits important features associated with Lagrangian coordinates, which we will address later. The essential point is that equilibrium has the geometric form $\text{div } \sigma = 0$, and writing it this way enables substantial analytical progress.

Even before discussing the full physical and mechanical content of continuum mechanics, this example shows why we must understand how to differentiate vectors and tensors.

1.2.4. Differentiation and the Covariant Derivative

Differentiating scalar fields in an arbitrary geometry is straightforward, because it only requires comparing scalar values at nearby points. Differentiating vectors or tensors is more subtle: one must first specify how to compare objects that belong to tangent spaces at different points.

In Euclidean space with Cartesian coordinates, we usually compare vectors by keeping them parallel as we move from one point to another. In curvilinear coordinates, and especially on curved spaces, this intuition must be made precise. The object that provides this rule is the connection. Its meaning becomes clear when we differentiate a vector field.

Consider a vector field written in the local basis as

$$V = V^\alpha e_\alpha.$$

Differentiating this vector field in the coordinate direction β gives two contributions:

$$\partial_\beta V = \partial_\beta (V^\alpha e_\alpha) = (\partial_\beta V^\alpha) e_\alpha + V^\alpha \partial_\beta e_\alpha.$$

The first term measures the spatial variation of the components V^α . The second term measures the spatial variation of the basis vectors e_α themselves.

We therefore define the connection by

$$\partial_\beta e_\alpha = \Gamma_{\alpha\beta}^\mu e_\mu,$$

and obtain

$$\partial_\beta V = \left(\partial_\beta V^\mu + \Gamma_{\alpha\beta}^\mu V^\alpha \right) e_\mu.$$

Thus the covariant derivative of the vector components is

$$\nabla_\beta V^\mu = \partial_\beta V^\mu + \Gamma_{\alpha\beta}^\mu V^\alpha.$$

The correction term involving $\Gamma_{\alpha\beta}^\mu$ accounts for the variation of the basis from point to point. In this way, the covariant derivative gives an intrinsic rule for differentiating vectors in surface coordinates.

Example 1.2: Polar Coordinates on the Plane

Consider the configuration $\phi : M \rightarrow \mathbb{R}^2 \subset \mathbb{R}^3$ given by

$$\phi(r, \theta) = (r \cos \theta, r \sin \theta, 0).$$

The coordinate basis vectors are

$$e_r = \partial_r \phi = (\cos \theta, \sin \theta, 0), \quad e_\theta = \partial_\theta \phi = (-r \sin \theta, r \cos \theta, 0).$$

Differentiating these basis vectors gives

$$\partial_r e_r = 0, \quad \partial_\theta e_r = (-\sin \theta, \cos \theta, 0) = \frac{1}{r} e_\theta,$$

$$\partial_r e_\theta = (-\sin \theta, \cos \theta, 0) = \frac{1}{r} e_\theta, \quad \partial_\theta e_\theta = (-r \cos \theta, -r \sin \theta, 0) = -r e_r.$$

Using

$$\partial_\beta e_\alpha = \Gamma_{\alpha\beta}^\mu e_\mu,$$

we identify the nonzero connection coefficients

$$\Gamma_{r\theta}^\theta = \Gamma_{\theta r}^\theta = \frac{1}{r}, \quad \Gamma_{\theta\theta}^r = -r.$$

Exercise 1.1: Covariant derivatives of covectors and tensors

Use the fact that the covariant derivative of a scalar coincides with its partial derivative to derive covariant-derivative formulas for covectors and higher-rank tensors. One way to do this is to form scalars by contracting tensors with arbitrary vectors; for example, consider $T_{\alpha\beta} U^\alpha V^\beta$.

Exercise 1.2: The Christoffel Symbols

Starting from the definition $\partial_\alpha e_\beta = \Gamma_{\alpha\beta}^\mu e_\mu$, with basis vectors derived from the configuration, show that the connection has two important properties: it is symmetric, $\Gamma_{\alpha\beta}^\mu = \Gamma_{\beta\alpha}^\mu$, and it is metric-compatible, $\nabla_\alpha g_{\beta\gamma} = 0$ (Convince yourself that this implies the conservation of inner product upon parallel transport). Then show that these two properties imply the Christoffel-symbol formula

$$\Gamma_{\alpha\beta}^\mu = \frac{1}{2} g^{\mu\nu} (\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}).$$

Hint: $\partial_\alpha g_{\beta\gamma} = \partial_\alpha (e_\beta \cdot e_\gamma)$.

1.2.5. Weingarten's equations

From now on, when considering a thin sheet in $\mathbb{R}^3$, we denote the metric by $a \equiv g$. The covariant derivative defined here is only the tangential component of the ordinary derivative. When the basis vectors e_α are tangent to a surface embedded in $\mathbb{R}^3$, their spatial derivatives need not remain tangent to the surface. Thus, differentiating a basis vector produces both tangential and normal contributions. Accounting for the normal component, we write

$$\partial_\beta e_\alpha = \Gamma_{\alpha\beta}^\mu e_\mu + b_{\alpha\beta} n, \quad (1.1)$$

where $\Gamma_{\alpha\beta}^\mu$ describes the tangential variation of the basis vectors and defines the intrinsic covariant derivative on the surface. The tensor $b_{\alpha\beta}$ is a bilinear map encoding the normal component of the variation of the tangent basis vectors. By definition, $n \cdot \partial_\alpha e_\beta = b_{\alpha\beta}$. Since $n \cdot e_\alpha = 0$, this relation can also be written as $b_{\alpha\beta} = -e_\alpha \cdot \partial_\beta n$. This expression assigns a simple geometric interpretation to b . The component $b_{\alpha\beta}$ represents the α component of the rate of change of the normal in the β direction. The conclusion is that the equations for $\partial_\alpha e_\beta$ are now supplemented with an equation

$$\partial_\alpha n = -b_\alpha^\beta e_\beta \quad (1.2)$$

Together, these equations form a set of ODEs for the basis $\{e_1, e_2, n\}$.

This raises a solvability question: given the fundamental forms a and b , what are the necessary and sufficient conditions for these PDEs to admit a solution? We discuss these conditions below, after first providing a visual interpretation of the extrinsic curvature.

1.2.6. Geometric interpretation of curvature

The second fundamental form measures how the surface normal rotates as one moves along the surface. If $t = t^\alpha e_\alpha$ is a unit tangent direction, then the directional curvature in that direction is

$$k(t) = b_{\alpha\beta} t^\alpha t^\beta.$$

Equivalently, $k(t)$ is the curvature of the normal section obtained by slicing the surface with the plane spanned by t and n . On a cylinder, for example, the curvature along the axial direction vanishes, while the curvature around the circular direction is nonzero.

The surface normal can also rotate in a direction transverse to the direction of motion. In that case, the curvature appears as a twist. This is illustrated in the figure below, which also shows that the same geometric content can appear as pure curvature in one coordinate system and as twist in another.

The extremal values of $k(t)$ are the principal curvatures k_1 and k_2 . These are the eigenvalues of the shape operator $b^\alpha_\beta = a^{\alpha\gamma} b_{\gamma\beta}$. Two scalar invariants are especially important:

$$K = \det(b^\alpha_\beta) = k_1 k_2, \quad H = \frac{1}{2} \text{tr}(b^\alpha_\beta) = \frac{k_1 + k_2}{2}.$$

The Gaussian curvature K distinguishes qualitatively different local shapes. A spherical cup has $K > 0$, since both principal curvatures have the same sign. A saddle, or Pringle-like shape,

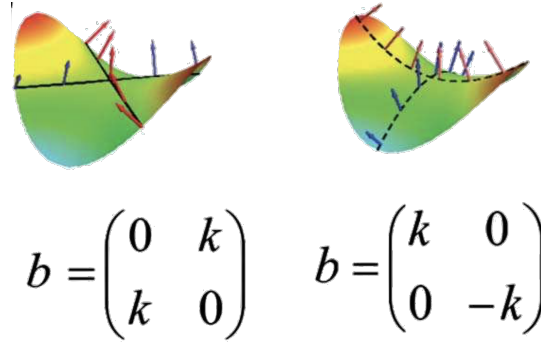


Figure 1.5.: The second fundamental form represented in different coordinate systems.

has $K < 0$, since the principal curvatures have opposite signs. The mean curvature H measures the average bending of the surface and changes sign when the orientation of the normal is reversed.

Remark: Intrinsic curvature

This classification already hints at a relation between intrinsic and extrinsic curvature. Imagine measuring, entirely within the surface, the perimeter $P(r)$ of a geodesic circle of radius r . In the plane, this perimeter grows linearly, $P(r) = 2\pi r$. Linear growth also occurs on a cone, although with a different slope that reflects the angular excess or deficit. On a positively curved surface, such as a sphere of radius R , the perimeter grows more slowly than linearly, with $P(r) = 2\pi R \sin(r/R)$. Thus an intrinsic measurement of distances can detect whether the surface carries Gaussian curvature.

1.2.7. Solvability of Weingarten and Compatibility Conditions

Given the first and second fundamental forms, the Weingarten equations specify the derivatives of the moving frame. They therefore constitute a first-order system of partial differential equations. Is there always a surface realizing these forms? This question reduces to the solvability of the Weingarten equations. We know that this cannot always be true. A configuration in three-dimensional space is described by three degrees of freedom, whereas the first and second fundamental forms contain six independent components. We therefore expect three constraints on the fundamental forms for a surface with these forms to exist. As for any system of PDEs, a solution exists only if mixed partial derivatives commute. The resulting compatibility conditions are precisely the Gauss and Codazzi compatibility equations, which we now derive explicitly.

Gauss's Theorema Egregium and the Mainardi–Codazzi Conditions from Commuting Euclidean Derivatives Partial derivatives commute in $\mathbb{R}^3$; that is, $\partial_2 \partial_1 \mathbf{e}_1 - \partial_1 \partial_2 \mathbf{e}_1 = 0$. From the Gauss–Weingarten equations, the derivatives are decomposed into tangential and normal parts according to

$$\partial_i \mathbf{e}_j = \nabla_i \mathbf{e}_j + b_{ij} \mathbf{n}, \quad (1.3)$$

$$\partial_i \mathbf{n} = -b_i^k \mathbf{e}_k, \quad (1.4)$$

1. First Lecture

where $\nabla_i \mathbf{e}_j = \Gamma_{ij}^k e_k$ is the connection with the Christoffel symbols, b_{ij} is the second fundamental form, and $b_i^k = a^{k\ell} b_{i\ell}$.

Substituting Eq. (1.3) into the commutativity condition and separating tangent and normal components, the tangential component becomes

$$(\nabla_2 \nabla_1 - \nabla_1 \nabla_2) \mathbf{e}_1 = (b_{11} b_2^k - b_{21} b_1^k) \mathbf{e}_k. \quad (1.5)$$

Taking the inner product with $\mathbf{e}_2$ gives

$$\langle (\nabla_2 \nabla_1 - \nabla_1 \nabla_2) \mathbf{e}_1, \mathbf{e}_2 \rangle = b_{11} b_{22} - b_{12}^2. \quad (1.6)$$

Finally, recalling that the Gaussian curvature is

$$K = \det(a^{-1}b), \quad (1.7)$$

we obtain

$$K = \frac{\langle (\nabla_2 \nabla_1 - \nabla_1 \nabla_2) \mathbf{e}_1, \mathbf{e}_2 \rangle}{\det a}. \quad (1.8)$$

This derivation provides a direct geometric interpretation of Gauss' theorem. The ambient Euclidean derivatives necessarily commute, but once each derivative is decomposed into tangential and normal components, the tangential derivatives no longer commute. Their failure to commute is precisely the intrinsic Gaussian curvature, while the normal component describes the extrinsic bending of the surface. The commutativity of the ambient derivatives therefore forces these two descriptions to agree, yielding Gauss' remarkable result that the Gaussian curvature can be computed entirely from the intrinsic metric, despite its apparent dependence on the embedding through the second fundamental form.

An immediate conclusion is that one cannot specify the fundamental forms of a surface arbitrarily, and this Gauss' theorem egregium is a necessary condition for solvability. But not a sufficient one as we still miss two additional constraints. These are obtained from the normal component of the same calculation immediately yields the Mainardi-Codazzi equations. Indeed, substituting the Gauss-Weingarten equations into the commutativity condition

$$\partial_i \partial_j \mathbf{e}_k - \partial_j \partial_i \mathbf{e}_k = 0,$$

the normal component is obtained from

$$\partial_i (\Gamma_{jk}^m \mathbf{e}_m + b_{jk} \mathbf{n}) - \partial_j (\Gamma_{ik}^m \mathbf{e}_m + b_{ik} \mathbf{n}).$$

Collecting the coefficients multiplying $\mathbf{n}$ gives

$$\partial_i b_{jk} + \Gamma_{jk}^m b_{im} - \partial_j b_{ik} - \Gamma_{ik}^m b_{jm} = 0.$$

Recognizing the covariant derivative of the second fundamental form,

$$\nabla_i b_{jk} = \partial_i b_{jk} - \Gamma_{ij}^m b_{mk} - \Gamma_{ik}^m b_{jm},$$

1. First Lecture

and using the symmetry of both the Christoffel symbols ($\Gamma_{ij}^k = \Gamma_{ji}^k$) and the second fundamental form ($b_{ij} = b_{ji}$), the previous equation becomes $\nabla_i b_{jk} = \nabla_j b_{ik}$, or

$$\nabla \times b = 0,$$

which are the Mainardi-Codazzi equations.

Thus, the Gauss and Mainardi-Codazzi equations arise from the same compatibility condition. The tangential projection of $\partial_i \partial_j \mathbf{e}_k = \partial_j \partial_i \mathbf{e}_k$ yields Gauss' equation, while the normal projection yields the Mainardi-Codazzi equations.

Exercise 1.3: Counting the Mainardi-Codazzi Equations

Show that, despite the three indices appearing in this expression, these equations contain only two independent scalar equations.

2. Second Lecture

Learning Objectives

- Shape selection and shape programming
- Geometric elasticity (zero co-dimension)
- The elastic Baron von Munchausen Effect

2.1. Summary of previous lecture

A configuration of a surface in $\mathbb{R}^3$ is a map $\phi : M \rightarrow \mathbb{R}^3$. Its tangent basis vectors and unit normal are

$$e_\alpha = \partial_\alpha \phi, \quad n = \frac{e_1 \times e_2}{|e_1 \times e_2|}. \quad (2.1)$$

The derivatives of this moving frame split into tangential and normal parts:

$$\partial_\beta e_\alpha = \Gamma_{\alpha\beta}^\mu e_\mu + b_{\alpha\beta} n, \quad (2.2)$$

$$\partial_\alpha n = -b_\alpha^\beta e_\beta. \quad (2.3)$$

We denote the tangent component of the tangents' derivatives by $\nabla_\beta e_\alpha \equiv \Gamma_{\alpha\beta}^\mu e_\mu$. These are the Weingarten equations. The first fundamental form $a_{\alpha\beta} = e_\alpha \cdot e_\beta$ measures intrinsic distances, while the second fundamental form $b_{\alpha\beta}$ measures how the normal changes along the surface. Their scalar invariants of b are

$$K_G = \det(a^{-1}b), \quad H = \frac{1}{2} \text{Tr}(a^{-1}b). \quad (2.4)$$

Finally, not every pair (a, b) comes from a surface. Requiring the mixed derivatives of the frame to commute gives the compatibility conditions

$$K_G(a) = \det(a^{-1}b), \quad (2.5)$$

$$\nabla_\alpha b_{\beta\gamma} = \nabla_\beta b_{\alpha\gamma}. \quad (2.6)$$

These are the Gauss and Mainardi–Codazzi equations.

2.2. Dimension reduction

Surfaces can be described as the thin limit of three-dimensional configurations. A standard way to relate the three-dimensional geometry to the geometry of the mid-surface is to use the

2. Second Lecture

thickness coordinate as a normal coordinate. Let $\phi_{\text{ms}}(u, v)$ denote the configuration of the mid-surface, and let $n(u, v)$ be its unit normal. We describe points through the thickness by

$$\phi(u, v, w) = \phi_{\text{ms}}(u, v) + wn(u, v),$$

where w measures signed distance from the mid-surface.

The tangent vectors to this three-dimensional configuration are

$$\partial_\alpha \phi = e_\alpha + w \partial_\alpha n, \quad \partial_w \phi = n,$$

where $e_\alpha = \partial_\alpha \phi_{\text{ms}}$. Using Weingarten's equation,

$$\partial_\alpha n = -b_\alpha^\gamma e_\gamma,$$

we obtain

$$\partial_\alpha \phi = (\delta_\alpha^\gamma - w b_\alpha^\gamma) e_\gamma.$$

The three-dimensional metric g induced by $\phi(u, v, w)$ therefore satisfies

$$g_{\alpha\beta}(u, v, w) = \partial_\alpha \phi \cdot \partial_\beta \phi = a_{\alpha\beta} - 2w b_{\alpha\beta} + w^2 b_\alpha^\gamma b_{\gamma\beta},$$

while

$$g_{\alpha w} = 0, \quad g_{ww} = 1.$$

Thus the first and second fundamental forms appear as the first terms in the Taylor expansion of the three-dimensional metric through the thickness:

$$g_{\alpha\beta}(w) = g_{\alpha\beta}(0) + w \partial_w g_{\alpha\beta}|_{w=0} + O(w^2).$$

In particular,

$$a_{\alpha\beta} = g_{\alpha\beta}(0), \quad b_{\alpha\beta} = -\frac{1}{2} \partial_w g_{\alpha\beta}|_{w=0}.$$

Dimension reduction therefore replaces the full three-dimensional metric near the mid-surface by the pair $(a_{\alpha\beta}, b_{\alpha\beta})$, together with higher-order corrections in the thickness coordinate. This is the geometric origin of theories in which stretching is controlled by the metric and bending is controlled by the second fundamental form. An immediate question is what happens when, for example, the midsurface metric is Euclidean while the second fundamental form is that of a sphere. No configuration realizes such a pair of fundamental forms, so the Weingarten equations are not solvable for these prescribed data.

2.3. Programming Shape

The Gauss and Mainardi-Codazzi equations reveal a remarkable fact: the global shape of a thin sheet can be programmed by prescribing its intrinsic geometry. There are two complementary strategies.

2.3.1. Programming the Metric

Gauss' Theorema Egregium states that the Gaussian curvature is completely determined by the metric. Consequently, if one can control the local distances within a sheet, one can prescribe its Gaussian curvature and therefore a central property of its global shape.

A particularly elegant realization of this idea is provided by responsive polymer gels. The local swelling ratio of N-isopropylacrylamide (NIPA) gels is determined by the local cross-linking density, which can be spatially controlled during fabrication using patterned ultraviolet illumination. Upon swelling, different regions of the sheet expand by different amounts, thereby prescribing a reference metric. For isotropic swelling, the reference metric takes the conformal form

$$\bar{a} = e^{2\varphi(x,y)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The corresponding reference Gaussian curvature is

$$\bar{K} = -e^{-2\varphi} \Delta \varphi. \quad (2.7)$$

Thus, designing a target Gaussian curvature amounts to solving for the conformal factor $\varphi(x, y)$ that satisfies this relation. The subsequent elastic relaxation determines the three-dimensional configuration of the sheet. This principle underlies numerous shape-programming techniques based on differential swelling, growth and thermal expansion.

Exercise 2.1: Metric Activation for a Spherical Cap

Calculate the Gaussian curvature and confirm Eq.2.7. Find a conformal factor φ for which the activated metric has constant positive Gaussian curvature $\bar{K} = 1/R^2$, corresponding locally to a spherical cap of radius R .

Exercise 2.2: Circular Nematic Pattern

Consider a thin nematic elastomer sheet polymerized in the nematic phase with a prescribed director field. Locally, if the director is along the coordinate direction u , the metric before activation is

$$ds^2 = du^2 + dv^2,$$

whereas heating to the isotropic phase changes the local metric to

$$ds^2 = \lambda^2 du^2 + \lambda^{-2\nu} dv^2.$$

Assume that the director field is circular, namely tangent to circles around the origin. In polar coordinates this means that the director is along the azimuthal direction. Show that the activated metric becomes

$$ds^2 = \lambda^{-2\nu} dr^2 + \lambda^2 r^2 d\theta^2.$$

Introduce the geodesic radial coordinate $\rho = \lambda^{-\nu} r$, and show that

$$ds^2 = d\rho^2 + \lambda^{2(1+\nu)} \rho^2 d\theta^2.$$

What shape is induced when $\lambda < 1$? Interpret your answer by comparing the circumference of a circle to its geodesic radius. What would change if the director were radial rather than circular?

2.3.2. Programming the Curvature Tensor

A more complete approach is to prescribe not only the metric, but also the second fundamental form. This is achieved naturally in bilayer structures.

A particularly versatile realization employs fused-filament 3D printing. During extrusion, polymer chains become aligned along the printing direction. After cooling, this alignment is frozen. When the printed sheet is subsequently heated above the glass transition temperature, the printed layer contracts primarily along the printing direction, whereas the transverse deformation is much weaker. By combining two such active layers, one obtains a bilayer with a prescribed spontaneous curvature determined by the mismatch between the two metrics associated with each layer.

Such a mechanism already appears in nature in the seed pods of *Bauhinia variegata*. Each pod valve is a bilayer made of fibrous tissue. The fibers in the two layers are oriented approximately perpendicular to one another, and upon dehydration each layer shrinks primarily in the direction transverse to its fibers. Thus the two layers would like to realize different anisotropic in-plane metrics. In coordinates aligned with these two preferred directions, a simple linearized description is

$$g_{\text{up}} = \begin{pmatrix} 1 - \alpha & 0 \\ 0 & 1 + \alpha \end{pmatrix}, \quad g_{\text{down}} = \begin{pmatrix} 1 + \alpha & 0 \\ 0 & 1 - \alpha \end{pmatrix}.$$

The average metric of the bilayer is therefore approximately Euclidean,

$$\bar{a} = \frac{1}{2} (g_{\text{up}} + g_{\text{down}}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so the sheet has no preferred Gaussian curvature at the level of its mid-surface metric. However, the difference between the two layer metrics produces a through-thickness gradient of strain. One principal direction prefers to bend with one sign, while the perpendicular direction prefers to bend with the opposite sign. The resulting reference second fundamental form is therefore hyperbolic,

$$\bar{b} = \begin{pmatrix} +k_0 & 0 \\ 0 & -k_0 \end{pmatrix},$$

up to the choice of normal orientation. This explains the shapes of the dehydrated *Bauhinia* seed pods. As we will see later, thin sheets prefer to satisfy the metric and compromise on the curvature. Therefore, the seed pod remains approximately Euclidean by adopting a nearly cylindrical shape: it curves in one direction while paying bending energy because it cannot satisfy the preferred curvature in the second principal direction.

2.4. Geometric elasticity: zero codimension

Baron von Munchausen was a fictionalized German nobleman famous for telling exaggerated and impossible stories about his adventures. In one story, he pulls himself out of a swamp by his own hair. In ordinary Euclidean mechanics, this is impossible because internal forces cannot change the total momentum of the center of mass. The image below, however, shows a larval insect moving away from the center of a water tank using internal forces alone. This is the capillary “Munchausen effect.” The term is inspired by Avron and Kenneth’s *Swimming in Curved Space or the Baron and the Cat*, which describes how cyclic motions can produce propulsion in curved spaces. Here we discuss the elastic Munchausen effect and develop a geometric theory of two-dimensional elastic materials embedded in a curved two-dimensional space.



Figure 2.1.: The elastic Munchausen effect.

In this second part of the lecture, we develop the elasticity of thin sheets embedded in $\mathbb{R}^3$, re-examine geometric frustration, and explain the shapes of rose petals and the dynamics of frustrated self-assembled macromolecules.

We begin with the simplest setting: a two-dimensional elastic body embedded in the Euclidean plane. In the notation of the previous lecture, this is a surface with vanishing second fundamental form, $b = 0$, whose metric is induced by the configuration $\phi(u, v) = (X(u, v), Y(u, v), 0)$. The results of this lecture extend directly to three-dimensional elastic solids, so in this section we use the more general notation g for the actual metric.

2.4.1. The strain

The basic assumption of elasticity is that the fundamental degrees of freedom are the positions of material elements. Elastic deformation is measured by the deviation of the actual distances between material elements from their rest values. The actual distances are quantified by the metric induced by the configuration,

$$d\ell^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \quad (2.8)$$

2. Second Lecture

while the stress-free or reference distances are encoded by the reference metric

$$d\bar{\ell}^2 = \bar{g}_{\alpha\beta} dx^\alpha dx^\beta. \quad (2.9)$$

A natural measure of strain is therefore

$$u_{\alpha\beta} = \frac{1}{2} (g_{\alpha\beta} - \bar{g}_{\alpha\beta}), \quad (2.10)$$

so that

$$d\ell^2 - d\bar{\ell}^2 = 2u_{\alpha\beta} dx^\alpha dx^\beta. \quad (2.11)$$

Exercise 2.3: Linearized Strain from the Metric

Assume a two-dimensional solid with rest configuration $\phi_0(u, v) = (u, v)$, and assume it is deformed so that every point is displaced according to

$$\phi(u, v) = (u, v) + (d_1(u, v), d_2(u, v)).$$

Calculate the rest metric, the actual metric, the strain, and the linearized strain. The expressions you find are the classical definitions for stress-free materials described in Lagrangian coordinates. The geometric formulation allows us to move beyond these assumptions and describe materials in a displacement-free formulation, in arbitrary coordinate systems, and potentially without a rest configuration.

2.4.2. Geometric nonlinearity

The essential difference between the linear and nonlinear theories is geometric nonlinearity: the metric is a nonlinear function of the configuration. Let φ_0 be a stress-free configuration and write

$$\varphi = \varphi_0 + d. \quad (2.12)$$

Then

$$g_{\alpha\beta} = \partial_\alpha \varphi \cdot \partial_\beta \varphi \quad (2.13)$$

$$= \partial_\alpha \varphi_0 \cdot \partial_\beta \varphi_0 + \partial_\alpha \varphi_0 \cdot \partial_\beta d + \partial_\alpha d \cdot \partial_\beta \varphi_0 + \partial_\alpha d \cdot \partial_\beta d. \quad (2.14)$$

Since

$$\bar{g}_{\alpha\beta} = \partial_\alpha \varphi_0 \cdot \partial_\beta \varphi_0, \quad (2.15)$$

the strain is

$$u_{\alpha\beta} = \frac{1}{2} (\partial_\alpha \varphi_0 \cdot \partial_\beta d + \partial_\beta \varphi_0 \cdot \partial_\alpha d + \partial_\alpha d \cdot \partial_\beta d). \quad (2.16)$$

The last term is the nonlinear contribution that survives even when the displacement gradients are small but not infinitesimal. The geometric approach naturally accounts for this nonlinearity. For $\varphi_0(u, v) = (u, v)$ we find

$$u_{\alpha\beta} = \frac{1}{2} (\partial_\beta d_\alpha + \partial_\alpha d_\beta + \partial_\alpha d \cdot \partial_\beta d). \quad (2.17)$$

2.4.3. Hyperelasticity and equilibrium

For a hyperelastic material one writes the energy as

$$E[\varphi] = \int_{\Omega} W(g, \bar{g}) dV_{\bar{g}} = \int_{\Omega} W(g, \bar{g}) \sqrt{|\bar{g}|} d^n x, \quad (2.18)$$

where

$$g_{\alpha\beta} = \partial_{\alpha}\varphi \cdot \partial_{\beta}\varphi, \quad (2.19)$$

and the energy density is positive definite with $W(g, \bar{g}) = 0$ iff $g = \bar{g}$.

Exercise 2.4: Small-Strain Equivalence of Length and Squared-Length Measures

For a one-dimensional elastic element with actual length ℓ and rest length ℓ_0 , one may define a scalar strain proportional to $1 - \ell/\ell_0$ and write an elastic energy of the form

$$E_0 = \frac{1}{2} \kappa_0 (\ell - \ell_0)^2.$$

The metric-based strain used above compares squared lengths, suggesting instead an energy of the form

$$E_1 = \frac{1}{2} \kappa_1 (\ell^2 - \ell_0^2)^2.$$

Show that for small strains, $|\ell - \ell_0| \ll \ell_0$, these two energies are equivalent up to a rescaling of the elastic constant. What relation between κ_0 and κ_1 makes their leading-order terms identical?

Define the contravariant stress by

$$\sigma^{\alpha\beta} := 2 \frac{\partial W}{\partial g_{\alpha\beta}}. \quad (2.20)$$

Then

$$\delta g_{\alpha\beta} = \partial_{\alpha}\varphi \cdot \partial_{\beta}\delta\varphi + \partial_{\beta}\varphi \cdot \partial_{\alpha}\delta\varphi. \quad (2.21)$$

Hence

$$\delta_{\varphi} E = \int_{\Omega} \frac{1}{2} \sigma^{\alpha\beta} \delta g_{\alpha\beta} \sqrt{|\bar{g}|} d^n x \quad (2.22)$$

$$= \int_{\Omega} \sigma^{\alpha\beta} \partial_{\alpha}\varphi \cdot \partial_{\beta}\delta\varphi \sqrt{|\bar{g}|} d^n x. \quad (2.23)$$

Integrating by parts gives, up to boundary terms,

$$\delta_{\varphi} E = - \int_{\Omega} \bar{\nabla}_{\beta} \left(\sigma^{\alpha\beta} \partial_{\alpha}\varphi \right) \cdot \delta\varphi dV_{\bar{g}}. \quad (2.24)$$

Here $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$. In zero codimension the second derivatives of the configuration can be expanded in the tangent basis,

$$\partial_{\alpha}\partial_{\beta}\varphi = \Gamma_{\alpha\beta}^{\mu}(g) \partial_{\mu}\varphi. \quad (2.25)$$

2. Second Lecture

Therefore the bulk equilibrium equation may be written as

$$\partial_\beta \sigma^{\alpha\beta} + \bar{\Gamma}_{\mu\beta}^\mu \sigma^{\alpha\beta} + \Gamma_{\mu\beta}^\alpha \sigma^{\mu\beta} = 0. \quad (2.26)$$

Equivalently,

$$\nabla_\beta^{(g)} \sigma^{\alpha\beta} + \left(\bar{\Gamma}_{\mu\beta}^\mu - \Gamma_{\mu\beta}^\mu \right) \sigma^{\alpha\beta} = 0, \quad (2.27)$$

or

$$\bar{\nabla}_\beta \sigma^{\alpha\beta} + \left(\Gamma_{\mu\beta}^\alpha - \bar{\Gamma}_{\mu\beta}^\alpha \right) \sigma^{\mu\beta} = 0, \quad (2.28)$$

This is sometimes a strange-looking form of force balance because it involves both the actual metric g and the reference metric $\bar{g}$. We denote it schematically by

$$\text{Div}_{\bar{g},g} \sigma = 0. \quad (2.29)$$

This covariant equilibrium equation was derived by Efrati et al.¹.

The linear case is recovered when the reference metric is Euclidean, the coordinates are Cartesian, and the strains are small, yielding $\partial_\beta \sigma^{\alpha\beta} = 0$.

2.4.4. Hookean Elasticity

The equilibrium equation derived above followed from variational principles and geometry, but it did not require specifying a particular elastic energy density. To obtain a closed stress formulation we must now choose a constitutive law. For small strains about a stress-free equilibrium, the energy density can be expanded in powers of the strain. The linear term vanishes at equilibrium, so the leading contribution is quadratic:

$$W(u) = \frac{1}{2} A^{\alpha\beta\gamma\delta} u_{\alpha\beta} u_{\gamma\delta} + O(u^3). \quad (2.30)$$

Here $A^{\alpha\beta\gamma\delta}$ is the elastic tensor. Differentiating the energy with respect to the strain gives the linear stress–strain relation

$$\sigma^{\alpha\beta} = A^{\alpha\beta\gamma\delta} u_{\gamma\delta}. \quad (2.31)$$

When the elastic tensor is nondegenerate on the space of symmetric strains, it can be inverted. We denote the inverse tensor by $A_{\alpha\beta\gamma\delta}$, so that the strain can equivalently be written in terms of the stress as

$$u_{\alpha\beta} = A_{\alpha\beta\gamma\delta} \sigma^{\gamma\delta}. \quad (2.32)$$

This inverse form is the one used below when the compatibility equation is rewritten as an equation for the stress.

¹E. Efrati, E. Sharon, and R. Kupferman, “Elastic theory of unconstrained non-Euclidean plates,” *Journal of the Mechanics and Physics of Solids* **57** (2009), 762–775.

2.4.5. Compatibility and Incompatibility

For a simply connected two-dimensional manifold, the first and second fundamental forms satisfy the Gauss and Mainardi–Codazzi equations. Since the second fundamental form vanishes, the metric tensor satisfies

$$K_G(g) = 0 . \quad (2.33)$$

A stress-free configuration corresponds to $g = \bar{g}$, and such a configuration exists if and only if

$$K_G(\bar{g}) = 0 . \quad (2.34)$$

Remark: Multiply Connected Domains

The condition $K_G(\bar{g}) = 0$ is sufficient locally, and on simply connected domains it is the relevant global compatibility condition. On multiply connected domains there can be additional global conditions. For example, on an annulus the metric

$$d\bar{\ell}^2 = dr^2 + (r_0 + (r - r_0)\kappa)^2 d\theta^2$$

with $\kappa = 1 + p \cos \theta$, has zero Gaussian curvature, and its total curvature over the annulus is also zero. Nevertheless, for $p \neq 0$ it cannot be embedded isometrically in the Euclidean plane as a single-valued planar annulus. We will discuss this later in the context of geometric defects.

The reference metric $\bar{g}$ is prescribed by the material, and therefore it need not satisfy the planar compatibility condition. If its Gaussian curvature is nonzero, $K_G(\bar{g}) \neq 0$, then no stress-free embedding into the Euclidean plane exists: any planar configuration must have an actual metric g with $K_G(g) = 0$, and therefore cannot coincide with $\bar{g}$. This makes the reference Gaussian curvature a natural local measure of incompatibility, in the same spirit as the classical notion of incompatibility in crystalline solids.

To connect this geometric statement with linear elasticity, write the actual metric as

$$g = \bar{g} + 2u.$$

The compatibility condition is then

$$K_G(\bar{g} + 2u) = 0.$$

For small strains this linearizes to

$$K_G(\bar{g}) + \nabla \times \nabla \times u = 0 . \quad (2.35)$$

In classical linear elasticity, compatibility of a strain field with a planar displacement is expressed by the vanishing of the double curl of the strain. The term $K_G(\bar{g})$ therefore plays the role of a source of incompatibility. Since defects are classical sources of incompatibility, this motivates interpreting the reference Gaussian curvature as a geometric defect density, or more generally as a source for elastic fields.

2.4.6. Airy stress function

In the linear planar theory, the equations of equilibrium consist of the force-balance equation for the stress together with the linearized compatibility condition for the strain. In Cartesian coordinates these are

$$\partial_\beta \sigma^{\alpha\beta} = 0, \quad (2.36)$$

and

$$\epsilon^{\alpha\gamma} \epsilon^{\beta\delta} \partial_\alpha \partial_\beta u_{\gamma\delta} = 0. \quad (2.37)$$

The first equation is solved identically by introducing the Airy stress function χ , defined by

$$\sigma^{\alpha\beta} = \epsilon^{\alpha\mu} \epsilon^{\beta\nu} \partial_\mu \partial_\nu \chi. \quad (2.38)$$

Thus the stress is represented in terms of a single scalar potential. The constitutive law expresses the strain in terms of the stress, $u_{\alpha\beta} = A_{\alpha\beta\mu\nu} \sigma^{\mu\nu}$. Substituting the Airy representation into the compatibility condition gives, for a homogeneous isotropic elastic solid,

$$\Delta^2 \chi = 0, \quad (2.39)$$

which is the biharmonic equation. This reduction of planar elasticity to a biharmonic equation is extremely useful: it turns the coupled equilibrium and compatibility equations into a single scalar PDE. Many classical problems can then be solved with powerful methods from complex analysis. For example, cracks may be treated by conformal maps in finite geometries, or as sharp-corner singularities in the semi-infinite limit.

At the same time, the construction has a limited domain of validity. The Airy stress function solves the linearized force-balance equation, $\partial_\beta \sigma^{\alpha\beta} = 0$, and the biharmonic equation follows only after substituting this linear stress representation into the linearized compatibility condition. It is therefore not obvious how to extend the same potential formulation to geometrically nonlinear elasticity, or to materials whose reference metric is intrinsically non-Euclidean.

2.4.7. Generalized Airy Stress Function

The equilibrium equation

$$\text{Div}_{\bar{g},g} \sigma = 0, \quad (2.40)$$

can be interpreted as an equation for the configuration, in which case the system is closed. Alternatively, it can be interpreted as an equation for the stress, strain, or metric, but then a compatibility condition must be imposed separately. In two dimensions, the actual metric must obey $K_G(g) = 0$. Interestingly, but not surprisingly, an exact representation of the stress in terms of a potential still applies:

$$\sigma^{\alpha\beta} = \frac{1}{\sqrt{|g|}\sqrt{|\bar{g}|}} \epsilon^{\alpha\mu} \epsilon^{\beta\nu} \nabla_{\mu\nu} \chi. \quad (2.41)$$

This result enables the calculation of geometrically nonlinear elastic fields. Such nonlinearities are not merely incremental corrections; they can form the basis of elastic instabilities. For now, we assume small strains and use the leading-order expression for the stress,

$$\sigma^{\alpha\beta} = \frac{1}{|\bar{g}|} \epsilon^{\alpha\mu} \epsilon^{\beta\nu} \bar{\nabla}_{\mu\nu} \chi. \quad (2.42)$$

2. Second Lecture

Now consider a thin elastic sheet with reference metric $\bar{g}$ constrained to a surface of constant Gaussian curvature K . The equation of motion and the potential representation still hold, but the compatibility condition becomes $K_G(\bar{g} + 2A^{-1}\sigma) = K$. Upon writing the Gaussian curvature explicitly we find

$$\frac{1}{Y}\Delta^2\chi = \bar{K}_G - K_G. \quad (2.43)$$

Exercise 2.5: Pressurized Annulus and Isotropic Inclusion

Consider a planar annulus with inner radius a and outer radius b , where $a \ll b$. A uniform pressure p is applied on the inner boundary $r = a$, while the outer boundary is traction-free. Solve the elastic problem using the Airy stress function. Then consider a circular domain in which a small disk of radius a at the center is isotropically inflated by a factor η . What is the corresponding reference metric? Solve Eq. 2.43 and show the duality between local pressure and isotropic inflation.

Exercise 2.6: Inverse Design by Internal Forces

The important conclusion of the pressure-inflation duality is that internal forces can effectively modify the reference metric. Suppose a desired reference state is prescribed, either as a reference metric $\bar{g}_{\alpha\beta}^{\text{des}}$ or as a reference Gaussian curvature $\bar{K}_G^{\text{des}}$. What internal force field f should be applied in order to generate this reference state effectively? Assume the equilibrium equation is $\bar{\nabla}_\beta \sigma^{\alpha\beta} = f^\alpha$ and impose constraints as you wish to recover the equilibrium equation

$$\frac{1}{Y}\Delta^2\chi = \bar{K}_G[f] \quad (2.44)$$

2.5. Curvature-induced stress and the Elastic Munchausen Effect

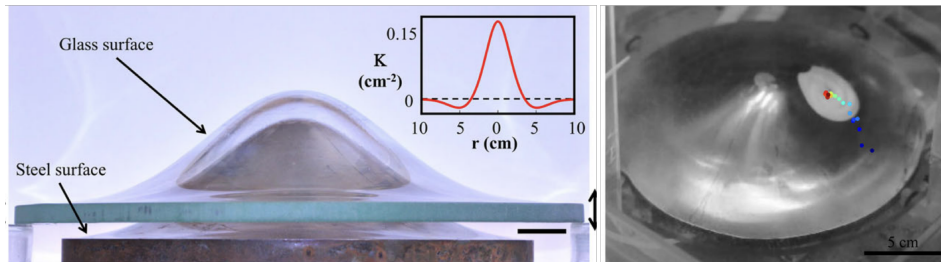


Figure 2.2.: Experimental demonstration of the migration of a two-dimensional frustrated gel in a nonuniformly curved space.

Consider a two-dimensional elastic body constrained to a surface of spatially varying Gaussian curvature K_0 . If the disk is small compared with the length scale over which K_0 varies, its stressed state is approximated by the result from the previous section:

$$\Delta^2\chi = Y\Delta K, \quad (2.45)$$

2. Second Lecture

where $\Delta K = \bar{K} - K_0$. For an axisymmetric solution,

$$\Delta \chi = \frac{1}{r} \frac{d}{dr} \left(r \frac{d\chi}{dr} \right), \quad (2.46)$$

and the biharmonic equation becomes

$$\frac{1}{r} \frac{d}{dr} \left[r \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} \left(r \frac{d\chi}{dr} \right) \right) \right] = Y \Delta K. \quad (2.47)$$

Integrating once gives

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d\chi}{dr} \right) = \frac{1}{4} Y \Delta K r^2 + C_1 \log r + C_2. \quad (2.48)$$

A general radial solution is therefore

$$\chi(r) = \frac{Y \Delta K}{64} r^4 + C_0 + C_1 \log r + C_2 r^2 \log r + C_3 r^2. \quad (2.49)$$

The constants are fixed by regularity, by the absence or presence of singular sources, and by the boundary conditions.

The stretching energy scales as

$$E \sim \int \frac{1}{Y} \sigma^2 r \, dr \, d\theta \quad (2.50)$$

$$\sim \frac{Y \Delta K^2}{16} \int r^5 \, dr \, d\theta \quad (2.51)$$

$$\sim 2\pi \frac{Y \Delta K^2}{96} R^6. \quad (2.52)$$

Upon accounting for the spatial variation of the ambient curvature, this produces an effective force

$$F \sim -C Y R^6 (K - \bar{K}) \nabla K. \quad (2.53)$$

This appears paradoxical only if one expects internal forces to generate no net force. The resolution is that the usual center-of-mass argument is Euclidean: it relies on translational symmetry. On a curved substrate there is no global translational symmetry and no unique Euclidean center of mass for an extended deformable body.

For the capillary version of the Munchausen effect, see “Surfing locomotion of elastic sheets driven by curvature mismatch,” PRL (2025).

3. Third Lecture: From Defects to Geometric Charges—Generalized Defects in Disordered Solids

Defects are central to the materials science of crystalline solids and are key to understanding plasticity, melting, and related phenomena. Their definition relies on the existence of an underlying order, which gives them a topological character. Their mechanical relevance, however, comes not only from their structural content but also from the stress fields they induce. For example, a 5–7 dislocation in a planar triangular lattice induces an elastic energy that diverges logarithmically with system size.

In crystalline materials there is a direct link between defects and their geometric constructions. For example, a five-fold or seven-fold disclination in a triangular lattice may be constructed by removing or inserting an angular sector. When such a construction is applied to a thin elastic sheet, the sheet may deform into a cone or an excess cone (e-cone). The five-fold defect corresponds to removing a sector of angle $\pi/3$, while the seven-fold defect corresponds to an excess angle of the same magnitude. This construction immediately shows an asymmetry: positive disclinations, interpreted as angular deficits, are bounded above by 2π , whereas negative disclinations can have arbitrarily large angular excess.

The three-dimensional configuration attempts to realize the reference metric $\bar{a}$. If a defective thin sheet is forced to remain planar, this reference metric generally cannot be realized in the plane, and residual stresses develop. For a non-Euclidean reference metric, the reference Gaussian curvature acts as a source of stress. In the linear Airy formulation this appears, schematically, as

$$\frac{1}{Y}\Delta^2\chi \sim \bar{K}. \quad (3.1)$$

Away from the defect core, however, any small piece of the material may be stress-free when removed from the body. The stress is generated by the singularity of the reference geometry at the defect. Thus the familiar Volterra construction can be reinterpreted geometrically: a defect is a singular source of reference Gaussian curvature.

3.1. The cone and the disclination

The Volterra construction provides a direct geometric description of a disclination. Begin with a Euclidean disk whose line element is

$$d\ell_0^2 = dr^2 + r^2 d\theta^2. \quad (3.2)$$

Remove a sector of opening angle $\Delta\theta$ and identify its exposed edges. Before identification, the angular coordinate spans an interval of length $2\pi - \Delta\theta$. To restore a 2π -periodic coordinate, define

$$\varphi = \frac{\theta}{\alpha}, \quad \alpha = 1 - \frac{\Delta\theta}{2\pi}. \quad (3.3)$$

The reference metric of the glued disk is therefore

$$d\bar{\ell}^2 = dr^2 + \alpha^2 r^2 d\varphi^2. \quad (3.4)$$

Although this metric is locally Euclidean away from the origin, a circle of radius r has circumference $2\pi\alpha r$. The angular deficit is thus encoded intrinsically in the metric and concentrated at the apex. This reference metric exposes the asymmetry between the possible magnitudes of positive and negative disclinations, with $0 < \alpha < \infty$.

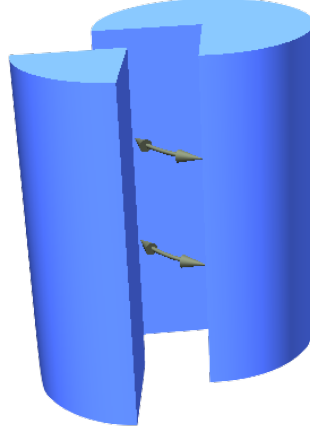


Figure 3.1.: Continuous geometric representation of a disclination.

This motivates the following working hypothesis. Singular sources of stress are singularities of the reference Gaussian curvature. Defects are particular realizations of this more general geometric notion.

3.2. Classification problem on a punctured disk

We therefore ask: what are all reference metrics $\bar{g}$ that are Euclidean everywhere except possibly at one point? Consider a punctured disk

$$D^* = D \setminus \{0\}. \quad (3.5)$$

By the two-dimensional uniformization theorem, there is a local coordinate system in which the metric takes the conformal form

$$\bar{g} = e^{2\varphi(x,y)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.6)$$

Equivalently, in polar coordinates,

$$\bar{g}_{\text{pol}} = e^{2\varphi(r,\theta)} \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \quad (3.7)$$

3. Third Lecture: Geometric Charges and Generalized Defects

The Gaussian curvature of this metric is

$$\bar{K} = -e^{-2\varphi} \Delta \varphi. \quad (3.8)$$

Thus flatness away from the puncture requires

$$\Delta \varphi = 0 \quad \text{on } D^*. \quad (3.9)$$

The singular part of a general harmonic function on a punctured disk has the multipole expansion

$$\varphi(r, \theta) = a \log r + \sum_{n=1}^{\infty} r^{-n} (b_n \cos n\theta + c_n \sin n\theta) + \text{regular terms}. \quad (3.10)$$

The logarithmic term is the monopole. The terms

$$\frac{\cos \theta}{r}, \quad \frac{\sin \theta}{r}, \quad (3.11)$$

are dipoles, and the terms

$$\frac{\cos 2\theta}{r^2}, \quad \frac{\sin 2\theta}{r^2}, \quad (3.12)$$

are quadrupoles.

The invariant delta function with respect to the metric area element satisfies

$$\delta_g^{(2)}(x) dA_g = \delta^{(2)}(x) dx dy. \quad (3.13)$$

Since

$$dA_g = e^{2\varphi} dx dy, \quad (3.14)$$

we have

$$\delta_g^{(2)}(x) = e^{-2\varphi} \delta^{(2)}(x). \quad (3.15)$$

Thus the conformal factor converts harmonic singularities into geometric curvature charges.

The solution in Eq. 3.10 implies a hierarchy of geometric charges and the corresponding multipole expansion

$$\bar{K} = M \delta^{(2)}(x) + P^\alpha \partial_\alpha \delta^{(2)}(x) + Q^{\alpha\beta} \partial_\alpha \partial_\beta \delta^{(2)}(x) + \dots. \quad (3.16)$$

For continuous charge distributions, the singular coefficients are replaced by smooth density fields:

$$\bar{K} = M + \partial_\alpha P^\alpha + \partial_\alpha \partial_\beta Q^{\alpha\beta} + \dots. \quad (3.17)$$

Exercise 3.1: Distributed Dislocations

Show that a rectangular domain with a Cartesian metric and a uniform distribution of geometric dipoles is equivalent to a polar domain.

3.3. Complex-coordinate interpretation

To interpret the different multipolar geometric charges, we seek a local isometric embedding of the metric in the plane. The conformal representation provides a convenient way to construct this embedding. Write

$$z = x + iy, \quad dz d\bar{z} = dx^2 + dy^2. \quad (3.18)$$

If

$$\varphi(z, \bar{z}) = \operatorname{Re} f(z) \quad (3.19)$$

for a holomorphic function f on the punctured disk, then

$$d\bar{\ell}^2 = e^{2\varphi(x,y)} (dx^2 + dy^2) = e^{2\varphi} dz d\bar{z} = e^{f(z)} e^{\overline{f(z)}} dz d\bar{z}. \quad (3.20)$$

Defining a new complex coordinate Z by

$$dZ = e^{f(z)} dz, \quad (3.21)$$

we obtain

$$d\bar{\ell}^2 = dZ d\bar{Z} = dX^2 + dY^2. \quad (3.22)$$

Thus the metric is Euclidean, and the configuration is then (X, Y) . It therefore remains to express these new variables in terms of the (x, y) or (r, θ) variables. The defect is encoded in the fact that $Z(z)$ may be multi-valued, singular, or require a branch cut.

For example,

$$\log r = \operatorname{Re} \log z. \quad (3.23)$$

Taking

$$f(z) = q \log z \quad (3.24)$$

leads to

$$dZ = z^q dz, \quad Z = \frac{z^{1+q}}{1+q}, \quad q \neq -1. \quad (3.25)$$

To construct the corresponding planar representation, one must choose a branch cut, which fixes the angular range of the map. The configuration is then

$$X = \operatorname{Re} Z = \frac{r^{1+q}}{1+q} \cos((1+q)\theta), \quad (3.26)$$

$$Y = \operatorname{Im} Z = \frac{r^{1+q}}{1+q} \sin((1+q)\theta). \quad (3.27)$$

Exercise 3.2: Recovering Volterra from Geometric Charges

Plot the planar embeddings associated with the dipole and quadrupole singularities. Note that the quadrupole is the first multipole that can be formed locally without tearing the material.

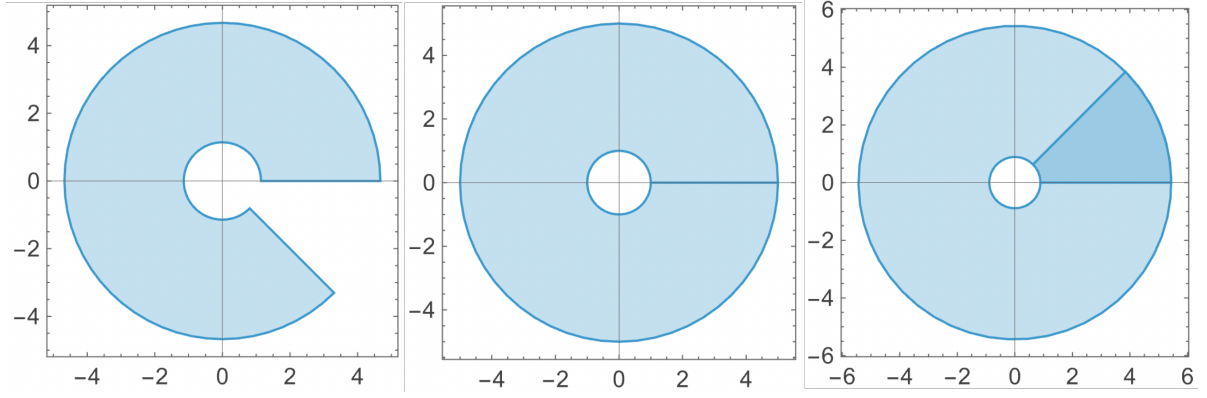


Figure 3.2.: Planar isometric embeddings of $\bar{g}$ for the conformal factor $\varphi = q \log r$, with $q = -0.125$ (left), $q = 0$ (middle), and $q = 0.125$ (right).

3.4. The impossible cone

Consider the conformal metric

$$d\bar{\ell}^2 = Ar^{2q} (dr^2 + r^2 d\theta^2). \quad (3.28)$$

Equivalently,

$$\bar{g} = Ar^{2q} \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \quad (3.29)$$

Define a new radial coordinate ρ by

$$d\rho = r^q dr. \quad (3.30)$$

For $q \neq -1$,

$$\rho = \frac{r^{q+1}}{q+1}. \quad (3.31)$$

Then

$$r^{2q} dr^2 = d\rho^2, \quad (3.32)$$

and

$$r^{2q+2} d\theta^2 = (q+1)^2 \rho^2 d\theta^2. \quad (3.33)$$

Thus the metric becomes

$$d\bar{\ell}^2 = A (d\rho^2 + (q+1)^2 \rho^2 d\theta^2). \quad (3.34)$$

This is a cone with angular factor

$$\alpha = q+1. \quad (3.35)$$

The total angle around the apex is

$$\Theta = 2\pi(q+1), \quad (3.36)$$

and the angular deficit, with one common sign convention, is

$$\Delta\theta = 2\pi - \Theta = -2\pi q. \quad (3.37)$$

3. Third Lecture: Geometric Charges and Generalized Defects

Therefore

$$q = -\frac{\Delta\theta}{2\pi}. \quad (3.38)$$

For the ordinary cone construction one has

$$-1 < q < \infty, \quad -\infty < \Delta\theta < 2\pi. \quad (3.39)$$

However, in the conformal representation q is not intrinsically limited by this Volterra construction. For example, take

$$q = -2. \quad (3.40)$$

Then

$$\Delta\theta = 4\pi \quad (3.41)$$

with the above sign convention, and

$$d\bar{\ell}^2 = Ar^{-4} (dr^2 + r^2 d\theta^2). \quad (3.42)$$

This metric is related to the Euclidean metric by inversion. Indeed, if

$$\rho = \frac{R_0^2}{r}, \quad d\rho = -\frac{R_0^2}{r^2} dr, \quad (3.43)$$

then

$$d\rho^2 + \rho^2 d\theta^2 = \frac{R_0^4}{r^4} (dr^2 + r^2 d\theta^2). \quad (3.44)$$

Thus inversion realizes a flat metric away from the origin, but the representation is singular at the origin. It can therefore be interpreted as an “impossible” conical defect: a geometric singularity that is natural in the conformal metric but not obtainable by an ordinary planar Volterra construction.

The main lesson is that geometry provides a non-Volterra construction of defects. Instead of starting from cutting and gluing operations, one starts from a singular reference metric $\bar{g}_{\alpha\beta}$ and identifies the corresponding singular reference curvature $\bar{K} = K_G(\bar{g})$. The price is that the construction is no longer directly tied to a microscopic order parameter or to the classical classification of ordered media.

In two dimensions, inversion already produces a singular conformal metric. In three dimensions the same idea becomes even more striking. The inversion

$$R = \frac{R_0^2}{r} \quad (3.45)$$

transforms the Euclidean metric into the conformally Euclidean metric

$$d\bar{\ell}^2 = \frac{R_0^4}{r^4} (dr^2 + r^2 d\Omega^2), \quad (3.46)$$

which is flat away from the origin but singular at the origin. Thus inversion creates a singular geometric solution without specifying an ordinary Volterra cut-and-glue operation.

3.5. Local changes of reference metric generate quadrupoles

We have classified the possible geometric charges and their corresponding reference metrics. We have also seen that monopoles and dipoles can be generated through Volterra constructions. Material response, however, is often governed by local plastic rearrangements that modify the reference metric. Whether these rearrangements are sparse and effectively isolated or dense and correlated is the subject of the next lecture on mechanical screening. Here, we ask which geometric charge is induced by a generic local perturbation of the reference metric. Let the reference metric be a small perturbation of the Euclidean metric,

$$\bar{g}_{\alpha\beta} = \delta_{\alpha\beta} + q_{\alpha\beta}. \quad (3.47)$$

To linear order in q , the Gaussian curvature is obtained by applying the linearized double-curl operator to the metric perturbation,

$$\bar{K} \simeq -\frac{1}{2}\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\partial_\alpha\partial_\beta q_{\gamma\delta}. \quad (3.48)$$

If we define the quadrupole tensor

$$Q^{\alpha\beta} := \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}q_{\gamma\delta}, \quad (3.49)$$

then the curvature takes the form

$$\bar{K} \simeq -\frac{1}{2}\partial_\alpha\partial_\beta Q^{\alpha\beta}. \quad (3.50)$$

Thus a localized change in the reference metric naturally produces a quadrupolar curvature source. In this sense, quadrupoles are the elementary local geometric charges.

A distribution of quadrupoles can generate effective dipoles and monopoles, particularly through boundary terms. This observation connects local internal rearrangements to macroscopic defect-like sources of stress.

3.6. Physical consequences

We now regard the defective metric $\bar{g}$ as the reference metric of the body. The elastic problem is to find, among all compatible metrics g , the one closest to $\bar{g}$. Residual stress arises because a defective reference metric with $\bar{K} \neq 0$ cannot satisfy $g = \bar{g}$.

Substituting the stress-potential representation into the Hookean energy, which is quadratic in the strain, and integrating twice by parts yields the interaction between defects. In analogy with electrostatics, let χ_1 be the Airy stress function generated by the first defect and let $\bar{K}_2$ be the curvature singularity, or geometric charge, of the second. Their interaction energy is, up to the sign and normalization set by convention,

$$E_{12}^{\text{int}} = \int_{\Omega} \chi_1(x) \bar{K}_2(x) dA. \quad (3.51)$$

3. Third Lecture: Geometric Charges and Generalized Defects

For two pure quadrupoles in an infinite elastic medium, with orientations $\theta_{1,2}$ relative to their connecting line, this gives

$$E_{12}^{QQ} = \frac{Q_1 Q_2}{\pi r} \cos(2\theta_1 + 2\theta_2), \quad (3.52)$$

where r is their separation and θ_i is the orientation of the i -th quadrupole relative to the line joining them. For fixed positive magnitudes, the interaction is minimized when $\theta_1 + \theta_2 = \pi/2$ modulo π . There is therefore a continuum of minimizers for pairwise quadrupole interactions.

4. Fourth Lecture: Anomalous Elasticity and Mechanical Screening

Anomalous elasticity refers to the behavior of elastic-like materials that support external stresses yet deform in ways that appear to contradict the standard principles of elasticity. Examples include an anomalously soft spring with vanishing stiffness (see Schenk and Guest, *On Zero Stiffness*, 2014) and a material with nonreciprocal interactions that appear to violate Newton’s third law (see Fruchart, Scheibner, and Vitelli, *Odd Viscosity and Odd Elasticity*, 2023). In this lecture, we show that mechanical screening provides a natural extension of elasticity for materials with internal degrees of freedom that can relax stress and strain. The theory predicts anomalous mechanical responses that were subsequently confirmed in experiments on disordered granular matter.

4.1. Motivation: Mechanical Screening in Holey Metamaterials

Screening—the reduction of energy through material response—is central to many physical systems. The canonical example is a dielectric, in which an applied electric field induces dipoles whose collective response renormalizes the effective dielectric constant.

A classical model considers an array of conducting spheres embedded in a perfectly insulating medium. The applied field polarizes each sphere, while the resulting dipoles alter the local field acting on every other sphere.

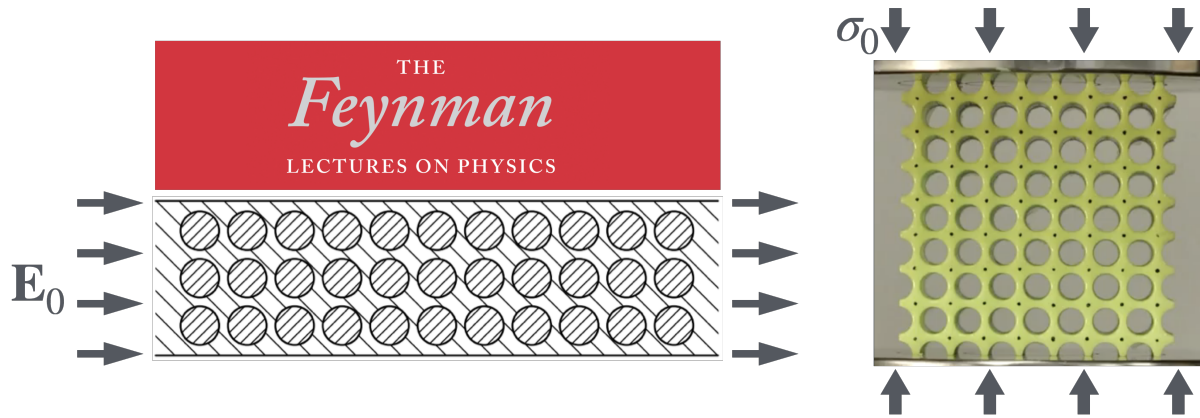


Figure 4.1.: Left: A model dielectric in which an array of perfectly conducting spheres becomes polarized in response to an external electric field (adapted from *The Feynman Lectures on Physics*). Right: A mechanical analogue in which the elastic matrix plays the role of the insulating medium and the holes play the role of the conducting spheres.

4. Fourth Lecture: Anomalous Elasticity and Mechanical Screening

This problem can be studied using the method of image charges. Each conducting sphere is replaced by an image dipole, placed at its center, whose field is chosen to balance the boundary condition of constant potential on the sphere. For an array of spheres, the dipole moments must respond both to the applied field and to the fields generated by all the other image dipoles. An approximate self-consistent solution is obtained by treating these dipole moments as variational degrees of freedom and minimizing the electrostatic energy with respect to them, yielding a constitutive relation between the applied field and the macroscopic polarization.

This theoretical picture has a direct analogue in two-dimensional perforated elastic metamaterials. The elastic matrix plays the role of the insulating medium, while the holes play the role of the conducting spheres. An applied load induces elastic fields around each traction-free hole, and these fields must satisfy both the external boundary conditions and the boundary conditions on all hole edges.

As in electrostatics, the holes can be replaced by image charges placed at their centers. The leading admissible elastic image charge is a quadrupole: elastic monopoles and dipoles correspond to topological charges, whereas a quadrupole can be induced locally without changing the material topology. At leading order, the Airy stress function is approximated by

$$\chi(x) = \chi_{\text{ext}}(x) + \sum_i Q_i^{\alpha\beta} \chi_{\alpha\beta}^{(Q)}(x - x_i), \quad (4.1)$$

where $Q_i^{\alpha\beta}$ is the image quadrupole associated with the i -th hole.

The unknown quadrupole magnitudes and orientations are determined variationally. Substituting this ansatz into the elastic energy produces an effective interaction energy $E_{\text{eff}}(\{Q_i\})$, which includes the coupling of each quadrupole to the applied load and to all other quadrupoles. The former produces an energy term that is linear in both the quadrupole charge and the external stress, whereas the latter is quadratic in the quadrupole charges. The approximate solution is obtained from

$$\frac{\partial E_{\text{eff}}}{\partial Q_i^{\alpha\beta}} = 0. \quad (4.2)$$

Thus a continuum problem with many hole boundaries is reduced to a finite problem of interacting elastic quadrupoles, as developed by Bar-Sinai et al. in *Geometric Charges and Nonlinear Elasticity of Two-Dimensional Elastic Metamaterials* (2020). As in the dielectric model, one can derive a linear constitutive relation between the inducing stress and the induced quadrupole, with the proportionality coefficient playing the role of a quadrupolarizability

$$Q^{\alpha\beta} = \Lambda^{\alpha\beta}_{\mu\nu} \sigma^{\mu\nu}. \quad (4.3)$$

There is, however, an essential difference between the two problems. Electrostatics is fully linear, so the induced charges remain proportional to the external field, and their mutual interactions cannot dominate their coupling to that field. Elasticity, by contrast, remains geometrically nonlinear even when the material response is Hookean. The ansatz in Eq. 4.1 is fully linear, so higher-order corrections are required to capture geometric nonlinearities. These corrections generate nonlinear load–quadrupole and quadrupole–quadrupole interactions that can drive mechanical instabilities. This mechanism produces the nonlinear screening relation

$$Q = \Lambda^{(1)} \sigma + \Lambda^{(3)} \sigma^3 + \dots \quad (4.4)$$

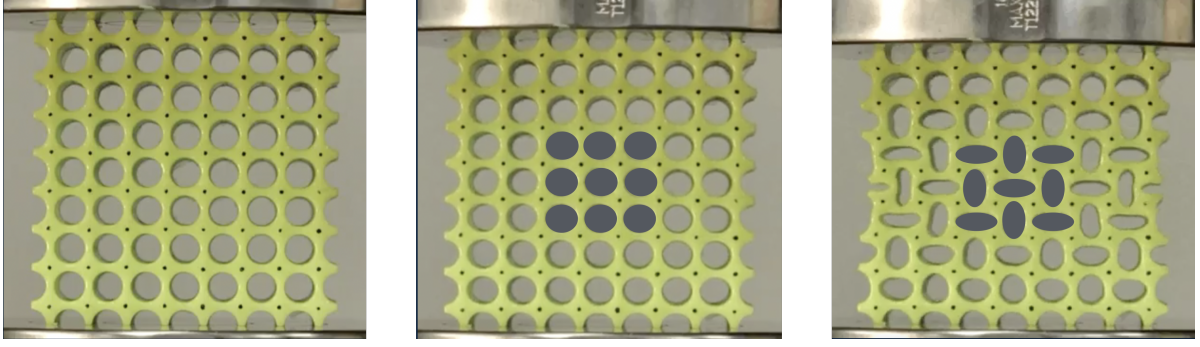


Figure 4.2.: Three deformation regimes of a holey elastic metamaterial under compression. Left: The undeformed structure, with circular holes arranged on a square lattice. Middle: At weak compression, the holes deform into ellipses with a uniform orientation selected by the external loading. Right: Beyond the instability, the hole orientations alternate to form a checkerboard pattern that minimizes hole–hole interactions. The holes are shaded gray only to emphasize their deformability.

and provides a natural explanation for the instability of holey metamaterials. At small compressive loads, the coupling between each quadrupole and the applied stress dominates, aligning all induced quadrupoles with the external loading (Fig. 4.2, middle panel). At larger compression, quadrupole–quadrupole interactions dominate and select a collective pattern. According to Eq. 3.52, the interaction between two quadrupoles is minimized when the sum of their orientations, measured relative to the line joining them, equals $\pi/2$. On a square lattice, alternating horizontal and vertical quadrupoles form a checkerboard state that minimizes both nearest-neighbor and next-nearest-neighbor interaction energies, reproducing the pattern observed beyond the instability (Fig. 4.2, right panel). A complete derivation appears in Bar-Sinai et al., *Geometric Charges and Nonlinear Elasticity of Two-Dimensional Elastic Metamaterials* (2020).

The quadrupolar image-charge construction is one realization of the broader concept of mechanical screening. Earlier work interpreted the buckling of two-dimensional defective membranes as the screening of structural defects by Gaussian curvature; see Seung and Nelson, *Defects in Flexible Membranes with Crystalline Order* (1988). Conversely, in curved self-assembled crystals, stresses generated by the imposed curvature can be screened through the nucleation and rearrangement of structural defects; see Nelson, *Defects and Geometry in Condensed Matter Physics* (2002), and Irvine, Vitelli, and Chaikin, *Pleats in Crystals on Curved Surfaces* (2010). These examples illustrate the dual role of curvature and defects: each can act as a source of stress, and each can screen the stress generated by the other.

The duality between curvature and defects—each acting both as a source and as a screening field—is reflected in the equation for the stress potential χ :

$$\frac{1}{Y} \Delta \Delta \chi = K_D - K_G . \quad (4.5)$$

Here K_G is the Gaussian curvature of the deformed configuration, while K_D represents singular or distributed structural defects (see Seung and Nelson, *Defects in Flexible Membranes with Crystalline Order*, 1988). The equation shows that stress is generated by the mismatch $K_D - K_G$.

Thus, when the curvature is prescribed, the material can reduce its elastic stress by nucleating and rearranging defects so that K_D partially compensates K_G . The defect patterns observed in curved colloidal crystals provide a direct realization of this screening mechanism. More broadly, Figure 4.3 illustrates four different screening mechanisms: (a) nonlinear screening by quadrupoles, (b) plastic screening by dislocations, (c) screening by localized plastic deformations, and (d) screening by localized wrinkles. In the next section, we derive a continuum screening theory in which defects, understood in the general sense developed in the previous chapter, nucleate to screen elastic fields.

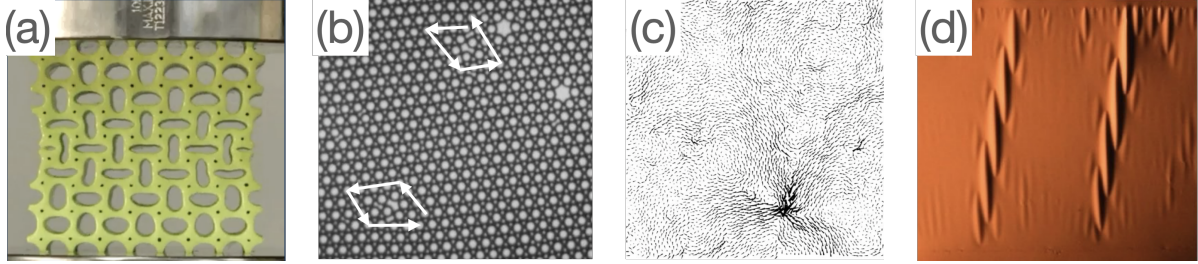


Figure 4.3.: Mechanical screening mechanisms. (a) Nonlinear quadrupole screening in holey metamaterials, developed by Bar-Sinai et al. in *Geometric Charges and Nonlinear Elasticity of Two-Dimensional Elastic Metamaterials* (2020). (b) Screening by dipoles through dislocation unbinding in a two-dimensional crystal bubble-raft model; see Bragg and Nye, *A Dynamical Model of a Crystal Structure* (1947), and Bowick et al., *Bubble-Raft Model for a Paraboloidal Crystal* (2008). (c) A quadrupolar Eshelby plastic event in an amorphous-solid model, adapted from Maloney and Lemaître, *Amorphous Systems in Athermal, Quasistatic Shear* (2006). (d) Screening by localized quadrupolar wrinkles, adapted from Timounay et al., *Crumples as a Generic Stress-Focusing Instability in Confined Sheets* (2020).

4.2. Continuum theory of mechanical screening

Screening theories are often derived from a detailed microscopic model. The Debye–Hückel theory, for example, starts from the statistical mechanics of mobile ions and derives a screening length from their temperature and concentration. For the athermal and frequently disordered systems of interest here, such a microscopic route is neither universal nor always available. We instead ask what can be deduced from symmetry, conservation laws, and the geometric character of the elementary relaxation events. This approach leads to a hierarchy of continuum screening theories without requiring that the microscopic screening element be identifiable as a defect in an ordered lattice; see Livne et al., *Geometric Theory of Mechanical Screening in Two-Dimensional Solids* (2023).

4.2.1. Screening Hierarchy in the Electrostatic Analogy

The logic is most familiar in electrostatics. In a dielectric, the electric field induces a polarization density P^α . Homogeneity, isotropy, and linearity give the constitutive relation

$$P^\alpha = \chi_e E^\alpha, \quad (4.6)$$

while Gauss' law reads

$$\partial_\alpha (\epsilon_0 E^\alpha + P^\alpha) = \rho_{\text{ext}}. \quad (4.7)$$

Eliminating P merely replaces the vacuum permittivity by $\epsilon = \epsilon_0 + \chi_e$. Dipoles therefore screen the field quantitatively, but the governing equation remains Poisson's equation.

A different regime is obtained when effective monopoles are mobile. Their density is the divergence of the polarization,

$$\rho_{\text{ind}} = -\partial_\alpha P^\alpha. \quad (4.8)$$

The lowest-order isotropic cost assigned to this density introduces a material length. Eliminating the polarization then gives a Helmholtz equation for the electric potential,

$$(\Delta - \ell_D^{-2}) \varphi = -\frac{\rho_{\text{ext}}}{\epsilon_0}, \quad (4.9)$$

where ℓ_D is the screening length. The important point for us is not its microscopic Debye–Hückel value, but the continuum logic: allowing successively lower multipoles changes the differential operator and introduces new material moduli with dimensions of length.

More generally, the hierarchy of screening modes can be derived from an energy functional rather than imposed directly on the field equations. The screening field is allowed to relax the original field, but this relaxation is accompanied by a material-dependent energy cost for nucleating the induced charges. The multipole expansion constrains the form of this cost. For an electric polarization (P^α), a term $(P_\alpha P^\alpha)$ penalizes the creation of dipoles, whereas $(\nabla_\alpha P^\alpha)^2$ penalizes effective monopoles produced by the divergence of a nonuniform dipole field. By contrast, $\nabla \times P$ does not appear in the charge multipole expansion. Consequently, one should not introduce an independent curl cost, or an unrestricted gradient cost that would include it. The central construction is therefore to permit field relaxation while assigning nucleation costs only to multipoles allowed by the relevant underlying mechanisms. These constraints determine the form of the energy and, after minimization, the resulting screening theory.

In electrostatics, charge conservation implies that the variation is performed with respect to the polarization field. The energy may nevertheless contain terms depending on the polarization itself or on its divergence, which represents the induced monopole density.

4.2.2. Geometric charges in ordered and disordered solids

The corresponding local relaxation field in a solid is a change of its reference metric. Let $\bar{g}_{\alpha\beta}^0$ be the fixed initial reference metric and write the locally relaxed reference metric as

$$\bar{g}_{\alpha\beta} = \bar{g}_{\alpha\beta}^0 + 2q_{\alpha\beta}. \quad (4.10)$$

4. Fourth Lecture: Anomalous Elasticity and Mechanical Screening

The symmetric tensor $q_{\alpha\beta}$ is a strain-like quadrupole field. If $u_{\alpha\beta}$ denotes the total strain measured from $\bar{g}^0$, the elastic strain that stores energy is

$$u_{\alpha\beta}^e = u_{\alpha\beta} - q_{\alpha\beta}. \quad (4.11)$$

A localized change of reference metric produces a quadrupolar distribution of reference curvature. Monopole and dipole elastic charges cannot be created by a strictly local rearrangement without changing global geometric invariants; the quadrupole is therefore the lowest elastic multipole that can be nucleated locally.

This observation gives a definition of a mechanical defect that does not rely on crystalline order. In a crystal, a dipole is visible as a dislocation and a quadrupole can be realized as a bound pair of dislocations. In an amorphous solid, the same quadrupolar field is generated by a localized particle rearrangement, even though no lattice is available with respect to which a missing or extra row of particles could be identified. What matters to the continuum theory is the elastic field produced by the rearrangement, not its structural appearance.

According to Eq. 3.50, the relaxation deformation q induces an elastic charge,

$$\bar{K} = \partial_\alpha \partial_\beta Q^{\alpha\beta}, \quad (4.12)$$

where $Q^{\alpha\beta} = \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} q_{\gamma\delta}$. Spatial derivatives of Q generate lower elastic multipoles:

$$P^\alpha = \partial_\beta Q^{\alpha\beta}, \quad M = \partial_\alpha P^\alpha = \partial_\alpha \partial_\beta Q^{\alpha\beta}, \quad (4.13)$$

where P^α and M are the effective dipole and monopole densities. For example, a finite line of correlated quadrupolar rearrangements can terminate in two opposite dipoles. In a crystal, this process appears as the nucleation and unbinding of a dislocation pair; in a disordered solid it appears as a sequence of local plastic events forming a shear band. Geometrically, these are two realizations of the same charge distribution.

The most general quadratic, isotropic, long-wavelength cost may consequently be organized as

$$E[u, q] = \frac{1}{2} \int_\Omega \left[A^{\alpha\beta\gamma\delta} (u_{\alpha\beta} - q_{\alpha\beta}) (u_{\gamma\delta} - q_{\gamma\delta}) + \Lambda_Q^{\alpha\beta\gamma\delta} Q_{\alpha\beta} Q_{\gamma\delta} + \Lambda_P^{\alpha\beta} P_\alpha P_\beta + \Lambda_M M^2 \right] dA. \quad (4.14)$$

The three core costs define quadrupole-, dipole-, and monopole-dominated screening regimes. If the local Q^2 term dominates, minimizing over Q gives a local relation between the induced quadrupole and the stress. As in a dielectric, the result is quasi-elastic: the elastic moduli are renormalized but the form of the equilibrium equation is unchanged. If the cost of a uniform quadrupole is small and the P^2 term dominates, gradients of the quadrupole field become the relevant degrees of freedom and the response is dipole screened. If the monopole cost becomes finite as well, a third screening regime follows. The hierarchy is summarized schematically below; the dipole-screening case is derived in detail in the following section:

screening mode	dominant charge cost	stress-potential equation
quadrupole	Q^2	$\frac{1}{\bar{V}} \bar{\Delta}^2 \chi = \bar{K}_0$
dipole	P^2	$\frac{1}{\bar{V}} \bar{\Delta} (\bar{\Delta} + \ell_P^{-2}) \chi = \bar{K}_0$
monopole	M^2	$\frac{1}{\bar{V}} (\bar{\Delta}^2 + \ell_M^{-4}) \chi = \bar{K}_0$

(4.15)

Numerical coefficients in the definitions of ℓ_P and ℓ_M depend on conventions and on the isotropic screening tensors. Their physical role is unambiguous: they are new mechanical moduli, beyond the Young modulus and Poisson ratio, that set intrinsic screening lengths.

4.2.3. Dipole screening in the stress-potential formulation

We now demonstrate the derivation for the dipole regime. We work to leading order about a flat reference metric and retain only the dipolar core cost,

$$E_{\text{dip}}[u, q] = \frac{1}{2} \int_{\Omega} \left[A^{\alpha\beta\gamma\delta} (u_{\alpha\beta} - q_{\alpha\beta}) (u_{\gamma\delta} - q_{\gamma\delta}) + \lambda_P P_{\alpha} P^{\alpha} \right] dA. \quad (4.16)$$

Variation with respect to the configuration gives ordinary force balance for the elastic stress,

$$\partial_{\beta} \sigma^{\alpha\beta} = 0, \quad \sigma^{\alpha\beta} \equiv A^{\alpha\beta\gamma\delta} (u_{\gamma\delta} - q_{\gamma\delta}). \quad (4.17)$$

In two dimensions this equation is solved identically by an Airy stress potential,

$$\sigma^{\alpha\beta} = \frac{1}{\det \bar{g}^0} \varepsilon^{\alpha\mu} \varepsilon^{\beta\nu} \bar{\nabla}_{\mu\nu}^0 \chi. \quad (4.18)$$

The stress-strain relation then expresses the metric in terms of the potential:

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta}^0 + 2q_{\alpha\beta} + 2A_{\alpha\beta\gamma\delta}^{-1} \sigma^{\gamma\delta}. \quad (4.19)$$

Requiring the actual configuration to be planar, and therefore the curvature of g to vanish, reduces the geometric compatibility condition to

$$\frac{1}{Y} \bar{\Delta} \bar{\Delta} \chi = \bar{K}_0 + K_Q, \quad (4.20)$$

where

$$K_Q = \frac{1}{\det \bar{g}^0} \epsilon^{\alpha\mu} \epsilon^{\beta\nu} \bar{\nabla}_{\mu\nu}^0 q_{\alpha\beta}. \quad (4.21)$$

The variation with respect to $q_{\alpha\beta}$, together with isotropy and the dipolar core cost, relates the induced charge to the hydrostatic part of the stress. Writing $\lambda_P = Y \ell_P^2$, the relation is

$$K_Q = -\frac{1}{Y \ell_P^2} \sigma^{\alpha}_{\alpha} = -\frac{1}{Y \ell_P^2} \bar{\Delta} \chi. \quad (4.22)$$

Thus, unlike quadrupole screening, the induced charge is not locally proportional to the full stress tensor. It is generated through spatial variations of the quadrupole field and is controlled by a new length-like modulus. Eliminating K_Q between Eqs. (4.20) and (4.22) yields

$$\boxed{\frac{1}{Y} \bar{\Delta} (\bar{\Delta} + \ell_P^{-2}) \chi = \bar{K}_0, \quad \ell_P^2 = \frac{\lambda_P}{Y}} \quad (4.23)$$

with possible isotropic numerical factors absorbed into ℓ_P . Ordinary elasticity is recovered when the screening length is irrelevant on the scale of the system. Otherwise the Airy potential contains a Helmholtz sector in addition to its harmonic sector, and stresses acquire an intrinsic spatial scale. This is the mechanical analogue of the change from dielectric to Debye–Hückel screening, although the mechanical sign permits spatially oscillatory rather than exponentially decaying responses.

Exercise 4.1: Pressurized Annulus with Dipole Screening

Revisit the pressurized-annulus problem of Exercise 2.5, but now assume that the material is in the dipole-screening regime. In the annular bulk take $\bar{K}_0 = 0$, so that the Airy stress function satisfies

$$\bar{\Delta} (\bar{\Delta} + \ell_P^{-2}) \chi = 0.$$

Assume an axisymmetric solution, apply a uniform pressure p at the inner boundary $r = a$, and leave the outer boundary $r = b$ traction-free. Derive the radial equation and its general solution, identify the harmonic and Helmholtz sectors, and state the additional boundary conditions required for the screening field. Determine the radial and azimuthal stresses and compare them with the ordinary elastic solution recovered as $\ell_P \rightarrow \infty$. How does the screening length alter the spatial transmission of stress across the annulus?

4.2.4. Equivalent displacement formulation

The same theory can be written directly in terms of the displacement field d . Assuming a Euclidean $\bar{g}_0$ and integrating out the quadrupolar relaxation field in an isotropic two-dimensional solid gives the screened equation

$$\Delta d + \lambda \nabla (\nabla \cdot d) = -\mu d, \quad \lambda = \frac{1 + \nu}{1 - \nu} \quad (4.24)$$

where μ has dimensions of inverse length squared and is determined by the dipole nucleation cost; see Mondal et al., *Experimental and Numerical Verification of Anomalous Screening Theory in Granular Matter* (2022). When $\mu = 0$, this is the usual scale-free equation of linear elasticity. For finite μ , solving for the induced dipole field requires fixing the displacement gauge, for example by imposing zero displacement on the boundary. Consequently, the equation is not invariant under adding a uniform displacement to a given solution.

The qualitative prediction is especially transparent for radial inflation. Consider an annulus $r_{\text{in}} < r < r_{\text{out}}$, with

$$d(r) = d_r(r) \hat{r}, \quad d_r(r_{\text{in}}) = d_0, \quad d_r(r_{\text{out}}) = 0. \quad (4.25)$$

For ordinary elasticity the solution is

$$d_r^{\text{el}}(r) = d_0 \frac{r_{\text{in}}}{r} \frac{r^2 - r_{\text{out}}^2}{r_{\text{in}}^2 - r_{\text{out}}^2}. \quad (4.26)$$

It is positive and decreases monotonically from the expanding inclusion to the fixed outer wall. Importantly, after lengths are normalized by r_{out} , this profile contains no adjustable elastic modulus.

In the dipole-screened regime Eq. (4.24) reduces to

$$d_r'' + \frac{1}{r} d_r' + \left(\kappa^2 - \frac{1}{r^2} \right) d_r = 0, \quad \kappa^2 = \frac{\mu}{1 + \lambda}. \quad (4.27)$$

The radial displacement is therefore

$$d_r(r) = d_0 \frac{Y_1(\kappa r)J_1(\kappa r_{\text{out}}) - J_1(\kappa r)Y_1(\kappa r_{\text{out}})}{Y_1(\kappa r_{\text{in}})J_1(\kappa r_{\text{out}}) - J_1(\kappa r_{\text{in}})Y_1(\kappa r_{\text{out}})}, \quad (4.28)$$

where J_1 and Y_1 are Bessel functions. Unlike the elastic solution, this profile can be nonmonotonic, cross zero, and oscillate in space. In particular, the screening theory predicts that a grain may move inward toward the center even though the central inclusion pushes outward. This counterintuitive motion is not a nonlinear correction to elasticity: it is a linear response governed by a new screening modulus.

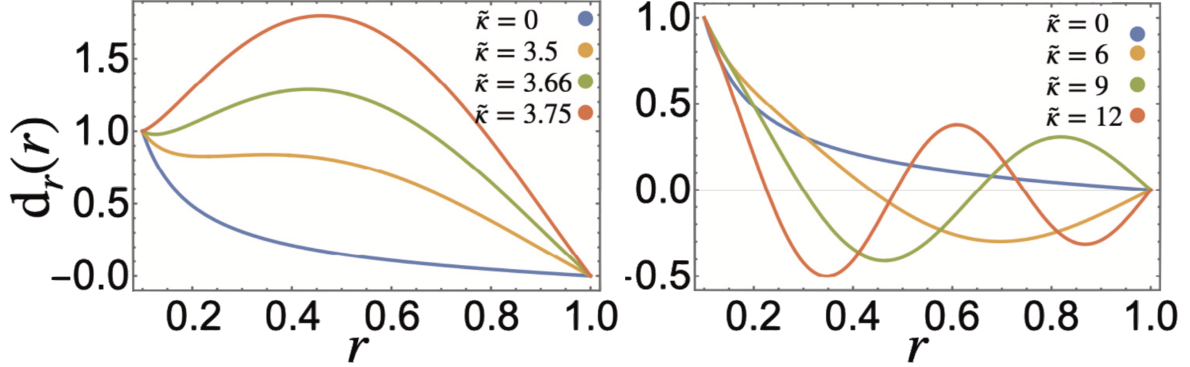


Figure 4.4.: Solutions of Eq. 4.28 for various values of the dimensionless screening parameter $\tilde{\kappa} = \kappa r_{\text{out}}$. The radial coordinate is normalized by r_{out} , and the displacement is normalized by d_0 . Left: Small values of $\tilde{\kappa}$ produce a weakly anomalous elastic response, for which the displacement profile may become nonmonotonic. Right: Large values of $\tilde{\kappa}$ produce a strongly anomalous response, including nonmonotonic and sign-changing displacement profiles. The number of nodes in the displacement field increases with $\tilde{\kappa}$.

4.2.5. Verification in granular matter

The radial prediction was tested in two-dimensional bidisperse frictional granular assemblies confined by a circular wall, as reported in *Experimental and Numerical Verification of Anomalous Screening Theory in Granular Matter* (2022). A conical pusher at the center acted as an inflating grain, and particle tracking measured the angle-averaged radial displacement. Parallel discrete-element simulations implemented the same protocol.

Confining pressure controls the availability of local rearrangements. At high dimensionless pressure, quadrupolar plastic events are sparse and their gradients are negligible. Both experiments and simulations then collapse onto the universal elastic profile in Eq. (4.26). At low pressure, local rearrangements are abundant and form a distributed dipole field. The measured displacement becomes nonmonotonic and changes sign over part of the sample: particles move toward the expanding center. Equation (4.28) reproduces this profile using only the dimensionless screening parameter κr_{out} as a fit parameter.

The experiment therefore distinguishes two solid-like regimes. In the quasi-elastic regime, screening renormalizes material properties without changing the form of elasticity. In the

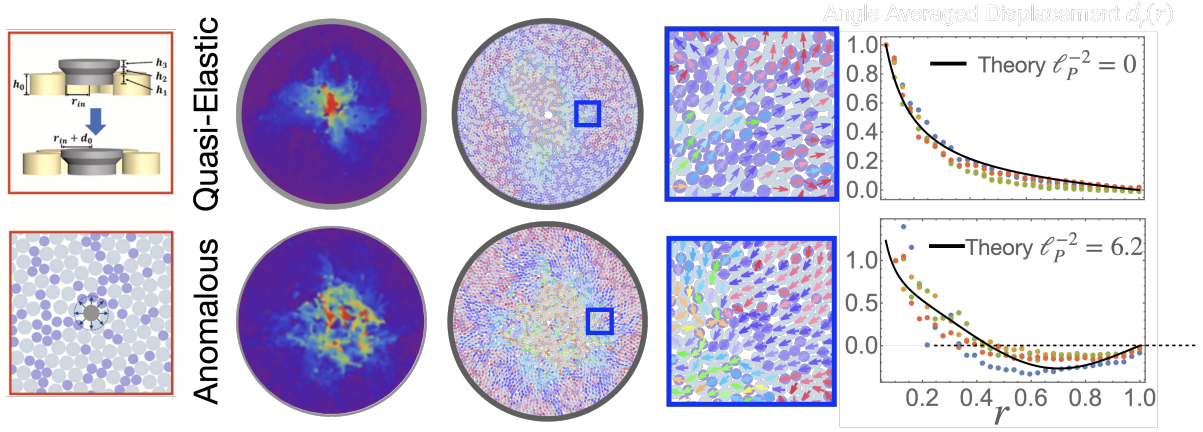


Figure 4.5.: Experimental and numerical tests of radial inflation. Left, red frames: Experimental setup. Top: At high pressure, quadrupolar rearrangements are costly and sparse, producing a quasi-elastic response. Bottom: At low pressure, quadrupolar rearrangements are inexpensive and abundant; their spatial organization produces effective dipole screening and anomalous mechanics. Adapted from *Experimental and Numerical Verification of Anomalous Screening Theory in Granular Matter* (2022).

dipole-screened regime, the material still supports static loads, but its response is governed by an intrinsic length and can be spatially oscillatory. This is the anomalous mechanical behavior predicted by the geometric screening theory.

4.2.6. Odd-dipole screening

Equation (4.24) follows from minimizing an energy and therefore assumes a reciprocal constitutive relation. Disordered solids, however, possess many metastable reference states. After a closed quasistatic deformation cycle the macroscopic shape may return to its initial state while particle rearrangements leave the system in a different microscopic state. Such a cycle can store or release a finite amount of mechanical work. A potential energy defined only by the macroscopic configuration is then insufficient.

The dipole screening relation may be written

$$P_\alpha = \Gamma_{\alpha\beta} d^\beta, \quad \Gamma = \kappa_e I. \quad (4.29)$$

Homogeneity and isotropy do not require Γ to be symmetric if the existence of a potential is abandoned. In two dimensions the minimal extension is

$$\Gamma = \begin{pmatrix} \kappa_e & -\kappa_o \\ \kappa_o & \kappa_e \end{pmatrix} = \kappa R(\theta_\kappa), \quad (4.30)$$

with

$$\kappa = \sqrt{\kappa_e^2 + \kappa_o^2}, \quad \tan \theta_\kappa = \frac{\kappa_o}{\kappa_e}. \quad (4.31)$$

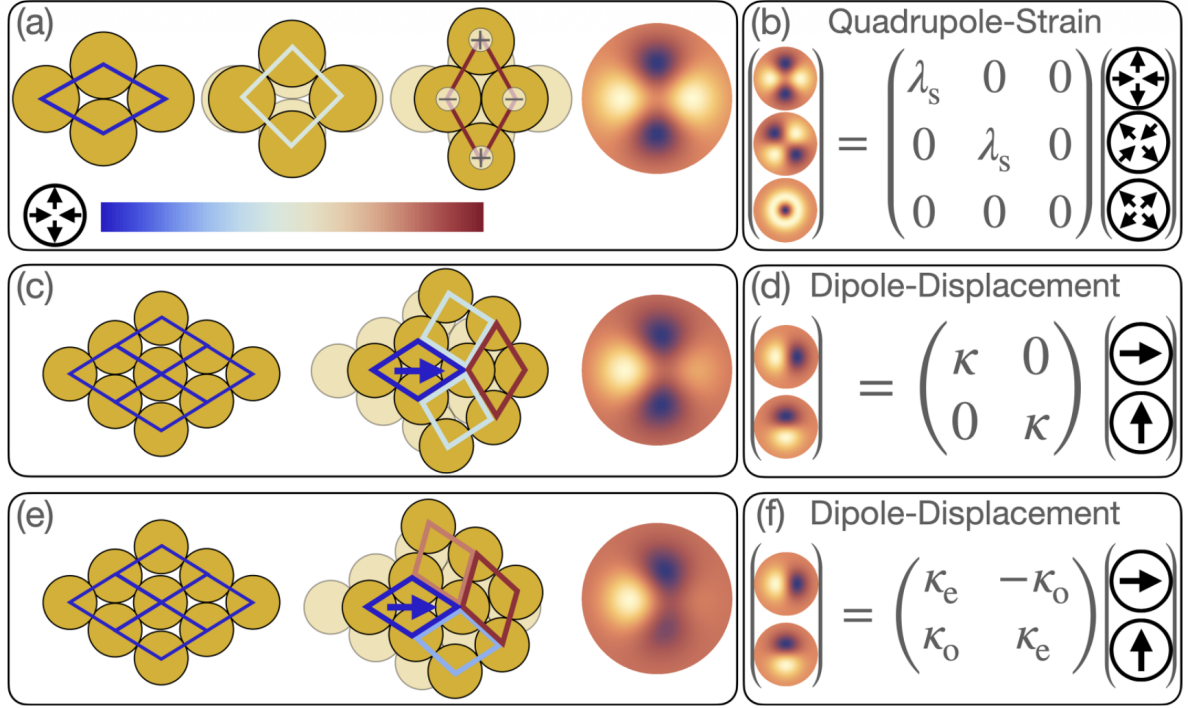


Figure 4.6.: Basic mechanical screening modes, their realization through particle rearrangements, and the corresponding geometric charges. (a) A strain-induced rearrangement produces a quadrupole of angular deficits and excesses, marked by $\pm$. Its magnitude is proportional to the applied strain, as indicated by the color bar; the heat map shows the induced monopole-charge distribution. (b) Linear relation between the applied strain modes and the induced quadrupole charges. (c) A displacement-induced rearrangement produces nonuniform quadrupoles whose coarse-grained description combines quadrupole and dipole charges. (d) Linear relation between the applied displacement and the induced dipole. (e) Under odd screening, the induced dipole is not parallel to the applied displacement. (f) Corresponding odd linear relation. In (c), the displacement and dipole are parallel, corresponding to $\kappa_o = 0$; in (e), the dipole is rotated by $\theta_\kappa = \arctan(\kappa_o/\kappa_e)$.

4. Fourth Lecture: Anomalous Elasticity and Mechanical Screening

The even modulus κ_e measures ordinary strain relaxation. The odd modulus κ_o rotates the induced dipole away from the inducing displacement and quantifies the nonconservative work associated with a closed deformation cycle. This is odd screening rather than odd elasticity: the elastic tensor itself may remain symmetric while nonreciprocity resides in the internal screening response; see Cohen et al., *Odd-Dipole Screening in Disordered Matter* (2025).

The screened Navier equation becomes the vector equation

$$\Delta d + \lambda \nabla(\nabla \cdot d) = -\Gamma d. \quad (4.32)$$

Its most direct consequence is coupling between orthogonal displacement components. For a rectangular sample of width L subjected to body shear, the imposed affine field is $d_y = u_{xy}x$. Even screening alone preserves $d_x = 0$, whereas odd screening predicts a transverse displacement. Using coordinates normalized by $L/2$, the leading solution takes the form

$$d_x(x) = \tau \left[x - \frac{\sin(\zeta x)}{\sin \zeta} \right], \quad \tau = u_{xy} \tan \theta_\kappa, \quad \zeta = \sqrt{\cos \theta_\kappa \kappa / (1 + \lambda)} L/2. \quad (4.33)$$

The functional form of d_x is therefore a direct signature of the odd modulus.

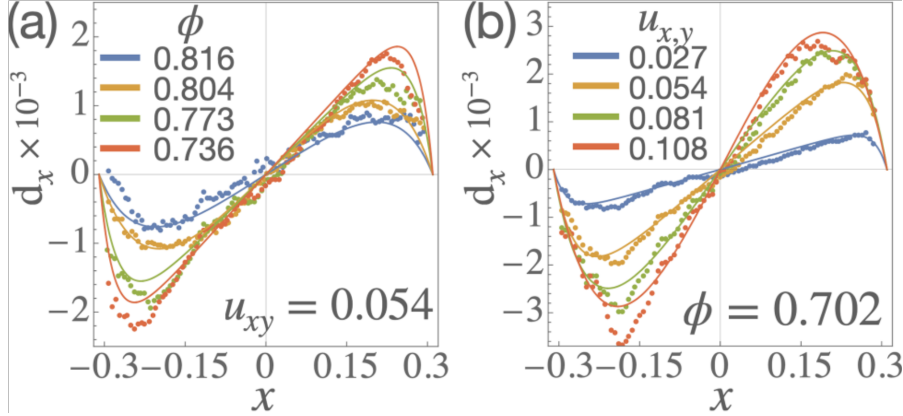


Figure 4.7.: Odd-displacement coupling in disordered granular matter. Theory (solid lines) and experimental data (dots) for the anomalous displacement $d_x(x)$ induced by the affine body shear $d_y = u_{xy}x$. The displacement is averaged along the y direction and fitted to Eq. 4.33 using κ and θ_κ . (a) Experimental and theoretical displacement fields for several packing fractions at fixed imposed shear. (b) Corresponding fields for several shear loads at fixed packing fraction.

This prediction was tested in a two-dimensional shear-jammed packing of bidisperse frictional disks by Cohen et al. in *Odd-Dipole Screening in Disordered Matter* (2025). The substrate imposed a quasistatic body shear, and the particle displacement was averaged along the direction of shear. A transverse component $d_x(x)$, forbidden by the even isotropic theory, was observed over packing fractions $0.692 < \phi < 0.816$ and imposed shear strains up to approximately 22%. Equation (4.33) accurately reproduced the measured profiles. At small strains the extracted even and odd screening parameters were reproducible across initial configurations and packing fractions, as required of continuum material parameters. At larger strains they became strain

dependent, yet followed reproducible relations and the same continuum equation continued to describe the displacement field.

The combined picture is thus broader than defect mechanics in a crystal. Local quadrupoles provide a common language for holes, dislocation pairs, Eshelby events, wrinkles, and cellular rearrangements. Their spatial organization produces dipole screening and anomalous length-dependent mechanics; nonreciprocal evolution among metastable reference states adds an odd screening modulus and transverse response. The continuum fields identify the mechanically relevant defects through the stresses they generate, whether or not those defects can be recognized by eye in the underlying structure.

Exercise 4.2: Pressurized Annulus with Odd-Dipole Screening

This problem is technically difficult, so a numerical solution is also valuable. Repeat the pressurized-annulus problem of Exercise 4.1, but now replace the even dipole-screening coefficient by the odd screening tensor and use the displacement formulation

$$\Gamma = \kappa_e I + \kappa_o \epsilon.$$

Although the loading and geometry are axisymmetric, allow the displacement to have both radial and azimuthal components,

$$d(r) = d_r(r)\hat{r} + d_\theta(r)\hat{\theta}.$$

Substitute this ansatz into Eq. (4.32) and derive the coupled radial equations and boundary conditions. Show that when $\kappa_o \neq 0$, radial pressurization generates a nonzero azimuthal displacement d_θ , so that a chiral response emerges from an achiral annulus and loading protocol. Verify that reversing the sign of κ_o reverses the handedness of the solution, and recover the even dipole-screened result when $\kappa_o = 0$.

A. Extra Material on Thin Sheets

A.1. Non-Euclidean Plates: Thin Sheets and Dimension Reduction

We now turn to thin sheets. Let w denote the coordinate through the thickness. The three-dimensional metric induced by a surface may be expanded as

$$g_{ij}(x, w) = \begin{pmatrix} a_{\alpha\beta} - 2wb_{\alpha\beta} + w^2c_{\alpha\beta} & 0 \\ 0 & 1 \end{pmatrix} + \dots \quad (\text{A.1})$$

Similarly, the reference metric may be expanded as

$$\bar{g}_{ij}(x, w) = \begin{pmatrix} \bar{a}_{\alpha\beta} - 2w\bar{b}_{\alpha\beta} + w^2\bar{c}_{\alpha\beta} & 0 \\ 0 & 1 \end{pmatrix} + \dots \quad (\text{A.2})$$

Here $a_{\alpha\beta}$ is the first fundamental form and $b_{\alpha\beta}$ is the second fundamental form of the mid-surface. We now substitute these metrics into the definition of strain and then into the energy,

$$E = \int_{\Omega} \frac{1}{8} A^{ijkl} (g_{ij} - \bar{g}_{ij}) (g_{kl} - \bar{g}_{kl}) dV_{\bar{g}}. \quad (\text{A.3})$$

After integrating through the thickness, the term that mixes the metric and curvature strains vanishes, and the elastic energy becomes

$$E = \frac{h}{8} \int_{\omega} A^{\alpha\beta\gamma\delta} (a_{\alpha\beta} - \bar{a}_{\alpha\beta}) (a_{\gamma\delta} - \bar{a}_{\gamma\delta}) \sqrt{|\bar{a}|} d^2x \quad (\text{A.4})$$

$$+ \frac{h^3}{24} \int_{\omega} A^{\alpha\beta\gamma\delta} (b_{\alpha\beta} - \bar{b}_{\alpha\beta}) (b_{\gamma\delta} - \bar{b}_{\gamma\delta}) \sqrt{|\bar{a}|} d^2x + \dots \quad (\text{A.5})$$

The first term is stretching and scales as h . The second term is bending and scales as h^3 .

Remark: Monge-patch representation

For a nearly flat sheet one may work directly with a configuration in Monge form,

$$\phi(x, y) = (x + u_x(x, y), y + u_y(x, y), w(x, y)). \quad (\text{A.6})$$

Then the metric and curvature are

$$a_{\alpha\beta} \simeq \delta_{\alpha\beta} + \partial_{\alpha}u_{\beta} + \partial_{\beta}u_{\alpha} + \partial_{\alpha}w \partial_{\beta}w, \quad b_{\alpha\beta} \simeq \partial_{\alpha}\partial_{\beta}w. \quad (\text{A.7})$$

Since a and b come from an actual map, the Gauss and Mainardi–Codazzi compatibility conditions are automatic. Minimizing the stretching–bending energy with respect to u_{α} and w gives the Föppl–von Kármán equations. In the absence of lateral or transverse loads,

$$\partial_{\beta}\sigma^{\alpha\beta} = 0, \quad B\Delta^2w = \sigma^{\alpha\beta}\partial_{\alpha}\partial_{\beta}w. \quad (\text{A.8})$$

A.1.1. Gauss and Mainardi–Codazzi compatibility

The intrinsic and extrinsic geometries of a surface are not independent. They must satisfy the Gauss equation

$$K_G(a) = \det(a^{-1}b), \quad (\text{A.9})$$

and the Mainardi–Codazzi equations

$$\nabla_\alpha b_{\beta\gamma} = \nabla_\beta b_{\alpha\gamma}. \quad (\text{A.10})$$

Unlike the actual fields, the reference fields need not satisfy the compatibility conditions. When they do not, the elastic sheet is frustrated because its preferred metric and curvature cannot be realized simultaneously. We distinguish between two kinds of incompatibility: Gauss incompatibility,

$$K_G(\bar{a}) \neq \det(\bar{a}^{-1}\bar{b}), \quad (\text{A.11})$$

and Mainardi–Codazzi incompatibility,

$$\bar{\nabla}_\alpha \bar{b}_{\beta\gamma} \neq \bar{\nabla}_\beta \bar{b}_{\alpha\gamma}. \quad (\text{A.12})$$

Bilayer of fibers. A simple model for a bilayer of fibers, as in *Bauhinia* seed pods, is

$$\bar{a} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 0 & \kappa \\ \kappa & 0 \end{pmatrix}. \quad (\text{A.13})$$

These reference fields violate the Gauss compatibility condition.

Rose petals. In polar coordinates a simplified reference geometry is

$$\bar{a} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix}. \quad (\text{A.14})$$

Then

$$K_G(\bar{a}) = 0, \quad \det(\bar{a}^{-1}\bar{b}) = 0, \quad (\text{A.15})$$

so the Gauss equation is satisfied. However,

$$\bar{\nabla}_\alpha \bar{b}_{\beta\gamma} \neq \bar{\nabla}_\beta \bar{b}_{\alpha\gamma}, \quad (\text{A.16})$$

so the Mainardi–Codazzi equations are violated. The symmetric solution may lose stability, and a lower-energy nonsymmetric configuration is obtained.

A.2. Curvature potentials for non-Euclidean plates

The stretching and bending strains are therefore

$$\varepsilon_{\alpha\beta}^{\text{st}} = \frac{1}{2}(a_{\alpha\beta} - \bar{a}_{\alpha\beta}), \quad \varepsilon_{\alpha\beta}^{\text{bn}} = \frac{1}{2\sqrt{3}}(b_{\alpha\beta} - \bar{b}_{\alpha\beta}). \quad (\text{A.17})$$

A. Extra Material on Thin Sheets

The elastic energy is

$$E = \frac{1}{2} \int_{\Omega} \left(h A^{\alpha\beta\gamma\delta} \varepsilon_{\alpha\beta}^{\text{st}} \varepsilon_{\gamma\delta}^{\text{st}} + h^3 A^{\alpha\beta\gamma\delta} \varepsilon_{\alpha\beta}^{\text{bn}} \varepsilon_{\gamma\delta}^{\text{bn}} \right) dS. \quad (\text{A.18})$$

Here h is the sheet thickness and $A^{\alpha\beta\gamma\delta}$ is the elastic tensor of the two-dimensional material.

The corresponding stretching stress and bending moment are

$$\sigma^{\alpha\beta} = \frac{\partial E}{\partial \varepsilon_{\alpha\beta}^{\text{st}}} = h A^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta}^{\text{st}}, \quad m^{\alpha\beta} = \frac{\partial E}{\partial \varepsilon_{\alpha\beta}^{\text{bn}}} = \frac{h^3}{2\sqrt{3}} A^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta}^{\text{bn}}. \quad (\text{A.19})$$

It is useful to define the effective in-plane stress

$$\Sigma^{\alpha\beta} = \sigma^{\alpha\beta} + m^{\mu\beta} s^{\alpha}_{\mu}, \quad s^{\alpha}_{\mu} = a^{\alpha\gamma} b_{\gamma\mu}, \quad (\text{A.20})$$

where $s = a^{-1}b$ is the shape operator.

The equilibrium equations are obtained by varying the energy with respect to the configuration ϕ . A virtual displacement can be decomposed into a tangential part and a normal part,

$$\delta\phi = \zeta^{\alpha} \partial_{\alpha} \phi + \eta \hat{n}. \quad (\text{A.21})$$

The tangential variation changes the parametrized distances along the surface, while the normal variation bends the surface. The corresponding variations of the fundamental forms have the schematic form

$$\delta a_{\alpha\beta} = \nabla_{\alpha} \zeta_{\beta} + \nabla_{\beta} \zeta_{\alpha} - 2\eta b_{\alpha\beta}, \quad (\text{A.22})$$

$$\delta b_{\alpha\beta} = \mathcal{L}_{\zeta} b_{\alpha\beta} + \nabla_{\alpha} \nabla_{\beta} \eta - \eta b_{\alpha\gamma} b^{\gamma}_{\beta}, \quad (\text{A.23})$$

where $\mathcal{L}_{\zeta}$ denotes the Lie derivative along the tangential field ζ . After integration by parts, the coefficients of the arbitrary fields ζ^{α} and η must vanish separately. The coefficient of ζ^{α} gives the tangential, or in-plane, force balance. The coefficient of η gives the normal force balance, in which bending moments are balanced against the normal projection of the in-plane stresses.

In the notation of geometric plate theory these two bulk equations can be written as

$$0 = (\text{div}_{a,\bar{a}})_{\beta} \Sigma^{\alpha\beta} + (\text{div}_{a,\bar{a}})_{\beta} m^{\mu\beta} a^{\gamma\alpha} b_{\gamma\mu}, \quad (\text{A.24})$$

$$0 = \bar{\nabla}_{\alpha} \left[(\text{div}_{a,\bar{a}})_{\beta} m^{\alpha\beta} \right] - \Sigma^{\alpha\beta} b_{\alpha\beta}. \quad (\text{A.25})$$

The mixed divergence operator is

$$(\text{div}_{a,\bar{a}})_{\mu} T^{\mu\nu} = \bar{\nabla}_{\mu} T^{\mu\nu} + (\Gamma^{\nu}_{\alpha\mu} - \bar{\Gamma}^{\nu}_{\alpha\mu}) T^{\alpha\mu}. \quad (\text{A.26})$$

The appearance of both a and $\bar{a}$ in this operator is one reason why the fully geometric equations are nonlinear and implicit: the differential operator itself depends on the unknown metric.

For the Föppl–von Kármán scaling, one assumes small stretching strain and small bending strain, with $\bar{b} = 0$, so that the difference $\Gamma - \bar{\Gamma}$ is higher order. To leading order the equilibrium equations reduce to

$$\bar{\nabla}_{\beta} \sigma^{\alpha\beta} = 0, \quad (\text{A.27})$$

$$\bar{\nabla}_{\alpha} \bar{\nabla}_{\beta} m^{\alpha\beta} - \sigma^{\alpha\beta} b_{\alpha\beta} = 0. \quad (\text{A.28})$$

The first equation is the in-plane force balance. The second is the normal force balance. It states that the transverse force generated by gradients of bending moments is balanced by the work done by in-plane stresses on a curved surface.

The equilibrium equations are not yet a closed theory for the fundamental forms a and b , because not every pair (a, b) comes from an embedded surface. They must also satisfy the compatibility equations

$$K_G(a) = \det(a^{-1}b), \quad (\text{A.29})$$

$$\nabla_\alpha b_{\beta\gamma} - \nabla_\beta b_{\alpha\gamma} = 0. \quad (\text{A.30})$$

The first equation is Gauss' theorem: the intrinsic Gaussian curvature of the metric equals the determinant of the shape operator. The second equation is the Mainardi–Codazzi compatibility condition, expressing the integrability of the normal field. The reference forms $(\bar{a}, \bar{b})$ need not satisfy these conditions. Their failure to do so is precisely the geometric frustration of a non-Euclidean plate. The actual forms (a, b) , however, must always satisfy these conditions.

The curvature-potential formulation is based on replacing the direct configuration variables by scalar potentials that solve the two differential compatibility problems. The first scalar is the Airy stress potential χ . In the FvK limit the in-plane equilibrium equation $\bar{\nabla}_\beta \sigma^{\alpha\beta} = 0$ is solved by writing the stress as a double curl,

$$\sigma^{\alpha\beta} = \bar{E}^{\alpha\gamma} \bar{E}^{\beta\delta} \bar{\nabla}_\gamma \bar{\nabla}_\delta \chi, \quad \bar{E}^{\alpha\beta} = \frac{\epsilon^{\alpha\beta}}{\sqrt{|\bar{a}|}}. \quad (\text{A.31})$$

For a flat reference metric this reduces to the usual expressions

$$\sigma^{xx} = \partial_y^2 \chi, \quad \sigma^{yy} = \partial_x^2 \chi, \quad \sigma^{xy} = -\partial_x \partial_y \chi. \quad (\text{A.32})$$

Thus the stress potential automatically enforces in-plane force balance.

The second scalar is the curvature potential ψ . In the small-strain regime, and locally for a Euclidean or nearly Euclidean reference metric, the Mainardi–Codazzi equation is solved by writing

$$b_{\alpha\beta} = \bar{\nabla}_\alpha \bar{\nabla}_\beta \psi. \quad (\text{A.33})$$

This is the key replacement of the classical height function by an intrinsic curvature-generating field. If the surface is a shallow graph $z = f(x, y)$, then $b_{\alpha\beta} \simeq \partial_\alpha \partial_\beta f$, so that ψ coincides with the height function f . In general, however, ψ is not a height. It is a scalar potential whose Hessian gives the curvature tensor, and the configuration is reconstructed only later from a and b by solving the Weingarten equations.

The Gauss compatibility condition then becomes the stretching equation. In the FvK approximation the metric is written as a small perturbation of the reference metric,

$$a_{\alpha\beta} = \bar{a}_{\alpha\beta} + 2u_{\alpha\beta}. \quad (\text{A.34})$$

Using the isotropic stress–strain relation to express $u_{\alpha\beta}$ in terms of $\sigma^{\alpha\beta}$, and then using the Airy representation of $\sigma^{\alpha\beta}$, the linearized intrinsic curvature of the metric gives

$$\frac{1}{Y} \bar{\Delta} \bar{\Delta} \chi = \det(\bar{a}^{-1}b) - \bar{K}_G. \quad (\text{A.35})$$

A. Extra Material on Thin Sheets

Here Y is the two-dimensional Young modulus and $\bar{K}_G = K_G(\bar{a})$. This equation says that Gaussian curvature is the source of stretching stress. For a non-Euclidean reference metric, $\bar{K}_G$ acts as a prescribed geometric source. Bending can partially screen this source through the extrinsic term $\det(\bar{a}^{-1}b)$.

For an isotropic plate with $\bar{b} = 0$, the bending moment in the FvK regime is

$$m^{\alpha\beta} = B \left[(1 - \nu)b^{\alpha\beta} + \nu\bar{a}^{\alpha\beta}b^\gamma_\gamma \right], \quad B = \frac{Yh^2}{12(1 - \nu^2)}. \quad (\text{A.36})$$

Substituting $b_{\alpha\beta} = \bar{\nabla}_\alpha \bar{\nabla}_\beta \psi$ into the normal force balance gives

$$B\bar{\Delta}\bar{\Delta}\psi = \sigma^{\alpha\beta}b_{\alpha\beta}. \quad (\text{A.37})$$

Equivalently,

$$Yh^2\bar{\Delta}\bar{\Delta}\psi = 12(1 - \nu^2)\sigma^{\alpha\beta}b_{\alpha\beta}. \quad (\text{A.38})$$

The curvature-potential form of the FvK equations is therefore

$$\boxed{\frac{1}{Y}\bar{\Delta}^2\chi = \det(\bar{a}^{-1}\bar{\nabla}\bar{\nabla}\psi) - \bar{K}_G} \quad (\text{A.39})$$

$$\boxed{\frac{Yh^2}{12(1 - \nu^2)}\bar{\Delta}^2\psi = \sigma^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\psi} \quad (\text{A.40})$$

with

$$\sigma^{\alpha\beta} = \bar{E}^{\alpha\gamma}\bar{E}^{\beta\delta}\bar{\nabla}_\gamma\bar{\nabla}_\delta\chi. \quad (\text{A.41})$$

These equations have the same structure as the classical FvK equations, but the field ψ is interpreted as a curvature potential rather than as a height function.

In flat coordinates, where $\bar{a}_{\alpha\beta} = \delta_{\alpha\beta}$, these equations become

$$\frac{1}{Y}\Delta^2\chi = \psi_{,xx}\psi_{,yy} - \psi_{,xy}^2 - \bar{K}_G, \quad (\text{A.42})$$

$$\frac{Yh^2}{12(1 - \nu^2)}\Delta^2\psi = \chi_{,yy}\psi_{,xx} + \chi_{,xx}\psi_{,yy} - 2\chi_{,xy}\psi_{,xy}. \quad (\text{A.43})$$

Using the bracket

$$[f, g] = f_{,xx}g_{,yy} + f_{,yy}g_{,xx} - 2f_{,xy}g_{,xy}, \quad (\text{A.44})$$

this can be written as

$$\frac{1}{Y}\Delta^2\chi = \frac{1}{2}[\psi, \psi] - \bar{K}_G, \quad \frac{Yh^2}{12(1 - \nu^2)}\Delta^2\psi = [\chi, \psi]. \quad (\text{A.45})$$

The advantage of the curvature-potential formulation is that it keeps the scalar-potential structure of FvK theory while removing the need to represent the surface as a single-valued height graph. Once χ and ψ are known, the curvature tensor follows from $b_{\alpha\beta} = \bar{\nabla}_\alpha\bar{\nabla}_\beta\psi$, and

A. *Extra Material on Thin Sheets*

the metric follows from the stress–strain relation. The actual configuration is then recovered from the Weingarten equations

$$\partial_\alpha \hat{n} = -b^\mu{}_\alpha \partial_\mu \phi, \quad \partial_\alpha \partial_\beta \phi = \Gamma^\mu_{\alpha\beta} \partial_\mu \phi + b_{\alpha\beta} \hat{n}. \quad (\text{A.46})$$

Thus the potentials determine the intrinsic and extrinsic geometry first, and the embedding is reconstructed only afterward. This is why the formulation can describe large rotations, multi-valued configurations, and non-Euclidean geometric frustration while retaining the analytic form of the FvK equations.