BSS 2026 Lecture Notes — Xiaoming Mao

Geometry & Topology of Waves in Soft & Structured Matter

Lecture 1 Multiscale physics of waves in soft and structured matter

1.1. Waves in soft matter across scales

Wave equation

$u(x,t)$: wave field

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

c : wave speed

Q1: Where is this equation from?

(a) In elasticity $\quad \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$

local acceleration = net force on element = restoring force from neighbors

(b) More general: Lagrangian $\rightarrow$ $\partial S = 0$ $\rightarrow$ wave eqn

(HW1) (Euler-Lagrangian equation)

Similar formulasm for EM fields

One might say (b) is better as it's more "fundamental". However, it's important to remember that elasticity is a theory for materials, so it is always a coarse-grained theory from smaller DOFs (degrees of freedom).

Therefore, we need to think about the multiscale physics of where this comes from

nm (10^{-9} m)	μm (10^{-6} m)	mm (10^{-3} m)

$\boxed{\text{Bulk}}$

atomic/molecular crystals | nanoparticle (<100nm) colloids assemblies, gels, foams, ... | metamaterials granular matter

fields : displacements | displacements | displacements/rotations

forces : chemical bonds | $\boxed{\text{coarse-grained}}$ | elasticity

examples:

$U_{DLVO} = U_{vdw} + U_{EDL}$ (electrostatic double layer)

$U_{depletion} = \underset{\substack{\uparrow \\ \text{osmotic pressure}}}{\Pi_d} V_{overlap}$

$\boxed{\text{Thin/anisotropic/low-D}}$

low D embeded in high D, or partial symmetry breaking (polymers, membranes, interfaces) (liquid crystals)

→ always some soft DOFs → strong entropic corrections

fields : displacements, tangents, orientations/directors, curvatures

forces : energetic + coarse-grained entropic forces

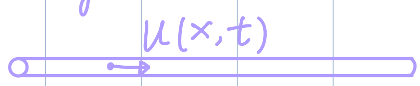
Bottom line: coefficients in the wave equation for elasticity come from $\sim 10^{23}$ interacting microscopic DOFs!

2. Ideal scalar and vector wave equations (brief review)

(1) Scalar : $\partial_t^2 u = c^2 \nabla^2 u$

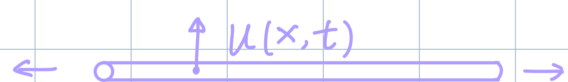
(This is a special limit.

Can describe longitudinal waves on rods :



$c^2 = \frac{E}{\rho}$

Or transverse waves on tensioned rods



$c^2 = \frac{\sigma}{\rho}$

More complete :

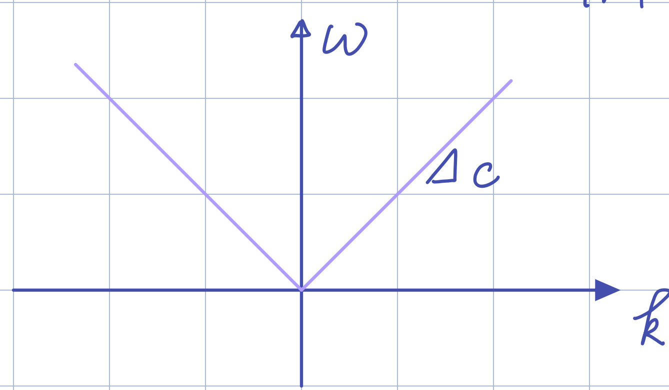
$$\rho A \, \partial_t^2 u = T \partial_x^2 u + EI \, \partial_x^4 u$$

)

General solution : $u(x,t) = u_0 e^{i(kx - \omega t)}$

plug into wave eqn : $\omega^2 = c^2 k^2 \rightarrow \omega = c|k|$

"dispersion relation"



Question :

What if I add a term $\partial_t^2 u = c \, \partial_x^2 u + k u$?

(2) Vector (3D elasticity)

$$\rho \, \partial_t^2 u_i = \partial_j \sigma_{ij} = \partial_j \left(C_{ijkl} \, \epsilon_{kl} \right)$$

strain tensor

$$\epsilon_{kl} = \frac{1}{2}\left(\partial_k u_l + \partial_l u_k \right)$$

in linear theory

If isotropic

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Question: what if chiral symmetry is broken, but keeping isotropy?

Answer: you need C_{ijklm} → strain gradient term. $C_{ijklm}\, \epsilon_{ij}\, \partial_k \epsilon_{lm}$

By symmetry (isotropy) $C_{ijklm} = \varepsilon_{kln}\, \delta_{ij}\, \delta_{mn}$ + permutations

$\uparrow$ Levi-Civita symbol

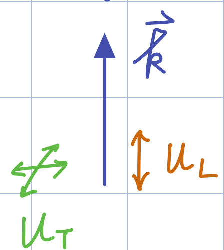
So $$\rho\, \partial_t^2 u_i = (\lambda+\mu)\, \partial_i \underbrace{\partial_j u_j} + \mu\, \underbrace{\partial_j \partial_j} u_i$$

or $$\rho\, \partial_t^2 \vec{u} = (\lambda+\mu)\, \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) + \mu \nabla^2 \vec{u} \underbrace{+ \chi \nabla^2 (\vec{\nabla}\times\vec{u})}_{\text{if chiral}}$$

(Hodge)

Helmholtz decomposition:

$u = u_L + u_T$ with $\nabla\cdot u_T = 0$, $\nabla\times u_L = 0$



$\omega_L^2 = c_L^2 |k|^2$, $c_L^2 = \dfrac{\lambda+2\mu}{\rho}$

$\omega_T^2 = c_T^2 |k|^2$, $c_T^2 = \dfrac{\mu}{\rho}$

Two interesting limits:

- $c_T = 0$: ordinary fluids
- $c_L = c_T = \dfrac{\mu}{\rho}$ in 2D (bulk modulus $B_{2D} = \mu + \lambda = 0$)

  Wave equation becomes vector Laplacian equation

  $$\rho\, \partial_t^2 \vec{u} = \mu \nabla^2 \vec{u}$$

3. Beyond simple wave equation

As we discussed, the wave equation is a simple coarse-grained picture. Many factors can take us out of this regime.

- Nonlinearity
- Dissipation, viscoelasticity
- Disorder
- Internal resonance

....

discuss

(1) Dissipation:

Simple approximation to add dissipation to the wave equ

$$\rho \partial_t^2 u = K \nabla^2 u + \underbrace{\eta \partial_t \nabla^2 u}_{\text{Viscoelasticity}} - \underbrace{\gamma \partial_t u}_{\text{friction/drag}}$$

Viscoelasticity: stress caused by strain rate

friction/drag $\propto$ velocity of element

We can still plug the plane wave solution

$$u(x,t) = u_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

Now we get complex ω:

$$\omega(k) = \underset{\text{oscillation}}{\omega'(k)} + i\underset{\text{decay}}{\omega''(k)}$$

General picture:

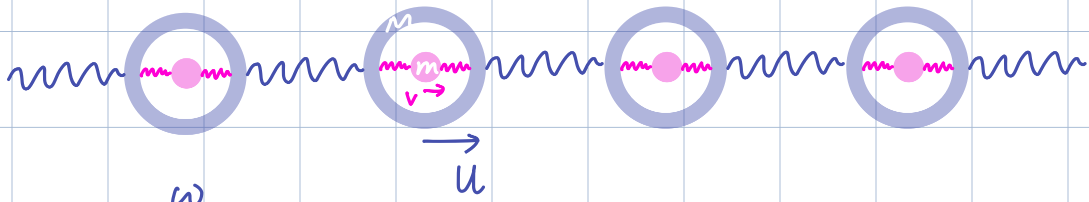
$\longrightarrow k$

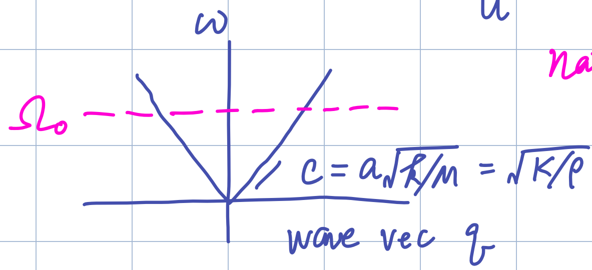
$\leftarrow$ overdamped due to friction | underdamped | overdamped due to viscoelasticity $\rightarrow$

(2) Internal resonance :

local osillatory DOF $\rightarrow$ $\boxed{\text{characteristic } \begin{cases}\text{freq} \\ \text{time}\end{cases} \text{ scale}}$

Start from discrete example





natural freq from resonators : $\Omega_0 = \sqrt{\frac{k_0}{m_r}}$

$$\begin{cases} M\ddot{u}_i = k(u_{i+1} + u_{i-1} - 2u_i) - k_0(u_i - v_i) \\ m_r\ddot{v}_i = -k_0(v_i - u_i) \end{cases}$$

$\downarrow$ continuum limit $\quad \rho = \frac{M}{a}, \; m = \frac{m_r}{a}, \; K = ka, \; \kappa = \frac{k_0}{a}$

$$\begin{cases} \rho\,\partial_t^2 u = K\partial_x^2 u - \kappa(u - v) \\ m\,\partial_t^2 v = -\kappa(v - u) \end{cases}$$

Plug in general solution $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} e^{i(qx - \omega t)}$

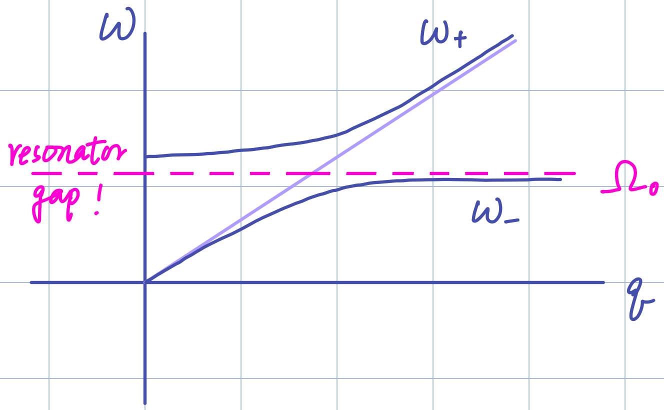
$$\underbrace{\begin{pmatrix} Kq^2 + \kappa - \rho\omega^2 & -\kappa \\ -\kappa & \kappa - m\omega^2 \end{pmatrix}}_{D \;\; (2\times 2 \text{ matrix})} \begin{pmatrix} U \\ V \end{pmatrix} = 0$$

$\det D = 0 \qquad$ 4-th order eqn for ω (2nd order eqn for ω^2)

$$\omega_\pm^2(q) = \frac{1}{2}\left[c^2q^2 + \Omega_*^2 \pm \sqrt{(c^2q^2 + \Omega_*^2)^2 - 4c^2\Omega_0^2 q^2}\right]$$

with $c^2 = \frac{K}{\rho}, \quad \Omega_0^2 = \frac{\kappa}{m}, \quad \Omega_*^2 = \Omega_0^2 + \frac{\kappa}{\rho}$

Simple picture :

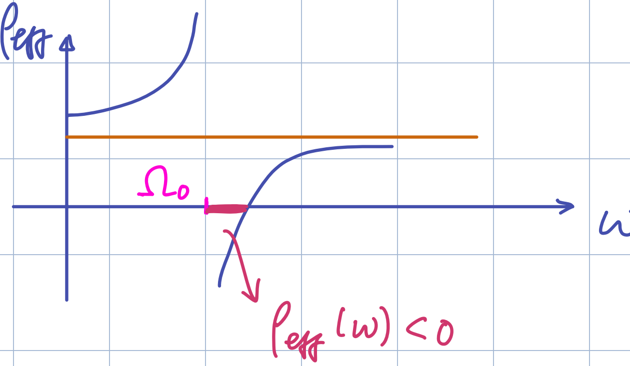


Interesting to mention : If eliminate V

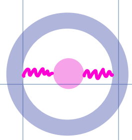
$$V = \frac{\kappa}{\kappa - m\omega^2} U \longrightarrow \text{sub to } U \text{ eqn}$$

$$K q^2 = \omega^2 \rho_{eff}(\omega)$$

$$\rho_{eff}(\omega) = \rho + \frac{m\Omega_0^2}{\Omega_0^2 - \omega^2}$$



(This doesn't need a chain. A single mass-in-mass object does this too



Similar phenomena : negative refractive index)

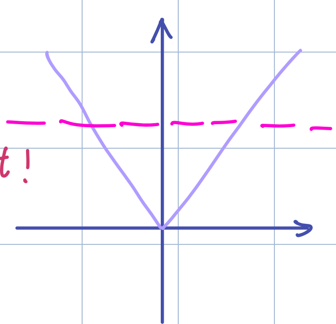
4. Periodic media : Bragg gaps and Bloch waves

Analogy :

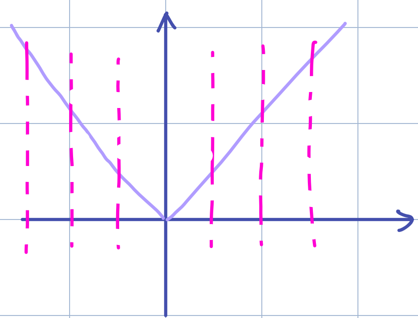
temporal periodicity

- resonators
- periodic driving

very different !



spatial periodicity



?

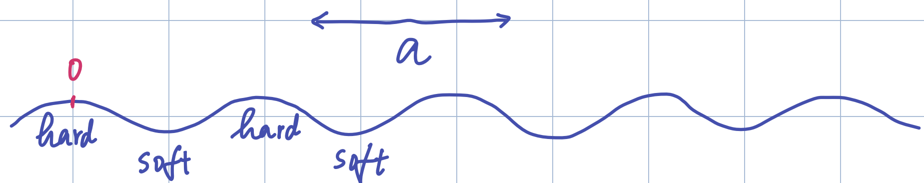
Mech eng. language : "metamatials" ↑ "phononic" ↑ crystals"

(1) Bragg gaps

Consider simple example : $G = 2\pi/a$

$\rho_0 \partial_t^2 u = \partial_x[K(x)\partial_x u]$ with $K(x) = K_0 + 2\delta K \cos(Gx)$

Stiffness modulation



How will a wave $e^{i(qx-\omega t)}$ get scattered by this modulation?

A few ways to think about it

① Elastic energy $E \sim \frac{1}{2}\int K(x)\,|\partial_x u|^2\,dx$ → resonant scattering

displacement

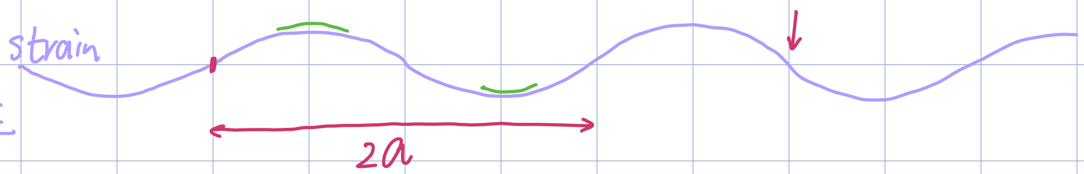
$u_c(x) = \cos\frac{Gx}{2}$

Strain

$\partial_x u_c \sim \sin\frac{Gx}{2}$

high $|\partial u|^2$ at soft

low $|\partial u|^2$ at hard



→ low E → $\omega_c^2 = \frac{G^2}{4\rho_0}(K_0 - \delta K)$ (will solve later)

displacement

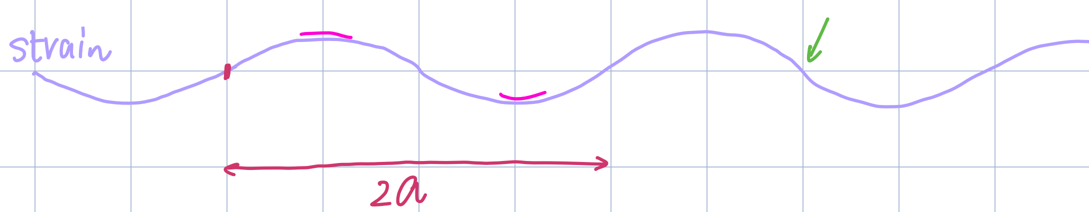
$u_s(x) = \sin\frac{Gx}{2}$

Strain

$\partial_x u_s \sim \cos\frac{Gx}{2}$

high $|\partial u|^2$ at hard

low $|\partial u|^2$ at soft



→ high E → $\omega_s^2 = \frac{G^2}{4\rho_0}(K_0 + \delta K)$ (will solve later)

② General translation sym of space $x \to x + \Delta$

is reduced to discrete translations $x \to x + a$

$$\boxed{\begin{array}{l} \Rightarrow \text{ Momentum conservation } q \\ \text{is reduced to } q \text{ Mod } G \end{array}}$$

$q \to q - G$ is allowed

$\Rightarrow$ Strong scattering when $q = \frac{G}{2} = \frac{\pi}{a}$

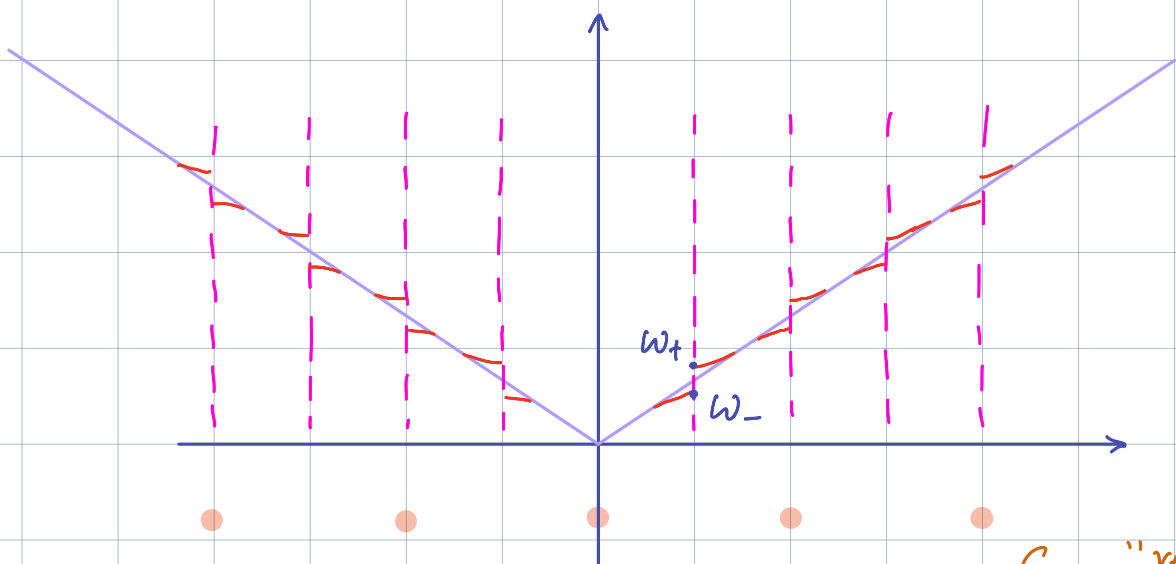
These are $\cos\left(\frac{G}{2}x\right)$ and $\sin\left(\frac{G}{2}x\right)$, composed of

$e^{i\frac{G}{2}x}$, $e^{-i\frac{G}{2}x}$

$\longrightarrow$ $\longleftarrow$

$q = \frac{G}{2}$ $q = -\frac{G}{2}$

lattice scattering



G : "reciprocal lattice vectors"

degeneracy $\begin{cases} e^{iqx} \\ e^{-iqx} \end{cases}$ break into $\begin{cases} \sin \frac{G}{2}x & \omega_+ \\ \cos \frac{G}{2}x & \omega_- \end{cases}$

$(q = \frac{G}{2})$

because of periodic $K(x)$!

(2) Bloch's theorem for continuum elastic waves

Consider a general 1D elastic wave problem

$$\rho\, \partial_t^2 u = \partial_x \left(K(x)\, \partial_x u \right)$$

$$K(x) = K(x+a) \quad \text{discrete periodicity}$$

Consider time-harmonic solutions (still have time translation symmetry)

$$u(x,t) = e^{-i\omega t}\, \psi(x)$$

$$\rho \omega^2 \psi(x) = -\partial_x \left[K(x)\, \partial_x \psi(x) \right]$$

$$\text{def:} \quad \mathcal{L}\psi = -\partial_x \left[K(x)\, \partial_x \psi(x) \right]$$

Also def translation operator $(T_a \psi)(x) = \psi(x+a)$

$$\rightarrow \quad T_a \mathcal{L} \psi(x) = -T_a\, \partial_x \left[K(x)\, \partial_x \psi(x) \right]$$

$$= -\partial_x \left[K(x+a)\, \partial_x \psi(x+a) \right]$$

$$= -\partial_x \left[K(x)\, \partial_x \psi(x+a) \right] = \mathcal{L}\, T_a \psi(x)$$

$\therefore\ [T_a, \mathcal{L}] = 0 \Rightarrow$ We can find a complete orthonormal basis for both simultaneously

$\therefore$ Eigenmodes of $\mathcal{L}$ can be chosen to translate as

$$\psi(x+a) = \lambda\, \psi(x) = e^{iqa}\, \psi(x)$$

$$\begin{cases} \text{must be unitary} \\ \psi(x+a+b) = \lambda_b \lambda_a\, \psi(x) \end{cases}$$

1D representation of translation group

$$\boxed{\psi_q(x+a) = e^{iqa}\, \psi_q(x)} \quad \text{Bloch's theorem}$$

This also means that we can write the wave function as

$$\psi_q(x) = e^{iqx}\, u_q(x) \quad \text{with} \quad u_q(x) = u_q(x+a)$$

periodicity of unit cells

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Geometry & Topology of Waves in Soft & Structured Matter

Lecture 2 Band structure and geometry of mechanical waves

1. Bloch waves

Two parallel formulasm

Discrete lattices "tight-binding model"

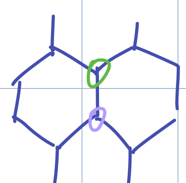
DoFs in unit cell — each md-dm

$$m\partial_t^2 \begin{pmatrix} u_{i,1} \\ u_{i,2} \\ \vdots \\ u_{i,md} \end{pmatrix} = \begin{pmatrix} F_{i,1}(u_{i-1}, u_i, u_{i+1}) \\ \vdots \\ F_{i,md}(\quad) \end{pmatrix}$$

m: # of sites in unit cell

d: space dim

examples



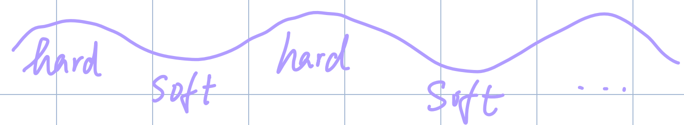
$\downarrow$ F.T.

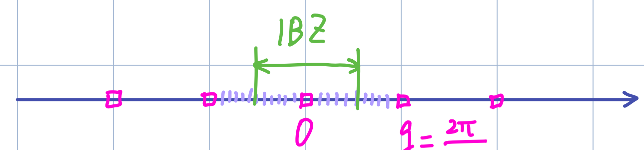
$$\begin{pmatrix} u_{i,1} \\ \vdots \\ u_{i,md} \end{pmatrix} = \sum_k e^{ikx} \begin{pmatrix} u_1(k) \\ \vdots \\ u_{md}(k) \end{pmatrix}$$

Continuum

"nearly-free electron model"

$$\rho\, \partial_t^2 u(x,t) = \partial_x [E(x)\, \partial_x u]$$





$G = \frac{2\pi n}{a}$, $k = \frac{2\pi n}{L}$, $g = \frac{2\pi}{a}$ (shortest G)

$\downarrow$ F.T.

$$E(x) = \sum_G E_G e^{iGx}$$

$$u(x) = \sum_k C_k e^{ikx}$$

$$\rho\omega^2 C_k = \sum_G k(k-G) E_G C_{k-G}$$

For each k, ∞ equations coupled

(block-diagonalized $\infty \to \frac{\infty}{L/a}$)

$$m\omega^2 \begin{pmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_{md}(k) \end{pmatrix} = \begin{pmatrix} \underset{md\times md}{D(k)} \end{pmatrix} \begin{pmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_{md}(k) \end{pmatrix}$$

↑ in basis of site disp ("orbitals")

↳ eigenvalue

$\omega_n(k)$ ↑ md bands

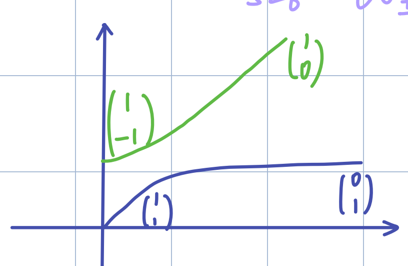
↳ eigen vec

$$\vec{u}^{(n)}(k) = \begin{pmatrix} u_1^{(n)}(k) \\ \vdots \\ u_{md}^{(n)}(k) \end{pmatrix}$$

wave: $e^{ikx}\,\vec{u}^{(n)}(k)$ ↑ md-dim

Example of resonator chain:

$$u_\pm(k) = \begin{pmatrix} 1 \\ \dfrac{\Omega_0^2}{\Omega_0^2 - \omega_\pm^2(k)} \end{pmatrix}$$



There are always real but not general

$$\rho\omega^2 \begin{pmatrix} \vdots \\ c_{k-g} \\ c_k \\ c_{k+g} \\ \vdots \end{pmatrix} = \begin{pmatrix} \underset{\infty\times\infty}{D(k)} \end{pmatrix} \begin{pmatrix} \vdots \\ c_{k-g} \\ c_k \\ c_{k+g} \\ \vdots \end{pmatrix}$$

↑ in basis of periodic fn in unit cell

↳ eigenvalue

$\omega_n(k)$ ↑ ∞ bands

↳ eigen vec

$$c_G^{(n)} \begin{pmatrix} \vdots \\ c_k^{(n)} \\ c_{k+g}^{(n)} \\ \vdots \end{pmatrix}$$

wave:

$$e^{ikx} \underbrace{\sum_G c_{k-G}\, e^{iGx}}_{\equiv \tilde{U}_k(x) = \tilde{U}_k(x+a)}$$

(Bloch's theorem) periodicity of unit cell

Full translation sym T → Full momentum conservation

↓ reduce ↓ reduce

Translation on Lattice T_a → Momentum Mod G Conserved

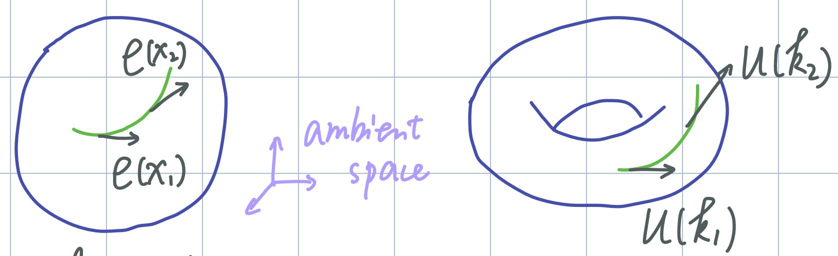
2. From Bloch waves to vector bundles

The Bloch waves we discussed above can be seen as complex vectors:

discrete: $$|u^{(n)}(k)\rangle = \begin{pmatrix} u_1^{(n)}(k) \\ \vdots \\ u_{md}^{(n)}(k) \end{pmatrix} \subset \text{ambient space } \mathbb{C}^{md}$$

continuum: $$|u^{(n)}(k)\rangle = \begin{pmatrix} \vdots \\ C_k^{(n)} \\ C_{k+g}^{(n)} \\ \vdots \end{pmatrix} \subset \text{ambient space } \mathbb{C}^{\infty}$$

analogy



real vector bundle

band bundle

A bit background:

- Vector bundle: $V = \{V_p \mid p \in B\}$
  - V_p: a vector space attached to point p
  - p: point
  - B: ∈ base

- Section: smooth choice of one vector from each fiber

$$v: B \to V, \quad v(p) \in V_p$$

- Inner product: $$\langle u | v \rangle = \sum_i u_i^* v_i$$

$$= \sum_G C^*_{u, k+G} C_{v, k+G} = \int_{\text{unit cell}} dr\, u(r)^* v(r)$$

- There is a $\boxed{\text{gauge degree of freedom}}$:

$$|u_n(k)\rangle \to e^{i\chi(k)}|u_n(k)\rangle$$

is the same wave physically.

Geometric concept	Vector bundle	Band bundle	Gauge transformation
Base	Manifold M	Brillouin zone T^d	
Coordinate	x^μ	k_μ	
Section	$V(x) = V^\mu(x)e_\mu$	$\lvert u_n(k)\rangle$	$\lvert u_n'(k)\rangle = e^{i\theta(k)}\lvert u_n(k)\rangle$
Connection	$\Gamma^a_{\mu b} = e^a \cdot \nabla_\mu e_b$	Berry connection $A_\mu^{(n)}(k) = i\langle u_n(k)\lvert\partial_{k_\mu} u_n(k)\rangle$ (gauge potential)	$A'^{(n)} = A^{(n)} - \nabla_k\theta(k)$
Curvature	$R^a_{b\mu\nu}$ (Riemann tensor)	Berry curvature $\Omega_{\mu\nu}^{(n)}(k) = \partial_{k_\mu}A_\nu^{(n)} - \partial_{k_\nu}A_\mu^{(n)}$ (field strength)	$\Omega'^{(n)}_{\mu\nu} = \Omega^{(n)}_{\mu\nu}$ invariant !
Metric	$g_{\mu\nu}$: $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$	Quantum geometric tensor $Q_{\mu\nu}(k) = \langle\partial_{k_\mu}u_n\lvert(\mathbb{I} - \lvert u_n\rangle\langle u_n\rvert)\rvert\partial_{k_\nu}u_n\rangle = \underset{\text{quantum metric}}{g_{\mu\nu}^{(n)}(k)} + i\,\underset{\text{Berry curvature}}{\Omega_{\mu\nu}^{(n)}(k)}$	$Q'^{(n)}_{\mu\nu} = Q^{(n)}_{\mu\nu}$ invariant

A bit background:

Vector bundle: $V = \{V_p \mid p \in B\}$

a vector space attached to point p

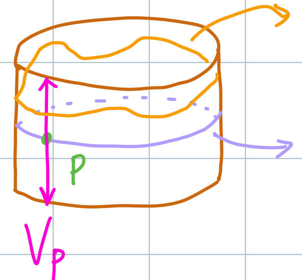
point $\in$ base

Section: smooth choice of one vector from each fiber

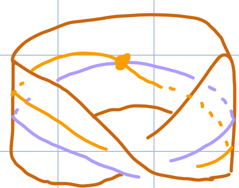
$v: B \to V, \quad v(p) \in V_p$

Example

trivial



nontrivial (twisted)



No-global non-vanishing section exists!

V_p rotates by π across the Mobius strip.

3. Wave topology

With this wave - geometry mapping, we can discuss topology of waves.

(1) The Zak phase

The simplest topological index

(Strictly speaking, the Zak phase is NOT topological unless protected by symmetry ← next lecture!)

We defined the Berry connection

$$A^{(n)}(k) = i \langle u^{(n)}(k) | \nabla_k u^{(n)}(k) \rangle$$

The <u>holonomy</u> of this connection over a 1D 1BZ :

→ the net transformation acquired by an object after it is parallel transported around a closed loop.

$$\gamma_n = \oint_{BZ} A^{(n)}(k)\, dk$$

is called the "Zak phase"

Note : $A^{(n)}(k)$ changes under a gauge transformation

$$A'^{(n)}(k) = A^{(n)}(k) - \nabla_k \chi$$

$$\Rightarrow \quad \gamma_n'(k) \to \gamma_n(k) + 2\pi n$$

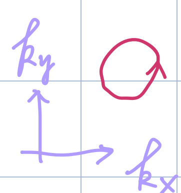
↑ winding of χ

↑ smooth but can change by $2\pi n$ across 1BZ

$\therefore$ $\gamma_n(k)$ is defined modulo 2π

You can also generally defined the "Berry phase" over any loop in a 2D k-space

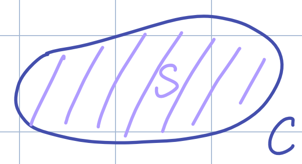
$$\gamma_c = \oint_c \vec{A} \cdot d\vec{k}$$



(2) The (first) Chern number

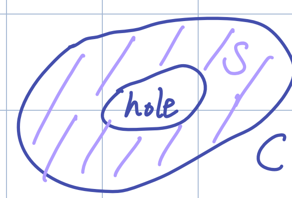
Apply Stoke's theorem to the Berry phase formula on a $\boxed{\text{contractable loop}}$

$$\gamma_c = \oint_C \vec{A}\cdot d\vec{k} = \int_S \Omega_{xy}(\vec{k})\, dk_x\, dk_y \qquad (\mathrm{mod}\ 2\pi)$$



This formula is NOT topological : you can contract the loop to 0

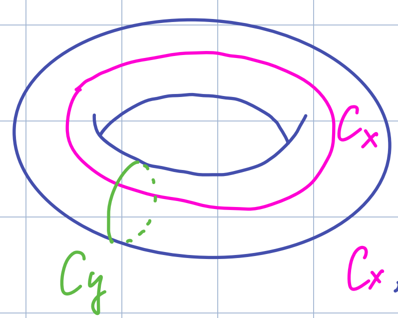
For a $\boxed{\text{non-contractable loop}}$,

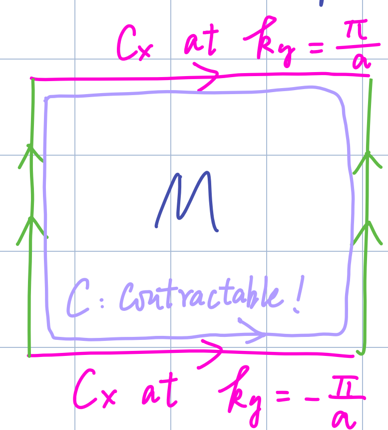


$$\oint_C \vec{A}\cdot d\vec{k} \underset{\text{in general}}{\neq} \int_S \Omega_{xy}\, dk_x\, dk_y$$

Important example of non-contractable loop

1BZ for 2D systems :





The Chern number is defined as

$$C = \frac{1}{2\pi}\int_M \Omega_{xy}(k)\, dk_x\, dk_y \in \mathbb{Z} \qquad \text{(integers)}$$

$$= \frac{1}{2\pi}\int_C \vec{A}\cdot d\vec{k}$$

To see why C has to be integer:

Consider Zak phase associated with k_x at given k_y

$$\gamma_x(k_y) = \oint_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk_x \, A_x(k_x, k_y)$$

k_y and $k_y + G = k_y + \frac{2\pi}{a}$ are the same line

$$\gamma_x(k_y + G) = \gamma_x(k_y) + 2\pi n$$

$\uparrow$ Chern number

$$C = \pm \frac{\gamma_x(k_y+G) - \gamma_x(k_y)}{2\pi} = \underbrace{\frac{1}{2\pi}\int_M \Omega_{xy} \, dk_x dk_y}_{\text{invariant under gauge transformations}}$$

(3) Analogy with Gauss-Bonnet theorem

In week 1 we learned that for closed oriented surfaces

$$\frac{1}{2\pi}\int_M K \, dA = \chi(M)$$

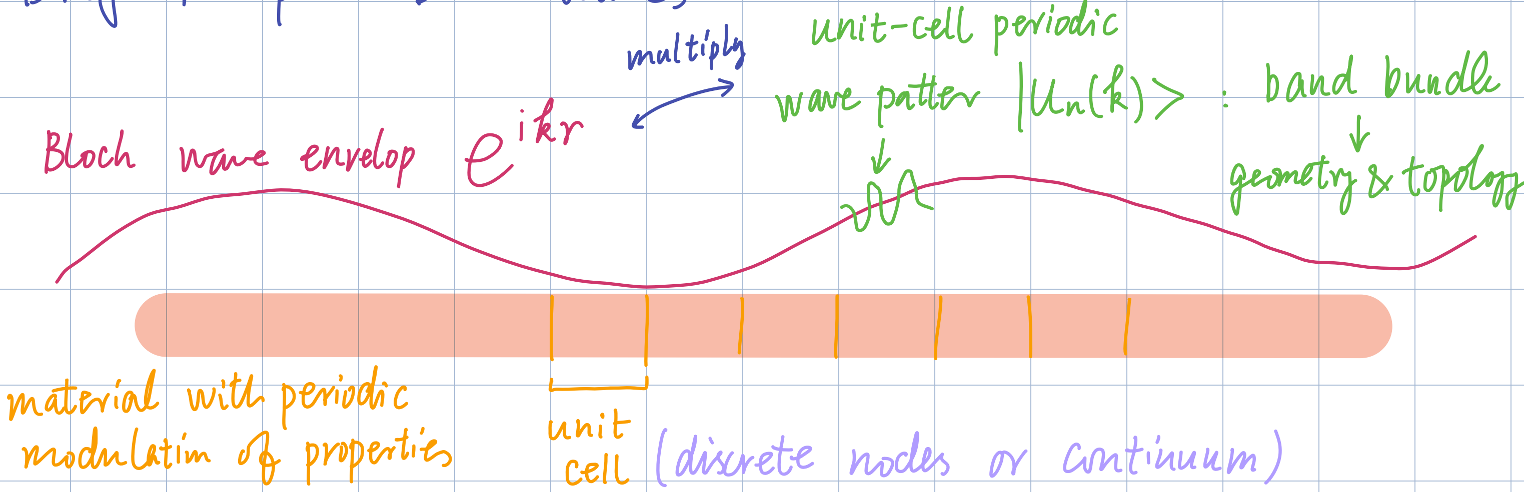
K: gaussian curvature ; $\chi(M)$: Euler characteristic

Here we have, for band bundles:

$$\frac{1}{2\pi}\int_{1BZ} \Omega \, dk = C$$

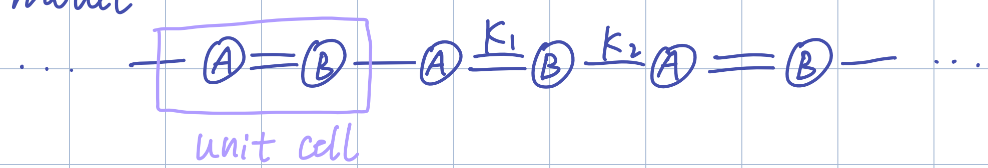
Lecture 3 Wave topology and topological mechanics

Brief recap: Bloch waves



1. Elastic Su-Schrieffer-Heeger (SSH) model

SSH model:



Real space equation of motion (EOS)

$$m\ddot{u}_{A,j} = k_1(u_{B,j} - u_{A,j}) + k_2(u_{B,j-1} - u_{A,j})$$

$$m\ddot{u}_{B,j} = k_1(u_{A,j} - u_{B,j}) + k_2(u_{A,j+1} - u_{B,j})$$

Wave form of solution (Bloch wave)

$$\begin{pmatrix} u_{A,j} \\ u_{B,j} \end{pmatrix} = \begin{pmatrix} u_A(k) \\ u_B(k) \end{pmatrix} e^{i(kja - \omega t)}$$

$$m\omega^2 \begin{pmatrix} u_A(k) \\ u_B(k) \end{pmatrix} = -\underbrace{\begin{pmatrix} K_1+K_2 & -K_1-K_2 e^{-ika} \\ -K_1-K_2 e^{ika} & K_1+K_2 \end{pmatrix}}_{D(k)} \begin{pmatrix} u_A(k) \\ u_B(k) \end{pmatrix}$$

$\uparrow$ dynamical matrix

So $\omega(k)^2$: eigenvalues of $D(k)$

It's often convenient to drop the $(K_1+K_2)\cdot I$ part and define the "chiral 2-band SSH hamiltonian"

$$h_{SSH}(k) = \begin{pmatrix} 0 & K_1+K_2 e^{-ika} \\ K_1+K_2 e^{ika} & 0 \end{pmatrix}$$

Define $d(k) = K_1+K_2 e^{ika}$, we have $h_{SSH}(k) = \begin{pmatrix} 0 & d^*(k) \\ d(k) & 0 \end{pmatrix}$

$$d(k) = |d(k)| e^{i\phi(k)}$$

Solve this 2x2 matrix $\rightarrow$

Eigenvalues $\lambda_\pm = \pm |d(k)|$

Eigenvectors $|u_\pm(k)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm e^{i\phi(k)} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ d/|d| \end{pmatrix}$

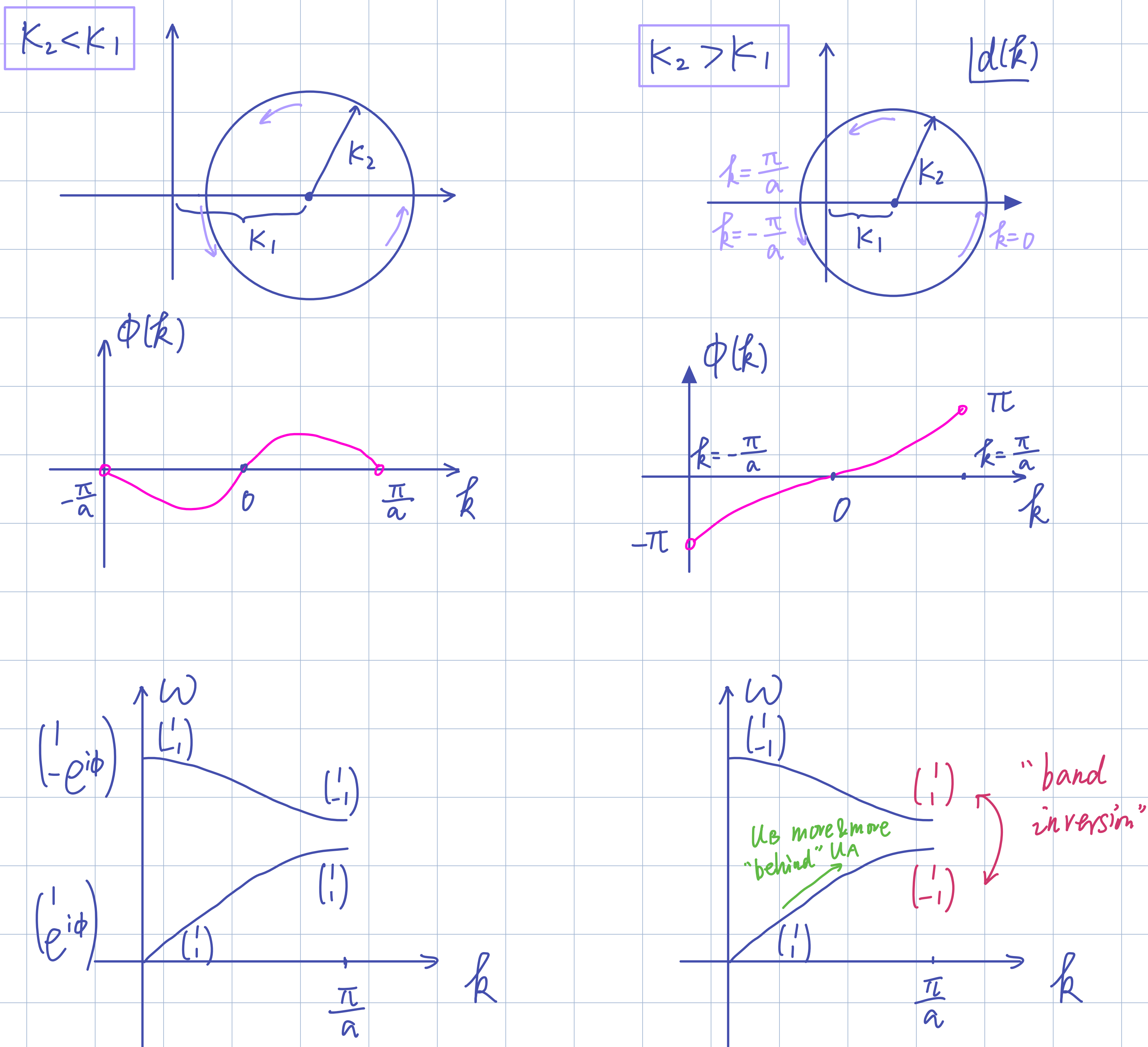
Actual frequencies of these eigenstates

$$\omega_\pm = \left[\frac{K_1+K_2 \pm |d(k)|}{m}\right]^{1/2}$$

To understand the eigen modes:

We analyze $d(k) = K_1+K_2 e^{ika}$ as k goes across the 1BZ: $k \in [-\frac{\pi}{a}, \frac{\pi}{a})$

Two cases :



2. Zak phase and interface modes of the SSH model

With this nice analytic solution we can compute the Zak phase introduced in Lecture 2.

Berry connection :

$$A_\pm(k) = i\langle u_\pm(k) | \partial_k u_\pm(k)\rangle$$
$$= \frac{i}{2}\begin{pmatrix}1 & \pm e^{-i\phi(k)}\end{pmatrix}\begin{pmatrix}0 \\ \pm i\phi'(k) e^{i\phi(k)}\end{pmatrix} = -\frac{1}{2}\phi'(k)$$

Zak phase:

$$\gamma_\pm = \int_{1BZ} A_\pm(k)\, dk = -\frac{1}{2}\oint_{1BZ} \phi'(k)\, dk = -\pi \underset{\substack{\uparrow \\ \text{winding number}}}{N_w} \mod 2\pi$$

$N_w = 0$ for $K_2 < K_1$ $\quad\quad$ $N_w = 1$ for $K_2 > K_1$

$\gamma = 0$ $\quad\quad$ $\gamma = \pi$

Important: $\phi(k)$ winds by 2π, but $\underline{\underline{\gamma = \pi}}$

A few points:

(1) "nontrivial" vs "trivial" $\quad (K_2 < K_1$ vs $K_2 > K_1)$

but it's only a shift of unit cell for the SSH!

- What matters is the change of γ at interfaces (example below)
- In contrast: 2D Chern number is truly "nontrivial".

(2). The Zak phase of this problem is protected by the chiral symmetry $\quad \{\sigma_z, h_{SSH}\} = 0$

To see this we write h_{SSH} as general 2-band:

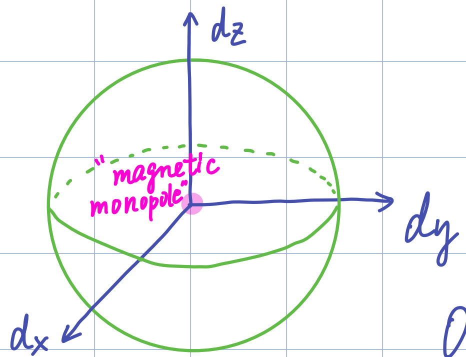
$$h(k) = d_0 I + d_x(k)\sigma_x + d_y(k)\sigma_y + d_z(k)\sigma_z$$

$$\sigma_x = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$ Pauli matrices for spin-$\frac{1}{2}$ particles

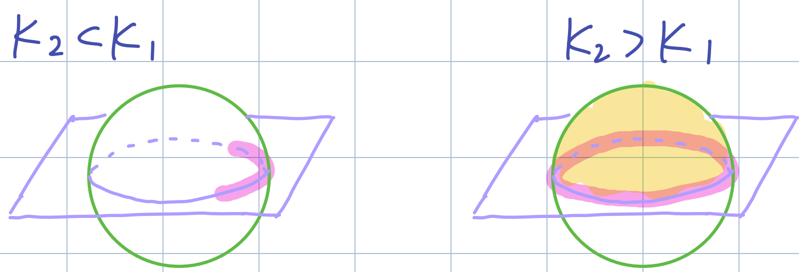
Any 2-band hermitian $\mathcal{H}$ can be written as this

$\rightarrow \vec{d}(k) = (d_x, d_y, d_z)$ is like a B field!

$\vec{k} \xrightarrow[\text{map}]{\vec{d}(\vec{k})}$ Bloch sphere



SSH model has $\boxed{d_z(k) = 0}$ chiral sym



One can show that on the Bloch sphere

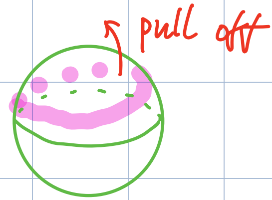
$$\gamma = \oint A = \frac{1}{2}\Omega_{\text{solid}} + 2\pi n$$

$\swarrow 0 \qquad \searrow \pi$

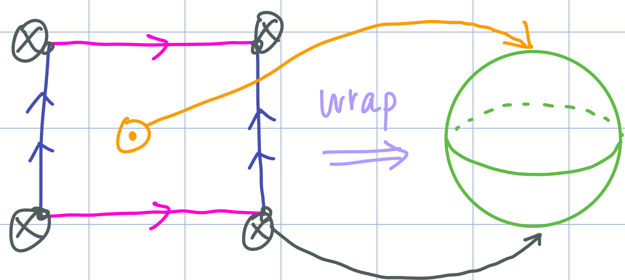
$\uparrow$ protected by chiral sym

If $d_z(k) \neq 0$ is allowed

γ is NOT $\boxed{\text{robust}}$



Quick mention of the Chern number



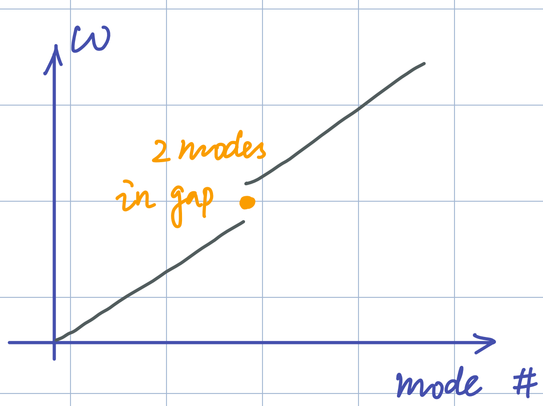
$\Rightarrow$ C as a gauge indep $\mathbb{Z}$

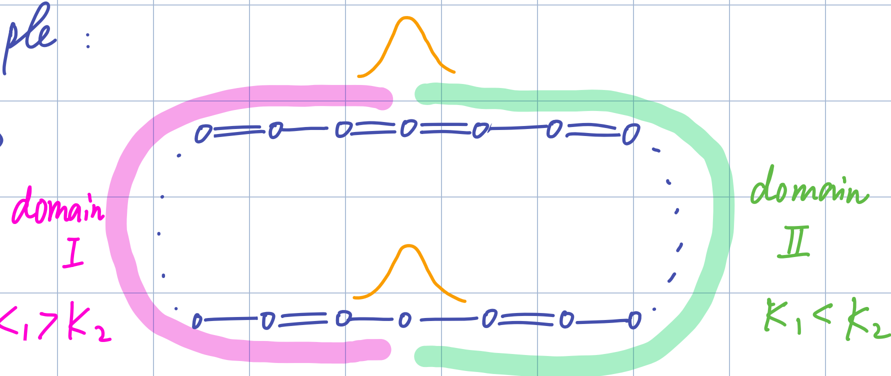
Why does γ matter? $\rightarrow$ Topological interface modes

"bulk-boundary correspondence"

Quick numerical example:

If you compute eigenstates





See lecture notes for theory

2. Maxwell lattices: topology of constraints

An important matrix for mechanics of networks

C : compatibility matrix

Kinematics

$$\underset{N_c \times Nd}{C} \; u = e$$

e: N_c-dim vec of edge extensions

u: $N \cdot d$-dim vec of node-disp

$Q = C^T$: equilibrium matrix

force balance

$$\underset{Nd \times N_c}{Q} \; t = -f$$

t: N_c-dim vec of edge tensions

f: $N \cdot d$-dim vec of node net force

Ker C : zero modes (ZM) — trivial (rigid-body motion of whole) / nontrivial ("soft modes")

Ker Q : states of self stress (SSS)

Rank-nullity theorem on C and C^T

$$\dim(\mathrm{Im}\, C) + \dim(\mathrm{Ker}\, C) = \dim(\mathrm{Domain}\, C)$$

$$\mathrm{Rank}\, C + N_0 = Nd$$

$$\dim(\mathrm{Im}\, Q) + \dim(\mathrm{Ker}\, Q) = \dim(\mathrm{Domain}\, Q)$$

$$\mathrm{Rank}\, Q + N_{sss} = N_c$$

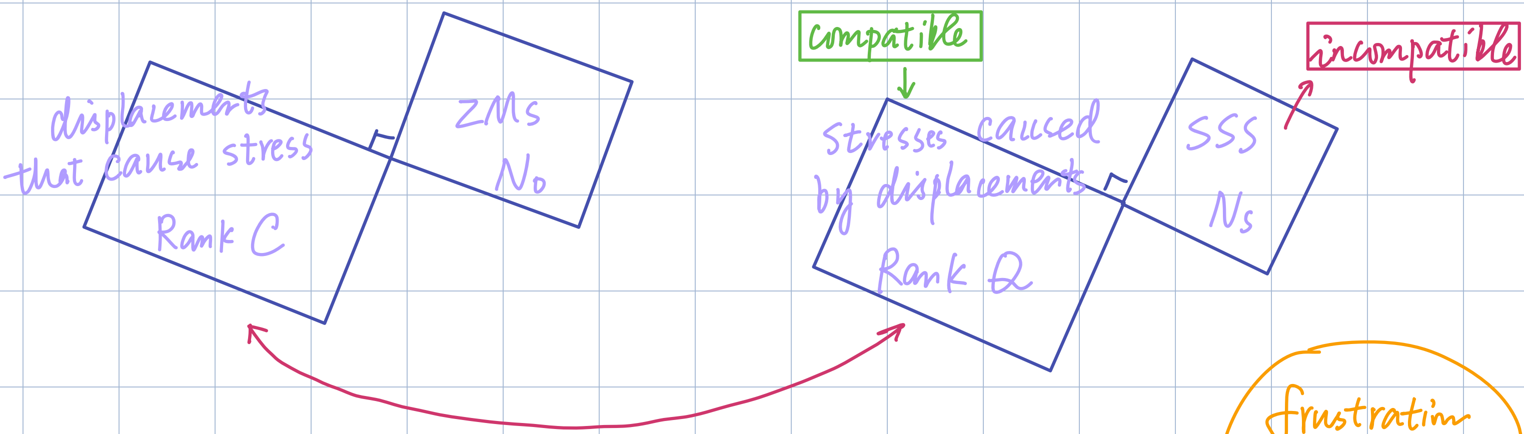
same

$$\Rightarrow \boxed{\underbrace{Nd}_{\text{total DOFs}} - \underbrace{N_c}_{\text{constraints}} = \underbrace{N_0}_{\text{ZMs}} - \underbrace{N_{sss}}_{\text{SSSs}} \quad : \text{Maxwell–Calladine index theorem}}$$

Same structure as Hodge decomposition we discussed in week 1

Domain C : all displacements U
($N_c \times N_d$)

Domain Q : all tensions t
($N_d \times N_c$)



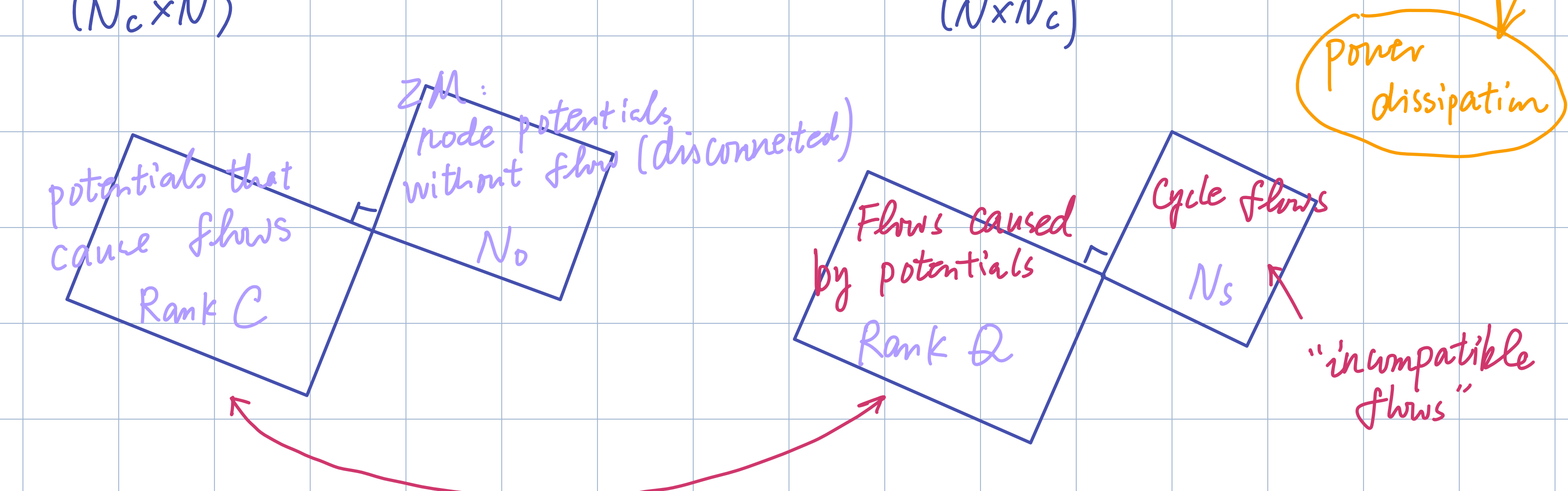
1-to-1 correspondence of
non-0 eigenstates of C and $Q = C^T$

Vector fields on nodes (U: disp, F: (net) force)

Scalar fields on nodes (V: potential, I: (net) flow)

Domain C : all potentials V
($N_c \times N$)

Domain Q : all flows i
($N \times N_c$)



1-to-1 correspondence of
non-0 eigenstates of C and $Q = C^T$

We can also call them :

C : ∂^T or $\hat{\Delta}$
co-boundary operator

Q : ∂ or $\hat{\Delta}^T$
boundary operator

So, what is the Hodge Laplacian?

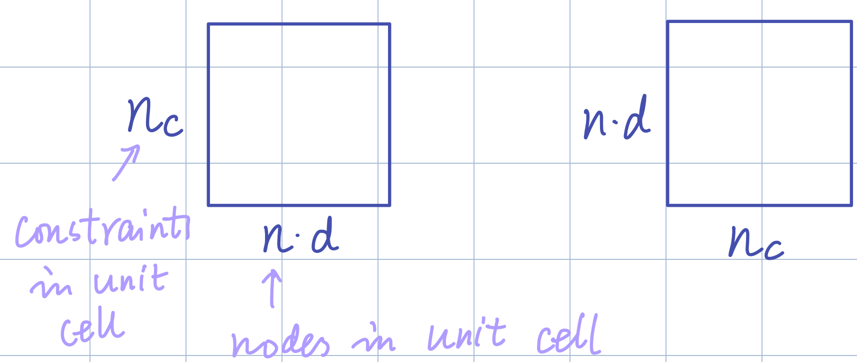
$$\mathcal{L} = \partial^T \partial + \partial \partial^T \qquad ??$$

The Kane-Lubensky theory for topological mechanics is an example of this.

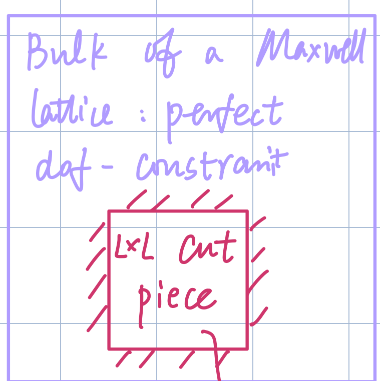
3. Topological mechanics of Maxwell lattices

We first write these matrices in k space

$C(k)$ $\qquad\qquad\qquad$ $Q(k)$



Maxwell lattices: $n \cdot d = n_c$ "verge of instability"



$N_0 \propto L$

(perimeter: sub-extensive)

ZM of a finite piece : $\ker C(k)$

SSS of an interface : $\ker Q(k)$

allowing $k = k' + i k''$

(spatial) oscillation $\qquad$ decay/growth

To solve for $\ker C$, $\ker Q$ we need to find $k \in \mathbb{C}$ that $\det Q(k) = \det C(k) = 0$

This turns out to be controlled by a Zak phase.

Kane & Lubensky defined the "chiral Hamiltonian"

$$H_c(k) = \begin{pmatrix} 0 & Q(k) \\ Q^\dagger(k) & 0 \end{pmatrix}$$

Note that this hamiltonian acts on $\begin{pmatrix} u \\ t \end{pmatrix}$ $\begin{matrix} \leftarrow \text{nods} \\ \leftarrow \text{edges} \end{matrix}$

$$H_c(k)^2 = \begin{pmatrix} 0 & Q \\ Q^\dagger & 0 \end{pmatrix}\begin{pmatrix} 0 & Q \\ Q^\dagger & 0 \end{pmatrix} = \begin{pmatrix} QQ^\dagger & 0 \\ 0 & Q^\dagger Q \end{pmatrix}$$

dynamical matrix D

"supersymmetry partner" of D : $\tilde{D}$

(1) This is an example of Hodge Laplacian (vector potential version)

$$\partial = \begin{array}{c|c} & \begin{matrix} \text{nodes} & \text{edges} \end{matrix} \\ \begin{matrix} \text{nodes} \\ \text{edges} \end{matrix} & \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} \end{array} \qquad \partial^\dagger = \begin{pmatrix} 0 & 0 \\ Q^\dagger & 0 \end{pmatrix}$$

$$H_c = \partial + \partial^\dagger$$

$$H_c^2 = (\partial + \partial^\dagger)^2 = \partial\partial^\dagger + \partial^\dagger\partial$$

(2) Maxwell lattice topological mechanics is then controlled by a Zak phase $\quad H_c(k) \to h_{SSH}(k)$

$$\gamma = -\frac{1}{2}\oint_{1BZ} \phi'(k)\, dk = -\pi N_W \mod 2\pi$$

$\phi(k)$: phase of $\det Q(k)$

(3) A few points to mention (link in LaTeX notes)

- This γ (or N_w) controls SSSs and ZMs localization : a lot more robust than the A-B spring chain. In Maxwell systems A, B are nodes and edges : another layer of topological protection !
- 2D, 3D lattices : N_w for each lattice direction
  $\longrightarrow \quad \vec{R}_T = n_1 \vec{a}_1 + n_2 \vec{a}_2 + n_3 \vec{a}_3$
- The Mod 2π gauge dependence can actually be removed at long wave-length using continuum elasticity ! "Polarized elasticity"
- Many applications : asymmetric surface stiffness
  transformation
  wave guide
  stress focusing
  ...

Topological Mechanics in Maxwell Lattices & Continuum

Xiaoming Mao
Department of Physics, University of Michigan

BSS 2026

Brief Review: Topological Mechanics in Maxwell Lattices

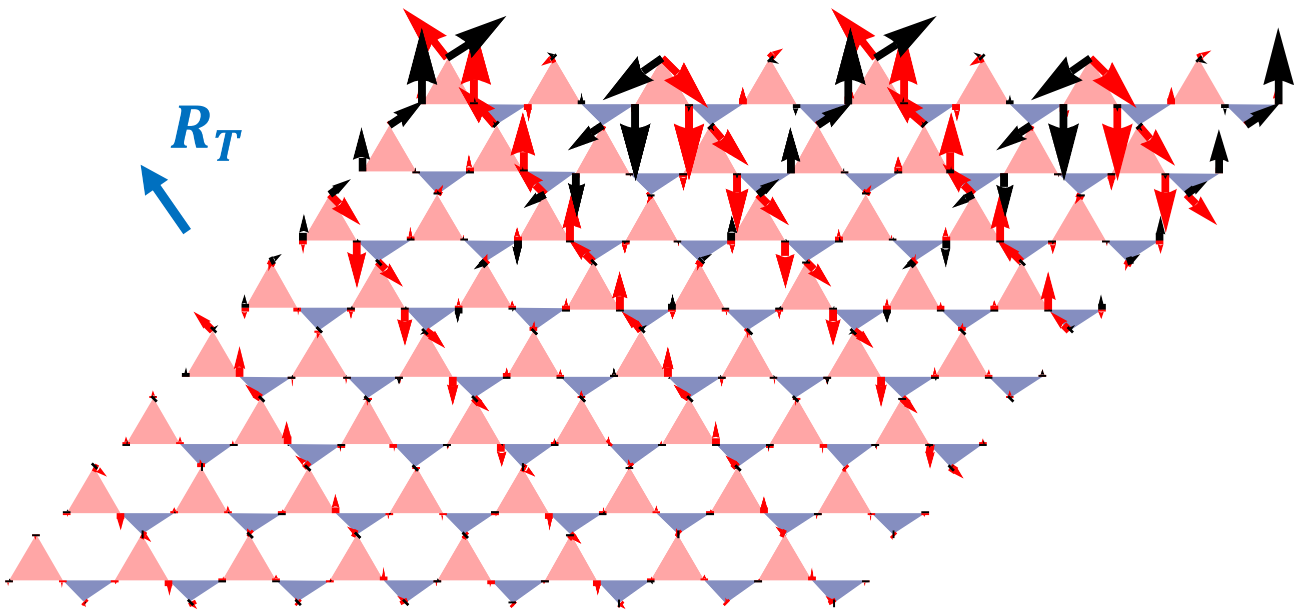
2D lattices:

Topological winding number:

$$n_i = \frac{1}{2\pi i} \oint_{C_i} d\mathbf{q} \cdot \nabla_{\mathbf{q}} \ln \det(C(\mathbf{q}))$$

Topological polarization

$$\boldsymbol{R}_T = n_1 \boldsymbol{a}_1 + n_2 \boldsymbol{a}_2$$



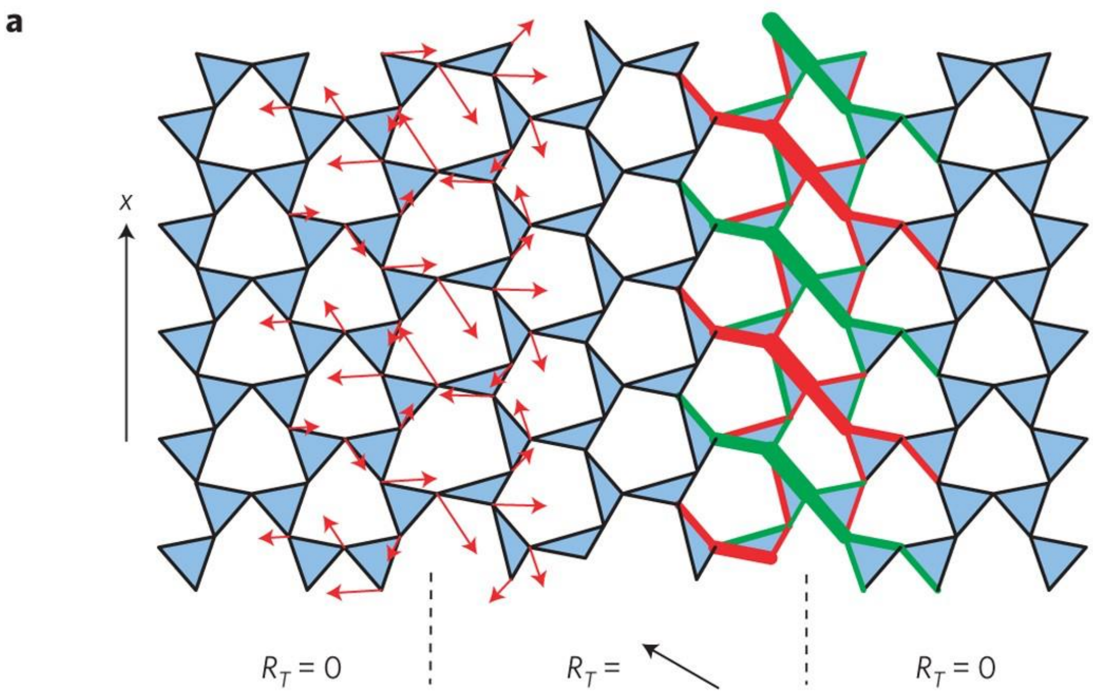
• C. L. Kane and T. C. Lubensky, Nat. Phys. 10, 39 (2014).

Brief Review: Topological Mechanics in Maxwell Lattices

Floppy modes vs. states of self-stress $N_0 - N_S = Nd - N_b$

Programmed locations

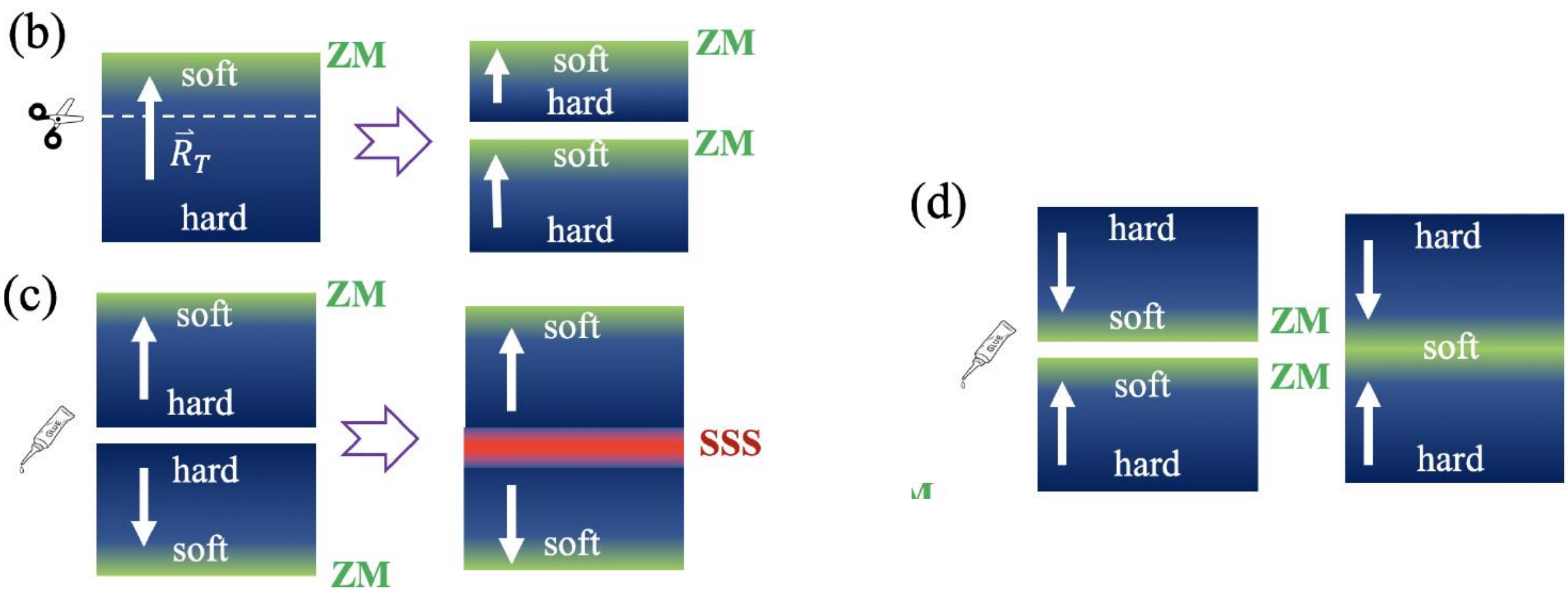
ZMs SSSs



• C. L. Kane and T. C. Lubensky, Nat. Phys. 10, 39 (2014).

Brief Review: Topological Mechanics in Maxwell Lattices

Analogy with magnetic polarization: ZM (N pole), SSS (S pole)



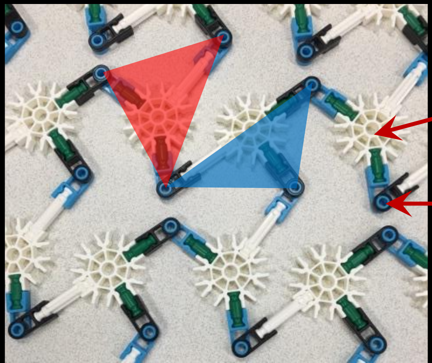
Property coming from bulk instead of edge

→ Highly robust against damage

Transformable topological mechanical metamaterials

D. Zeb Rocklin, Shangnan Zhou,
Kai Sun, Xiaoming Mao

University of Michigan

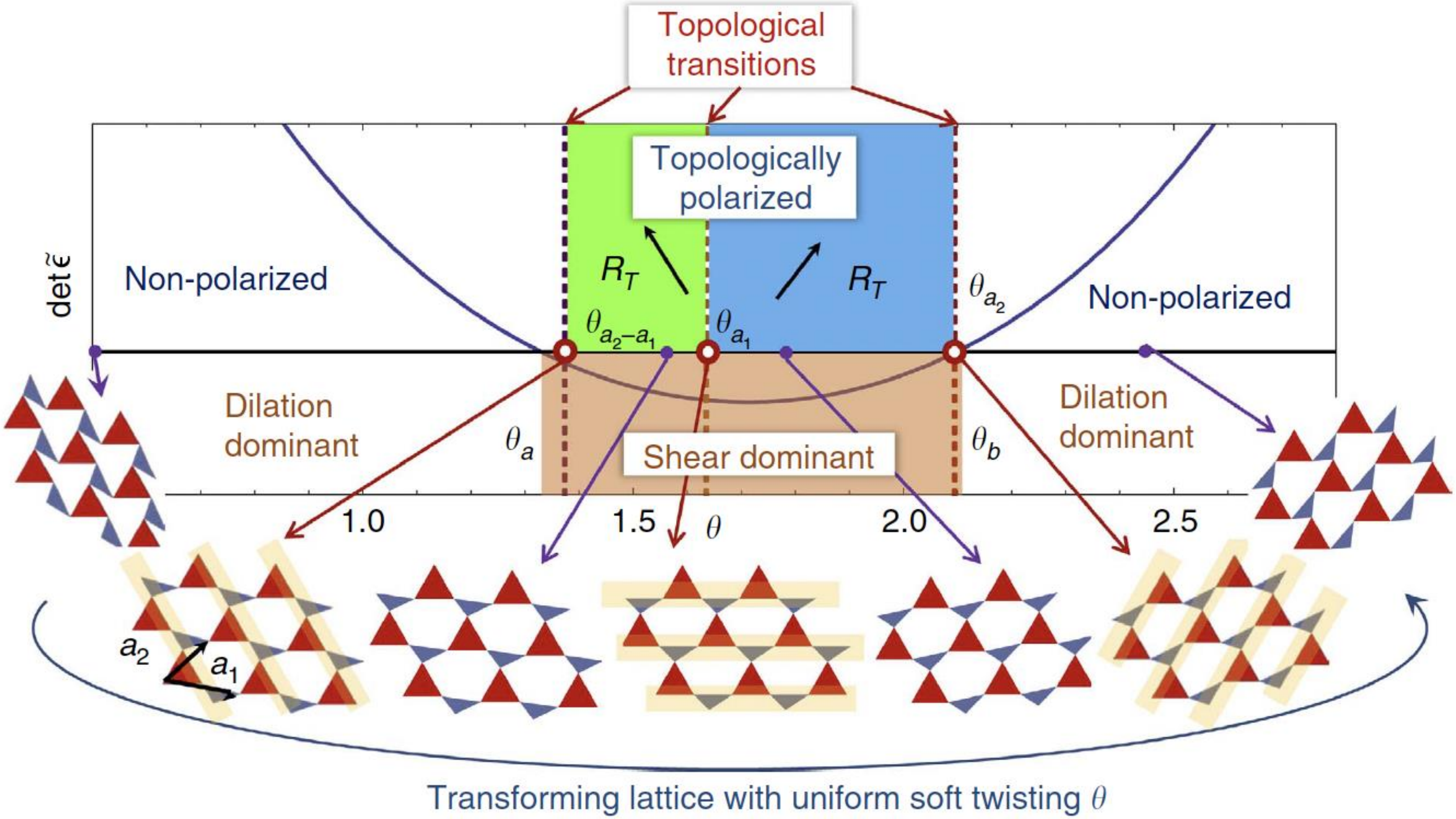


Rigid connector

Hinge

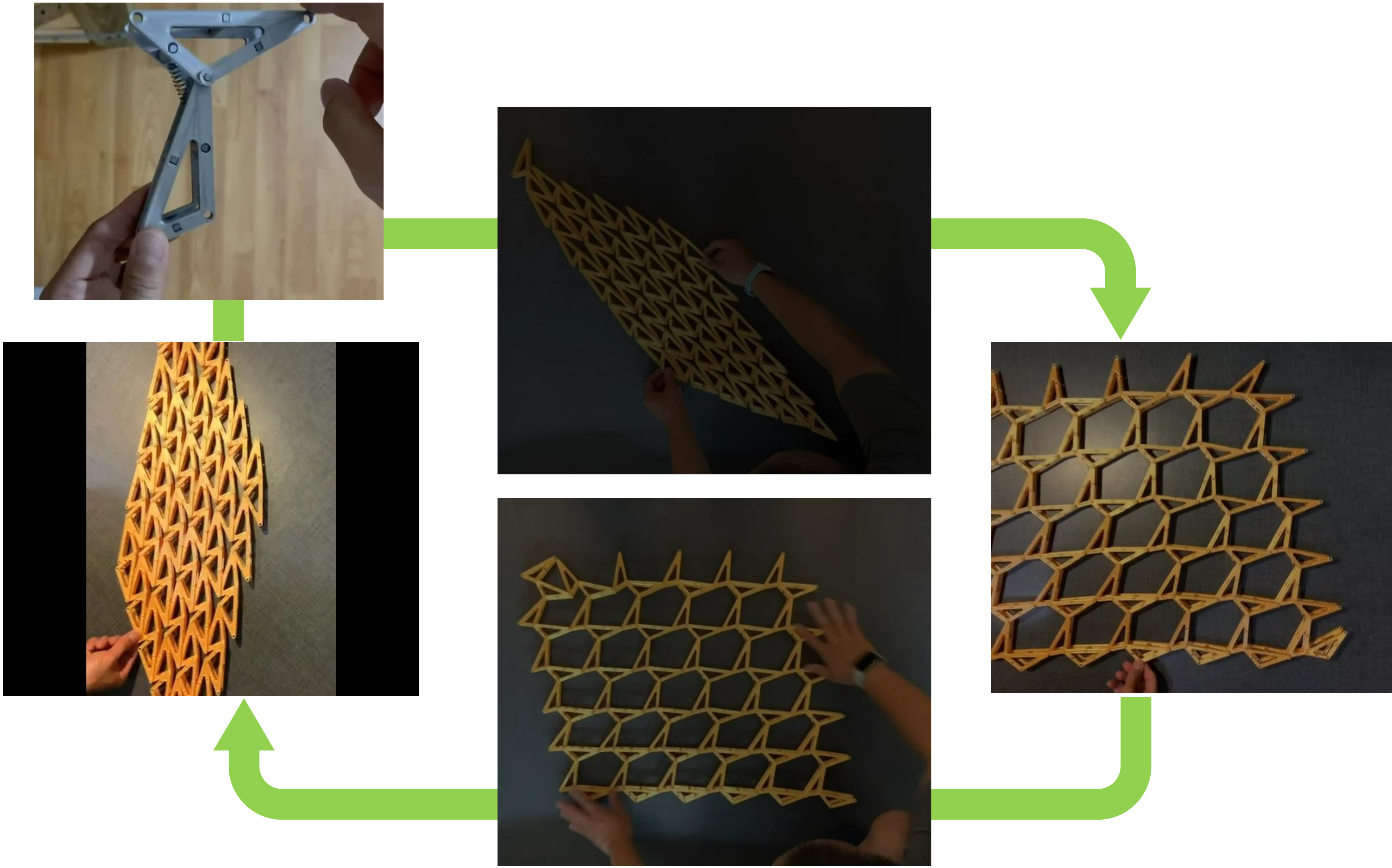
Webpage of video:
https://sites.lsa.umich.edu/xiaoming-mao/

Maxwell Topological Mechanics + Nonlinearity



- D. Z. Rocklin, S. Zhou, K. Sun, X. Mao, Nat. Comm. 8, 14201 (2017).

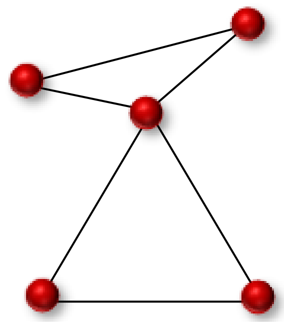
TTMM + Bistability



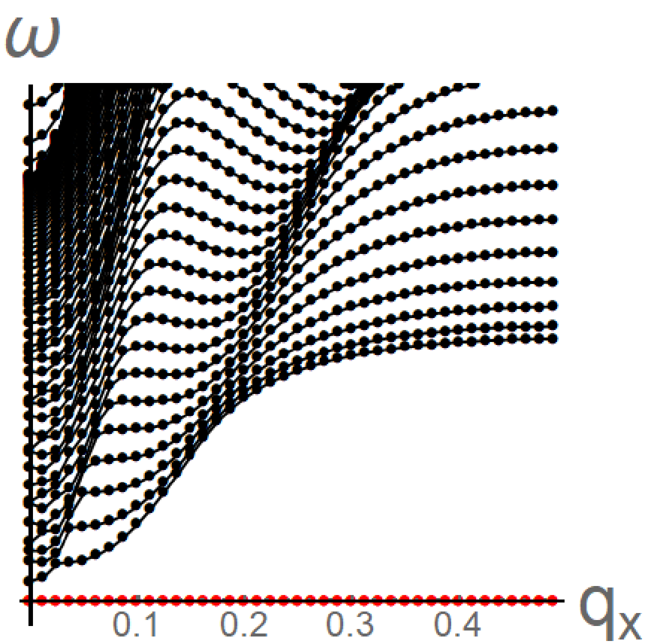
Xiu, et al, PNAS 119, e2211725119 (2022)

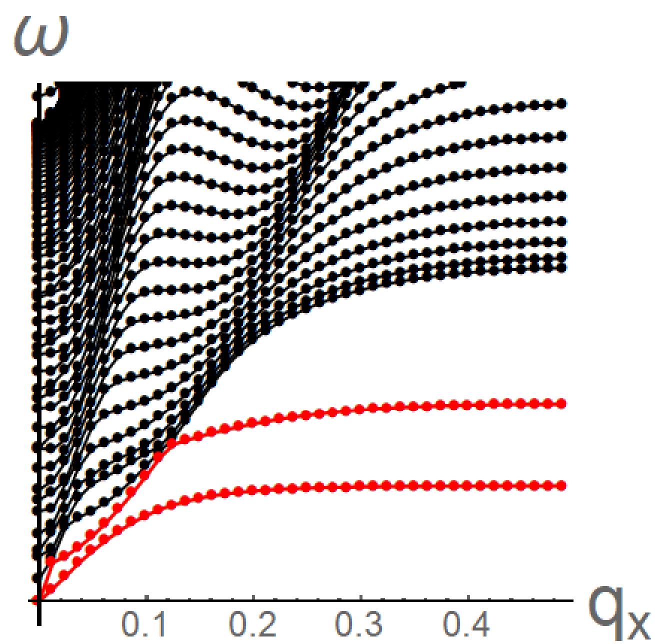
Realistic Maxwell Lattices

- Ball-and-spring model
- Topological edge modes at $\omega = 0$



- Non-ideal hinges providing bending stiffness
- Topological edge modes at $\omega > 0$
- **Transport!**

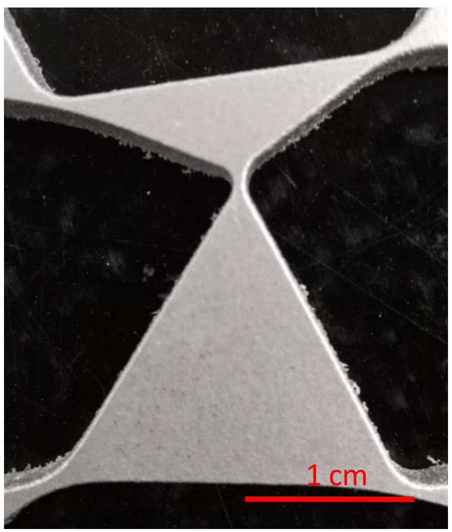




Ma, Zhou, Sun, Mao, and Gonella, PRL 121, 094301 (2018)

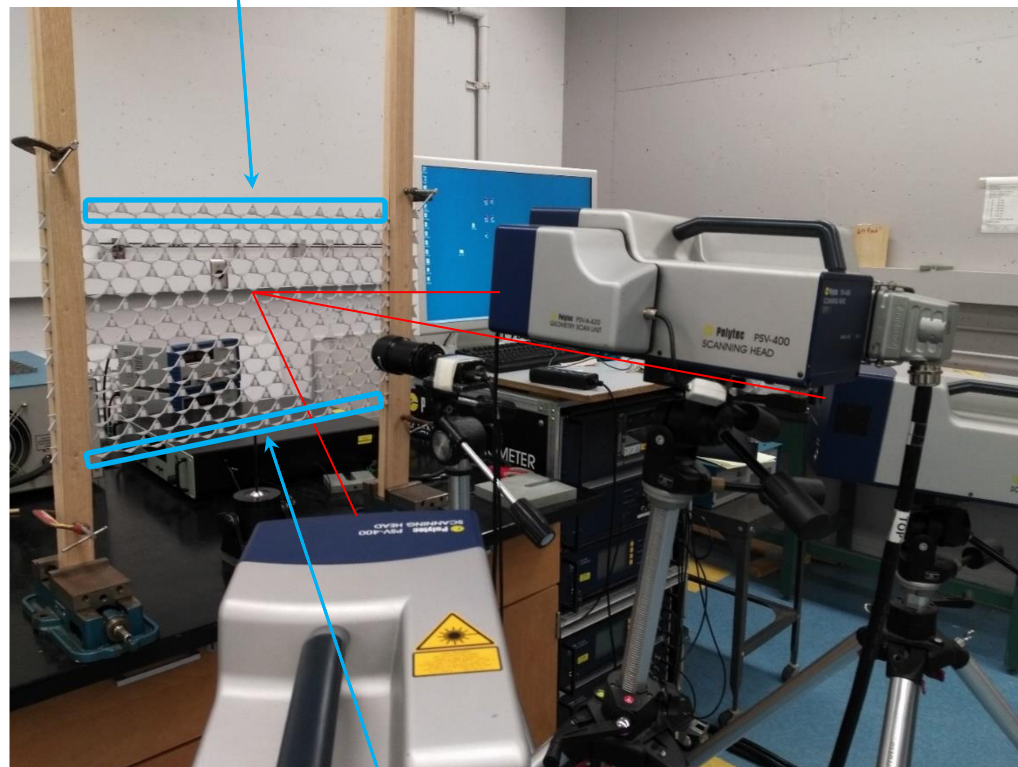
Experimental Setup

- Material: ABS with reflective paint
- Cutting method: waterjet
- Thin sheets: in-plane mechanics
- A point shaker for excitation
- Measurement: 3D Scanning Laser Doppler Vibrometer



Zoom-in figure of the experimental specimen

Floppy side



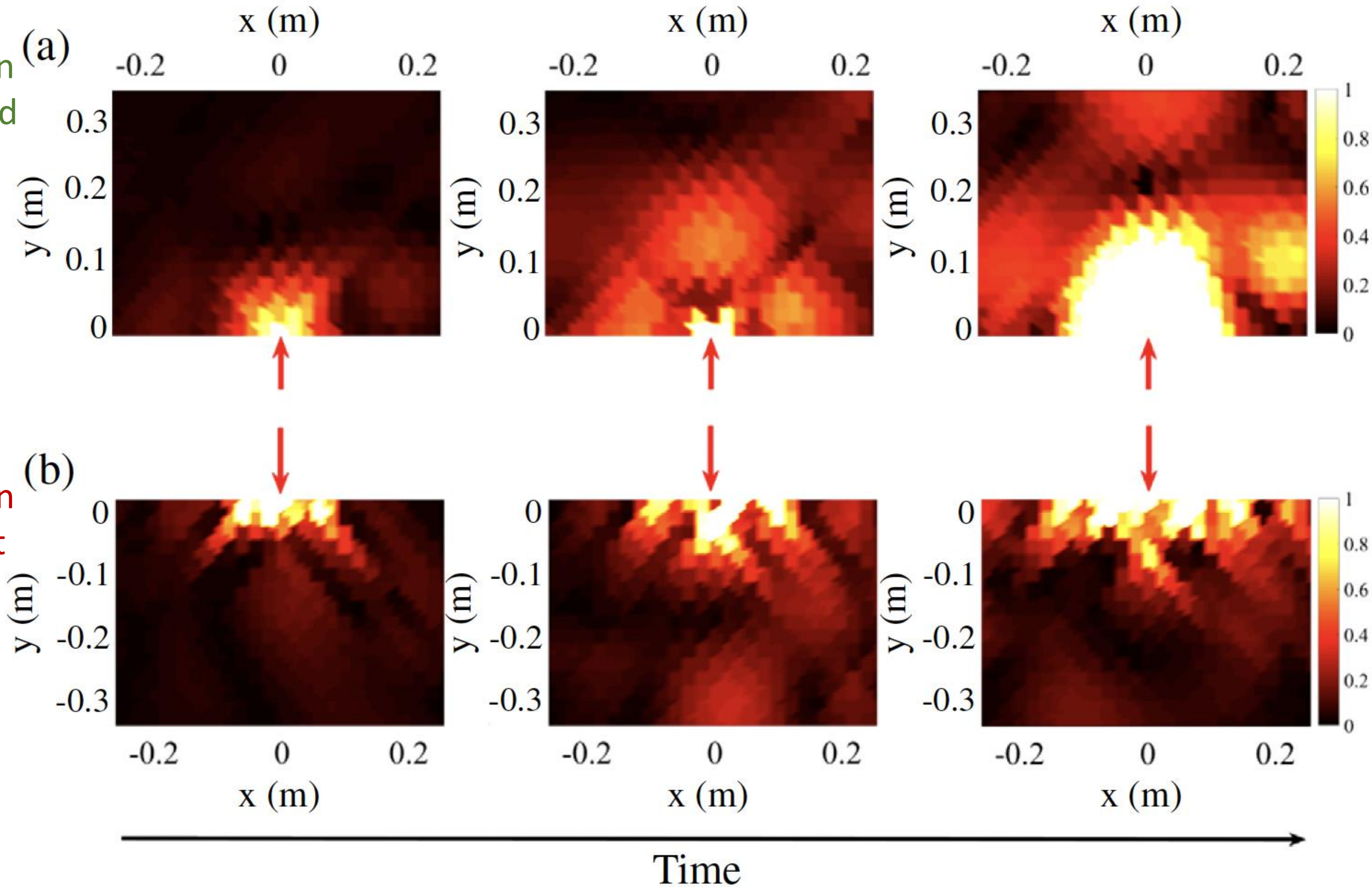
Non-floppy side

Ma, Zhou, Sun, Mao, and Gonella, PRL 121, 094301 (2018)

Observing Topological Edge Modes

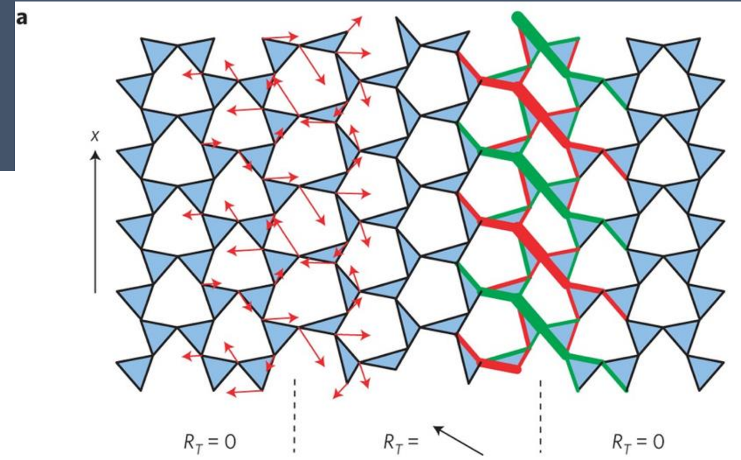
Excitation from hard edge

Excitation from soft edge

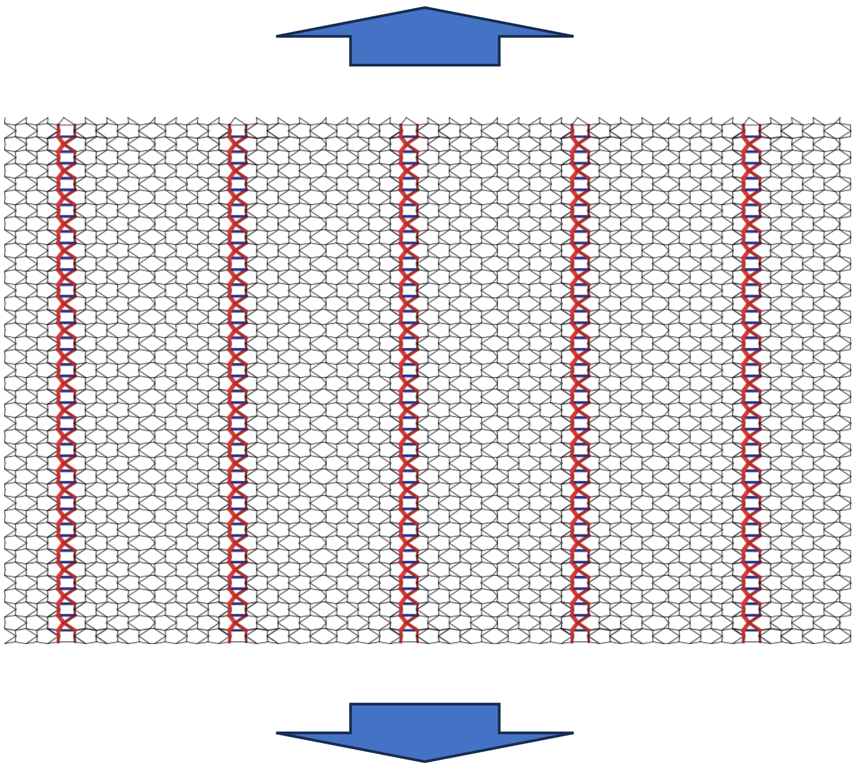


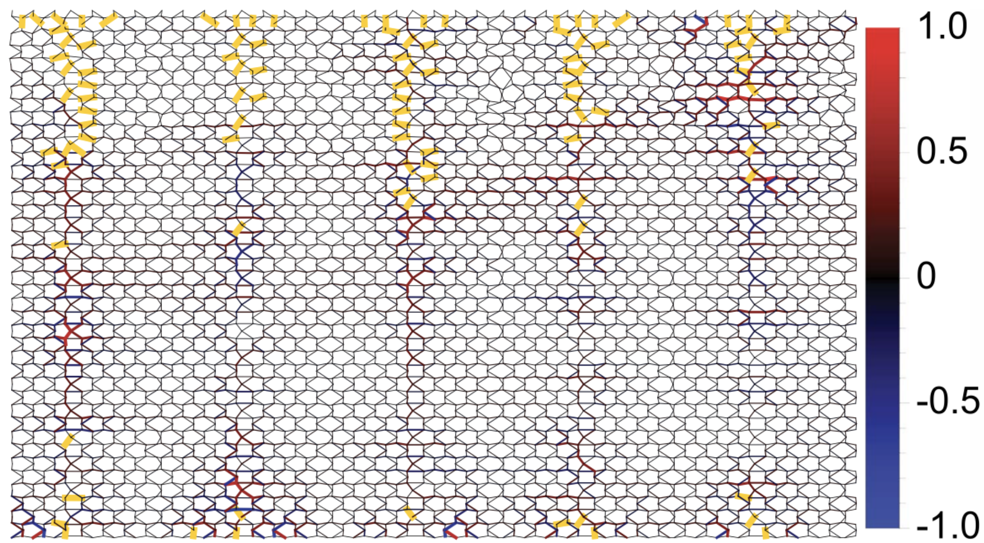
Ma, Zhou, Sun, Mao, and Gonella, PRL 121, 094301 (2018)

Effects of states of self stress



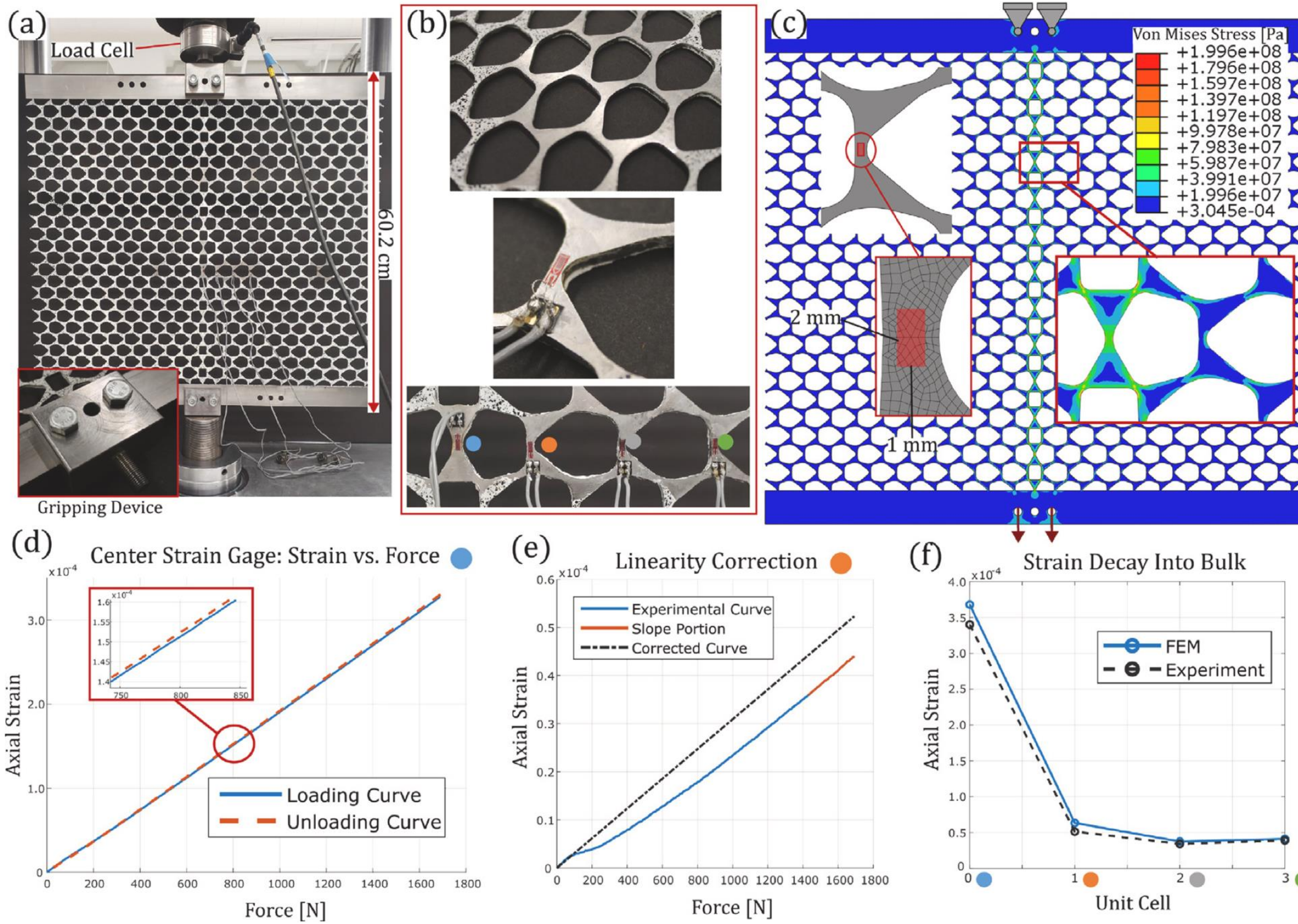
Stress focusing on SSSs:





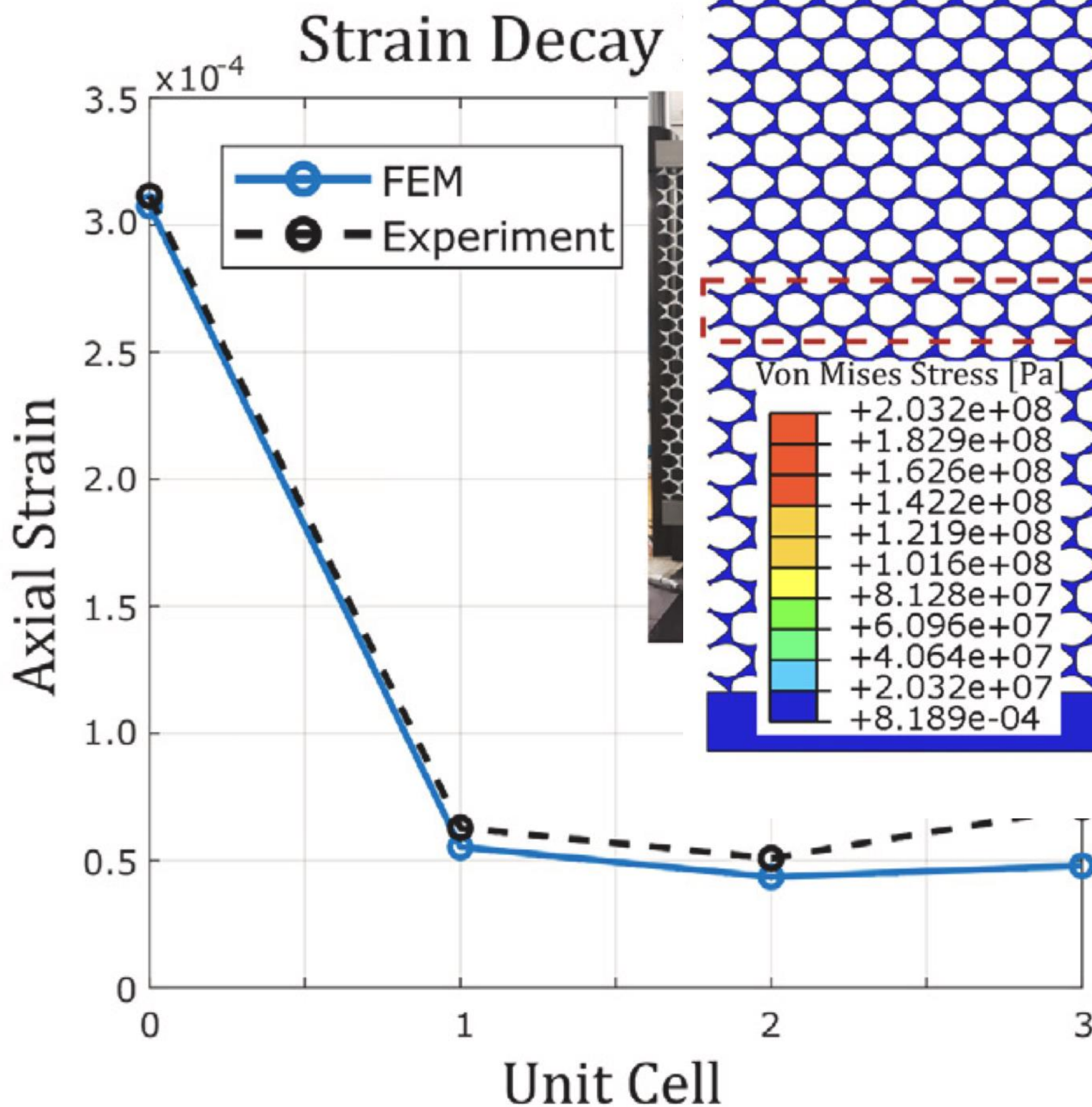
Zhang and Mao, New Journal of Physics 20, 063034 (2018)

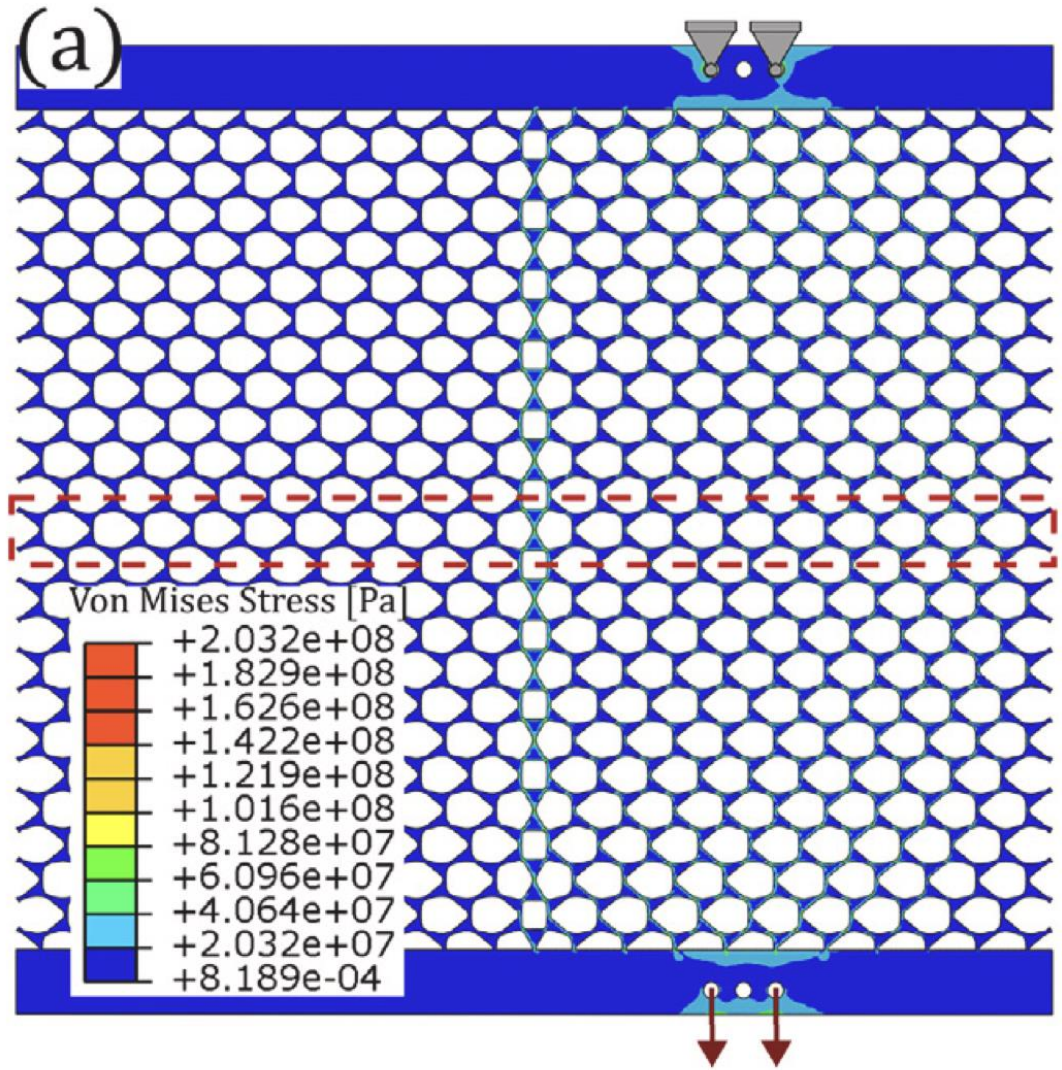
Effects of states of self stress



Widstrand et al, IJSS 274, 112268 (2023)

Effects of states of self stres

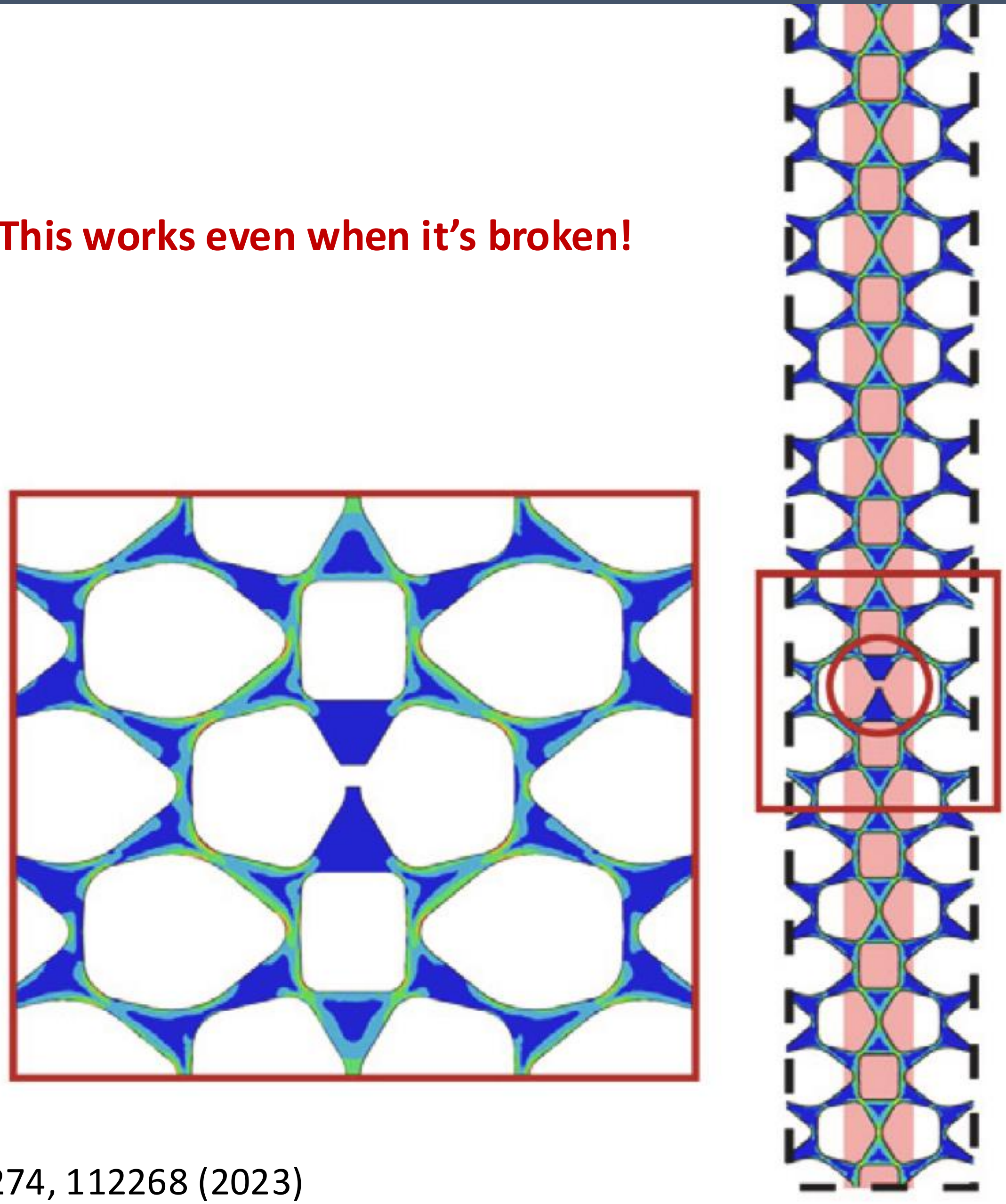




ınd et al, IJSS 274, 112268 (2023)

Effects of states of self stress

This works even when it's broken!



Widstrand et al, IJSS 274, 112268 (2023)

From Maxwell lattices to Maxwell continuum

Elastic energy in linear materials

$$E = \int \mathbf{dr}\, \frac{1}{2} K_{ijkl}\epsilon_{ij}\epsilon_{kl}$$

strain tensor ($d \times d$ symmetric)

$$\epsilon_{ij} = (\partial_i u_j + \partial_j u_i)/2$$

Voigt notation ($d(d+1)/2$ vector)

$$\epsilon^{(V)} = (\epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy})$$

$$E = \int \mathbf{dr}\, \frac{1}{2} K_{ij}\epsilon_i^{(V)}\epsilon_j^{(V)}$$

diagonalize

$$E = \int \mathbf{dr} \sum_{i=1}^{d(d+1)/2} \frac{1}{2}\lambda_i \tilde{\epsilon}_i^2$$

"Maxwell continuum":

elastic constants

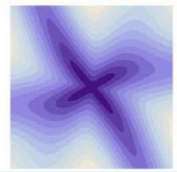
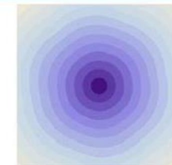
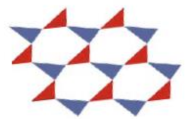
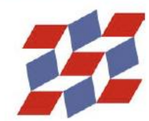
$$\{\boldsymbol{\lambda}_i\} = \{\underbrace{\boldsymbol{\lambda}_1, \cdots, \boldsymbol{\lambda}_d}_{d \text{ positive}}, \underbrace{\mathbf{0}, \cdots, \mathbf{0}}_{d(d-1)/2}\}$$

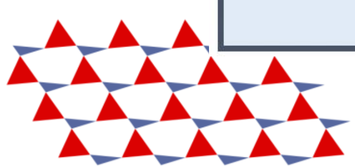
d dim displacement vector u
Marginal stability:
degrees of freedom = constraints

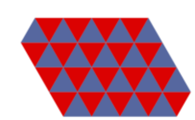
Mechanism of marginal stability: $\tilde{\epsilon}$

K Sun and X Mao, PRL 124, 207601 (2020)

Classification Based on Soft Strain $\tilde{\epsilon}$

Soft twisting characteristic		Shear dominant $\det \tilde{\epsilon} < 0$		Dilation dominant $\det \tilde{\epsilon} > 0$	
Spatially varying floppy modes		$k_y = \lambda_\pm k + O(k^2)$ with $\lambda_\pm \in \mathbb{R}$		$k_y = \mathrm{Re}(\lambda_\pm)\, k + i\, \mathrm{Im}(\lambda_\pm)\, k + O(k^2)$	
Bulk phonon spectra		Vanishing speed of sound in two directions 		Positive speed of sound in all directions 	
Floppy edge phonons (FEP)	Example lattices	$z = 2d$	$z > 2d$	$z = 2d$	$z > 2d$
					
	Frequency	$\omega = 0$	FEP NOT guaranteed to exist	$\omega = 0$	$\omega = O(k^2)$
	Decay rate	$O(k^2)$ Can be 0 in some cases (bulk phonon)		$O(k)$	$O(k)$
	FEP features	Can be topologically polarized		FEP appear in pairs at opposite edges and can be described by generalized conformal transformations	





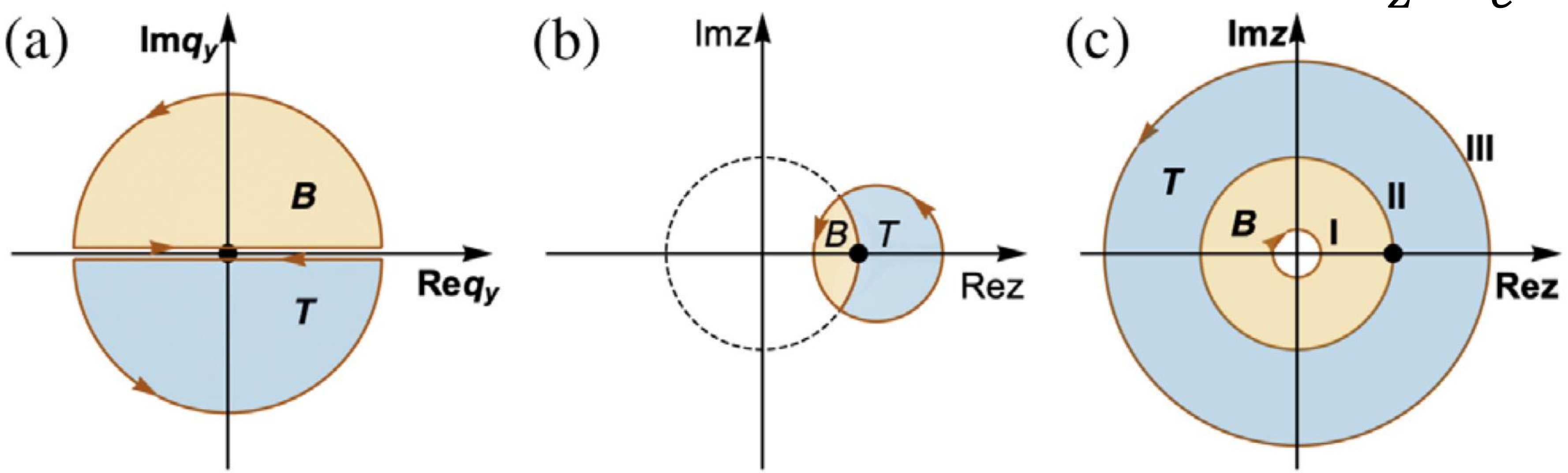
Rocklin et al, Nat Comm 8, 14201 (2017)

Continuum theory

"Long wavelength theory" for topological mechanics

$$E = \int \mathbf{dr} \sum_{i=1}^{d(d+1)/2} \frac{\lambda_i}{2} (\mathcal{M}^{(0)}_{ijk} \partial_j u_k + \mathcal{M}^{(1)}_{ijk\ell} \partial_j \partial_k u_\ell +, \ldots)^2 + O(\partial\partial u)^2, \quad (2)$$

$z = e^{ika}$



K Sun and X Mao, PRL 124, 207601 (2020)

Summary

- Topological mechanics in Maxwell lattices & Continuum
  - Topological modes
  - Transformable topological metamaterials
  - Backscattering-free transport

Thanks!