Problem Set

These exercises are intended to reinforce material the full lecture series and range in scope and difficulty. You are not expected to complete every problem. Instead, choose the exercises that best match your interests and background; the more advanced or time-consuming parts may be treated as optional extensions.

Forward homogenization

1. **Cell problem uniqueness.** For the periodic cell problem (2), identify the trivial nullspace and formulate a constraint that removes it. Assuming the resulting equilibrium is otherwise locally unique, explain why this makes ω^* locally unique.

   Now, suppose $\overline{F}$ is included as an optimization variable as in the uniaxial stretching formulation in (7). What additional nullspace is introduced, and how can it be removed?

2. **Average deformation gradient.** Justify the name *average deformation gradient* for $\overline{F}$ by showing

$$\frac{1}{|Y|}\int_Y \nabla\Phi \,\mathrm{d}\mathbf{X} = \overline{F} \quad \text{when} \quad \nabla\Phi = \overline{F} + \nabla\boldsymbol{\omega}.$$

   To avoid technicalities, you may assume that $\boldsymbol{\omega}$ is defined and continuously differentiable throughout the entire unit cell. For example, for the purposes of this problem, you can imagine Y as being completely filled with a smooth heterogeneous solid rather than containing voids.

3. **Linear elasticity homogenization.** Describe the computational steps that can be used to evaluate explicit entries of the fourth order tensor $\overline{C}$ given a microstructure geometry Ω in unit cell Y. In other words, how can we extract these values using (6)?

4. **Uniaxial stress.** Prove that the solution to (7) is a state of uniaxial homogenized stress $\overline{\psi}'$.

5. **Pressure potential.** Derive the first variation of $-p\,\mathrm{Vol}[\mathcal{S}]$ where $\mathrm{Vol}[\mathcal{S}]$ is the volume enclosed by a smooth surface S. Hint: apply the Reynolds transport theorem to $\int_{\mathcal{S}_{\text{int}}} 1\,\mathrm{d}\mathbf{X}$ where $\mathcal{S}_{\text{int}}$ is the interior volume (so that $\mathcal{S} = \partial\mathcal{S}_{\text{int}}$). The physical interpretation of this variation is the force exerted by the pressure term on the surface $\mathcal{S}$.

   Optional: do the same for a triangulated surface, differentiating with respect to the vertex positions. Comparing the two results, what definition of discrete surface normal arises?

6. **Periodic volume.** Prove that any choice of endcaps inserted to delimit the volume enclosed by an inflatable unit, provided it respects periodicity, yields the same definition of enclosed volume. Please reference Figure 6.

7. **Schur complement.** Recall from Section 8.1 that the energy in a unit cell can be written as $E([\mathbf{x}_\mu, \mathbf{x}_M])$. The Hessian of this energy has the block structure:

$$\frac{\partial^2 E}{\partial[\mathbf{x}_\mu, \mathbf{x}_M]^2} = \begin{bmatrix} H_{\mu\mu} & H_{\mu M} \\ (H_{\mu M})^\top & H_{MM} \end{bmatrix}, \qquad H_{\mu\mu} := \frac{\partial^2 E}{\partial \mathbf{x}_\mu^2}, \quad H_{\mu M} := \frac{\partial^2 E}{\partial \mathbf{x}_\mu \partial \mathbf{x}_M}, \quad H_{MM} := \frac{\partial^2 E}{\partial \mathbf{x}_M^2}.$$

   What is the Schur complement of $H_{\mu\mu}$ in this block matrix? The requested expression can be obtained by running a block Gaussian elimination, or simply taken from the linked Wikipedia article. Does it have a physical interpretation based on Section 8.1?

Inverse design

8. **Adjoint equations for linear elasticity.** This problem asks you to derive sensitivity analysis formulas for a linear elasticity problem $K(\mathbf{p})\mathbf{u} = \mathbf{f}$ with design-dependent stiffness matrix $K(\mathbf{p})$ and constant load $\mathbf{f}$. Take the design objective to be $J(\mathbf{u}, \mathbf{p})$ and determine an expression for $\nabla_{\mathbf{p}} J(\mathbf{u}(\mathbf{p}), \mathbf{p})$, where $\mathbf{u}(\mathbf{p})$ is defined implicitly as the linear system solution. Rather than starting with the nonlinear sensitivity analysis formulas from the lecture notes, please proceed from scratch by differentiating the equilibrium equation and identify an adjoint problem whose solution yields an efficient design gradient expression.

   Finally, relate your solution to the generic nonlinear elasticity formulation. Why might higher-order derivatives appear? Why is the adjoint solve especially cheap?

9. **Self-adjoint optimization problems.** Now, consider the special case where $J(\mathbf{u}, \mathbf{p}) = \mathbf{u} \cdot \mathbf{f}$. This is a popular structural optimization formulation for designing stiff structures referred to as the *compliance minimization*. What is special about this objective and the resulting adjoint problem that can make sensitivity analysis even cheaper?

10. **Derivative of the homogenized elasticity tensor.** Derive sensitivity analysis formulas for the homogenized elasticity tensor $\overline{C}(\mathbf{p})$ in the setting of topology optimization, where the cell problem takes the form:

$$\int_Y \left[C(\mathbf{p}) : \left(\overline{\varepsilon} + \varepsilon_{\text{lin}}(\boldsymbol{\omega}^*)\right)\right] : \varepsilon_{\text{lin}}(\boldsymbol{\phi})\,\mathrm{d}\mathbf{X} = 0 \qquad \forall\, \boldsymbol{\phi} \text{ periodic},$$

   and the homogenized tensor is obtained via:

$$\overline{C}(\mathbf{p}) : \overline{\varepsilon} = \frac{1}{|Y|}\int_Y C(\mathbf{p}) : \left(\overline{\varepsilon} + \varepsilon_{\text{lin}}(\boldsymbol{\omega}^*)\right)\mathrm{d}\mathbf{X}.$$

   (This formulation differs from our original solid-void one in (5) and (6) in that the microstructure geometry is fixed, filling the entire domain Y, but the fabrication material properties $C(\mathbf{p})$ are taken to vary with the design parameters $\mathbf{p}$.) You'll need a concrete expression for $\overline{C}$ before attempting this problem, so please first finish Problem 3. Your solution should be expressed in terms of $\frac{\partial C}{\partial \mathbf{p}}$, which you can assume is known.
   Hint: introduce $\varepsilon^{kl} = \operatorname{sym}(\mathbf{e}_k \otimes \mathbf{e}_l)$, the canonical basis for symmetric tensors, and define $\boldsymbol{\omega}^{kl}$ as the corresponding solution to the cell problem with $\overline{\varepsilon} = \varepsilon^{kl}$. Then one can show:

$$\varepsilon^{ij} : \overline{C}(\mathbf{p}) : \varepsilon^{kl} = \frac{1}{|Y|}\int_Y (\varepsilon^{ij} + \varepsilon_{\text{lin}}(\boldsymbol{\omega}^{ij})) : C(\mathbf{p}) : \left(\varepsilon^{kl} + \varepsilon_{\text{lin}}(\boldsymbol{\omega}^{kl})\right)\mathrm{d}\mathbf{X}.$$

   Notice the insertion of $\varepsilon_{\text{lin}}(\boldsymbol{\omega}^{ij})$, which is valid (does not change the integral) because $\boldsymbol{\omega}^{kl}$ satisfies the weak form of the cell problem. This apparently more complicated expression turns out to simplify the process of differentiating with respect to $\mathbf{p}$. Why?