Notes on fluid interfaces and membranes: Boulder 2026 lectures

Petia M. Vlahovska, Northwestern University
petia.vlahovska@northwestern.edu

Warning: These notes have not been proofread and may contain typos and errors.

1 Dynamics

1.1 Boundary conditions: fluid-structure interaction

I. Velocity

- continuity
$$\boldsymbol{u}_1 = \boldsymbol{u}_2 \,, \quad atS$$
]
- kinematic condition(for evolving interfaces) F- shape function. For example, in the Monge representation $F = z - h(x, y)$
$$\frac{DF}{Dt} = 0 \tag{1}$$
the normal vector $\mathbf{n} = \nabla F/|\nabla F|$, so the kinematic condition becomes
$$\frac{1}{|\nabla F|}\frac{\partial F}{\partial t} = -\boldsymbol{u} \cdot \mathbf{n} \tag{2}$$

II. Stresses:

$$\mathbf{n} \cdot (\mathbf{T}_1 - \mathbf{T}_2) = \mathbf{f}_s \tag{3}$$

Note $\nabla \cdot \mathbf{n}$ is the mean curvature H, $\mathbf{n}$ points into fluid 1.

For a simple fluid/fluid interface

$$\mathbf{f}_s = \sigma(\nabla \cdot \mathbf{n})\mathbf{n} - \nabla_s \sigma$$

For a membrane

$$\mathbf{f}_s = \sigma(\nabla \cdot \mathbf{n})\mathbf{n} - \nabla_s \sigma - \kappa \left(\frac{1}{2} H^2 - 2HK + \nabla_s^2 H \right) \mathbf{n}$$

here $H = \nabla \cdot \mathbf{n}$

1.2 Fluctuating membrane

Consider tensionless planar membrane, with profile $h(x, t) = \sum_k h_k(t) e^{\mathrm{i}kx}$ The mode amplitudes evolve as
$$\frac{dh_k}{dt} = -s(k)h_k(t) + \zeta_k(t) \,,$$

where ζ_k is a (Gaussian white) thermal random force that satisfies the fluctuation- dissipation theorem.

$$\langle \zeta_k(t)\zeta_{k'}(t')\rangle = 2D\delta_{k+k',0}\delta(t-t') = \frac{k_BT}{2\eta k}\delta_{k+k',0}\delta(t-t'), \quad \langle \zeta_k(t)\rangle = 0$$

$$\langle h_k^2\rangle = \frac{2D}{s(k)}$$

In the next Section 1.3, we will derive the expression for the relaxation rate for a tensionless membrane

$$s(k) = \frac{\kappa k^3}{4\eta}$$

Thus

$$\langle h_k^2\rangle = \frac{k_BT}{\kappa k^4} \Longrightarrow D = \frac{k_BT}{4\eta k}$$

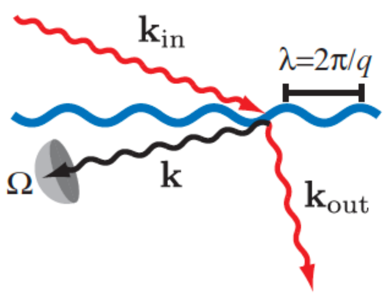

FIG. 6. The scattering wave vector $\mathbf{k}$ is defined as the difference between the incident and outgoing wave vectors $\mathbf{k} \equiv \mathbf{k}_{\text{out}} - \mathbf{k}_{\text{in}}$. In order to calculate $S(k, t)$, we integrate $S(\mathbf{k}, t)$ over all solid angles Ω. The membrane (blue) is shown with wavenumber $q = 2\pi/\lambda$.

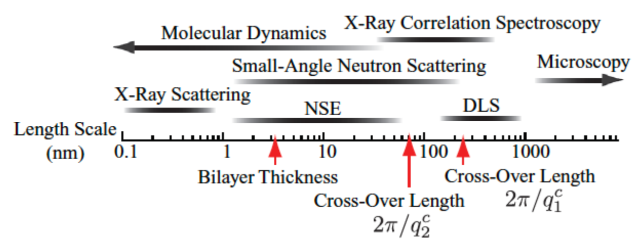

FIG. 1. Membrane fluctuations have been studied across several length scales using a wide variety of techniques. The ranges of applicability for several techniques are shown above. Using the constants from Fig. 5, we also show the bilayer thickness and cross-over wavelengths $q_{1,2}^c$ associated with DMPC (see Sec. II).

Watson and Brown (J. Chem Phys., 2011)

Figure 1: Scattering is used to study the structure of soft materials

Dynamic structure factor, Zilman-Granek power-law References: F. Brown, annual Review of Physical Chemistry (2008), Zilman and Granek, PRL (1996), Watson and brown, J Phys. Chem. (2011)

The dynamic structure factor is

$$S(q,t) \approx S(q)\exp\left(-\frac{1}{2}q^2\langle(\Delta h)^2\rangle\right)$$

The roughness

$$\langle(\Delta h)^2\rangle = ?$$

For a fluctuating membrane

$$\langle h_q(t)h_q(0)\rangle = \langle |h_q(0)|^2\rangle e^{-s(q)t}$$

$$\langle(\Delta h)^2\rangle \equiv \langle (h(t)-h(0))^2\rangle = 2\left(\langle h(0)^2\rangle - \langle h(0)h(t)\rangle\right) = 2\sum_q \langle |h_q|^2\rangle\left(1-e^{-s(q)t}\right)$$

$$\sim \int 2\pi q dq \frac{k_BT}{\kappa q^4}\left(1-e^{-\frac{\kappa q^3}{4\eta}t}\right) = const\frac{k_BT}{\kappa}\left(\frac{\kappa t}{4\eta}\right)^{2/3}$$

Thus the structure factor is

$$S(q,t) = S(q)e^{-(\Gamma_{ZG}t)^{2/3}}\,, \quad \Gamma_{ZG} = 0.025\left(\frac{k_B T}{\kappa}\right)^{1/2}\frac{k_B T}{\eta}q^3$$

1.3 Capillary waves, membrane fluctuations/flickering

see Mathematica script for the solution

Consider a membrane separating a fluid 1 with viscosity η_1 (top) and fluid 2 with viscosity η_2 (bottom). (viscosity ratio is defined as $\lambda = \eta_2/\eta_1$). All variables are scaled by the bending rigidity of the membrane κ, bulk viscosity η_1 and size of the membrane ℓ. The time scale is $\eta_1\ell^3/\kappa$, the velocity scale is $\kappa/\eta_1\ell^2$, the pressure (and stresses) scale is κ/ℓ^3

The shape of the deformed interface represented as (WLG assume only fluctuation in one direction)

$$h(x,t) = \sum_{q=1}^{\infty} h_q e^{st+\mathrm{i}qx}\,. \tag{4}$$

Accprdngly, all variables are expanded in modes

$$\boldsymbol{v} = \boldsymbol{v}_q(z)e^{st+\mathrm{i}qx}\,, \quad p = p_q(z)e^{st+\mathrm{i}qx}\,, \quad \sigma = \sigma_0 + \sigma_q e^{st+\mathrm{i}qx} \tag{5}$$

Summation over the wavenumber q is implied.

If the magnitude of the deformation is small, $h \ll 1$, any variable g is expanded $g = g^{(0)}(z,t) + g^{(1)}(z,x,t) + ...$ The superscript (0) corresponds to the base state of a flat membrane, $h = 0$. $g^{(1)} = \sum_q g_q(z)\exp(st + \mathrm{i}qx)$ is a correction accounting for the effect of small membrane curvature.

In order to determine the velocity and pressure fields around a weakly curved membrane, we need to apply the boundary conditions at the deformed interface. All variables are expanded in Taylor series, and the boundary conditions are evaluated at $z = 0$

$$\begin{aligned} g(z=h) =& g(z=0) + h\frac{\partial g}{\partial z}|_{z=0} \\ &= g^{(0)}(z=0) + \left[g^{(1)}(z=0) + h\frac{\partial g^{(0)}}{\partial z}|_{z=0}\right] \end{aligned} \tag{6}$$

Geometry The interface is defined by a shape fundtion $F = z - h(x) = 0$

$$\nabla F = \frac{\partial F}{\partial x}\hat{\mathbf{x}} + \frac{\partial F}{\partial z}\hat{\mathbf{z}} = \hat{\mathbf{z}} + \frac{\partial h}{\partial x}\hat{\mathbf{x}} \tag{7}$$

The unit normal vector pointing into the top fluid is then

$$\mathbf{n} \quad = \quad \frac{\nabla F}{|\nabla F|} = \hat{\mathbf{z}} - \frac{\partial h}{\partial x}\hat{\mathbf{x}}$$

- Then, the tangential vector is $\mathbf{t} = \hat{\mathbf{x}} + \frac{\partial h}{\partial x}\hat{\mathbf{z}}$, obtained using the orthogonality condition $\mathbf{t}\cdot\mathbf{n} = 0$

Dynamics .

In the base state (planar membrane), there is no flow.

Flow:

in the case of a membranes - area-incompressibility

$$\nabla \cdot \boldsymbol{v} = 0 \Longrightarrow v_{q,x} = -\frac{1}{\mathrm{i}q}\frac{dv_{z,q}}{dz} \tag{8}$$

TO DO. show that the Stokes equations yield the following 4th order ODE for $v_{z,q}$

$$\nabla^2 \boldsymbol{v} = \nabla p \Longrightarrow \left(\frac{d^2}{dz^2} - q^2\right)^2 v_{z,q} = 0 \tag{9}$$

Show that the solutions that decay to zero at infinity are (note (2) and (1) denote the bottom and top fluid)

$$v_{q,z}^{(1)} = (B_1 + C_1 z)\exp(-qz)\ , \quad v_{q,z}^{(2)} = (B_2 + C_2 z)\exp(qz) \tag{10}$$

$$p_{q,z}^{(1)} = C_1 \exp(-qz)\ , \quad v_{q,z}^{(2)} = C_2 \exp(qz) \tag{11}$$

Boundary conditions:

- Continuity of velocity (normal $\boldsymbol{v} \cdot \mathbf{n}$ and tangential $\boldsymbol{v} \cdot (\mathbf{I} - \mathbf{nn})$ component)

$$\begin{aligned} v_{q,z}^{(2)} = v_{q,z}^{(1)} \quad \text{at} \quad z = 0 \quad &\Longrightarrow B_2 = B_1 \\ v_{q,x}^{(2)} = v_{q,x}^{(1)} \quad \text{at} \quad z = 0 \quad &\Longrightarrow C_2 = C_1 - 2B_1 k \end{aligned} \tag{12}$$

- membrane incompressibility $\nabla_s \cdot \boldsymbol{v} = 0$

$$\frac{\partial v_x}{\partial x} = 0 \quad \Longrightarrow v_{q,x} = 0 \quad \text{at} \quad z = 0 \quad \Longrightarrow C_1 = 2B_1 k \tag{13}$$

- normal and tangential stress balance. In general,

$$\mathbf{n} \cdot \mathbf{T} = \mathbf{n}\left[\bar{\sigma}H - \left(\frac{1}{2}H^3 - 2HK + \nabla_s^2 H\right)\right] - \nabla_s \sigma \quad \text{at} \quad z = h \tag{14}$$

$H = \nabla \cdot \mathbf{n}$. Here, $\bar{\sigma} = \sigma \ell^2/\kappa$. Note that only the normal component is needed to find s; the tangential component serves to determine the nonunifom part of the tension σ_q (recall that the tension enforces area incompressibility)

To leading order (linear with the perturbation h) the normal stress is $\sim (-\nabla^2 H + \bar{\sigma}H)\hat{\mathbf{z}}$

$$f_{z,q}^{\mathrm{mm}} = h_q\left(\bar{\sigma}q^2 + q^4\right) \quad \text{at} \quad z = 0 \tag{15}$$

(note that the tension is dimensionless $\bar{\sigma} = \sigma \ell^2/\kappa$) to leading order,

$$[[\mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n}]] = -p_q + 2\frac{dv_{q,z}}{dz}$$

and the stress balance is

$$-p_q^{(1)} + 2\frac{dv_{q,z}^{(1)}}{dz} - \lambda\left(-p_q^{(2)} + 2\frac{dv_{q,z}^{(2)}}{dz}\right) = f_{z,q}^{\mathrm{mm}}\,, \qquad z = 0$$

and this becomes

$$-2B_1 k(1 + \lambda) = h_q\left(\bar{\sigma}q^2 + q^4\right)$$

- Kinematic condition (for the shape function $F = z - h(x,t)$, $\frac{DF}{Dt} = 0$). This serves to determine the growth rate s(correlation time $t = 1/s$)

$$\frac{\partial h}{\partial t} = v_z \quad \text{at} \quad z = 0 \quad \Longrightarrow sh_q = B_1 \tag{16}$$

leading to

$$s^{-1}(q) = -\frac{2(1+\lambda)}{\bar{\sigma}q + q^3} \tag{17}$$

In dimensional form, the relaxation time

$$\tau(k) = -\frac{2(\eta_1 + \eta_2)}{\sigma k + \kappa k^3} \tag{18}$$

if the fluids are the same, $\lambda = 1$, we get the results derived by Brochard and Lennon (1975).

1.4 Capillary waves

in this case, the interfacial force is $\sigma H \hat{\mathbf{z}}$. The interface is compressible, hence the BCs are velocity continuity (x and z components), and both the tangential (x) and normal (z) stress balance.

$$s^{-1}(k) = -\frac{4\eta}{\sigma k} \tag{19}$$

1.5 EXTRA: Capillary Instability of a Liquid Thread/Jet: Rayleigh–Plateau instability

based on Drazin's book "Introduction to Hydrodynamic Stability"

1.5.1 Physical problem

A cylindrical liquid jet can spontaneously break into droplets because of surface tension. This phenomenon is known as the *Rayleigh–Plateau instability.* We consider an infinitely long cylindrical jet of radius a, density ρ, and surface tension γ. The surrounding fluid is assumed to have negligible density and pressure p_∞.

The liquid is assumed to be

- incompressible,
- inviscid,
- initially cylindrical,
- at rest after a Galilean transformation.

We use cylindrical coordinates (x, r, θ), where x is the coordinate along the jet axis.

1.5.2 Governing equations

The motion of the inviscid liquid is governed by Euler's equation

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p, \tag{20}$$

together with incompressibility,

$$\nabla\cdot\boldsymbol{u} = 0. \tag{21}$$

Let the perturbed interface be

$$r = \zeta(x,\theta,t). \tag{22}$$

The dynamic boundary condition is the Young–Laplace condition,

$$p = p_\infty + \gamma\,\nabla\cdot\boldsymbol{n} \qquad \text{at } r=\zeta, \tag{23}$$

where $\boldsymbol{n}$ is the outward unit normal.

The exact kinematic boundary condition states that the interface is a material surface:

$$u_r = \frac{D\zeta}{Dt} \qquad \text{at } r=\zeta. \tag{24}$$

1.5.3 Base state

For an undisturbed liquid "tube",

$$\boldsymbol{U} = 0, \qquad \zeta = a. \tag{25}$$

The curvature of a cylinder is

$$\nabla\cdot\boldsymbol{n} = \frac{1}{a}, \tag{26}$$

so that the pressure inside the jet is

$$P = p_\infty + \frac{\gamma}{a}. \tag{27}$$

1.5.4 Linearization

Introduce small perturbations

$$\boldsymbol{u} = \boldsymbol{u}', \qquad p = P + p', \qquad \zeta = a + \zeta'. \tag{28}$$

Neglecting terms quadratic in the perturbations gives

$$\rho\frac{\partial \boldsymbol{u}'}{\partial t} = -\nabla p', \tag{29}$$

and

$$\nabla\cdot\boldsymbol{u}' = 0. \tag{30}$$

At $r=a$, the linearized kinematic condition is

$$u_r' = \frac{\partial \zeta'}{\partial t}. \tag{31}$$

To linear order, the curvature is

$$\nabla\cdot\boldsymbol{n} = \frac{1}{a} - \frac{\zeta'}{a^2} - \frac{\partial^2\zeta'}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2\zeta'}{\partial\theta^2}. \tag{32}$$

Subtracting the base-state pressure jump gives the linearized dynamic condition

$$p' = -\gamma\left(\frac{\zeta'}{a^2} + \frac{\partial^2\zeta'}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2\zeta'}{\partial\theta^2}\right), \qquad r=a. \tag{33}$$

1.5.5 Pressure equation

Taking the divergence of the linearized momentum equation,

$$\rho \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{u}') = -\nabla^2 p', \tag{34}$$

and using incompressibility gives

$$\nabla^2 p' = 0. \tag{35}$$

In cylindrical coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{36}$$

1.5.6 Normal-mode analysis

Seek perturbations of the form

$$(\boldsymbol{u}', p', \zeta') = \left(\hat{\boldsymbol{u}}(r), \hat{p}(r), \hat{\zeta}\right) e^{st+i(kx+n\theta)}. \tag{37}$$

Here k is the axial wavenumber, n is the azimuthal mode number, and s is the growth rate. Substitution into Laplace's equation gives

$$\frac{d^2\hat{p}}{dr^2} + \frac{1}{r}\frac{d\hat{p}}{dr} - \left(k^2 + \frac{n^2}{r^2}\right)\hat{p} = 0. \tag{38}$$

This is the modified Bessel equation. Its solutions are

$$I_n(kr), \qquad K_n(kr). \tag{39}$$

Since $K_n(kr)$ is singular at the jet axis, boundedness at $r = 0$ requires

$$\hat{p}(r) = A I_n(kr). \tag{40}$$

From the radial component of the momentum equation,

$$\rho s \hat{u}_r = -\frac{d\hat{p}}{dr}, \tag{41}$$

we obtain

$$\hat{u}_r = -\frac{Ak}{\rho s} I_n'(kr). \tag{42}$$

1.5.7 Dispersion relation

Define the dimensionless wavenumber

$$\alpha = ak. \tag{43}$$

The dynamic boundary condition at $r = a$ becomes

$$A I_n(\alpha) = -\frac{\gamma}{a^2}(1 - \alpha^2 - n^2)\hat{\zeta}. \tag{44}$$

The kinematic condition gives

$$-\frac{A\alpha}{a\rho s} I_n'(\alpha) = s\hat{\zeta}. \tag{45}$$

Eliminating A and $\hat{\zeta}$ yields Rayleigh's dispersion relation:

$$\boxed{s^2 = \frac{\gamma}{\rho a^3}\frac{\alpha I_n'(\alpha)}{I_n(\alpha)}\left(1 - \alpha^2 - n^2\right)}. \tag{46}$$

1.5.8 Stability criterion

For $\alpha > 0$,

$$\frac{\alpha I_n'(\alpha)}{I_n(\alpha)} > 0. \tag{47}$$

Therefore the sign of s^2 is determined by

$$1 - \alpha^2 - n^2. \tag{48}$$

For non-axisymmetric disturbances, $n \geq 1$, and hence

$$1 - \alpha^2 - n^2 \leq 0. \tag{49}$$

Thus these modes are stable.

For the axisymmetric mode $n = 0$,

$$s^2 = \frac{\gamma}{\rho a^3} \frac{\alpha I_1(\alpha)}{I_0(\alpha)} (1 - \alpha^2). \tag{50}$$

The mode is unstable when

$$0 < \alpha < 1, \tag{51}$$

or equivalently

$$ka < 1. \tag{52}$$

Since $\lambda = 2\pi/k$, the instability criterion can be written as

$$\boxed{\lambda > 2\pi a}. \tag{53}$$

Thus an axisymmetric disturbance grows only if its wavelength exceeds the circumference of the undisturbed jet.

1.5.9 Physical interpretation

Consider an axisymmetric "varicose" perturbation consisting of alternating bulges and necks. Surface tension tends to reduce surface area. For sufficiently long wavelengths, transferring liquid from the necks into the bulges decreases the total interfacial area. The pressure differences generated by curvature therefore amplify the disturbance.

For wavelengths shorter than the jet circumference, surface tension instead acts as a restoring mechanism.

The most rapidly growing mode occurs approximately at

$$k_m a \simeq 0.7. \tag{54}$$

Hence the characteristic breakup wavelength predicted by linear theory is

$$\lambda_m = \frac{2\pi}{k_m} \simeq 9a. \tag{55}$$

This wavelength gives an estimate of the typical spacing between droplets produced during jet breakup.

1.5.10 Characteristic time scale

The dispersion relation identifies the capillary–inertial time scale

$$t_\gamma = \sqrt{\frac{\rho a^3}{\gamma}}. \tag{56}$$

Thus

$$s = \frac{1}{t_\gamma}\,\sigma(\alpha), \tag{57}$$

where σ is a dimensionless growth rate.

Smaller jets therefore generally evolve on shorter capillary time scales.