Quantum Noise and Measurement

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Gurus: Michel Devoret, Steve Girvin, Aash Clerk

And many discussions with D. Prober, K. Lehnert, D. Esteve, L. Kouwenhoven, B. Yurke, L. Levitov, K. Likharev, …

Thanks for slides: L. Kouwenhoven, K. Schwab, K. Lehnert,…

And God said:

$$[a, a^{\dagger}] = 1$$

"Go forth, be fruitful, and multiply (but don't commute)"

And there was light, and quantum noise…

Manifestations of Quantum Noise

Well-known:

Spontaneous emission
Casimir effect
Lamb shift
g-2 of electron

Mesoscopic and solid-state examples (less usual?):

Shot noise
Minimum noise temperature of an amplifier
Measurement induced dephasing of qubit
Environmental destruction of Coulomb blockade
Quasiparticle renormalization of SET's capacitance
…

Overview of Lectures

Lecture 1: Equilibrium and Non-equilibrium Quantum Noise in Circuits

Reference: "Quantum Fluctuations in Electrical Circuits," M. Devoret Les Houches notes.

Lecture 2: Quantum Spectrometers of Electrical Noise

Reference: "Qubits as Spectrometers of Quantum Noise," R. Schoelkopf et al., cond-mat/0210247

Lecture 3: Quantum Limits on Measurement

References: "Amplifying Quantum Signals with the Single-Electron Transistor," M. Devoret and RS, Nature 2000.
"Quantum-limited Measurement and Information in Mesoscopic Detectors," A.Clerk, S. Girvin, D. Stone PRB 2003.

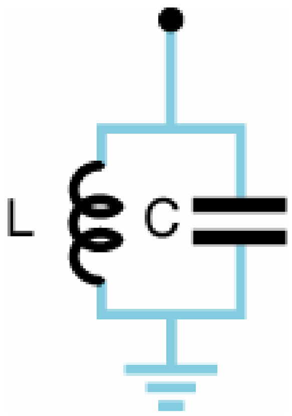
And see also upcoming RMP by Clerk, Girvin, Devoret, & RS.

Outline of Lecture 1

- Quantum circuit intro and toolbox
- Electrical quantum noise of a harmonic oscillator (L-C)
- How to make a quantum resistor (= the vacuum!)
- Noise of a resistor:
  the quantum Fluctuation-Dissipation Theorem (FDT)
- Experiments on the zero point noise in circuits
- Shot noise and the nonequilibrium FDT (time permitting)

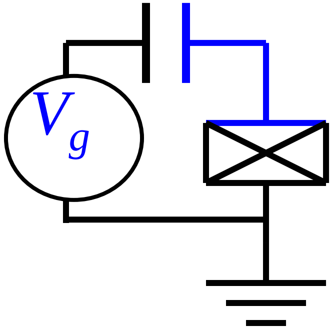
Quantum Circuit Toolbox

L-C Resonator



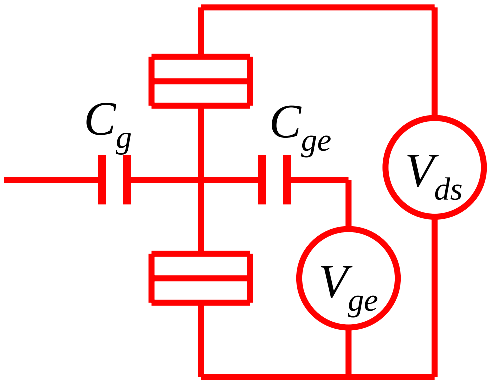
Harmonic oscillator

Cooper-Pair Box



Two-level system (qubit)

Single Electron Transistor

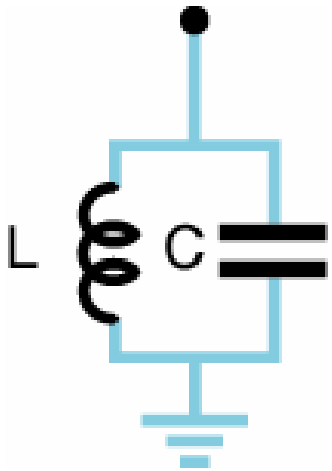


Voltage/Charge amplifier

Superconductors: quality factor 10^6 or greater – levels sharp

$\hbar\omega > kT$ 1 GHz = 50 mK, very few levels populated

The Electrical Harmonic Oscillator



$$\boxed{\omega_0 = 1/\sqrt{LC}}$$

$$\boxed{Z_0 = \sqrt{\frac{L}{C}}}$$

$$Z_{HO} = \frac{1}{1/i\omega L + i\omega C} = \frac{i\omega L}{1-(\omega/\omega_0)^2}$$

$$\phi(t) = LI(t) = \int_{-\infty}^{t} V(\tau)d\tau$$

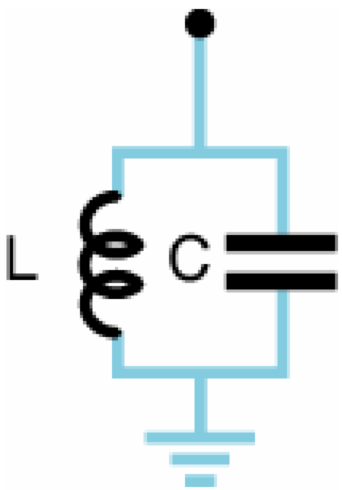
$$\ell = \frac{1}{2}C\dot{\phi}^2 - \frac{1}{2L}\phi^2$$

$C \Leftrightarrow$ mass $\quad 1/L \Leftrightarrow$ spring constant

$Q = C\dot{\phi} \Leftrightarrow$ momentum

Thermal equilibrium: $\left\langle Q^2 \right\rangle \sim kTC$

The Quantum Electrical Oscillator



$$H = \frac{Q^2}{2C} + \frac{\phi^2}{2L} = \hbar\omega_0\left(a^\dagger a + \frac{1}{2}\right)$$

"p"

$$\boxed{Q = \frac{1}{i}\sqrt{\frac{\hbar}{2Z_0}}\left(a - a^\dagger\right)}$$

"x"

$$\boxed{\phi = \sqrt{\frac{\hbar Z_0}{2}}\left(a + a^\dagger\right)}$$

$$\left[Q,\phi\right] = -i\hbar\left[a, a^\dagger\right] = -i\hbar$$

$$\left[Q,H\right] \neq 0$$

$$\left[\phi,H\right] \neq 0$$

Q and ϕ are not constants of motion!

$$A(t) = e^{iHt/\hbar}A(0)e^{-iHt/\hbar}$$

$\Rightarrow$

$$\left[Q(t),Q(0)\right] \neq 0$$

$$\left[\phi(t),\phi(0)\right] \neq 0$$

Noise of Quantum Oscillator

What about correlation functions of ϕ and Q ?

e.g. for thermal equilibrium

!?

$$\left\langle \phi(t)\phi(0) \right\rangle = \frac{\hbar Z_0}{2}\left[\coth\left(\frac{\hbar\omega_0}{2kT}\right)\cos\left(\omega_0 t\right) \boxed{- i\sin\left(\omega_0 t\right)} \right]$$

1) Correlator not real, how to define/interpret a spectral density?

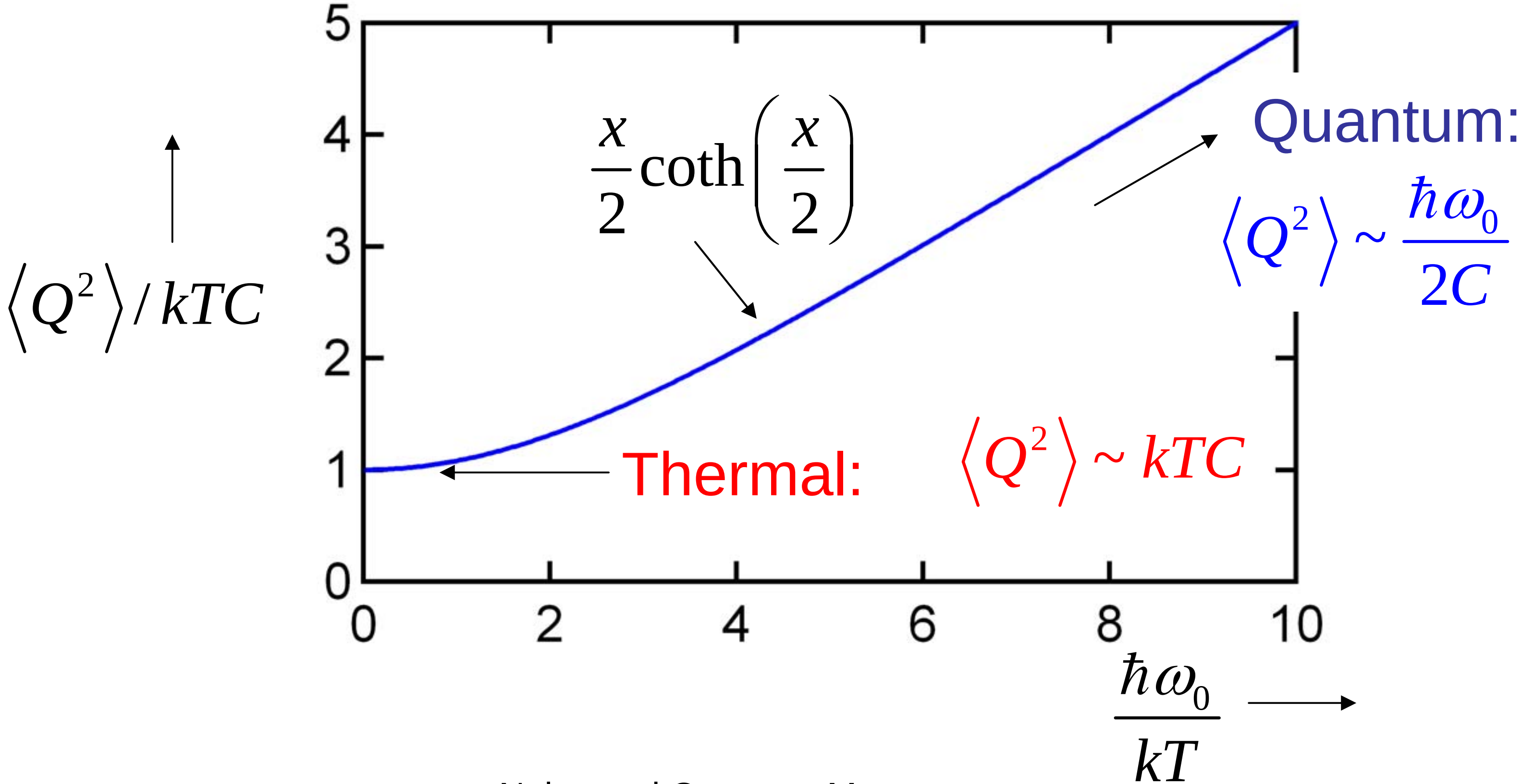
2) Non-zero variances even at T=0

$$\left\langle \phi(0)\phi(0) \right\rangle = \frac{\hbar Z_0}{2}\coth\left(\frac{\hbar\omega_0}{2kT}\right)$$

$$\left\langle Q(0)Q(0) \right\rangle = \frac{\hbar}{2Z_0}\coth\left(\frac{\hbar\omega_0}{2kT}\right)$$

Quantum Fluctuations of Charge

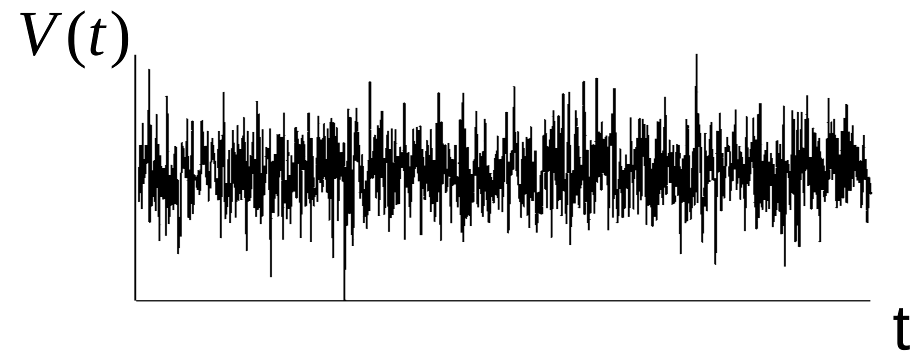
$$\left\langle Q^2 \right\rangle = \frac{\hbar}{2Z_0}\coth\left(\frac{\hbar\omega_0}{2kT}\right) = kTC\,\frac{h\omega_0}{2kT}\coth\left(\frac{\hbar\omega_0}{2kT}\right)$$



Noise and Spectral Densities Classically

Random variable

$$V(t)$$



Auto-correlation function $C_{VV}(t-t') = \langle V(t)V(t') \rangle$

Fourier transform $\tilde{V}(\omega) = \dfrac{1}{\sqrt{T}} \int_{-T/2}^{T/2} dt\, e^{i\omega t}\, V(t)$

Spectral density $S_{VV}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} \langle V(t)V(t') \rangle = \langle \tilde{V}(\omega)\tilde{V}(-\omega) \rangle$

Since $V(t)$ is classical and real, $\langle V(t)V(t') \rangle = \langle V(t')V(t) \rangle$

And so: $\boxed{S_{VV}(\omega) = S_{VV}(-\omega)}$

Spectral Density of Classical Oscillator

for mechanical harmonic oscillator in thermal equil.:

mass, m
resonant freq, ω_0

$$x(t) = x(0)\cos(\omega_0 t) + p(0)\frac{1}{m\omega_0}\sin(\omega_0 t)$$

$$p(t) = p(0)\cos(\omega_0 t) - x(0)m\omega_0{}^2\sin(\omega_0 t)$$

position correlation function:

0 in equil.

$$C_{xx}(t) = \langle x(t)x(0)\rangle = \langle x(0)x(0)\rangle\cos(\omega_0 t) + \cancel{\langle p(0)x(0)\rangle}\frac{1}{m\omega_0}\sin(\omega_0 t)$$

equipartition thm: $\frac{1}{2}k\langle x^2\rangle = \frac{1}{2}m\omega_0^2\langle x^2\rangle = \frac{1}{2}kT$

$$C_{xx}(t) = \frac{kT}{m\omega_0^2}\cos(\omega_0 t) \quad \Longrightarrow \quad \text{F.T.}$$

$$S_{xx}(\omega) = \pi\frac{kT}{m\omega_0{}^2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

symmetric in ω!

Spectral Density of Quantum Oscillator - I

$$\hat{x}(t) = \hat{x}(0)\cos(\omega_0 t) + \hat{p}(0)\frac{1}{m\omega_0}\sin(\omega_0 t)$$

$$\hat{p}(t) = \hat{p}(0)\cos(\omega_0 t) - \hat{x}(0)m\omega_0^{\;2}\sin(\omega_0 t)$$

$$C_{xx}(t) = \left\langle \hat{x}(0)\hat{x}(0)\right\rangle \cos(\omega_0 t) + \left\langle \hat{p}(0)\hat{x}(0)\right\rangle \frac{1}{m\omega_0}\sin(\omega_0 t)$$

but because $\left[x(0), p(0)\right] = i\hbar$

$$\left\langle x(0)\,p(0)\right\rangle - \left\langle p(0)\,x(0)\right\rangle = i\hbar$$

and $\left\langle p(0)\,x(0)\right\rangle = -i\hbar/2 \neq 0$

Spectral Density of Quantum Oscillator - II

using $\hat{x} = x_{RMS}\left(a^{\dagger} + a\right)$

with $x_{RMS}^2 = \left\langle 0\left|\hat{x}^2\right|0\right\rangle = \dfrac{\hbar}{2m\omega_0}$

$$a^{\dagger}(t) = e^{iHt/\hbar}a^{\dagger}(0)e^{-iHt/\hbar} = e^{i\omega_0 t}a^{\dagger}(0)$$

$$C_{xx}\left(t\right) = \left\langle \hat{x}\left(t\right)\hat{x}\left(0\right)\right\rangle = x_{RMS}^2\left\langle e^{i\omega_0 t}a^{\dagger}(0)a(0) + e^{-i\omega_0 t}a(0)a^{\dagger}(0)\right\rangle$$

$$C_{xx}\left(t\right) = x_{RMS}^2\left(n_{BE}\left(\hbar\omega_0\right)e^{+i\omega_0 t} + \left[n_{BE}\left(\hbar\omega_0\right)+1\right]e^{-i\omega_0 t}\right)$$

where $n_{BE}\left(\hbar\omega_0\right) = \dfrac{1}{e^{\hbar\omega_0/kT}-1}$ is the Bose-Einstein occupation

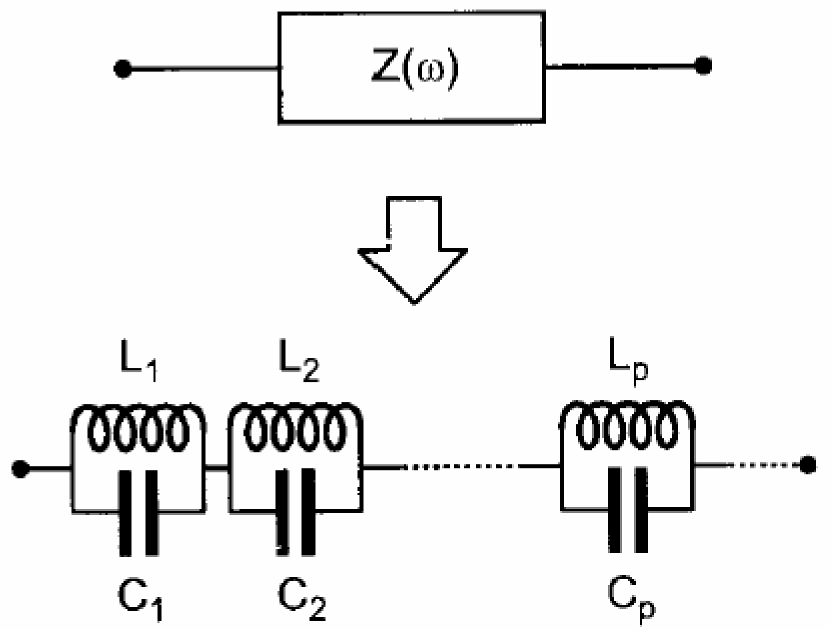
$$S_{xx}\left(\omega\right) = 2\pi x_{RMS}^2\left[n_{BE}\left(\hbar\omega_0\right)\delta\left(\omega+\omega_0\right) + \left[n_{BE}\left(\hbar\omega_0\right)+1\right]\delta\left(\omega-\omega_0\right)\right]$$

asymmetric in frequency!!

How to Make a Resistor - 1

Caldeira-Legget prescription:

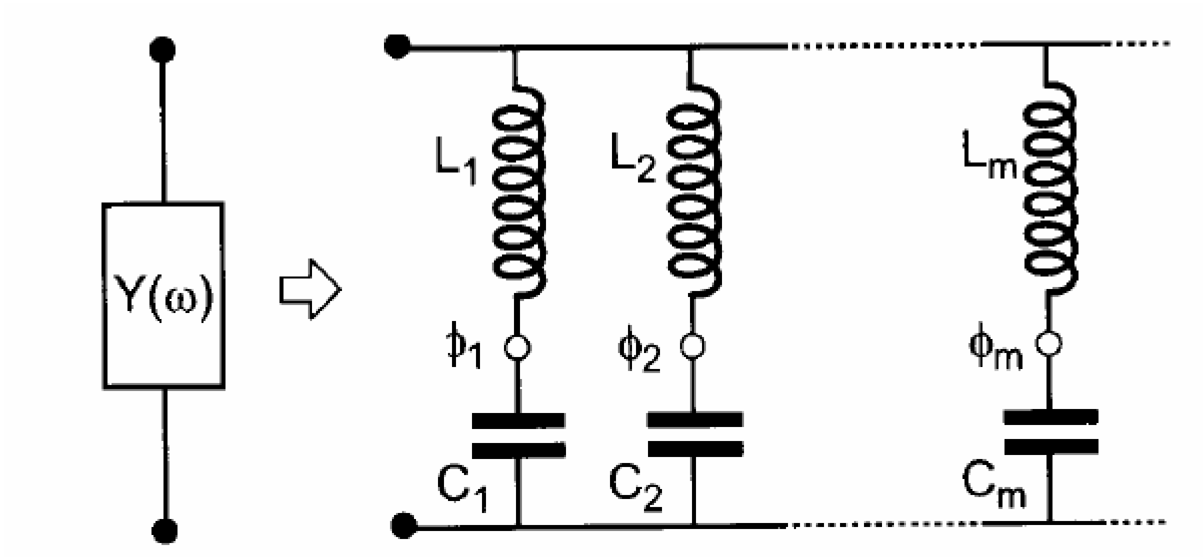
“Sum infinite number of oscillators to make continuum”



Caldeira and Leggett, Ann. Phys. **149**, 374 (1983).

How to Make a Resistor - 2

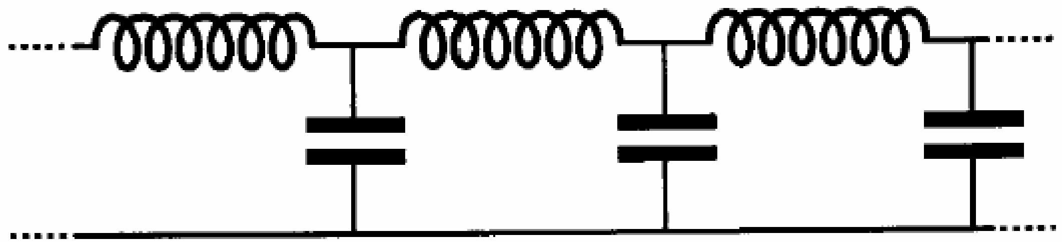
Admittance = parallel sum of series resonances



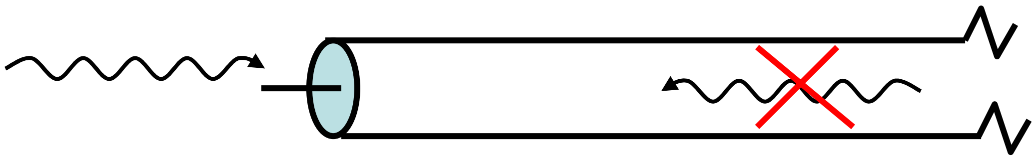
L's and C's chosen to give dense comb of frequencies and the correct value of impedance/admittance

How to Make a Resistor - 3

Transmission line = infinite LC ladder



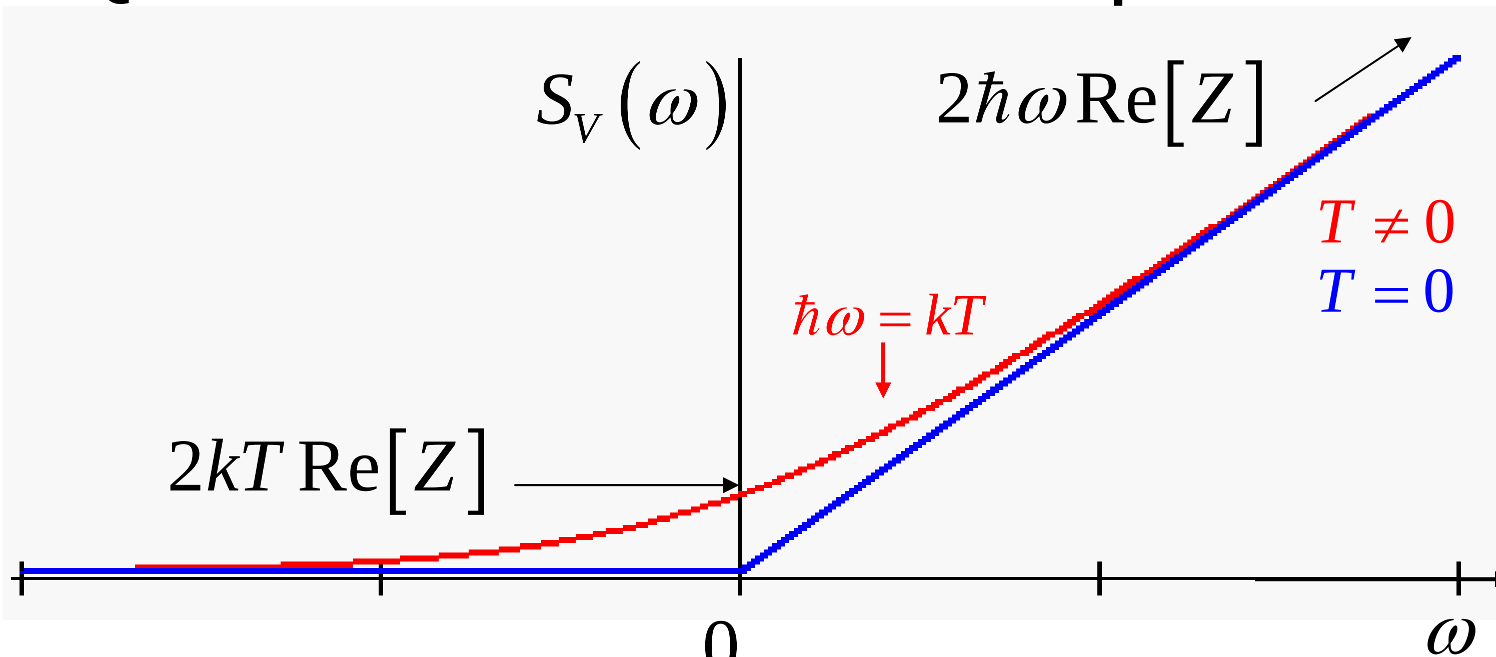
L and C are constants (all same) =
to the inductance and capacitance per unit length,
calculated from electro/magneto-statics of the
particular transmission line



the "vacuum"
or a perfect
blackbody!

Line needs to be infinite – no reflections/memory and
infinite number of d.o.f. to make reservoir

Quantum Noise of an Impedance



The quantum fluctuation-dissipation relation:

$$S_V(\omega) = \frac{2\hbar\omega \mathrm{Re}[Z]}{1 - e^{-\hbar\omega/kT}}$$

<u>Three limiting cases:</u>

$|\hbar\omega| \ll kT \qquad S_V = 2kT\,\mathrm{Re}[Z]$

$\hbar\omega \gg kT \qquad S_V = 2\hbar\omega \mathrm{Re}[Z]$

$\hbar\omega \ll -kT \qquad S_V = 0$

Symmetrized and Antisymmetrized Noise

$$S_V(\omega) = \frac{2\hbar\omega R}{1-e^{-\hbar\omega/kT}} \qquad \langle n\rangle = \frac{1}{e^{\hbar|\omega|/kT}-1}$$

stim. emission

$\boxed{\omega > 0:}$ $$S_V(+\omega) = 2\hbar\omega\mathrm{R}\left(\langle n\rangle + 1\right)$$ spont. emission

$\boxed{\omega < 0:}$ $$S_V(-\omega) = 2\hbar\omega\mathrm{R}\langle n\rangle$$ absorption

Symmetrized noise spectrum:

$$S_V^S(|\omega|) = S_V(+\omega) + S_V(-\omega) \sim 2\hbar\omega\left(2\langle n\rangle + 1\right)\mathrm{R}$$

$$S_V^S(|\omega|) = 2\hbar|\omega|\mathrm{R}\coth\left[\hbar|\omega|/2kT\right]$$

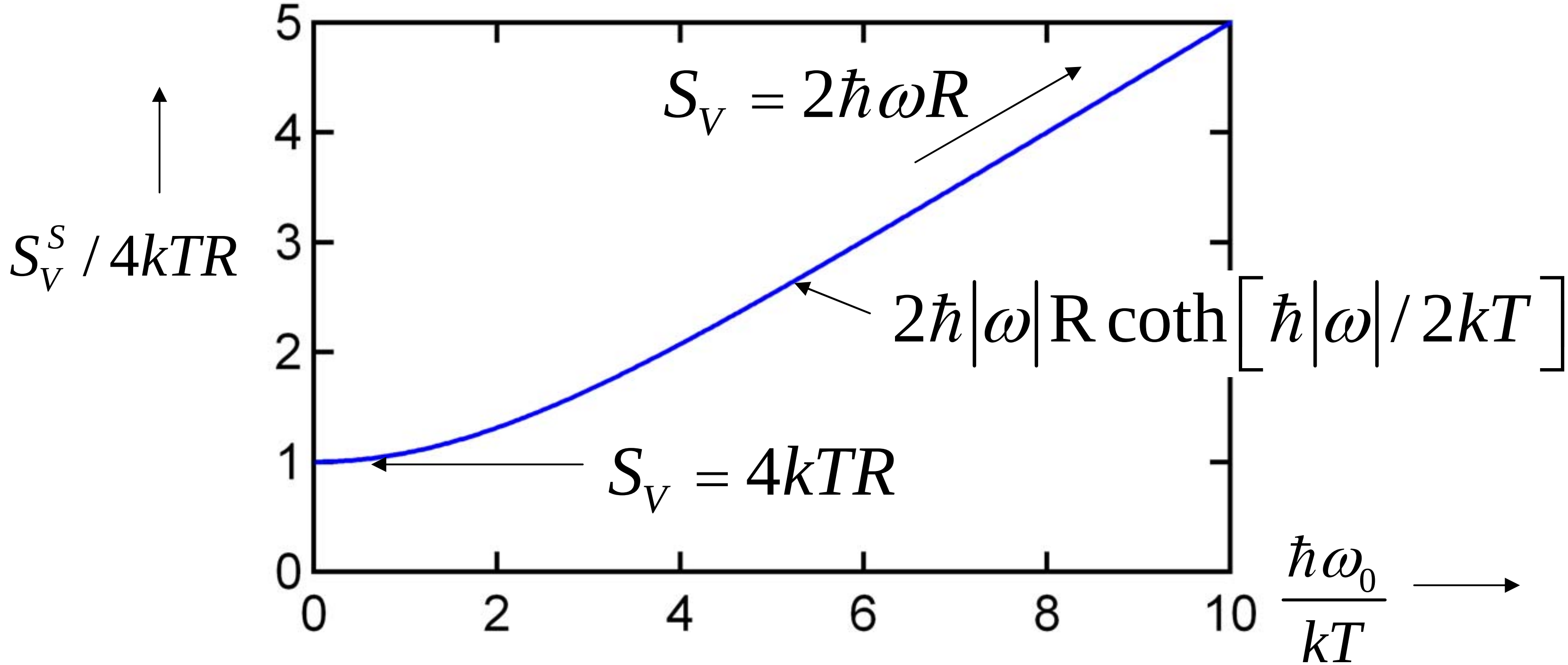
Callen and Welton, Phys. Rev. **83**, 34 (1951)

Anti-Symmetrized noise spectrum:

$$S_V^A(|\omega|) = S_V(+\omega) - S_V(-\omega) \sim 2\hbar\omega\mathrm{R}$$

dissipation (T indep.!)

Symmetrized (One-Sided) Johnson Noise



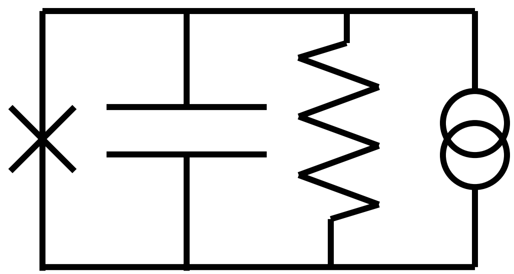
$$T_{RJ} = \frac{S_V^S(\omega \to 0)}{4kR} = T \qquad T_Q = \frac{S_V^S(\omega \to \infty)}{4kR} = \frac{\hbar\omega}{2k}$$

"energy per mode = ½ photon"

Experiments On Quantum Johnson Noise

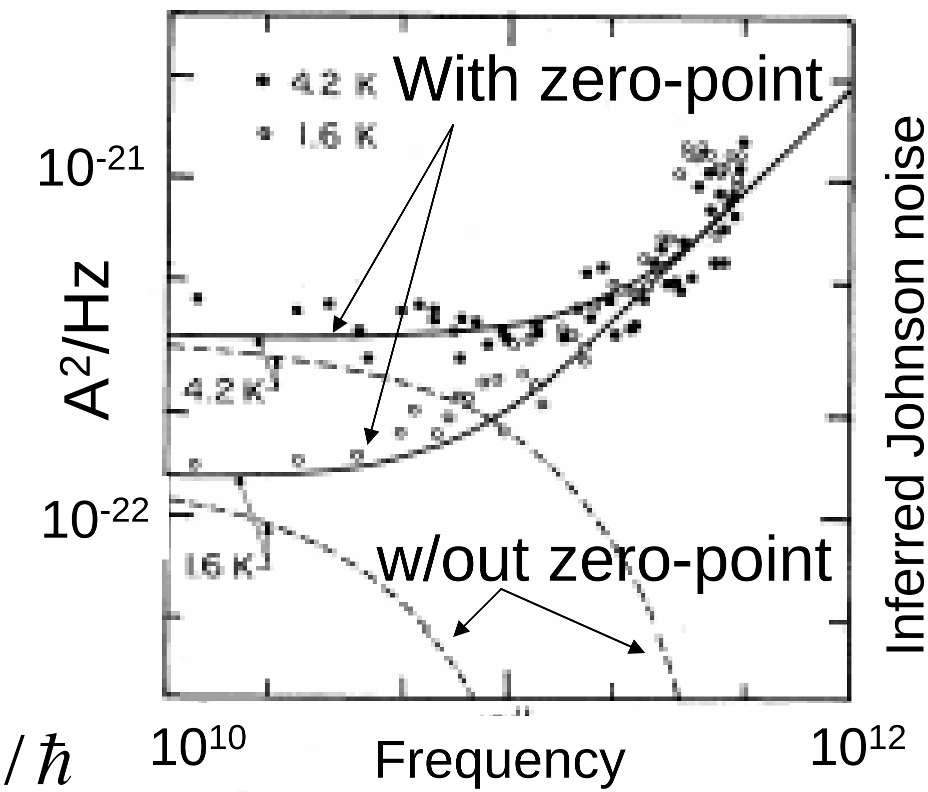
Method: measure low-freq. noise of resistively-shunted JJ

$$I = I_C \sin\phi$$



$$\frac{S_v(0)}{R_D^{\,2}} = \frac{4k_BT}{R} + \boxed{\frac{2eV}{R}\left(\frac{I_0}{I}\right)^2 \coth\left(\frac{eV}{k_BT}\right)}$$

Rectified noise from $\omega = 2eV_{DC}/\hbar$



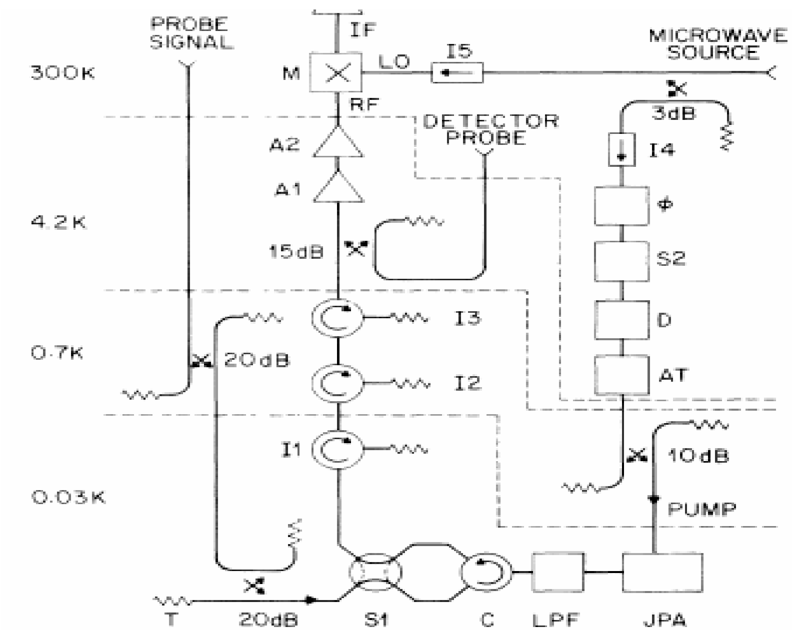
JJ “mixes down” noise from THz frequencies to audio

Koch, van Harlingen, and Clarke, PRL **47**, 1216 (1981)

Experiments On Quantum Johnson Noise

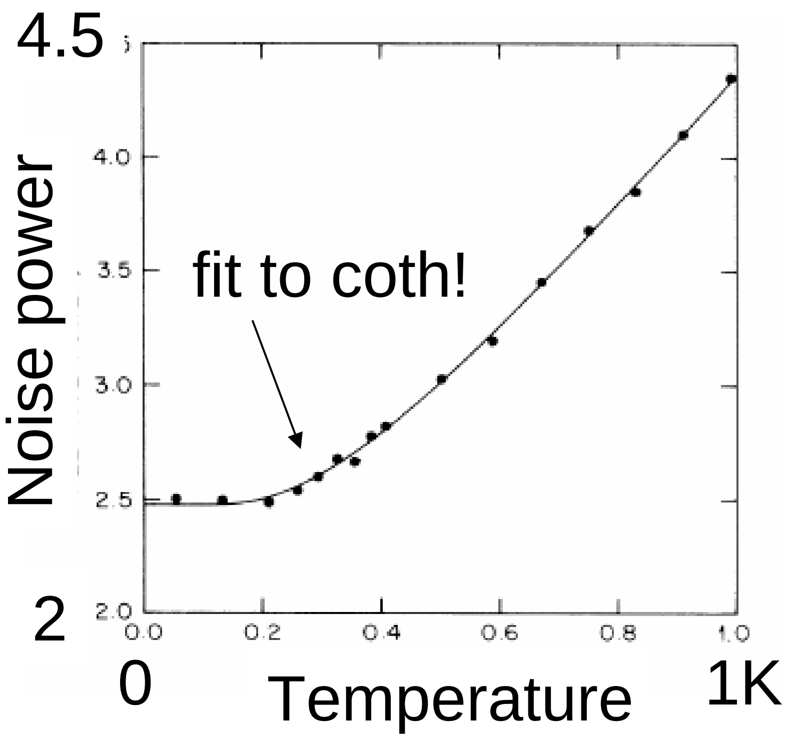
Work by Bernie Yurke et al. at Bell Labs

Josephson parametric amplifier:
19 GHz and 30 mK



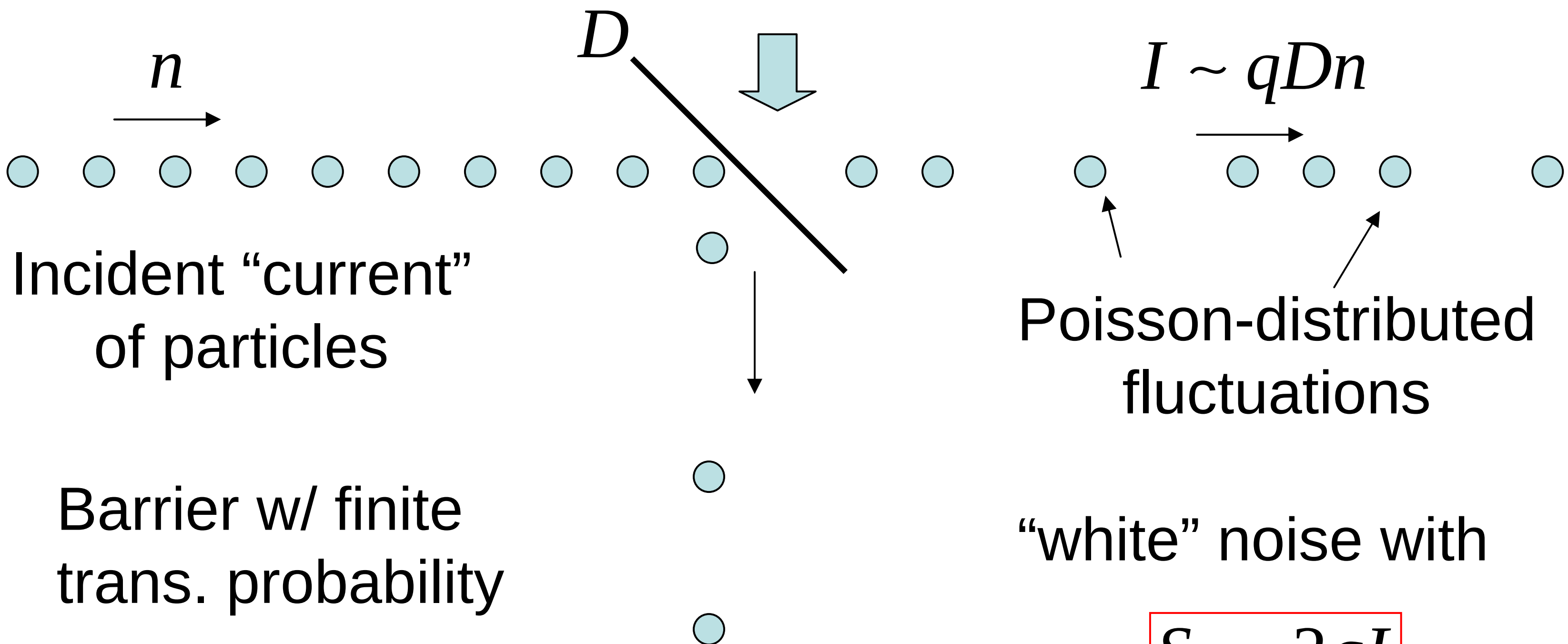
T_{amp} = 0.45K = $h\nu/2k$
quantum limited amplifier!

observed zero-point part of waveguide's noise (and then squeezed it!)



Movshovich et al., PRL **65**, 1419 (1990)

Shot Noise – "Classically"



Incident "current" of particles

Barrier w/ finite trans. probability

Poisson-distributed fluctuations

"white" noise with

$$\boxed{S_I = 2qI}$$

Shot Noise is Quantum Noise

Einstein, 1909: Energy fluctuations of thermal radiation

"Zur gegenwartigen Stand des Strahlungsproblems," Phys. Zs. **10** 185 (1909)

$$\left\langle (\Delta E)^2 \right\rangle = \left[\underset{\text{particle term = shot noise!}}{\underbrace{\hbar\omega\rho(\omega)}} + \frac{\pi^2 c^3}{\omega^2} \underset{\text{wave term}}{\underbrace{\rho^2(\omega)}} \right] V d\omega$$

particle term = shot noise!   wave term

first appearance of wave-particle complementarity?

Can show that "particle term" is a consequence of $\left[a, a^\dagger\right] = 1$

(see Milloni, "The Quantum Vacuum," Academic Press, 1994)

$$\left\langle \Delta n^2 \right\rangle = \left\langle a^\dagger a a^\dagger a \right\rangle - \left\langle a^\dagger a \right\rangle^2$$

$$= \left\langle a^\dagger (a^\dagger a + 1) a \right\rangle - \left\langle a^\dagger a \right\rangle^2$$

$$= \left\langle a^\dagger a^\dagger a a \right\rangle + \left\langle a^\dagger a \right\rangle - \left\langle a^\dagger a \right\rangle^2$$

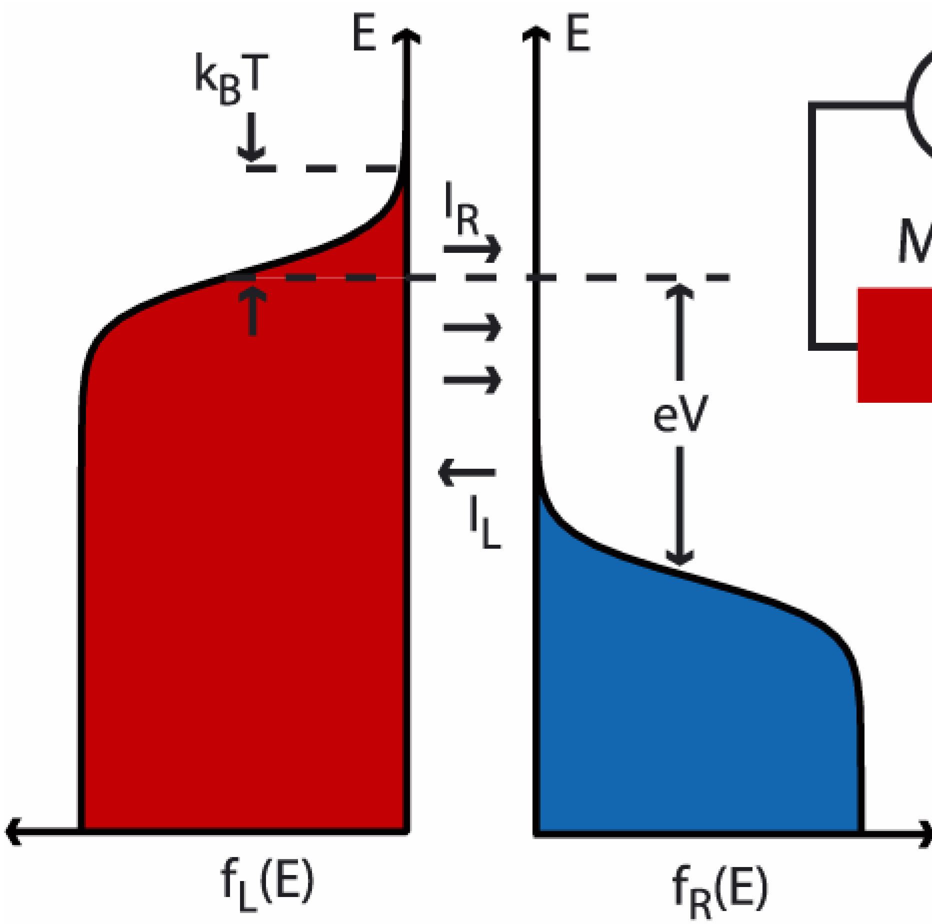
$$\boxed{\left\langle \Delta n^2 \right\rangle = \bar{n}^2 + \bar{n}}$$

$$\bar{n} = \left\langle n \right\rangle = \left(e^{\hbar\omega/kT} - 1 \right)^{-1}$$

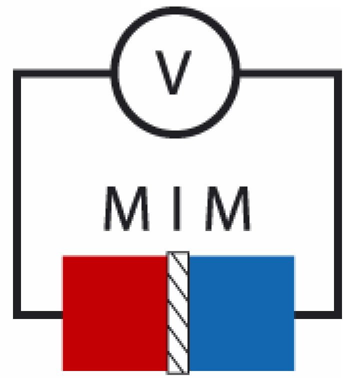
$$P_n = \bar{n}^n / \left(\bar{n} + 1 \right)^{n+1}$$

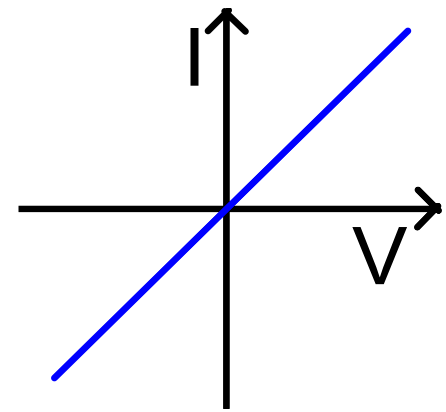
$$\left\langle a^\dagger a^\dagger a a \right\rangle = \sum n(n-1) P_n = 2\bar{n}^2$$

Conduction in Tunnel Junctions



Fermi functions





$$I_{L\to R} = \frac{G}{e}\int f_L(1-f_R)dE$$

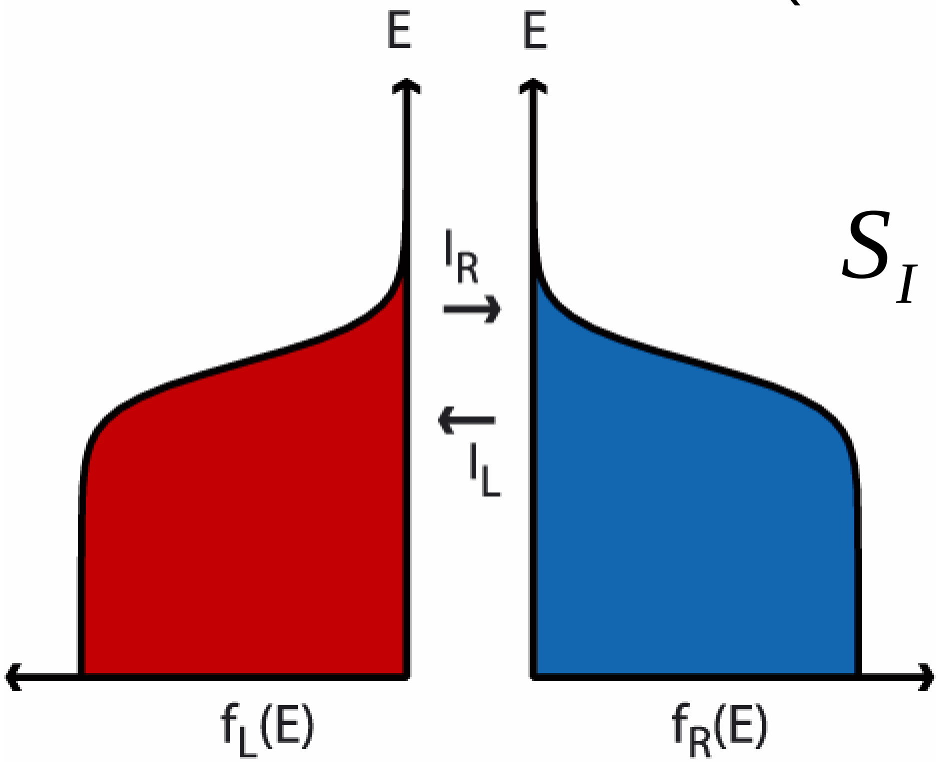
$$I_{R\to L} = \frac{G}{e}\int f_R(1-f_L)dE$$

Difference gives current:

$$I = I_{L\to R} - I_{R\to L} = GV$$

Assume: Tunneling amplitudes and
D.O.S. independent of energy ⟶ Conductance (G) is constant
Fermi distribution of electrons

Non-Equilibrium Noise of a Tunnel Junction



(Zero-frequency limit)

Sum gives noise:

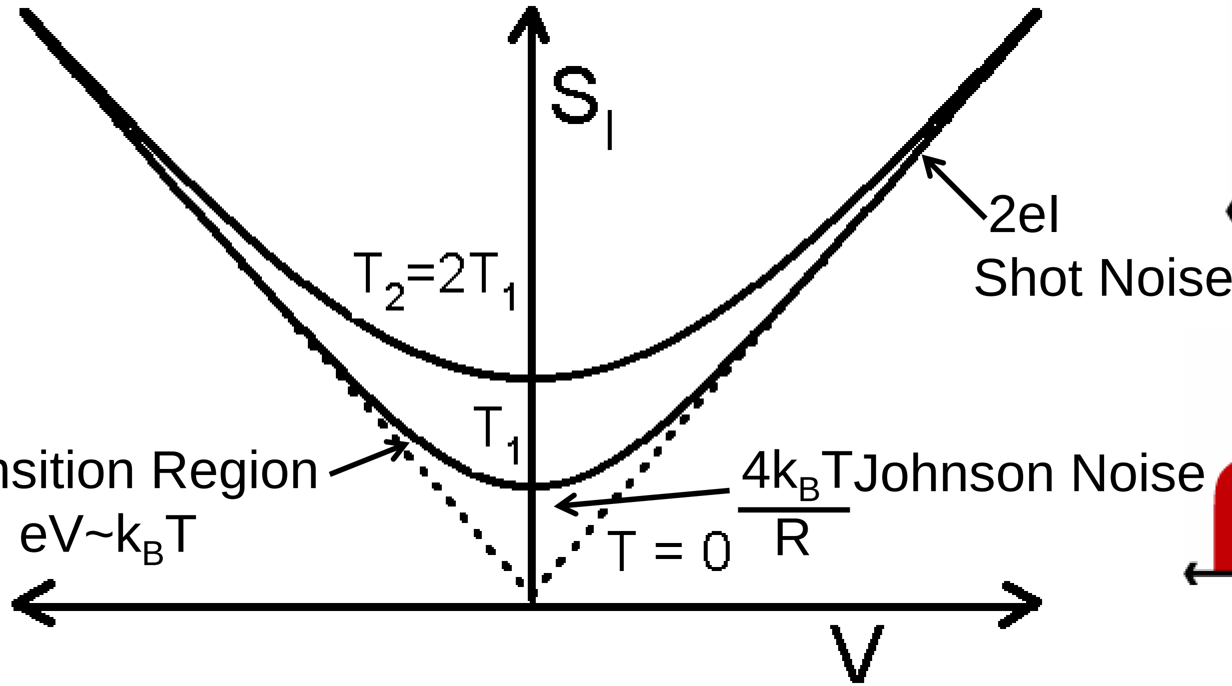
$$S_I(f) = 2e(I_{L\to R} + I_{R\to L})$$

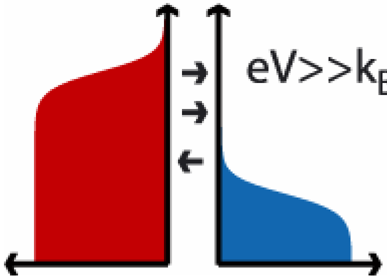
$$\boxed{S_I(f) = 2eI\coth\left(\frac{eV}{2k_BT}\right)}$$

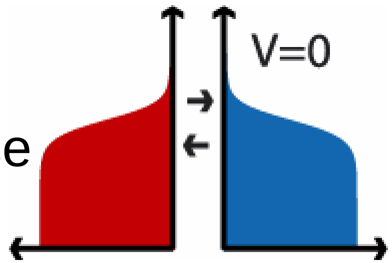
$$\boxed{I = V/R}$$

*D. Rogovin and D.J. Scalpino, Ann Phys. **86**,1 (1974)

Non-Equilibrium Fluctuation Dissipation Theorem

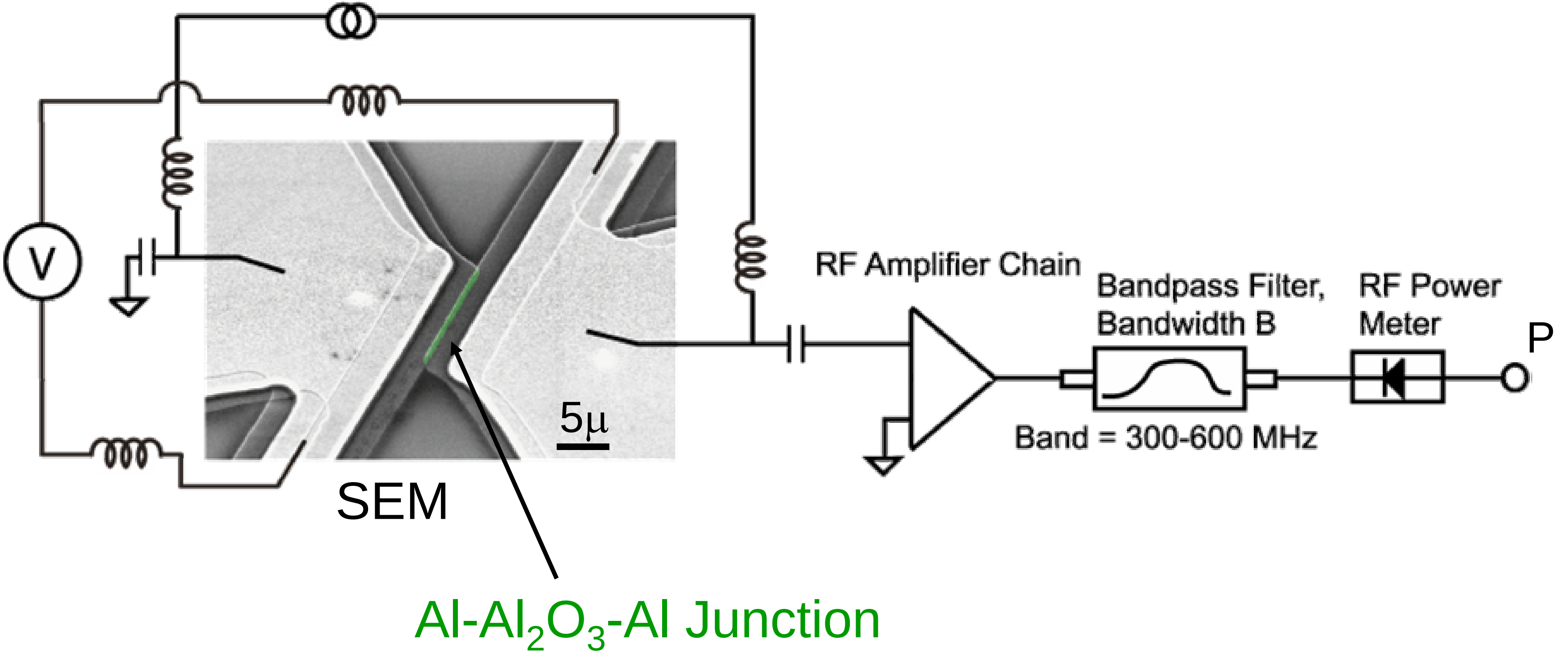






$$\boxed{S_I(f) = 2eV/R \coth\left(\frac{eV}{2k_BT}\right)}$$

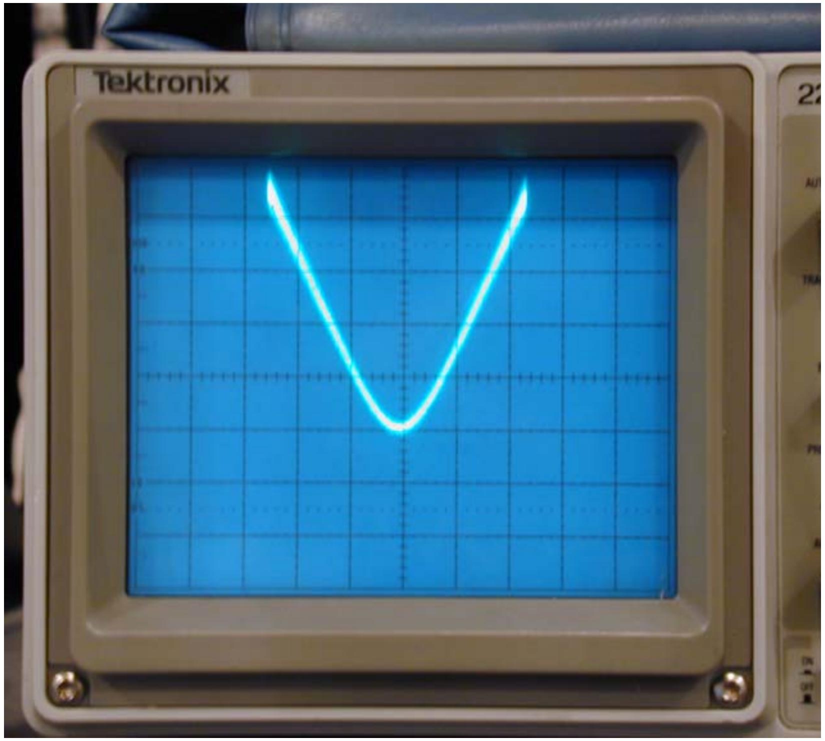
Noise Measurement of a Tunnel Junction



Measure symmetrized noise spectrum at $\hbar\omega < kT$

Seeing is Believing

$$\frac{\delta P}{P} = \frac{1}{\sqrt{B\tau}}$$

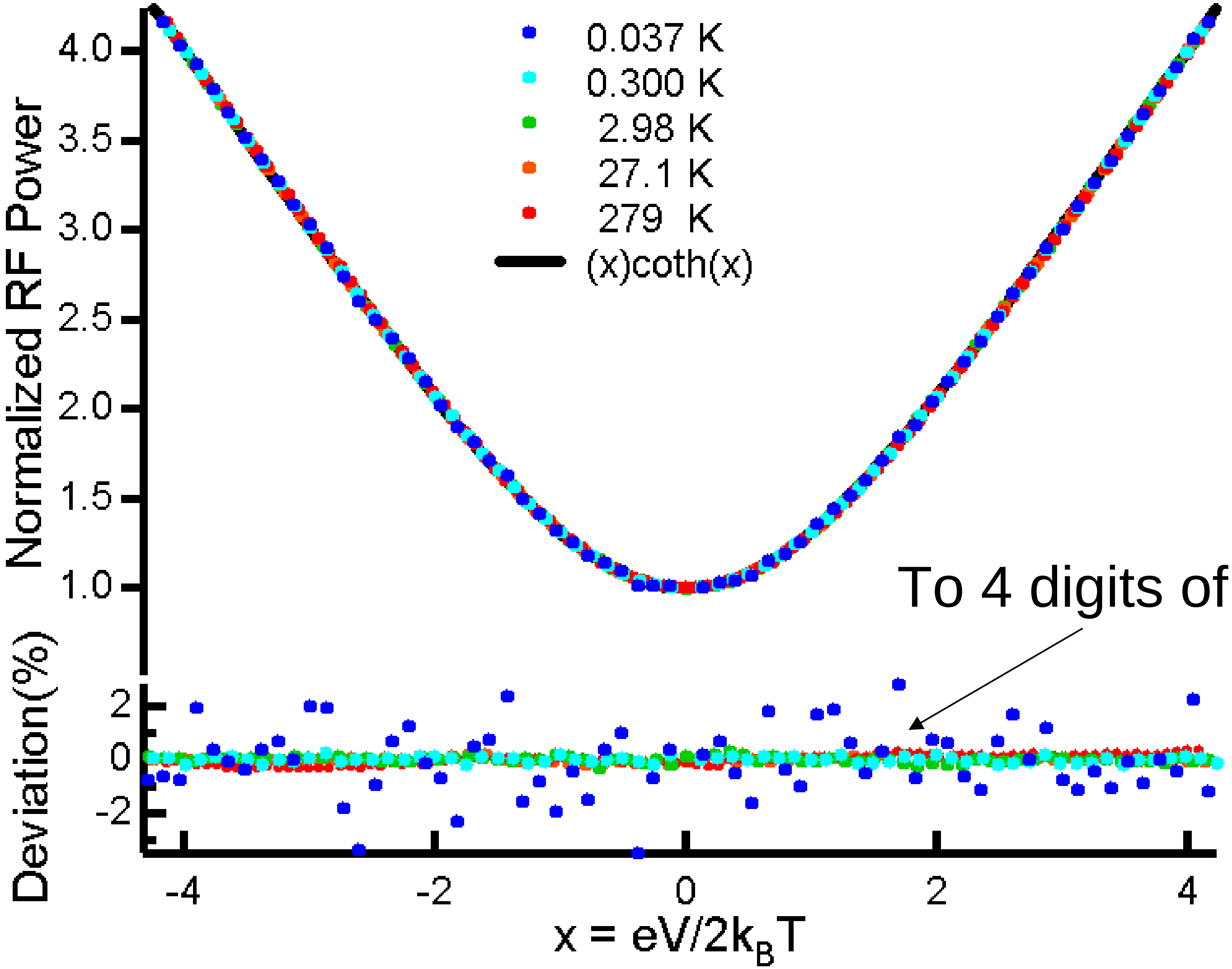


High bandwidth measurements of noise

$B \sim 10^8\,Hz, \tau$ = 1 second $\longrightarrow \dfrac{\delta P}{P} = 10^{-4}$

Test of Nonequilibrium FDT

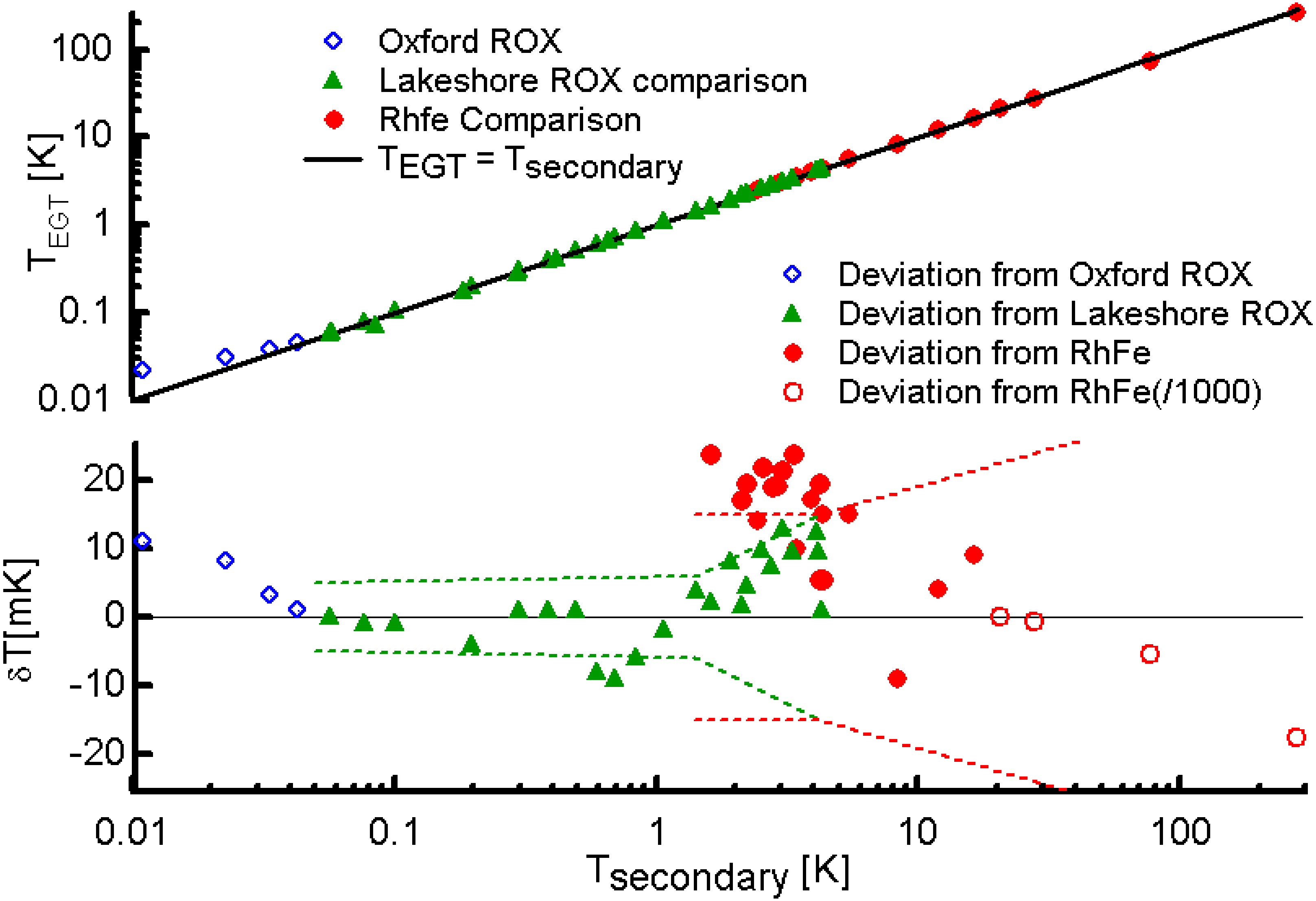
Agreement over four decades in temperature



To 4 digits of precision

L. Spietz et al., Science **300**, 1929 (2003)

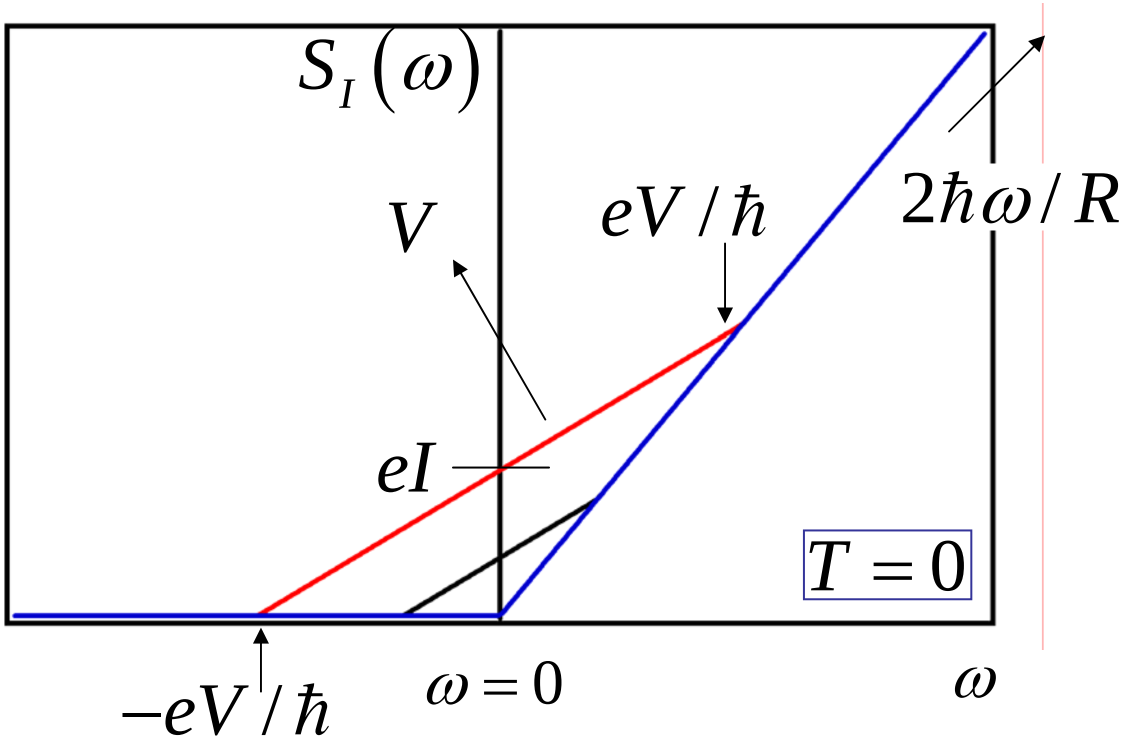
Comparison to Secondary Thermometers



Two-sided Shot Noise Spectrum

(Quantum, non-equilibrium FDT)

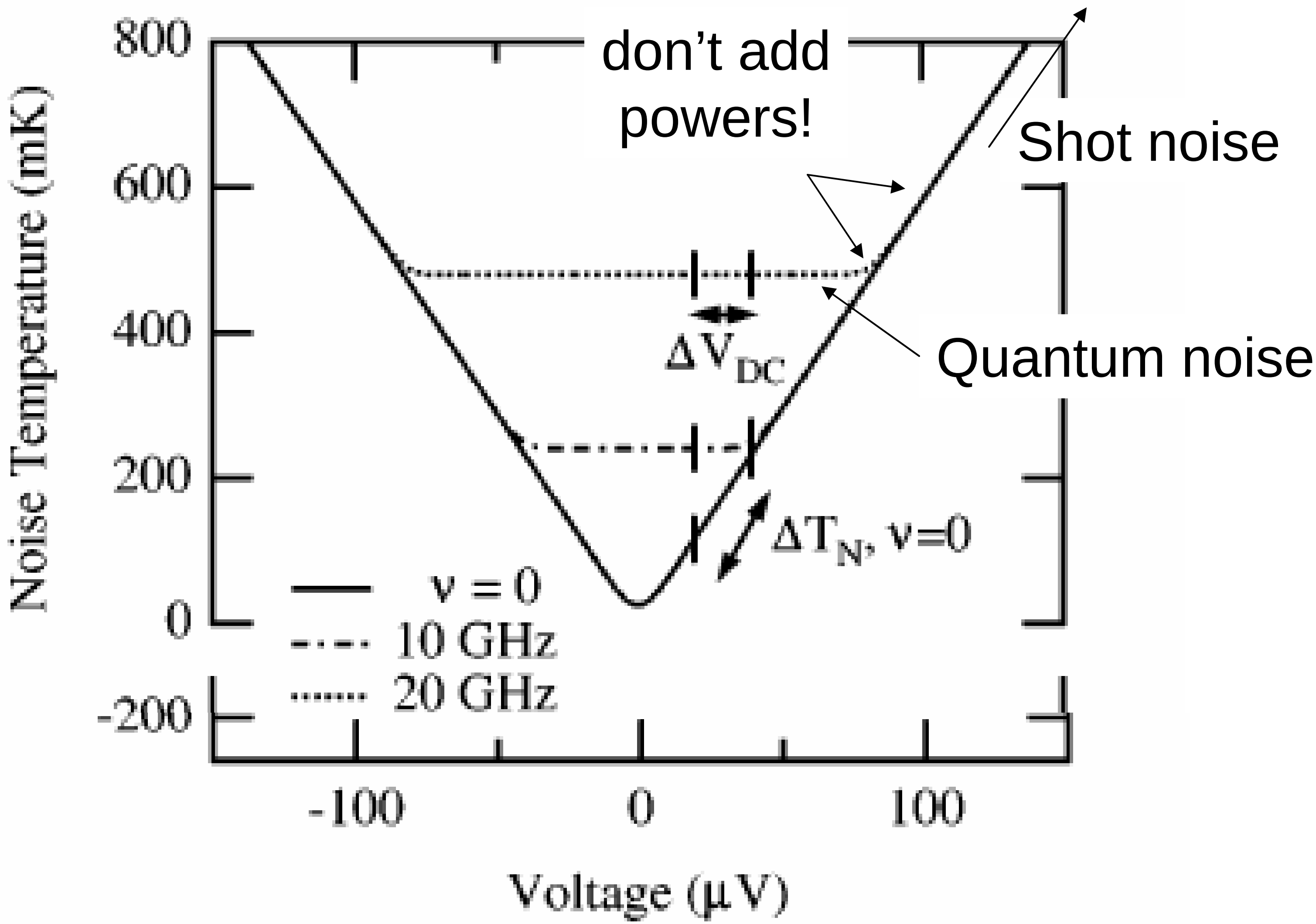
$$S_I(\omega) = \frac{\left(\hbar\omega + eV\right)/\mathrm{R}}{1 - e^{-\left(\hbar\omega + eV\right)/kT}} + \frac{\left(\hbar\omega - eV\right)/\mathrm{R}}{1 - e^{-\left(\hbar\omega - eV\right)/kT}}$$



Aguado & Kouwenhoven, PRL **84**, 1986 (2000).

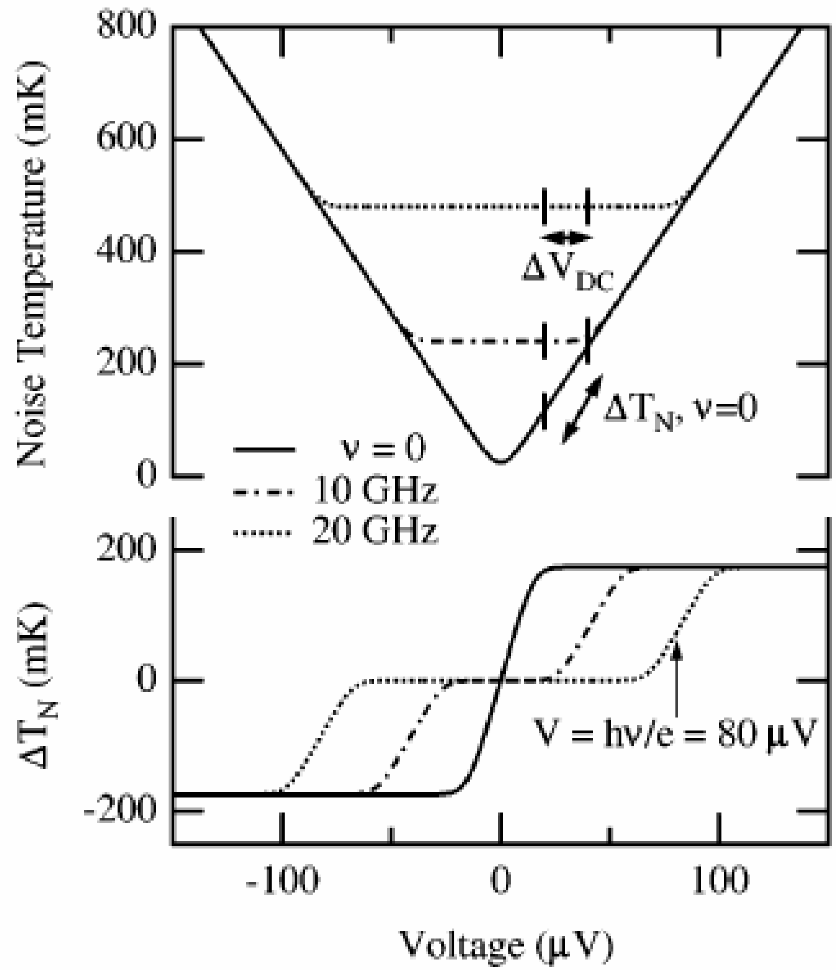
Finite Frequency Shot Noise

Symmetrized Noise: $S_{sym} = S(+\omega) + S(-\omega)$

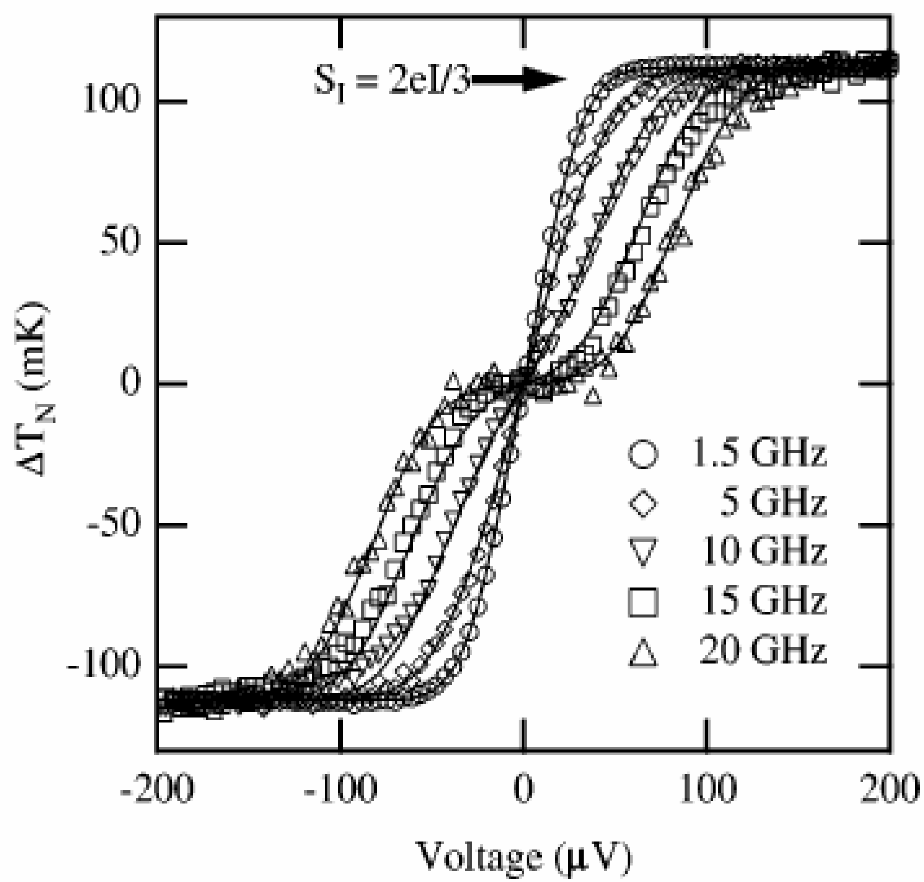


Measurement of Shot Noise Spectrum

Theory

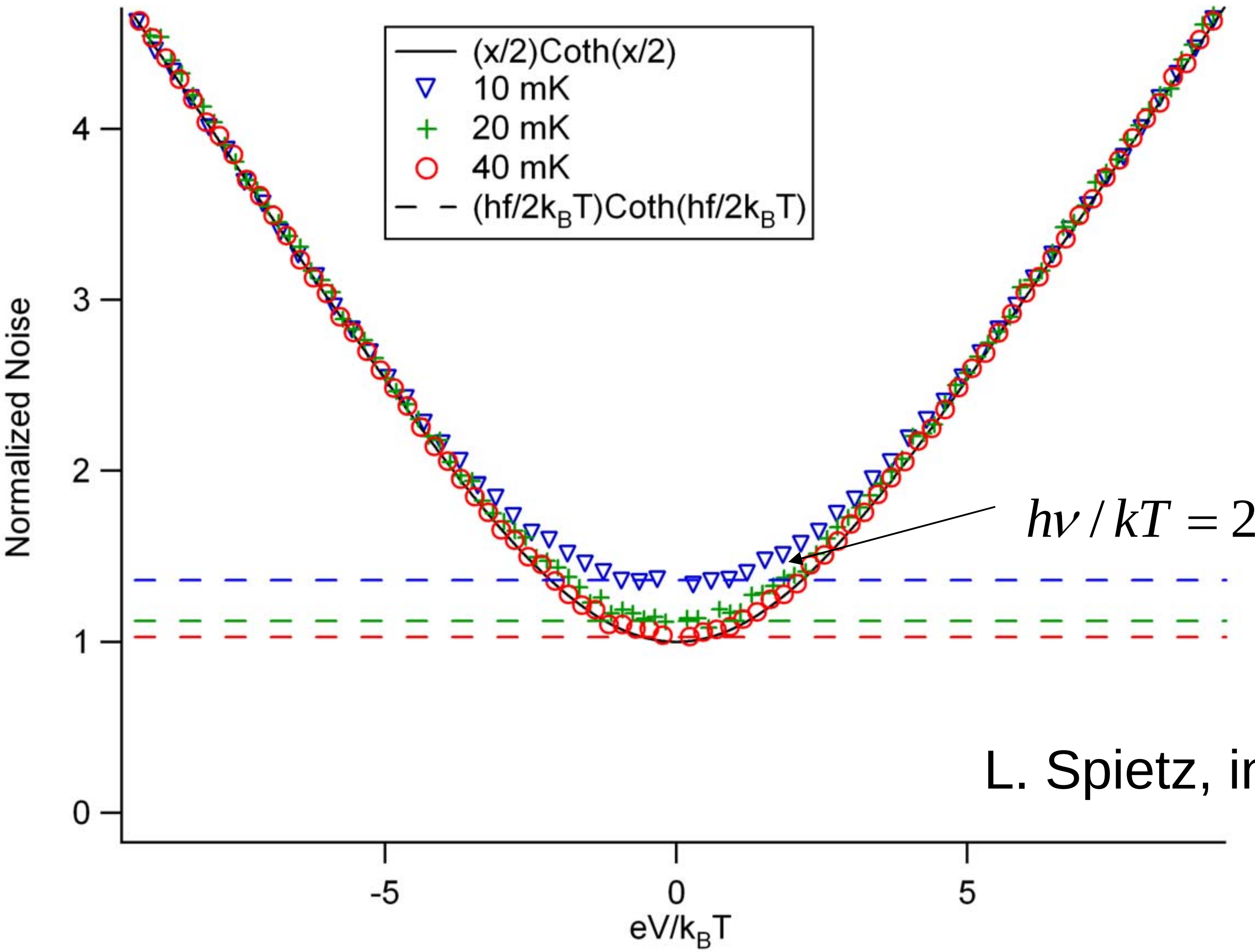


Expt.



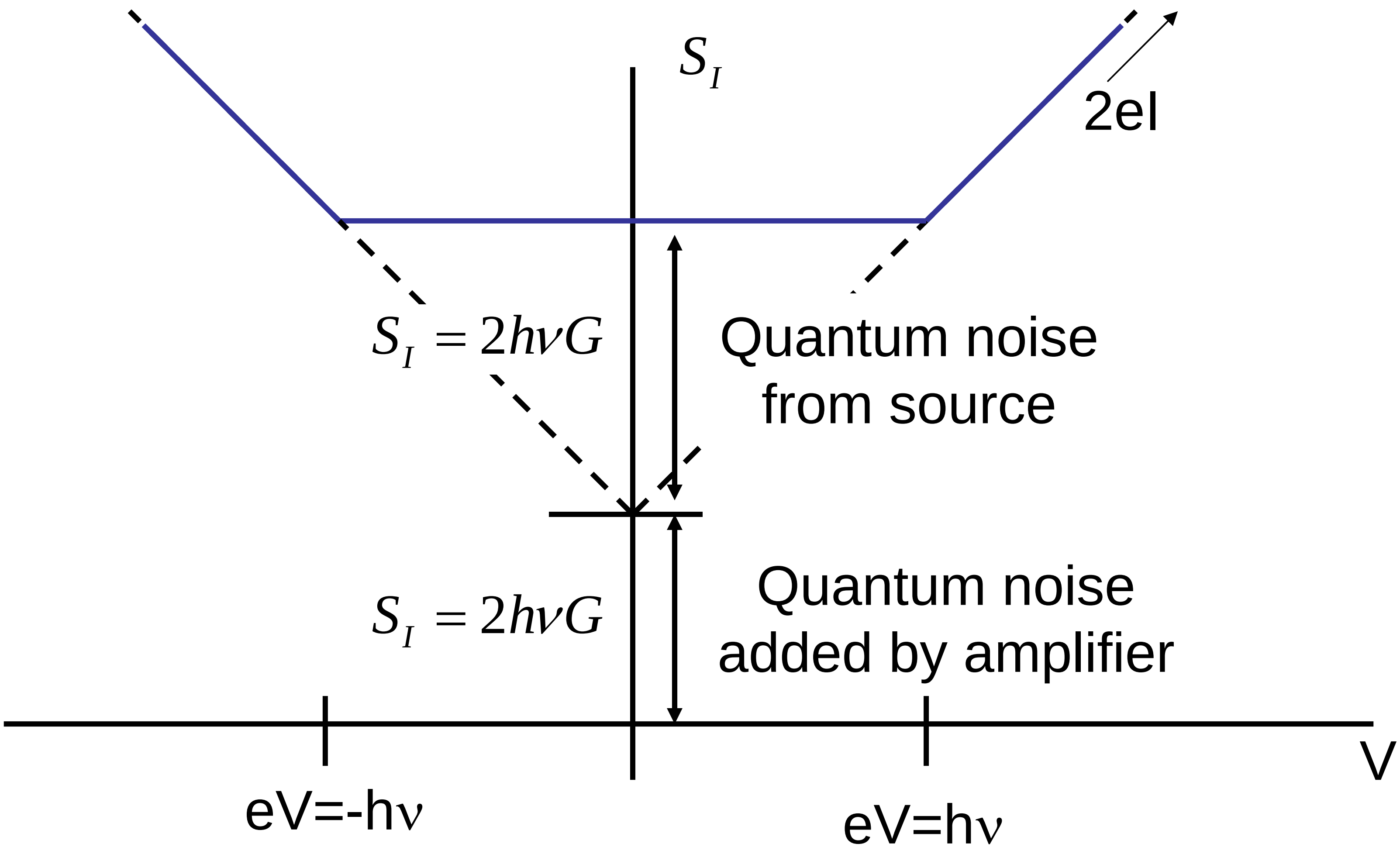
Schoelkopf et al., PRL **78**, 3370 (1998)

Shot Noise at 10 mK and 450 MHz



L. Spietz, in prep.

With An Ideal Amplifier and T=0



Summary – Lecture 1

- Quantizing an oscillator leads to quantum fluctuations present even at zero temperature.
- This noise has built in correlations that make it very different from any type of classical fluctuations, and these cannot be represented by a traditional spectral density- requires a “two-sided” spectral density.
- Quantum systems coupled to a non-classical noise source can distinguish classical and quantum noise, and allow us to measure the full density – next lecture!