Introduction to Theory of Mesoscopic Systems

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Lecture 3

Beforehand — Weak Localization and Mesoscopic Fluctuations

Today — Random Matrices, Anderson Localization, and Quantum Chaos

Later — Interaction between electrons in mesoscopic systems

ORIGINS

E.P. Wigner, Conference on Neutron Physics by Time of Flight, November 1956

P.W. Anderson, "*Absence of Diffusion in Certain Random Lattices*"; Phys.Rev., 1958, v.109, p.1492

L.D. Landau, "*Fermi-Liquid Theory*" Zh. Exp. Teor. Fiz.,1956, v.30, p.1058

J. Bardeen, L.N. Cooper & J. Schriffer, "*Theory of Superconductivity*"; Phys.Rev., 1957, v.108, p.1175.

RANDOM MATRIX THEORY

Spectral statistics

$N \times N$ **ensemble of Hermitian matrices with random matrix element** $N \to \infty$

E_α - spectrum (set of eigenvalues)

$\delta_1 \equiv \langle E_{\alpha+1} - E_\alpha \rangle$ - mean level spacing

$\langle \ldots\ldots \rangle$ - ensemble averaging

$s \equiv \frac{E_{\alpha+1} - E_\alpha}{\delta_1}$ - spacing between nearest neighbors

$P(s)$ - distribution function of nearest neighbors spacing between

Spectral Rigidity

Level repulsion

$P(s=0)=0$

$P(s \ll 1) \propto s^\beta \qquad \beta=1,2,4$

Noncrossing rule (theorem) $P(s=0)=0$

Suggested by Hund (*Hund F. 1927 Phys. v.40, p.742*)

Justified by von Neumann & Wigner (*v. Neumann J. & Wigner E. 1929 Phys. Zeit. v.30, p.467*) . . . . .

Usually textbooks present a simplified version of the justification due to Teller (*Teller E., 1937 J. Phys. Chem 41 109).*

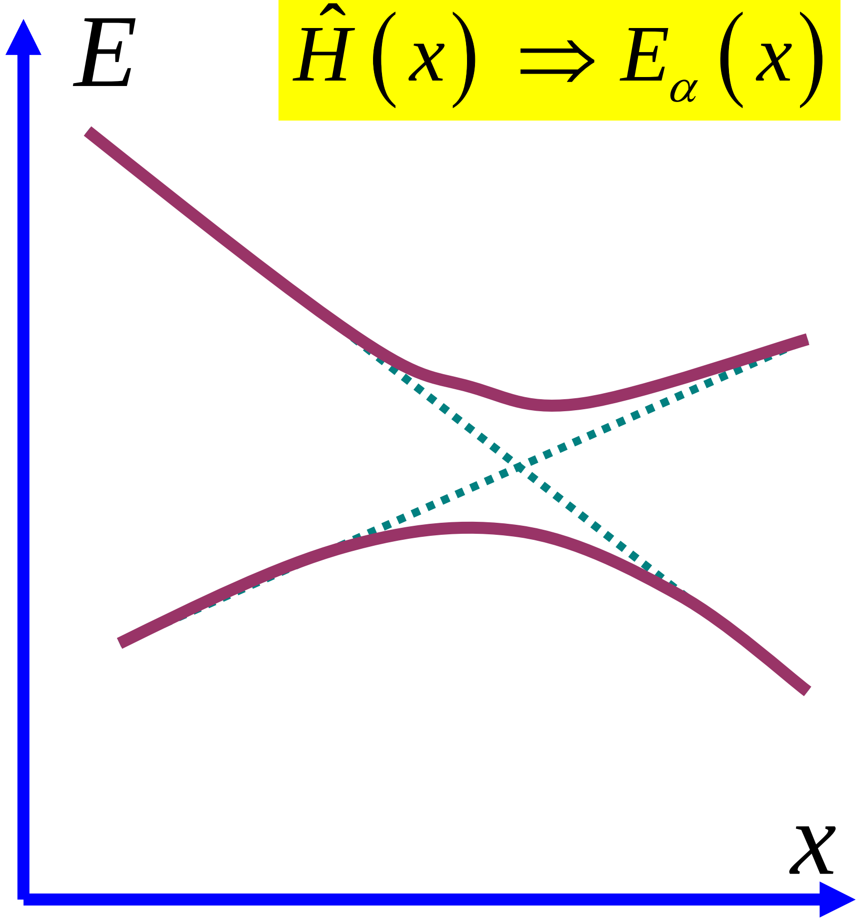
Arnold V. I., 1972 Funct. Anal. Appl.v. 6, p.94

Mathematical Methods of Classical Mechanics (Springer-Verlag: New York), Appendix 10, 1989

Arnold V.I., *Mathematical Methods of Classical Mechanics (Springer-Verlag: New York), Appendix 10, 1989*

In general, a multiple spectrum in typical families of quadratic forms is observed only for two or more parameters, while in one-parameter families of general form the spectrum is simple for all values of the parameter. Under a change of parameter in the typical one-parameter family the eigenvalues can approach closely, but when they are sufficiently close, it is as if they begin to repel one another. The eigenvalues again diverge, disappointing the person who hoped, by changing the parameter to achieve a multiple spectrum.

$$\hat{H}(x) \Rightarrow E_\alpha(x)$$

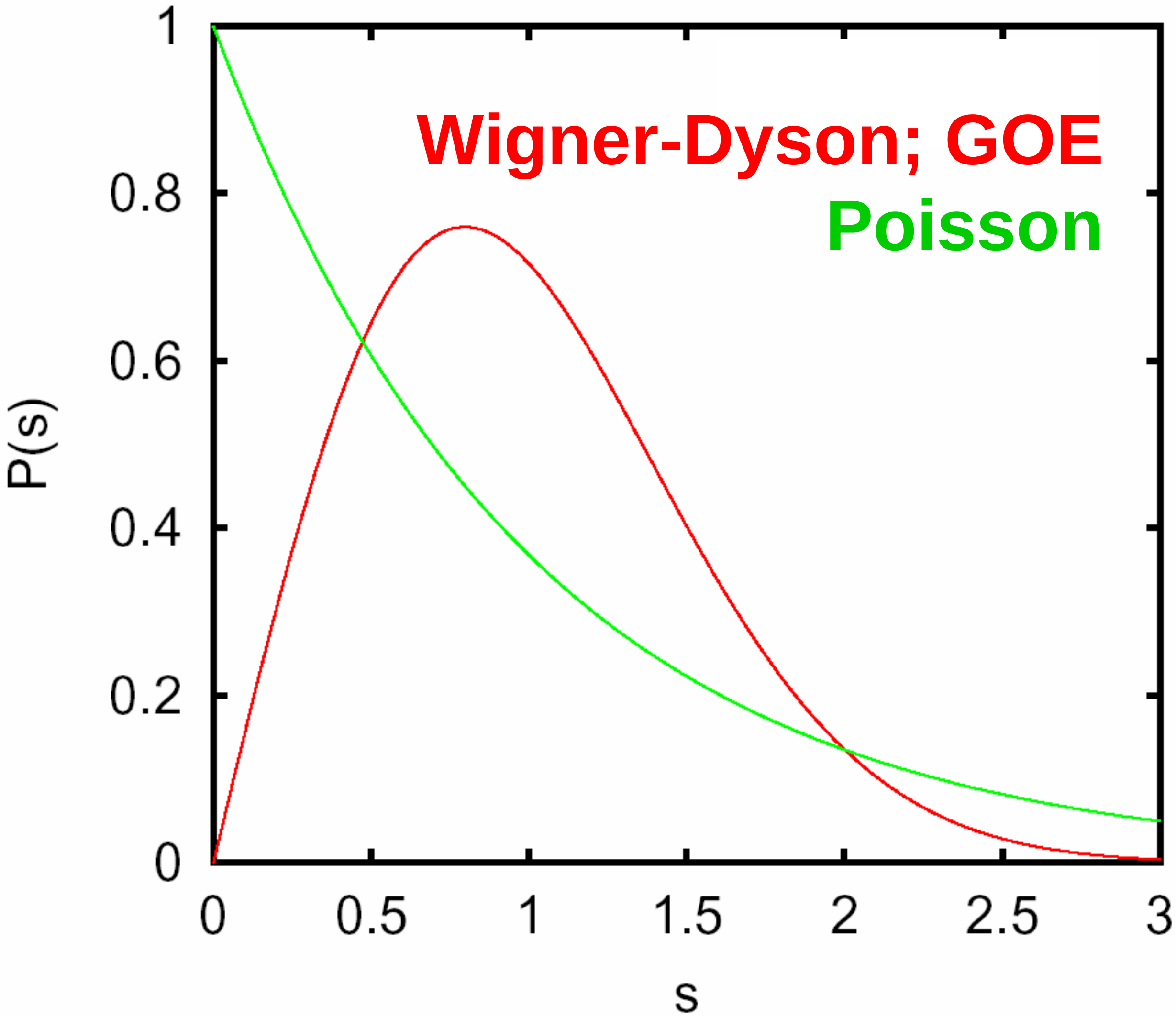


RANDOM MATRICES

$N \times N$ matrices with random matrix elements. $N \to \infty$

Dyson Ensembles

Matrix elements	Ensemble	β	realization
real	orthogonal	1	T-inv potential
complex	unitary	2	broken T-invariance (e.g., by magnetic field)
2×2 matrices	simplectic	4	T-inv, but with spin-orbital coupling

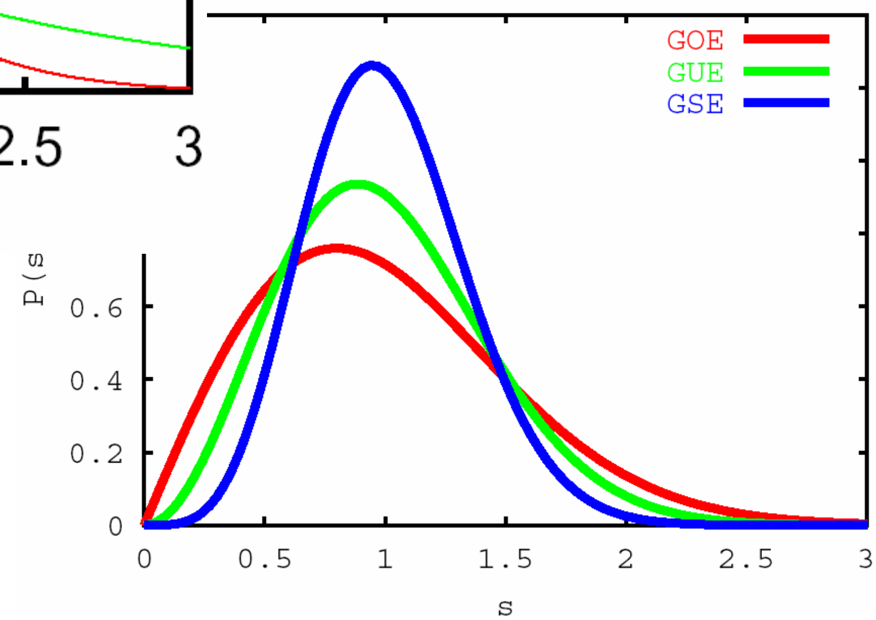


Gaussian Orthogonal Ensemble

Orthogonal
β=1

Unitary
β=2

Simplectic
β=4



Poisson – completely uncorrelated levels

ATOMS

Main goal is to classify the eigenstates in terms of the quantum numbers

NUCLEI

For the nuclear excitations this program does not work

E.P. Wigner:

Study spectral **statistics** of a **particular** quantum system – a given nucleus

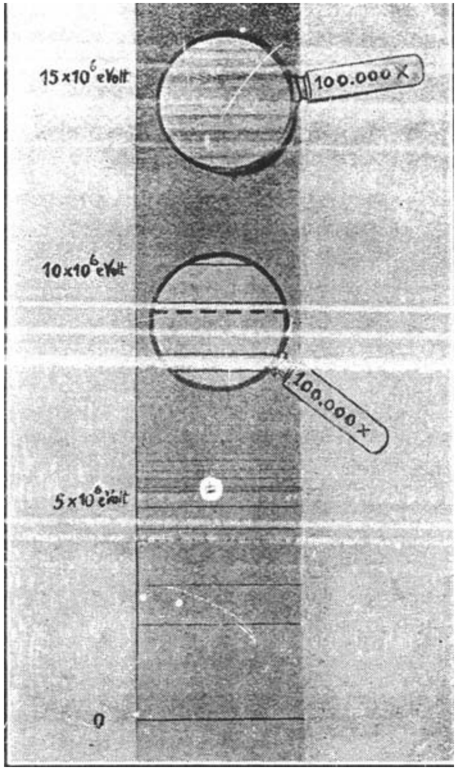
ATOMS Main goal is to classify the eigenstates in terms of the quantum numbers

NUCLEI For the nuclear excitations this program does not work

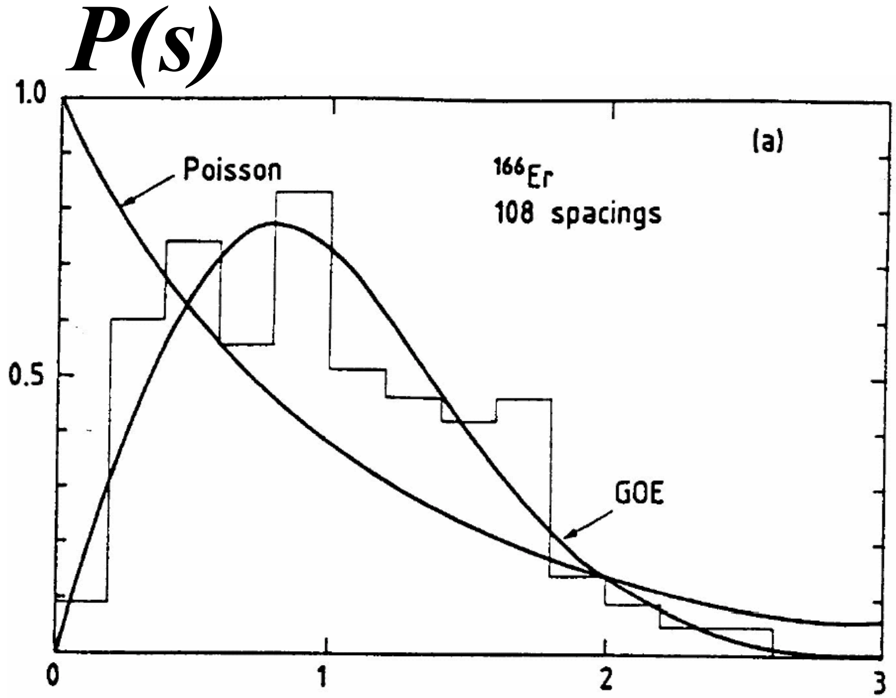
E.P. Wigner: Study spectral **statistics** of a **particular** quantum system – a given nucleus

Random Matrices	Atomic Nuclei
• *Ensemble*	• ***Particular quantum system***
• *Ensemble averaging*	• ***Spectral averaging (over*** α***)***

Nevertheless Statistics of the nuclear spectra are almost exactly the same as the Random Matrix Statistics

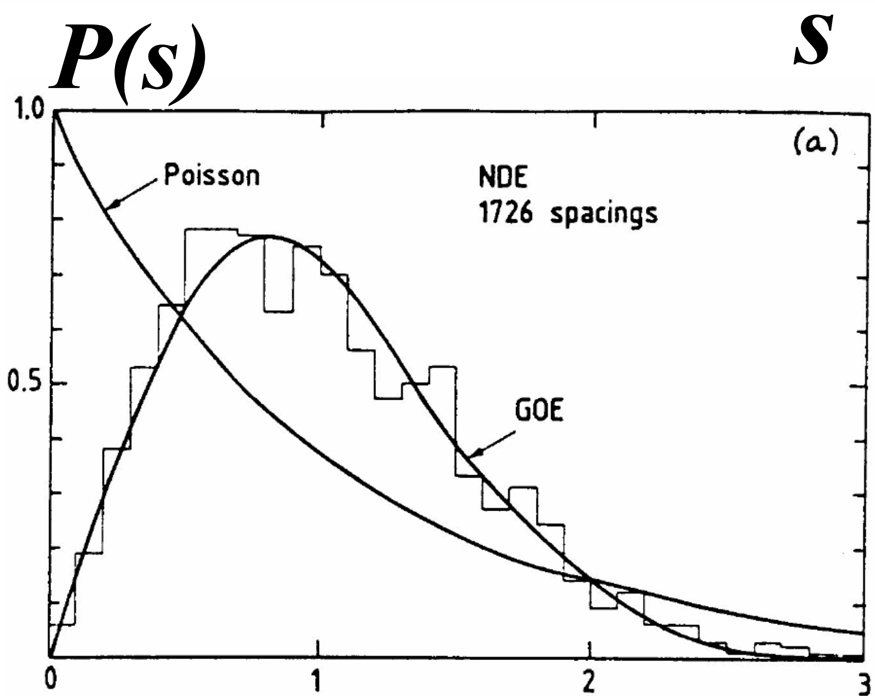


N. Bohr, Nature 137 (1936) 344.



Particular nucleus

^{166}Er



Spectra of several nuclei combined (after spacing) rescaling by the mean level

Q: *Why the random matrix theory (RMT) works so well for nuclear spectra* ?

Original answer:

These are systems with a large number of degrees of freedom, and therefore the "complexity" is high

Later it became clear that

there exist very "simple" systems with as many as 2 degrees of freedom (d=2), *which demonstrate RMT - like spectral statistics*

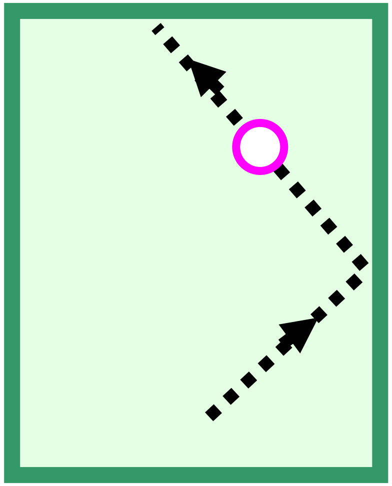
Classical ($\hbar$ =0) Dynamical Systems with *d* degrees of freedom

Integrable Systems

The variables can be separated and the problem reduces to ***d*** one-dimensional problems ⇒ ***d*** integrals of motion

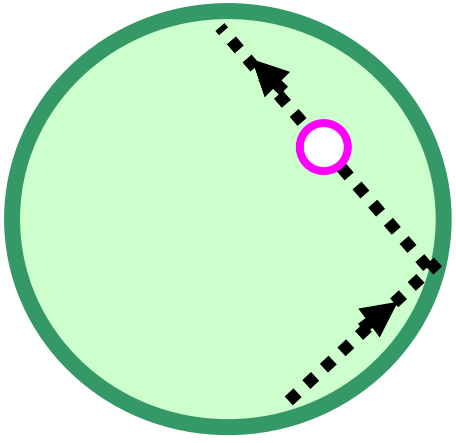
Examples

1. A ball inside rectangular billiard; *d=2*



- Vertical motion can be separated from the horizontal one
- Vertical and horizontal components of the momentum, are both integrals of motion

2. Circular billiard; *d=2*



- Radial motion can be separated from the angular one
- Angular momentum and energy are the integrals of motion

Classical Dynamical Systems with *d* degrees of freedom

Integrable Systems

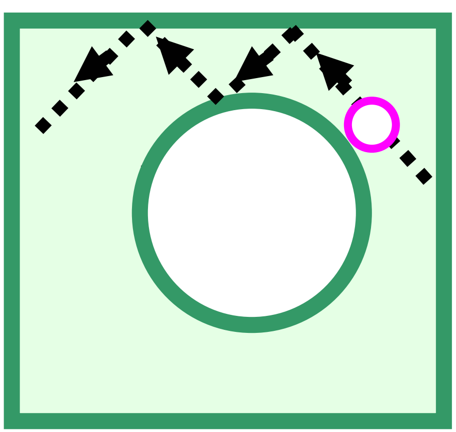
The variables can be separated ⇨ *d* one-dimensional problems ⇨ *d* integrals of motion

Rectangular and circular billiard, Kepler problem, . . . ,
1d Hubbard model and other exactly solvable models, . .

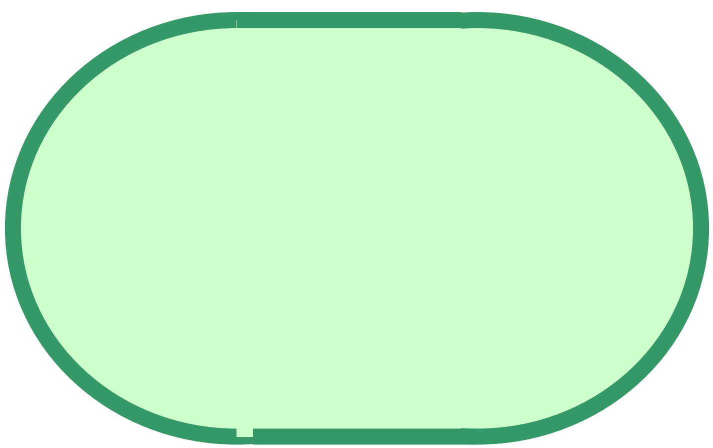
Chaotic Systems

The variables can not be separated ⇨ there is only one integral of motion - energy

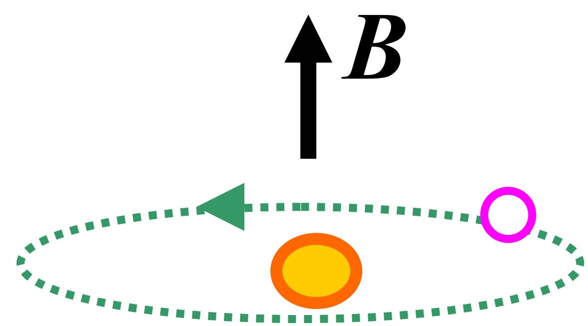
Examples



Sinai billiard



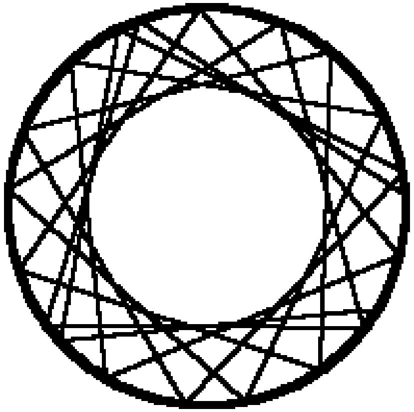
Stadium

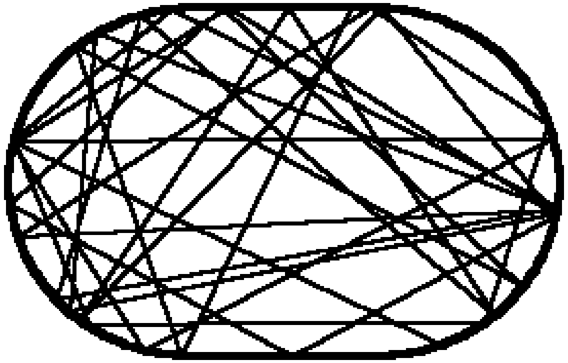


Kepler problem in magnetic field

Classical Chaos
$\hbar = 0$

- *Nonlinearities*
- *Exponential dependence on the original conditions (Lyapunov exponents)*
- *Ergodicity*





Quantum description of any System with a finite number of the degrees of freedom is a linear problem – Shrodinger equation

Q: What does it mean Quantum Chaos ?

$\hbar \neq 0$ Bohigas – Giannoni – Schmit conjecture

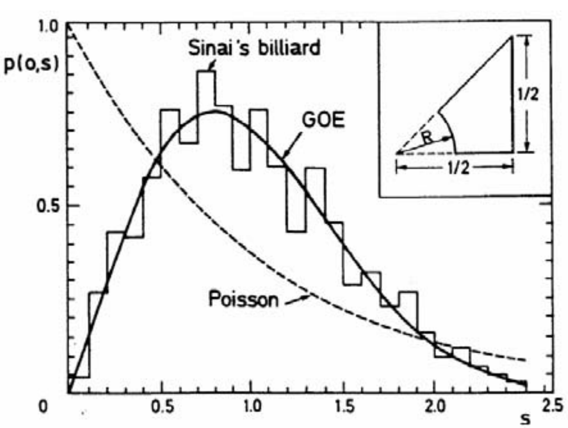
VOLUME 52 2 JANUARY 1984 NUMBER 1

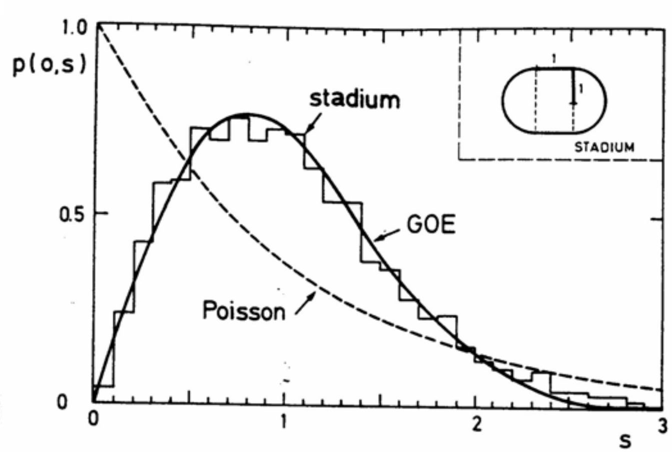
Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws

O. Bohigas, M. J. Giannoni, and C. Schmit
Division de Physique Théorique, Institut de Physique Nucléaire, F-91406 Orsay Cedex, France
(Received 2 August 1983)

It is found that the level fluctuations of the quantum Sinai's billiard are consistent with the predictions of the Gaussian orthogonal ensemble of random matrices. This reinforces the belief that level fluctuation laws are universal.

In summary, the question at issue is to prove or disprove the following conjecture: Spectra of time-reversal-invariant systems whose classical analogs are K systems show the same fluctuation properties as predicted by GOE





Chaotic classical analog

Wigner- Dyson spectral statistics

No quantum numbers except energy

Q: What does it mean Quantum Chaos ?

Two possible definitions

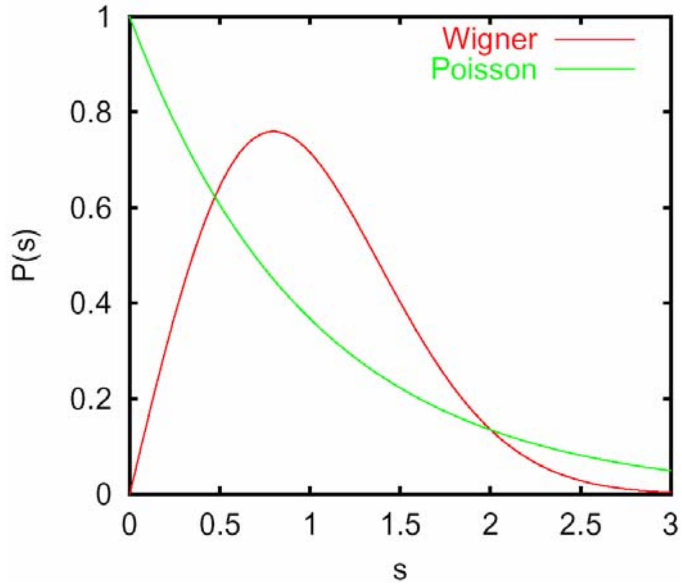
Chaotic classical analog

Wigner - Dyson-like spectrum

Classical Quantum

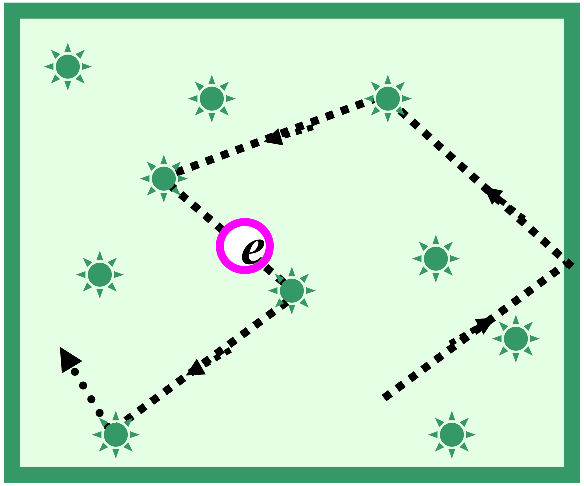
Integrable ? ⟷ **Poisson**

Chaotic ? ⟷ **Wigner-Dyson**



Poisson to Wigner-Dyson crossover

Important example: quantum particle subject to a random potential – disordered conductor



✷ *Scattering centers, e.g., impurities*

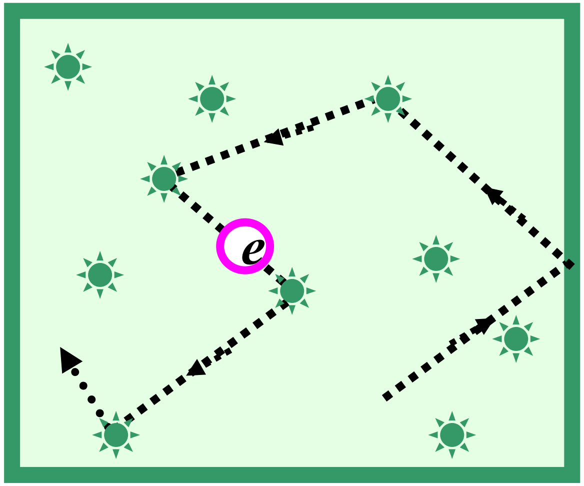
- As well as in the case of Random Matrices (RM) there is a luxury of ensemble averaging.
- The problem is much richer than RM theory
- There is still a lot of universality.

Anderson localization (1958)

At strong enough disorder all eigenstates are localized in space

Poisson to Wigner-Dyson crossover

Important example: quantum particle subject to a random potential – disordered conductor



Scattering centers, e.g., impurities

Models of disorder:

Randomly located impurities

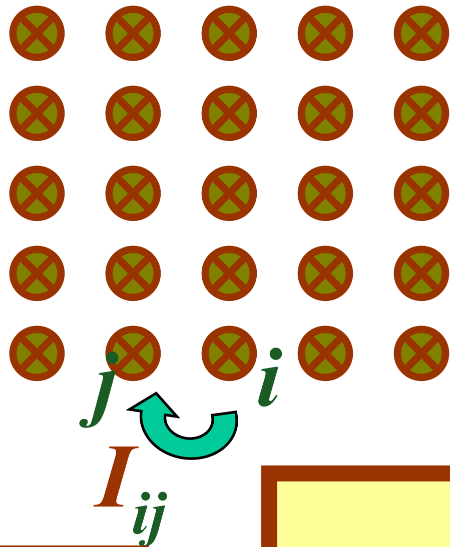
White noise potential

Lattice models

Anderson model

Lifshits model

Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* ε_i - *random*
- *Hopping matrix elements* I_{ij}

$$-W < \varepsilon_i < W$$

uniformly distributed

$$I_{ij} = \begin{cases} I & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

Anderson Transition

$I < I_c$	$I > I_c$
Insulator	***Metal***
All eigenstates are localized	*There appear states extended all over the whole system*
Localization length ξ	

Anderson Transition

Strong disorder

$I < I_c$

Insulator

All eigenstates are localized

Localization length ξ

The eigenstates, which are localized at different places will not repel each other

↓

Poisson spectral statistics

Weak disorder

$I > I_c$

Metal

There appear states extended all over the whole system

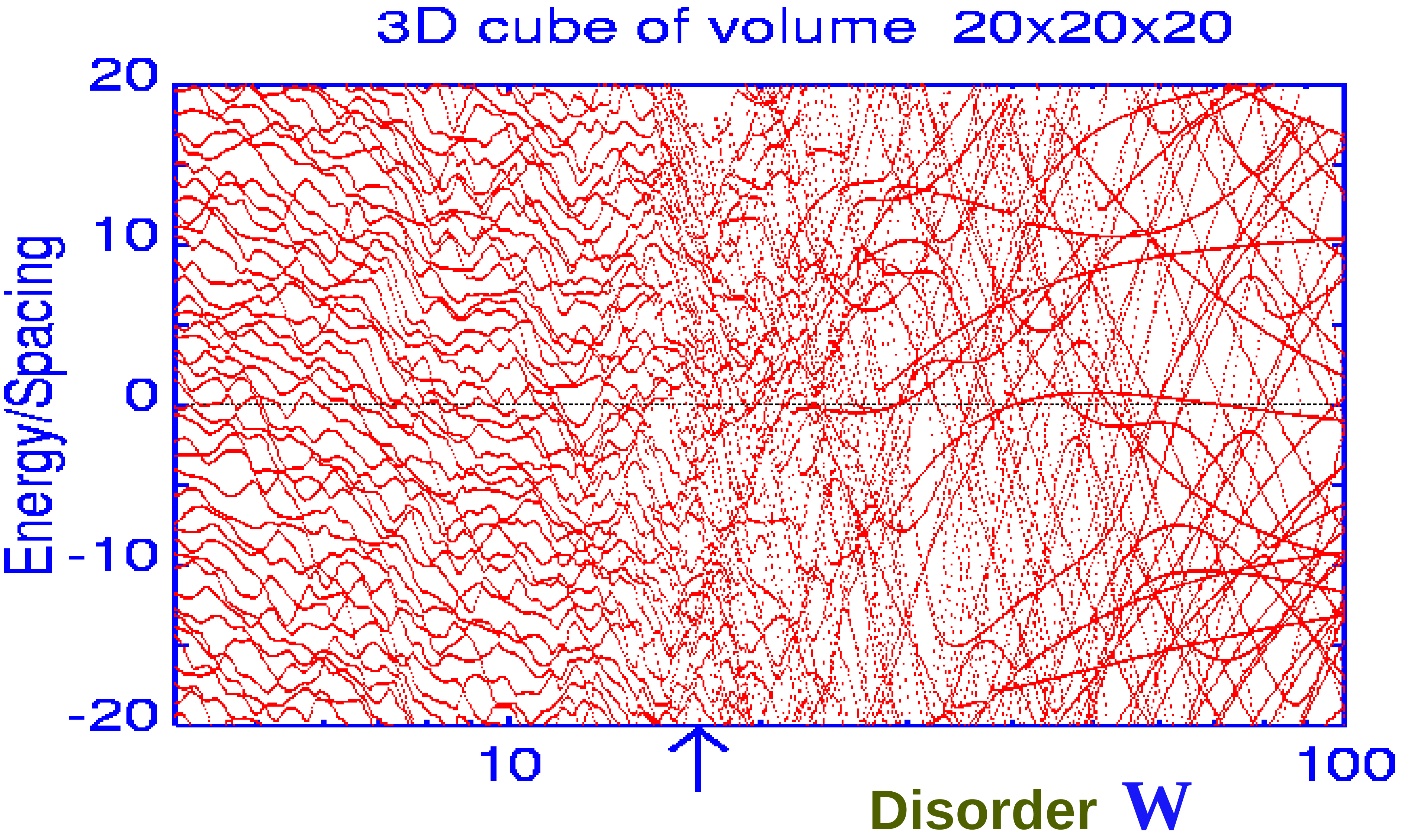
Any two extended eigenstates repel each other

↓

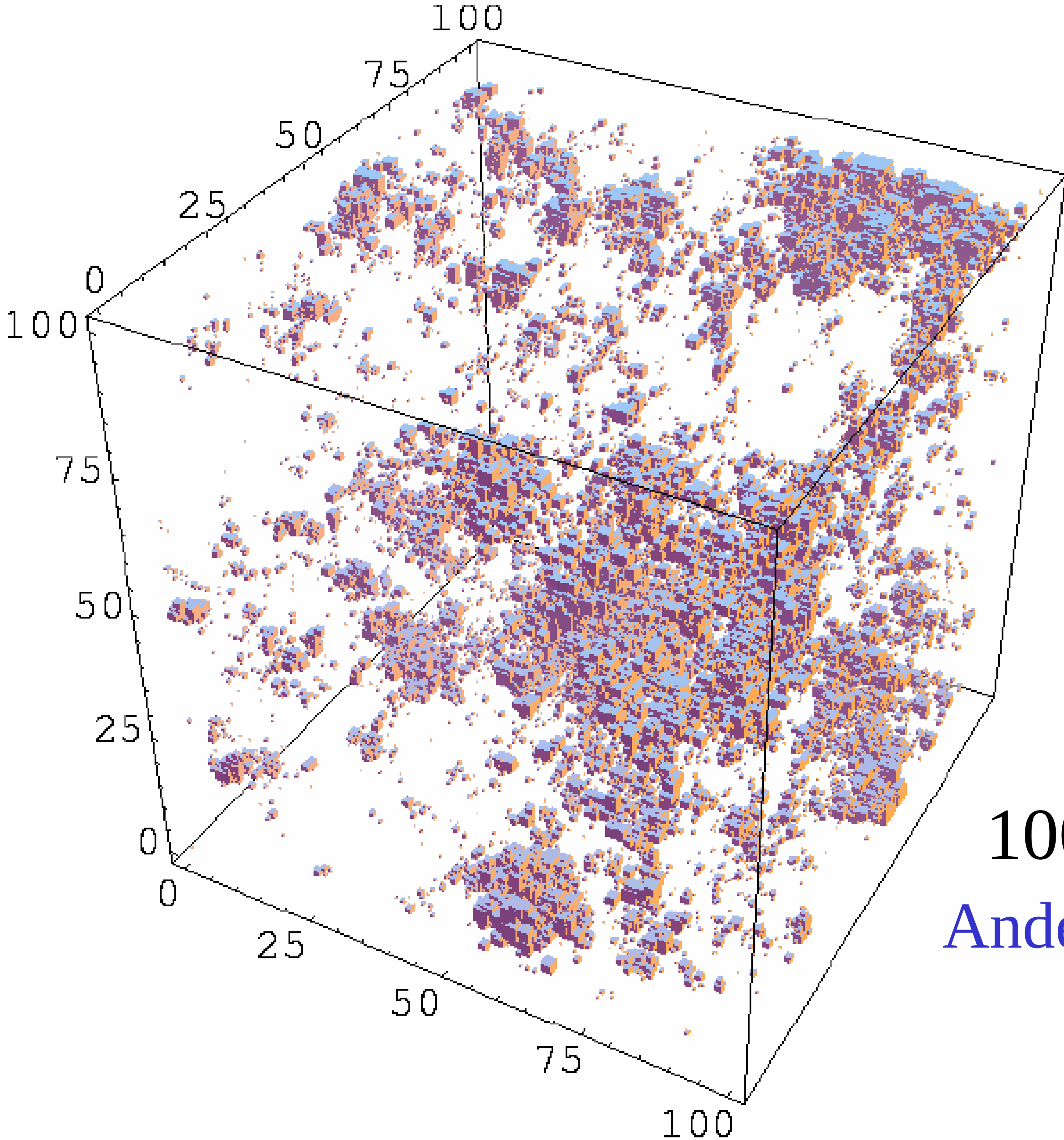
Wigner – Dyson spectral statistics

Zharekeschev & Kramer.

Exact diagonalization of the Anderson model



Critical electron eigenstate at the Anderson transition



Zharekeshev, Computer Phys. Commun. 1999

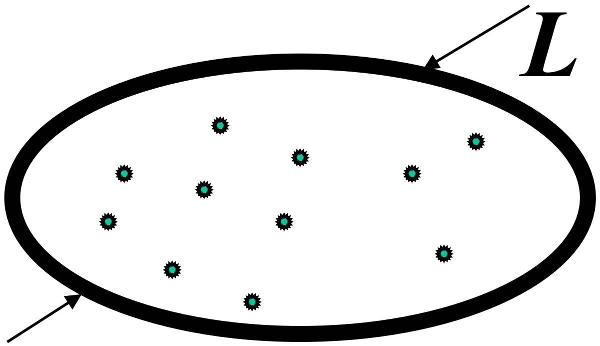
$100 \times 100 \times 100$

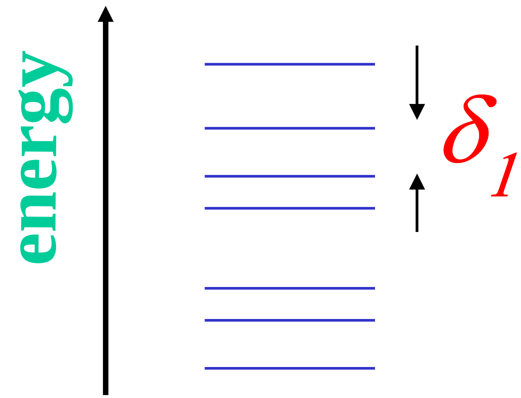
Anderson model cube

Energy scales (*Thouless, 1972*)

1. Mean level spacing

$$\delta_1 = 1/\nu \times L^d$$





L *is the system size;*

d *is the number of dimensions*

2. Thouless energy

$$E_T = hD/L^2$$

D *is the diffusion const*

E_T *has a meaning of the inverse diffusion time of the traveling through the system or the escape rate (for open systems)*

$$g = E_T / \delta_1$$

dimensionless Thouless conductance

$$g = Gh/e^2$$

Thouless Conductance and One-particle Spectral Statistics

Localized states
Insulator

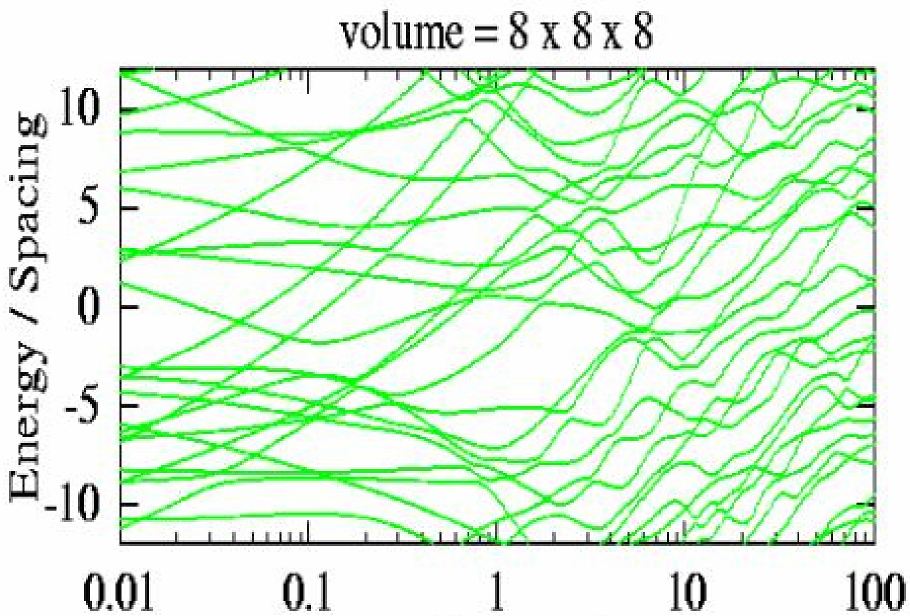
Poisson spectral statistics

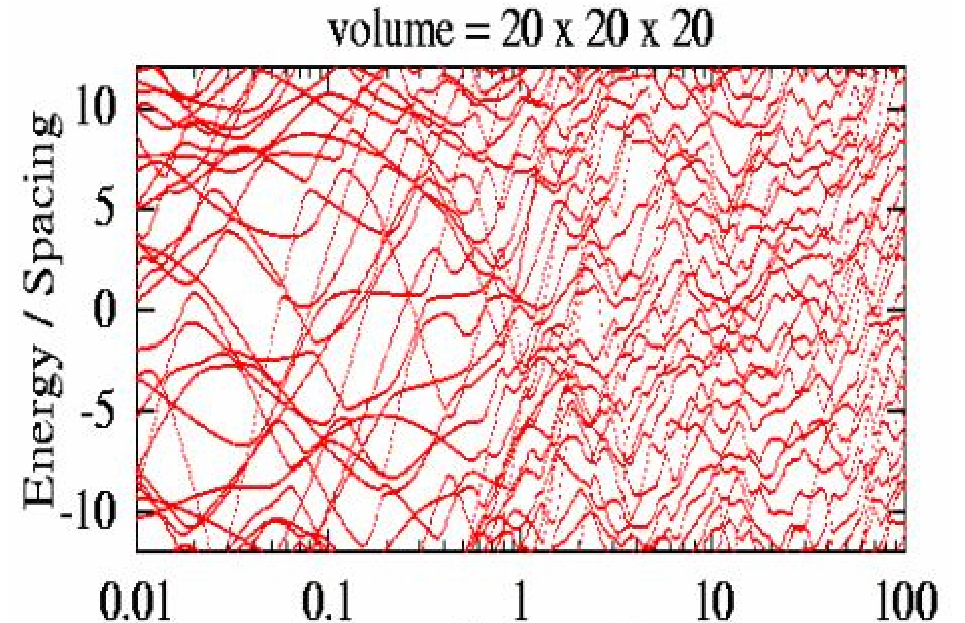
Extended states
Metal

Wigner-Dyson spectral statistics

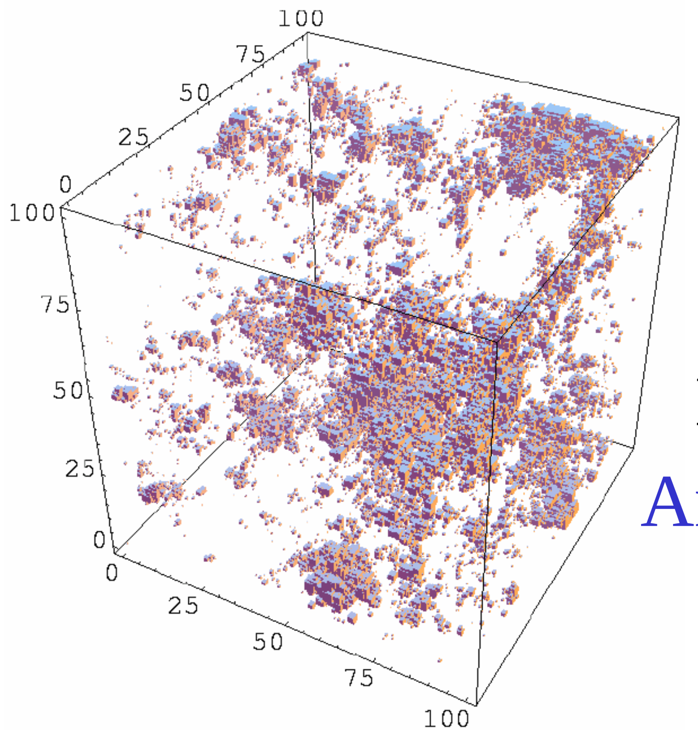
Transition at $g \sim 1$.

Is it sharp?





Critical electron eigenstate at the Anderson transition



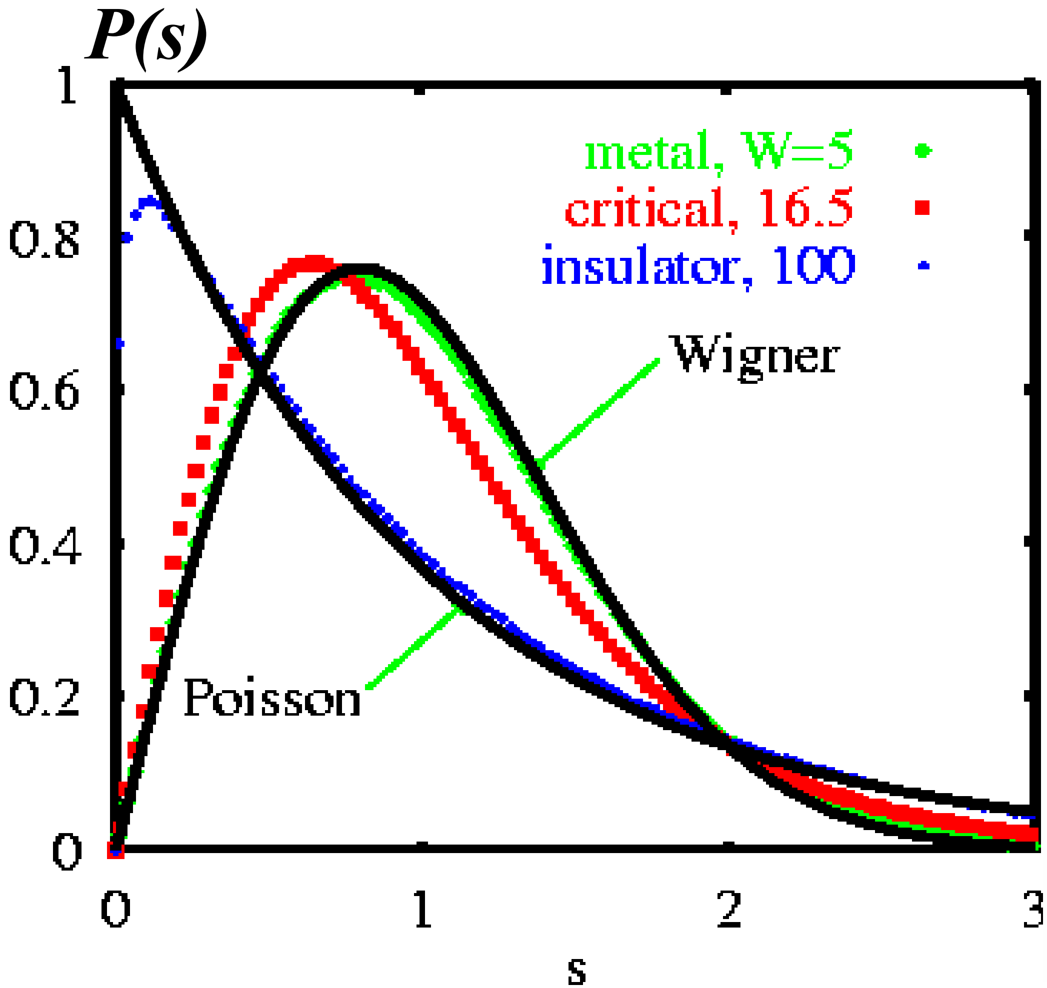
Zharekeshev, Computer Phys. Commun. 1999

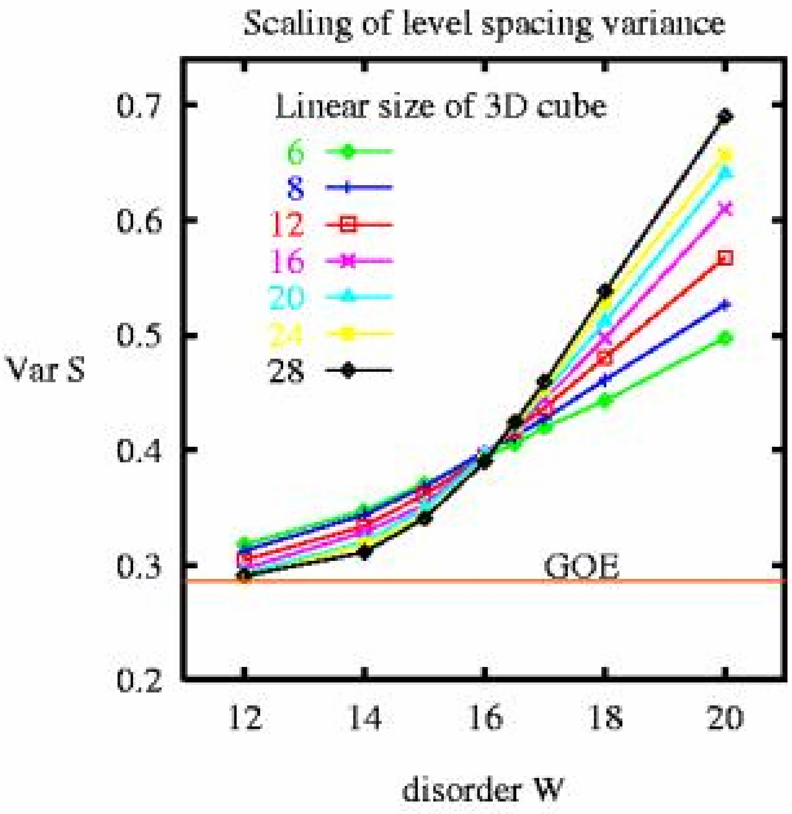
Conductance g

$$100 \times 100 \times 100$$

Anderson model cube

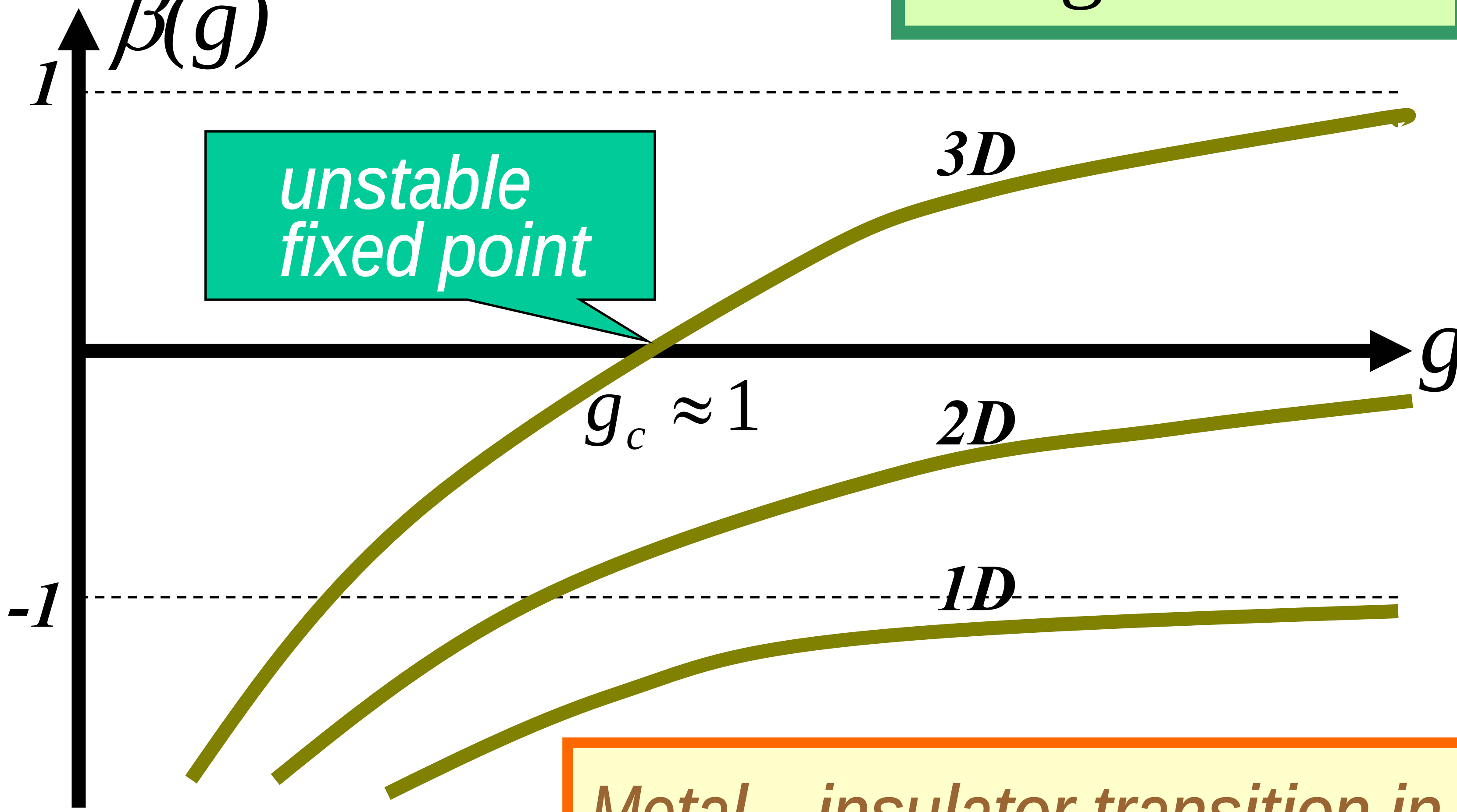
Anderson transition in terms of pure level statistics





β - function

$$\frac{d \log g}{d \log L} = \beta(g)$$



Metal – insulator transition in **3D**

All states are localized for ***d=1,2***

Thouless Conductance and One-particle Spectral Statistics

0 ——— **1** ——— ***g*** →

Localized states	*Extended states*
Insulator	*Metal*
Poisson spectral statistics	**Wigner-Dyson spectral statistics**

$N \times N$
Random Matrices

Quantum Dots with Thouless conductance g

$N \to \infty$

$g \to \infty$

The same statistics of the random spectra and one-particle wave functions (eigenvectors)

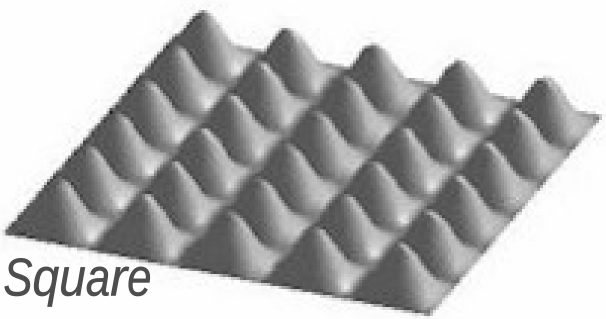
VOLUME 85, NUMBER 11 PHYSICAL REVIEW LETTERS 11 SEPTEMBER 2000

Correlations due to Localization in Quantum Eigenfunctions of Disordered Microwave Cavities

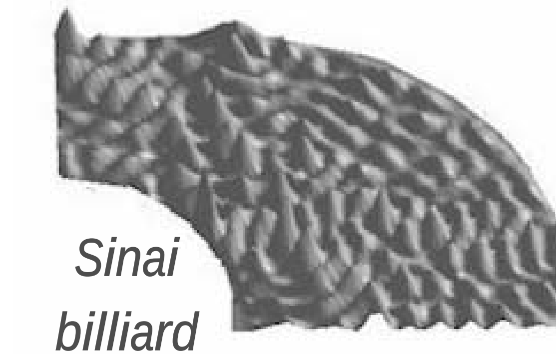
Prabhakar Pradhan and S. Sridhar
Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 28 February 2000)

Integrable

Chaotic

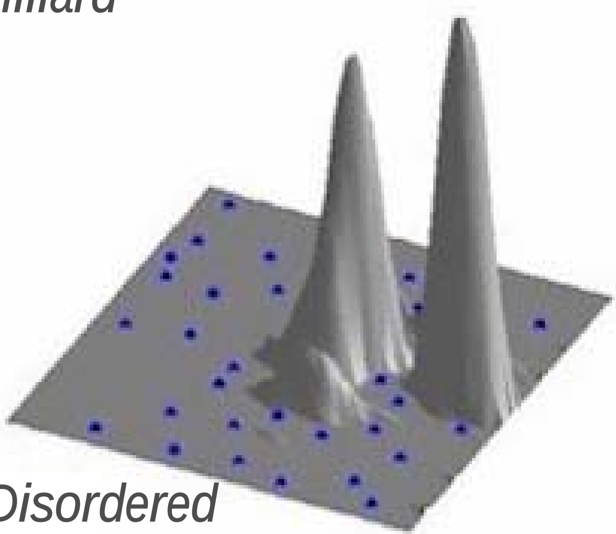


Square billiard



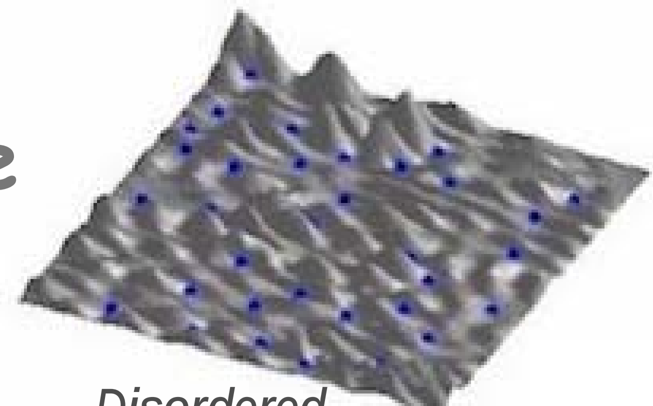
Sinai billiard

All chaotic systems resemble each other.



Disordered localized

All integrable systems are integrable in their own way



Disordered extended

Disordered Systems:

*Anderson **metal**;*
***Wigner-Dyson** spectral statistics*

*Anderson **insulator**;*
***Poisson** spectral statistics*

Q: *Is it a generic scenario for the Wigner-Dyson to Poisson crossover* **?**

Speculations

*Consider an **integrable** system. Each state is characterized by a **set of quantum numbers**.*

*It can be viewed as a point in the **space of quantum numbers**. The whole set of the states forms a **lattice** in this space.*

*A **perturbation** that violates the integrability provides matrix elements of the **hopping** between different sites (**Anderson model !?**)*

Q: ***Does Anderson localization provide a generic scenario for the Wigner-Dyson to Poisson crossover*** **?**

Consider an integrable system. Each state is characterized by a set of quantum numbers.

It can be viewed as a point in the space of quantum numbers. The whole set of the states forms a lattice in this space.

A perturbation that violates the integrability provides matrix elements of the hopping between different sites (Anderson model !?)

Weak enough hopping - Localization - Poisson
Strong hopping - transition to Wigner-Dyson

The very definition of the localization is **not invariant** – one should specify in which space the eigenstates are localized.

Level statistics **is invariant**:

Poissonian statistics $\quad \exists$ basis where the eigenfunctions are localized

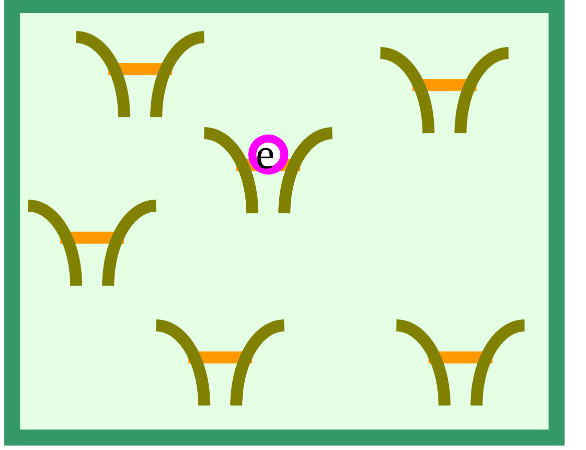
Wigner -Dyson statistics $\quad \forall$ basis the eigenfunctions are extended

Example 1

Doped semiconductor

Low concentration of donors ⇒ Electrons are localized on donors ⇨ Poisson

Higher donor concentration ⇒ Electronic states are extended ⇨ Wigner-Dyson



Example 1

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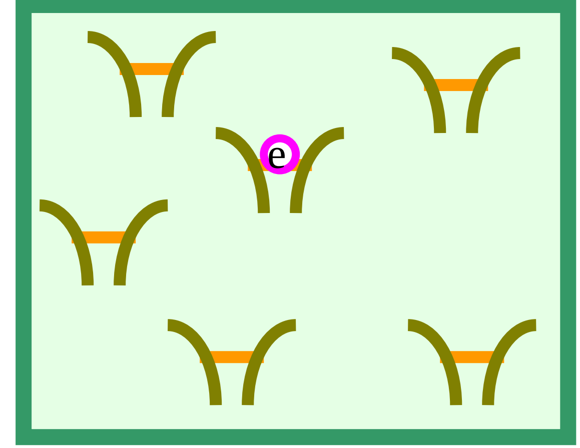
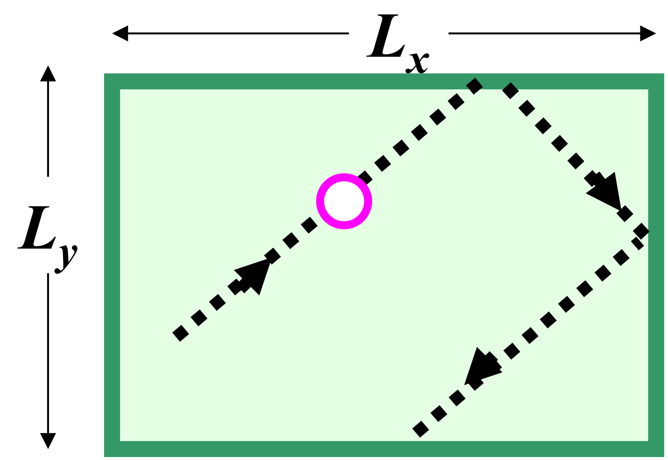


Example 2

Rectangular billiard

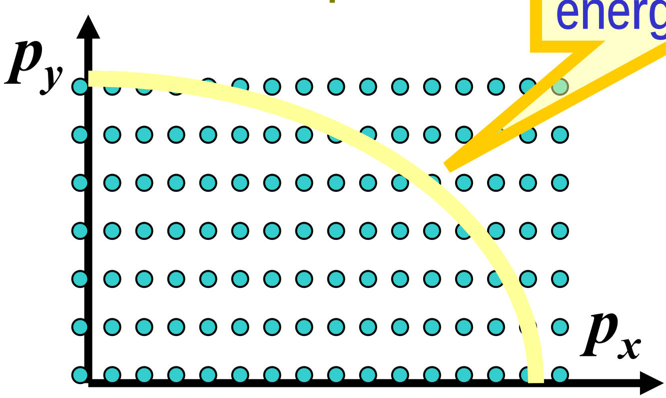
Two integrals of motion

$$p_x = \frac{\pi n}{L_x}; \quad p_y = \frac{\pi m}{L_x}$$



Lattice in the momentum space

Line (surface) of constant energy



Ideal billiard – localization in the momentum space ⇨ Poisson

Deformation or smooth random potential – delocalization in the momentum space ⇨ Wigner-Dyson

VOLUME 77, NUMBER 23 PHYSICAL REVIEW LETTERS 2 DECEMBER 1996

Diffusion and Localization in Chaotic Billiards

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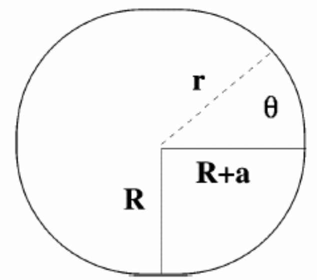
(Received 29 July 1996)

Localization and diffusion in the angular momentum space

$$\varepsilon \equiv \frac{a}{R}$$



$\varepsilon > 0$ Chaotic stadium

$\varepsilon \to 0$ Integrable circular billiard

Angular momentum is the integral of motion

$$\hbar = 0; \quad \varepsilon << 1$$

Angular momentum is not conserved

VOLUME 77, NUMBER 23 PHYSICAL REVIEW LETTERS 2 DECEMBER 1996

Diffusion and Localization in Chaotic Billiards

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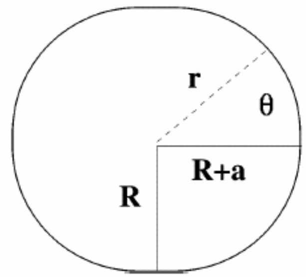
Localization and diffusion in the angular momentum space

$$\varepsilon \equiv \frac{a}{R}$$



$\varepsilon > 0$ Chaotic stadium

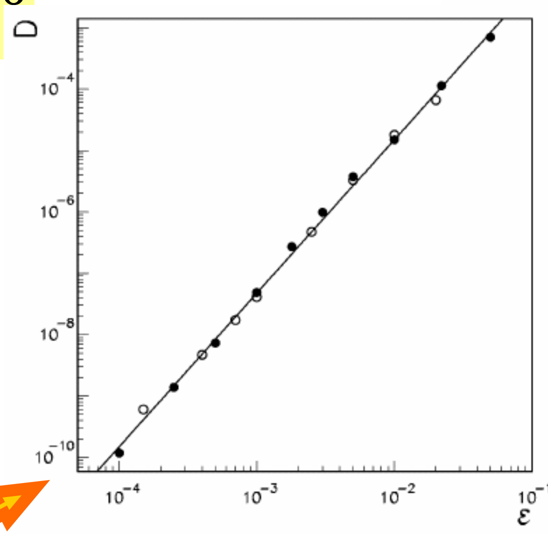
$\varepsilon \to 0$ Integrable circular billiard

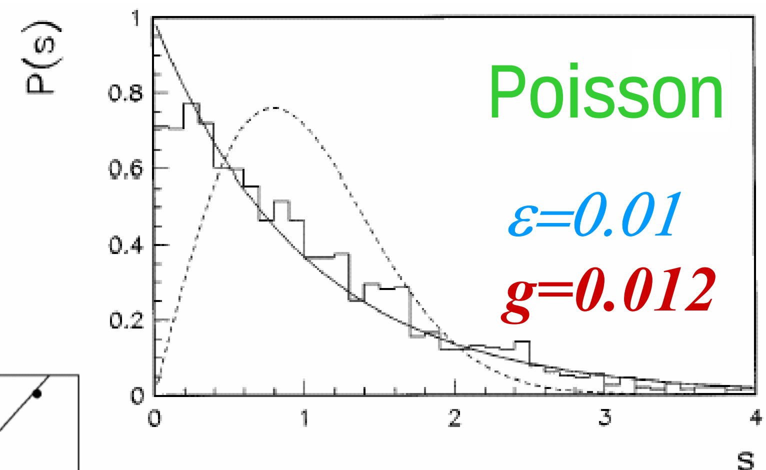
Angular momentum is the integral of motion $\frac{dL}{dt} = 0$

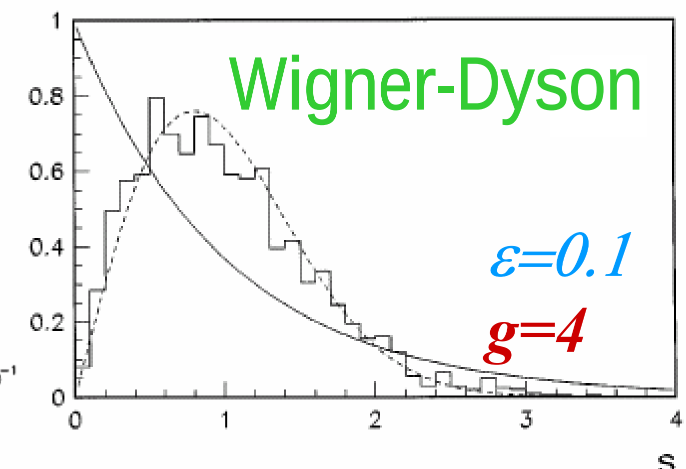
$\hbar = 0$; $\varepsilon << 1$ Diffusion in the angular momentum space

$$\frac{dP(L)}{dt} = D\frac{d^2P(L)}{dL^2}$$

$$D \propto \varepsilon^{5/2}$$







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Europhysics Letters, v.22, p.537, 1993

1D Hubbard Model on a periodic chain

$$H = t\sum_{i,\sigma}\left(c^{+}_{i,\sigma}c_{i+1,\sigma} + c^{+}_{i+1,\sigma}c_{i,\sigma}\right) + U\sum_{i,\sigma} n_{i,\sigma}n_{i,-\sigma} + V\sum_{i,\sigma,\sigma'} n_{i,\sigma}n_{i+1,\sigma'}$$

Onsite interaction

n. neighbors interaction

$V = 0$ Hubbard model integrable

$V \neq 0$ extended Hubbard model nonintegrable

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Onsite interaction

n. neighbors interaction

$V = 0$ Hubbard model — integrable

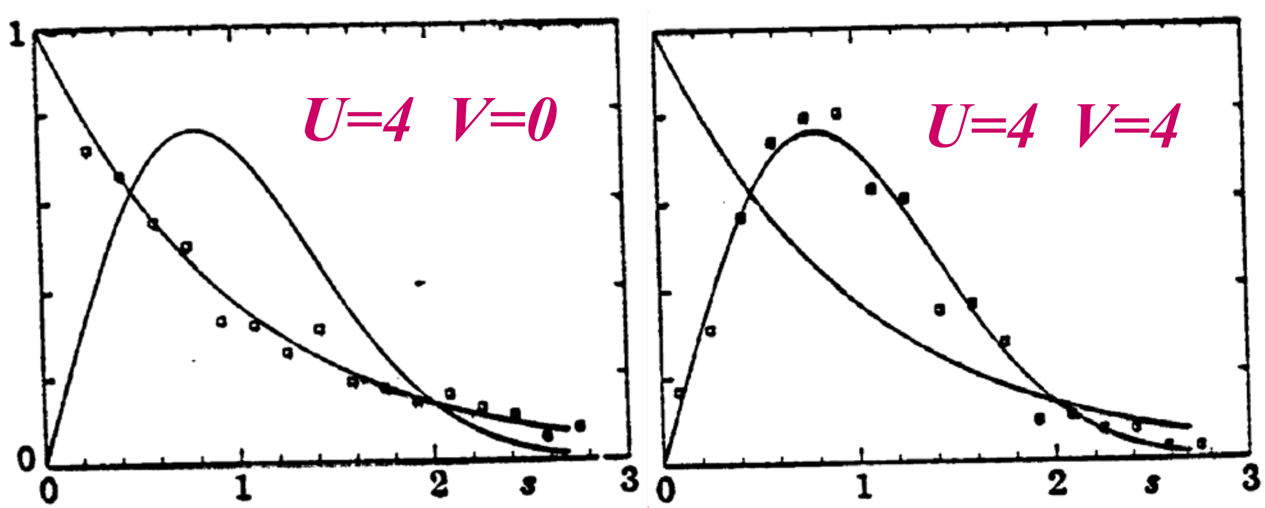
$V \neq 0$ extended Hubbard model — nonintegrable

12 sites

3 particles

Zero total spin

Total momentum $\pi/6$



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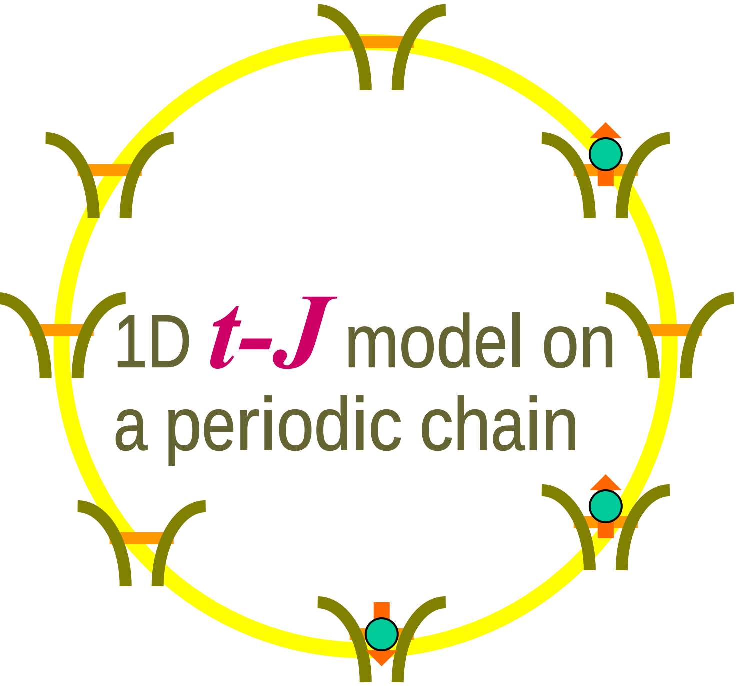

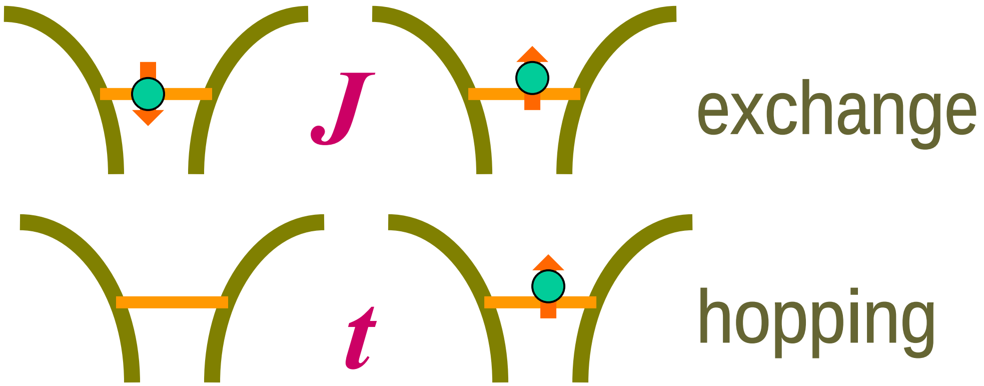

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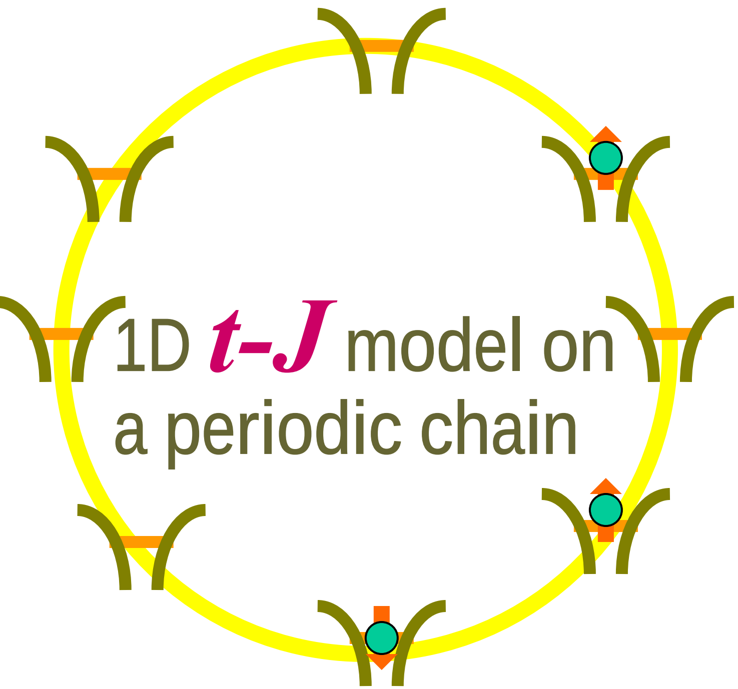
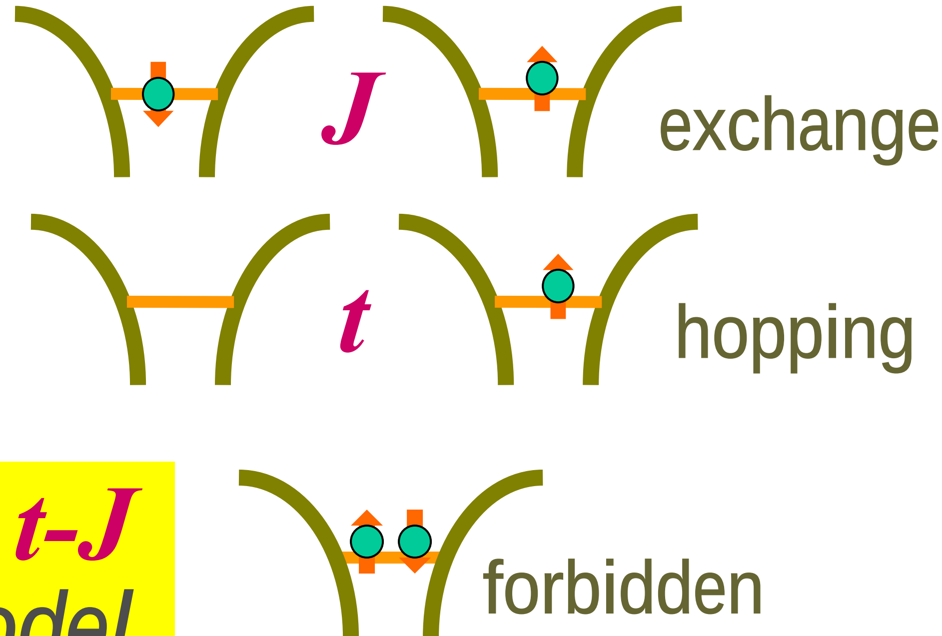


1d *t–J* *model*



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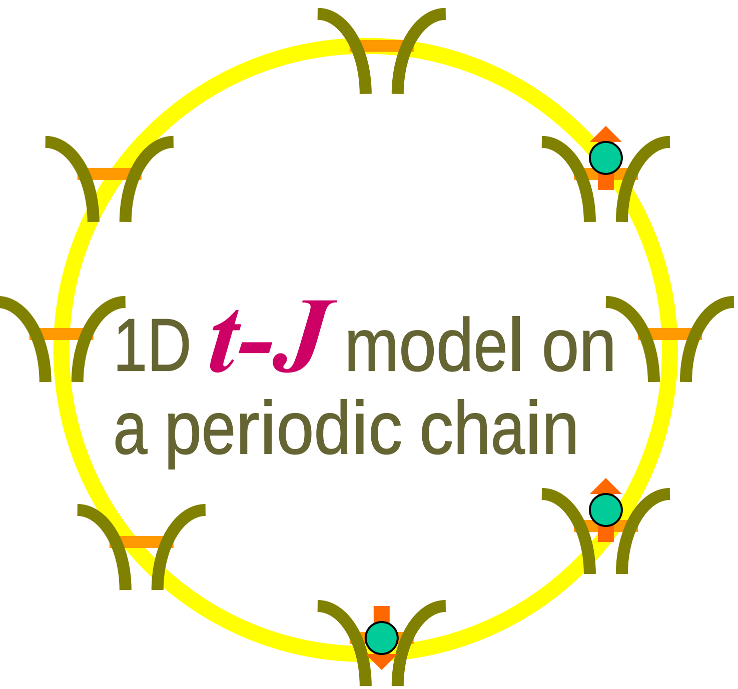
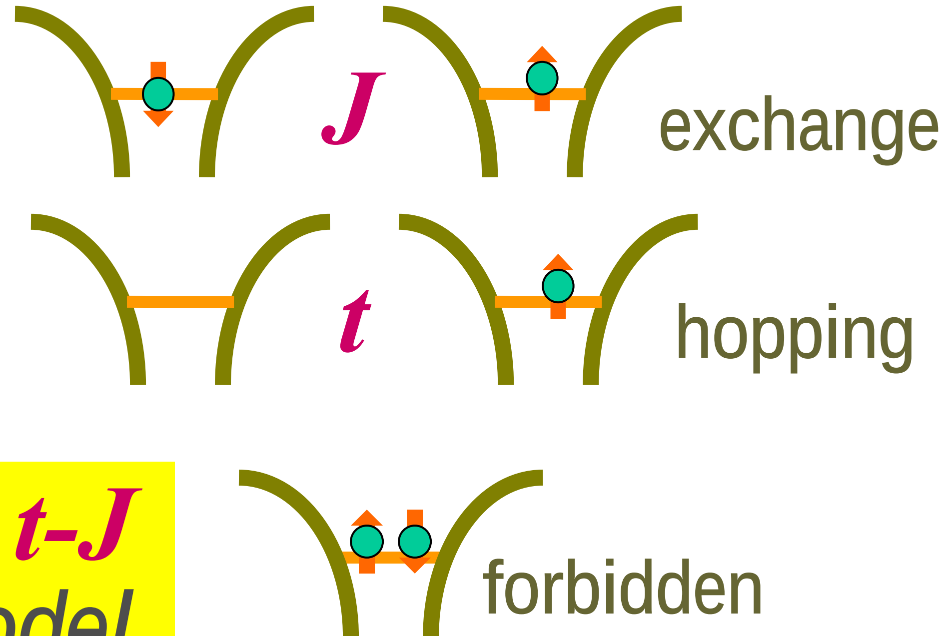
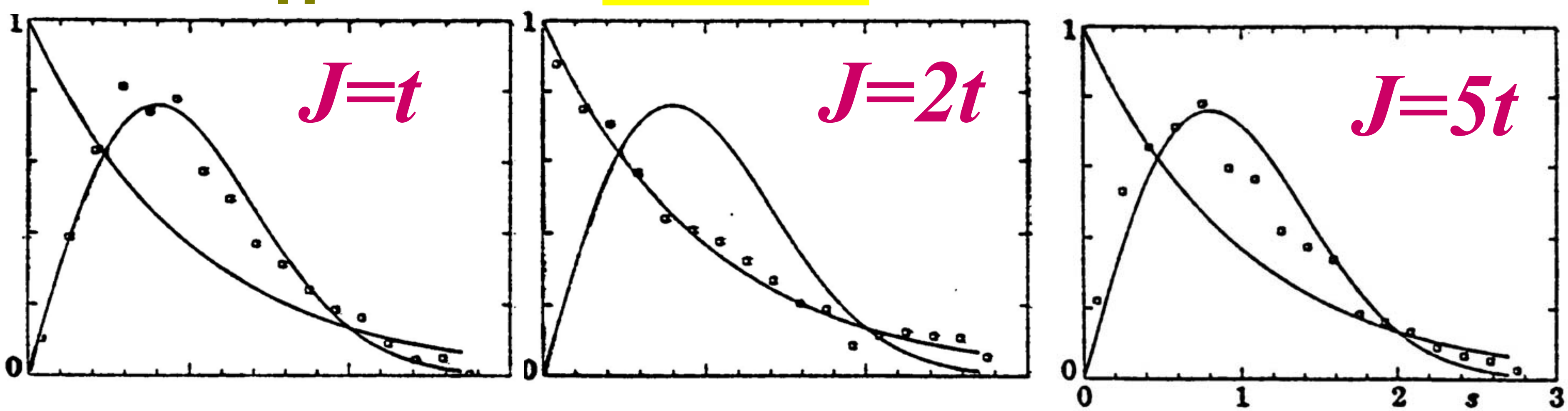




1d *t-J*
model



N=16; one hole

Q: ***Why the random matrix theory (RMT) works so well for nuclear spectra*** **?**

Chaos in Nuclei – Delocalization?

Delocalization in Fock space