Introduction to Theory of Mesoscopic Systems

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Lecture 2

Diffusion Equation

$$\frac{\partial \rho}{\partial t} - D\nabla^2 \rho = 0$$

Lessons from the Einstein's work:

- Universality: the equation is valid as long as the process is marcovian

- Can be applied to the probability and thus describes both fluctuations and dissipation

- There is a universal relation between the diffusion constant and the viscosity

- Studies of the diffusion processes brings information about micro scales.

Lesson 1:

Beyond Markov chains:

Anderson Localization

and

Magnetoresistance

Т. 18 Журнал экспериментальной и теоретической физики. Вып.
1948

ОБ ИЗМЕНЕНИИ ЭЛЕКТРИЧЕСКОГО СОПРОТИВЛЕНИЯ ТЕЛЛУРА В МАГНИТНОМ ПОЛЕ ПРИ НИЗКИХ ТЕМПЕРАТУРАХ

Р. А. Ченцов

R.A. Chentsov “***On the variation of electrical conductivity of tellurium in magnetic field at low temperatures***”, *Zh. Exp. Theor. Fiz. v.18, 375-385, (1948).*

Таблица 2

Уменьшение сопротивления теллура в магнитном поле

Образец	Температура (°K)	Максимальное уменьшение сопротивления
Te-1	2,13	$0,7 \cdot 10^{-3}$
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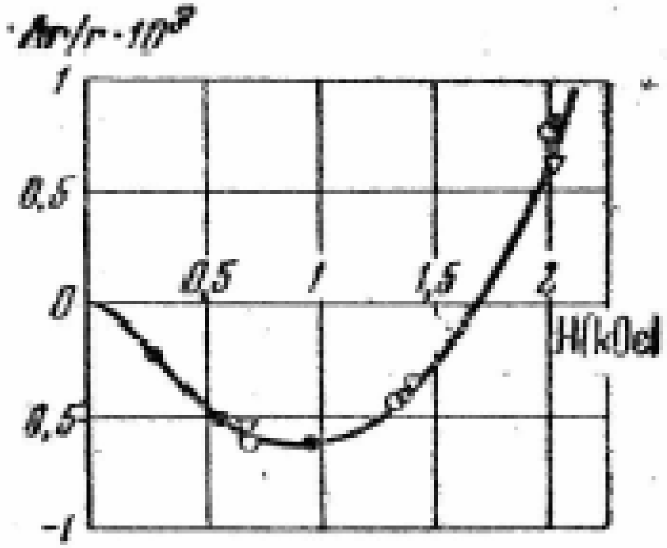
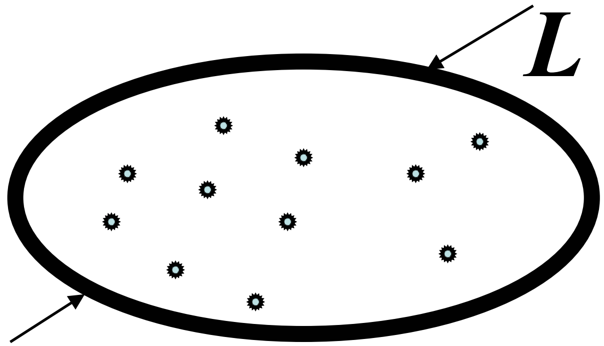


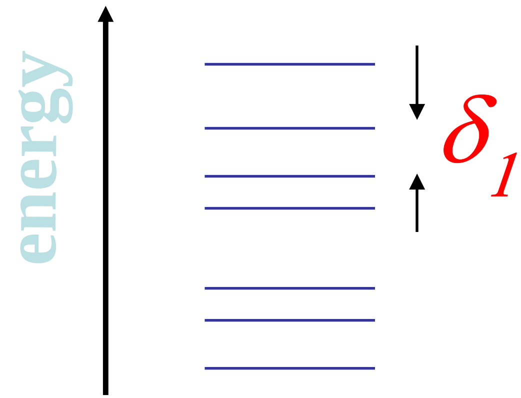
Рис. 2

Energy scales (*Thouless, 1972*)

1. Mean level spacing

$$\delta_1 = 1/\nu \times L^d$$





L *is the system size;*

d *is the number of dimensions*

2. Thouless energy

$$E_T = hD/L^2$$

D *is the diffusion const*

E_T *has a meaning of the* ***inverse diffusion time*** *of the traveling through the system or the* ***escape rate*** *(for open systems)*

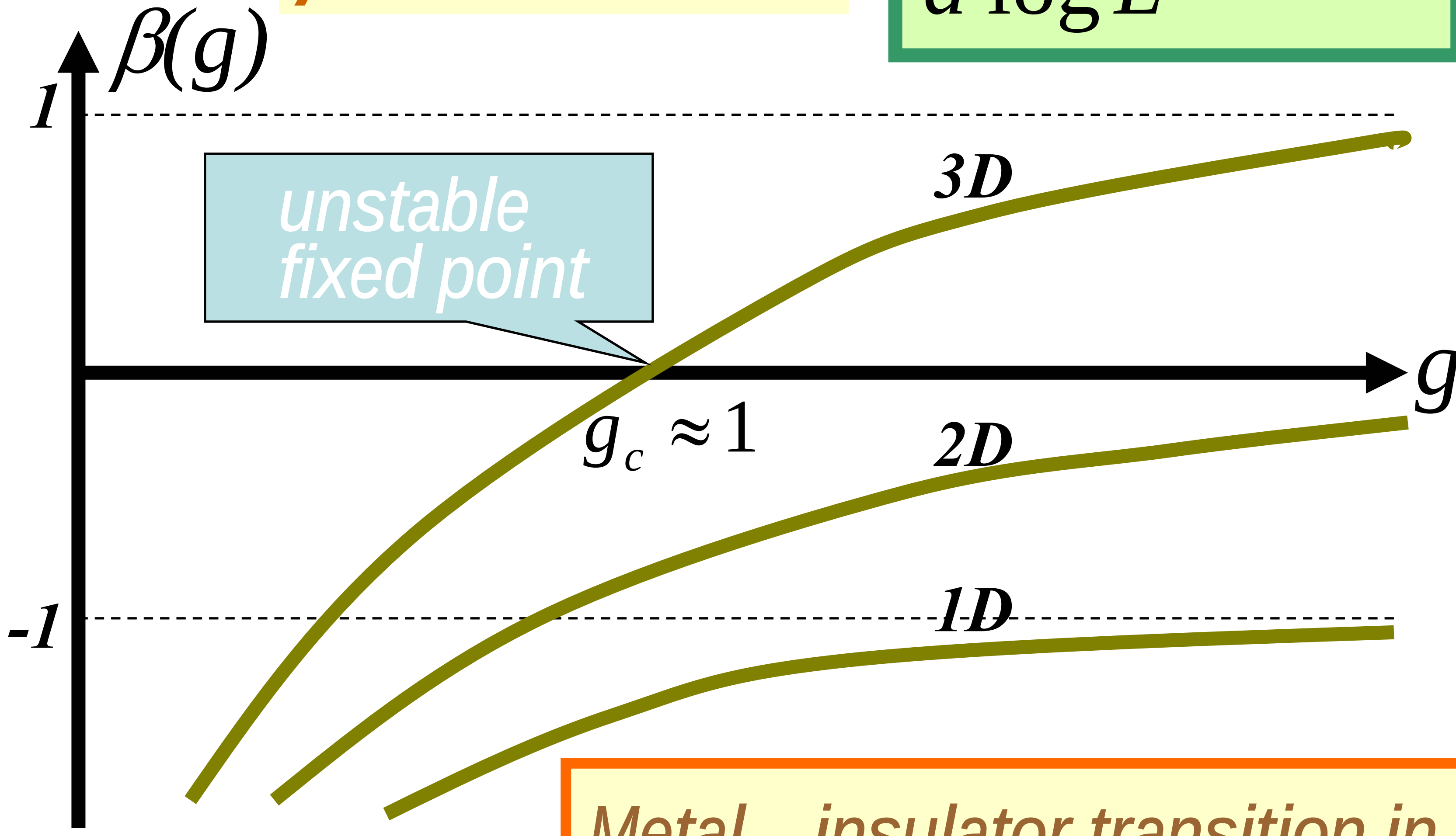
$$g = E_T / \delta_1$$

dimensionless
Thouless
conductance

$$g = Gh/e^2$$

β - function

$$\frac{d\log g}{d\log L} = \beta(g)$$



Metal – insulator transition in ***3D***

All states are localized for ***d=1,2***

Questions:

Why

- the scaling theory is correct?
- the corrections of the diffusion constant and conductance are negative?

Why diffusion description fails at large scales ?

Diffusion description fails at large scales

Why?

Einstein: there is no diffusion at too short scales – there is memory, i.e., the process is not marcovian.

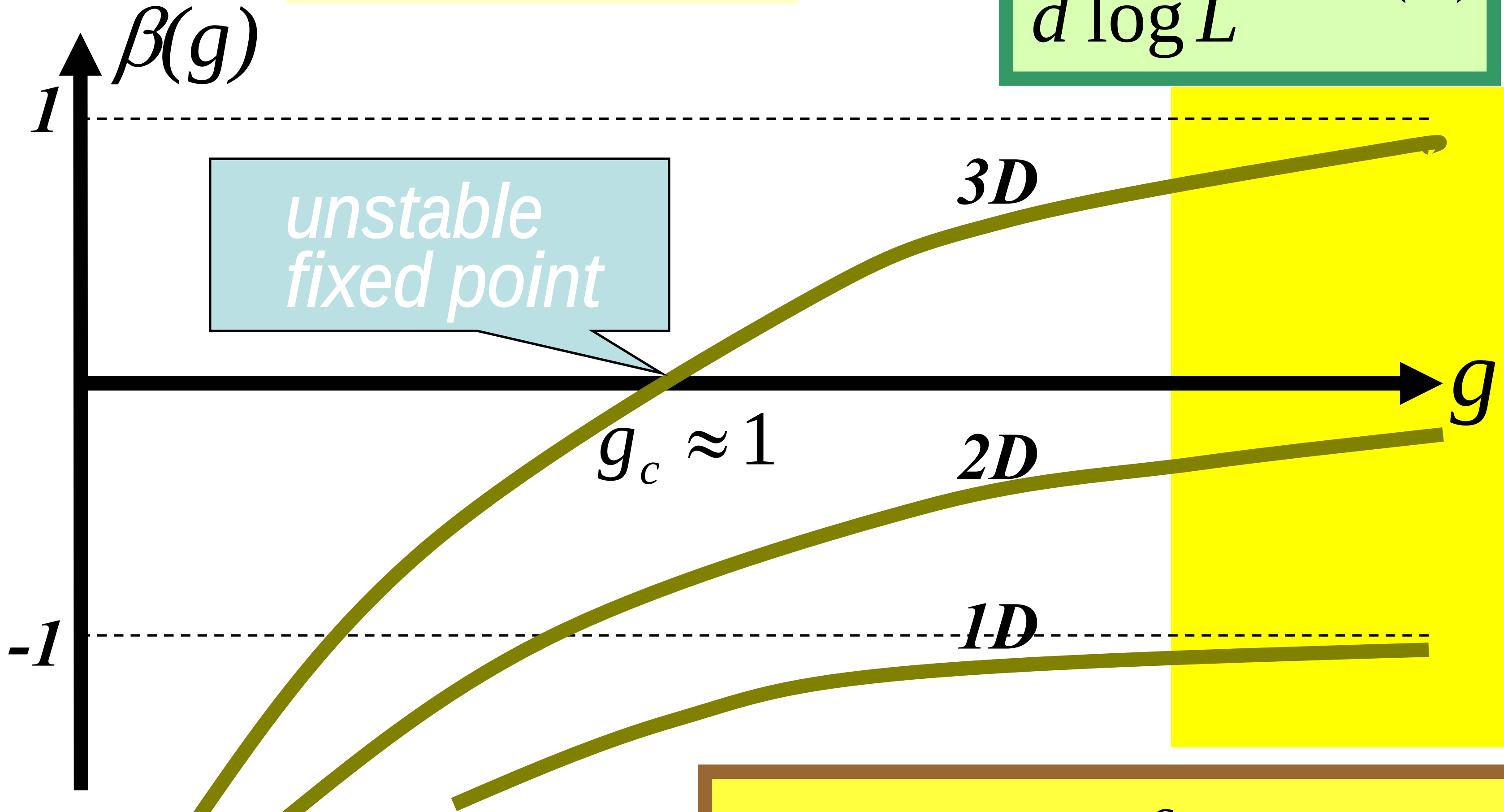
Why there is memory at large distances in quantum case ?

Quantum corrections at large Thouless conductance – weak localization

Universal description

β - function

$$\frac{d\log g}{d\log L} = \beta(g)$$



$$\beta(g) = d - 2 + \frac{c_d}{g}$$

$c_d = ? \quad \pm ?$

$$g(L) = \sigma_{cl} L^{d-2} - \begin{cases} \dfrac{c_d}{d-2} & d \neq 2 \\ c_2 \log\left(\dfrac{L}{l}\right) & d = 2 \end{cases}$$

Quantum corrections

$$\beta(g) = d - 2 + \frac{c_d}{g}$$

$c_d = ? \quad \pm ?$

$$g(L) = \sigma_{cl} L^{d-2} - \begin{cases} \dfrac{c_d}{d-2} & d \neq 2 \\ c_2 \log\left(\dfrac{L}{l}\right) & d = 2 \end{cases}$$

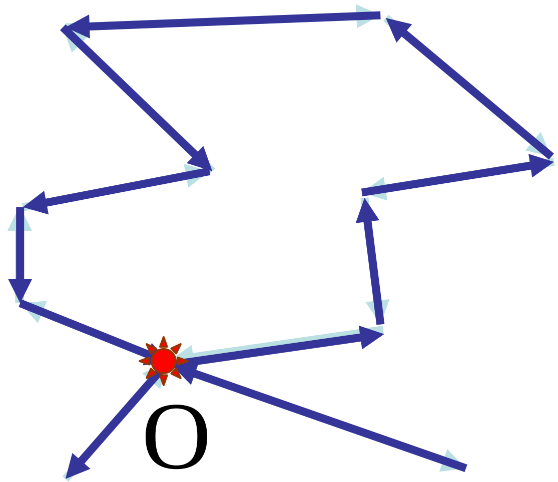
Suggested homework:

1. Derive the equation for $g(L)$ from this limit of the β-function
2. Suppose you know $\beta(g)$ for some number of dimensions d. Let g at some size of the system L_0 be close to the critical value: $g(L_0) = g_c + \delta g; \quad |\delta g| \ll 1$ Estimate the localization length ξ (for $\delta g < 0$) and the conductivity σ in the limit $L \to \infty$ (for $\delta g > 0$)

WEAK LOCALIZATION

$$\varphi = \oint \vec{p}\, d\vec{r}$$

Phase accumulated when traveling along the loop



The particle can go around the loop in two directions

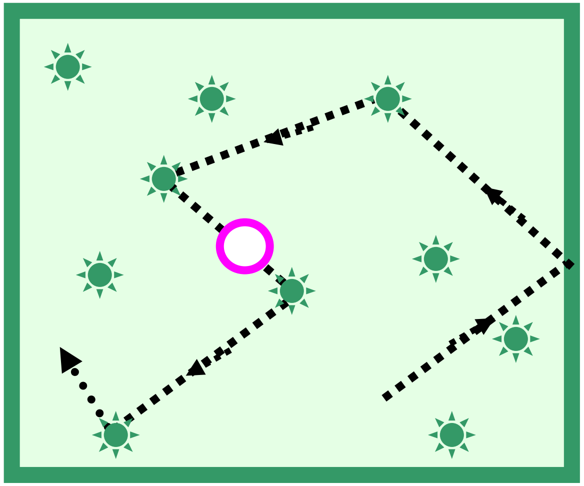
Memory!

$$\varphi_1 = \varphi_2$$

Constructive interference ⟹ ***probability to return to the origin gets enhanced*** ⟹ ***diffusion constant gets reduced. Tendency towards localization***

β ***- function is negative for d=2***

Diffusion



Random walk

Density fluctuations $\rho(\boldsymbol{r},t)$ at a given point in space $\boldsymbol{r}$ and time t.

$$\frac{\partial\rho}{\partial t} - D\nabla^2\rho = 0$$

Diffusion Equation

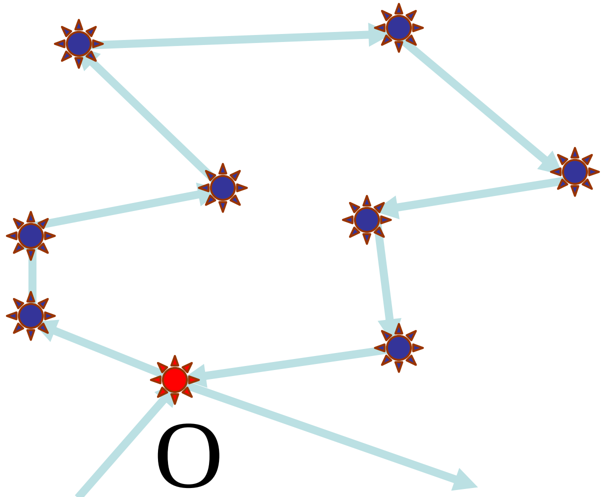
D - Diffusion constant

Mean squared distance from the original point at time t

$$\langle r(t)^2 \rangle = Dt$$

Probability to come back (to the element of the volume dV centered at the original point)

$$P\left(r\left(t\right)=0\right)dV = \frac{dV}{\left(Dt\right)^{d/2}}$$



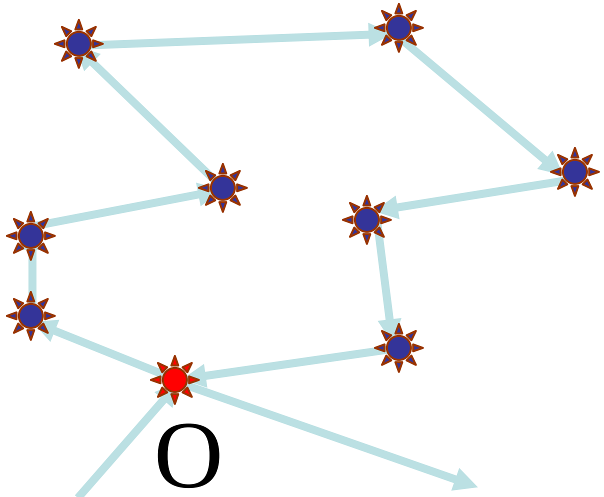
What is the probability ***P(t)*** that such a loop is formed within a time t ?

Probability to come back (to the element of the volume dV around the original point)

$$P\left(r\left(t\right)=0\right)dV=\frac{dV}{\left(Dt\right)^{d/2}}$$

Q: $dV=?$

A: $dV=\lambda^{d-1}v_F dt$



What is the probability *P(t)* that such a loop is formed within a time t ?

Probability to come back (to the element of the volume dV around the original point)

$$P\left(r\left(t\right)=0\right)dV=\frac{dV}{\left(Dt\right)^{d/2}}$$

Q: $dV=?$

A: $dV=\lambda^{d-1}v_F\,dt$

$$P\left(t\right)=-\lambda^{d-1}\int_{\tau}^{t}\frac{v_F\,dt'}{\left(Dt'\right)^{d/2}}$$

$$\frac{\delta g}{g}\approx P\left(t_{\max}\right)$$

$$P(t) = -\lambda^{d-1} \int_{\tau}^{t} \frac{v_F dt'}{(Dt')^{d/2}}$$

$$\frac{\delta g}{g} \approx P(t_{\max})$$

Q: $t_{\max} = ?$

A: $t_{\max} \sim \min\left\{ \frac{L^2}{D}, \frac{1}{\omega}, \tau_{\varphi,\ldots} \right\}$

$$P(t) = -\lambda\!\!\!^{-}\,^{d-1}\int_{\tau}^{t} \frac{v_F\,dt'}{(Dt')^{d/2}}$$

$$\frac{\delta g}{g} \approx P(t_{\max})$$

$$t_{\max} \sim \frac{L^2}{D} = \frac{h}{E_T}$$

$$d = 2$$

$$\frac{\delta g}{g} \approx -\frac{\lambda\!\!\!^{-}\,v_F}{D}\log\frac{L^2}{D\tau}$$

$$P(t) = \bar{\lambda}^{d-1} \int_{\tau}^{t} \frac{v_F \, dt'}{(Dt')^{d/2}}$$

$$\frac{\delta g}{g} \approx P(t_{\max})$$

$$\frac{\delta g}{g} \approx -\frac{\bar{\lambda} v_F}{D} \log \frac{L^2}{D\tau} = -\frac{2\bar{\lambda} v_F}{D} \log \frac{L}{l}$$

$$\bar{\lambda} v_F = \frac{1}{\pi \nu}$$

$$g = \nu D \hbar$$

$$\delta g = -\frac{2}{\pi} \log \frac{L}{l}$$

$$\beta(g) = -\frac{2}{\pi g}$$

Universal !!!

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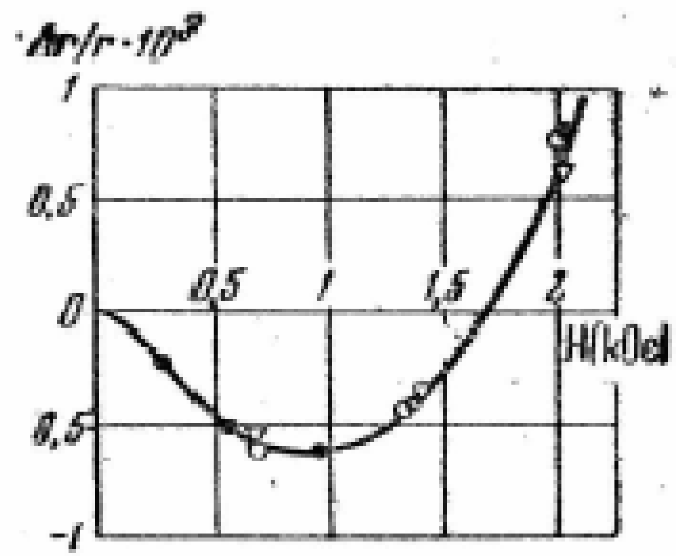
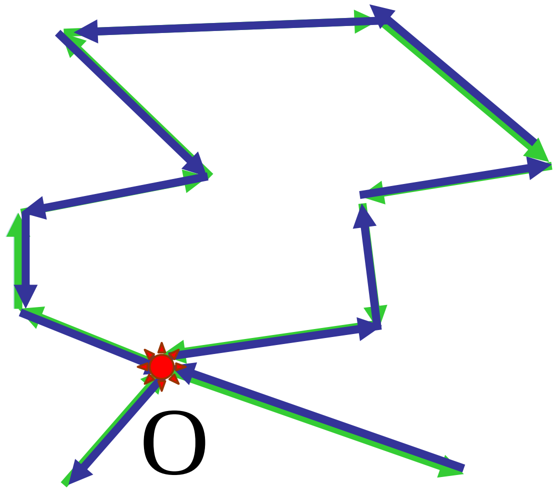
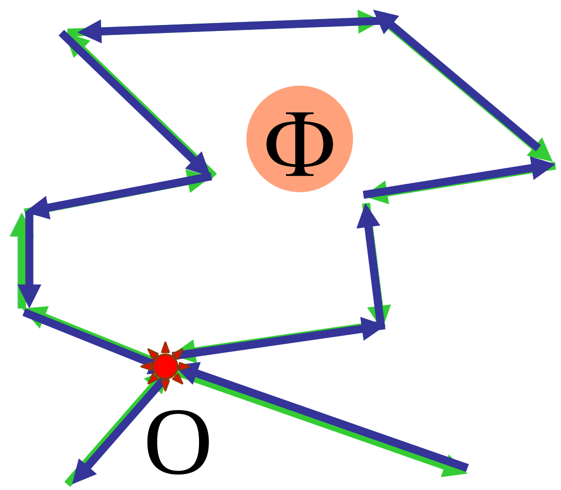


Рис. 2

Magnetoresistance





No magnetic field

$$\varphi_1 = \varphi_2$$

With magnetic field H

$$\varphi_1 - \varphi_2 = 2 * 2\pi\, \Phi/\Phi_0$$

Length Scales

Magnetic length $L_H = (hc/eH)^{1/2}$

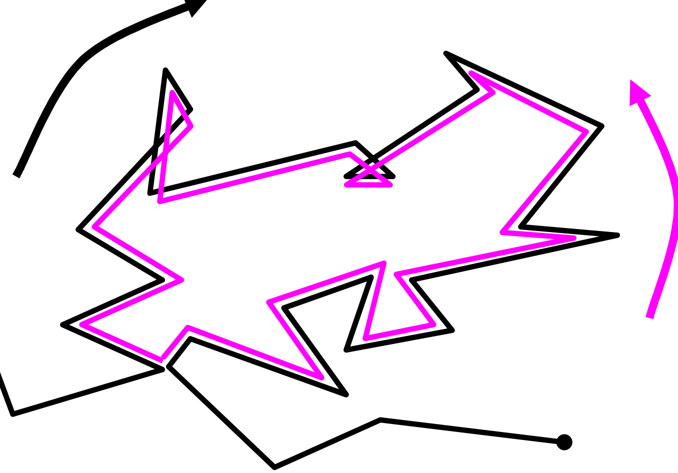
Dephasing length $L_\varphi = (D\,\tau_\varphi)^{1/2}$

$$\delta g(H) = f_d\left(\frac{L_H}{L_\varphi}\right)$$

Universal functions

Magnetoresistance measurements allow to study inelastic collisions of electrons with phonons and other electrons

Weak Localization



Negative Magnetoresistance

Chentsov (1949)

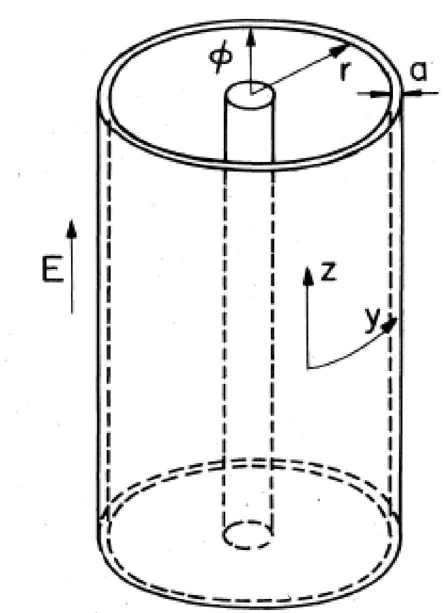
Aharonov-Bohm effect

Theory

B.A., Aronov & Spivak (1981)

Experiment

Sharvin & Sharvin (1981)



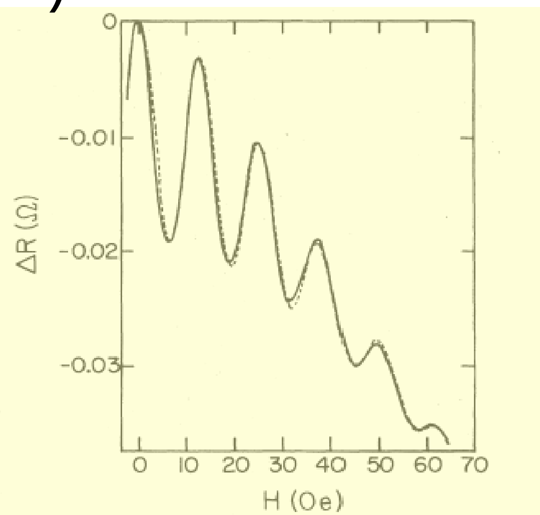


FIG. 8. Longitudinal magnetoresistance $\Delta R(H)$ at $T=1.1$ K for a cylindrical lithium film evaporated onto a 1-cm-long quartz filament. $R_{4.2}=2$ kΩ, $R_{300}/R_{4.2}=2.8$. Solid line: averaged from four experimental curves. Dashed line: calculated for $L_{\varphi}=2.2$ μm, $\tau_{\varphi}/\tau_{so}=0$, filament diameter $d=1.31$ μm, film thickness 127 nm. Filament diameter measured with scanning electron microscope yields $d=1.30\pm0.03$ μm (Altshuler *et al.*, 1982; Sharvin, 1984).

Q: What does it mean $d=2$ **?**

A: Transverse dimension is much less than

$$\sqrt{Dt_{\max}}$$

Lesson 2:

Brownian Particle as a mesoscopic system

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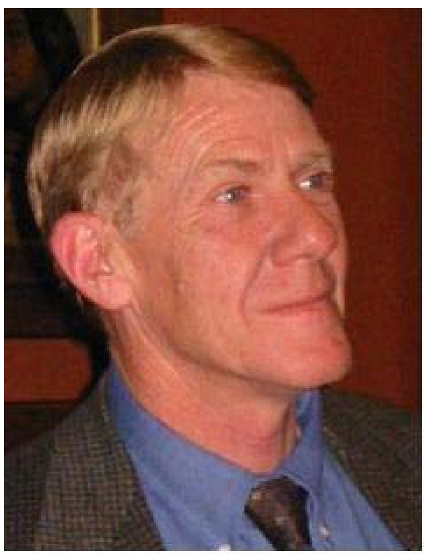
Magnetoresistance of small, quasi-one-dimensional, normal-metal rings and lines

C. P. Umbach, S. Washburn, R. B. Laibowitz, and R. A. Webb
IBM Thomas J. Watson Research Center, P. O. Box 218,
Yorktown Heights, New York 10598
(Received 6 July 1984)

The magnetoresistance of sub-0.4-μm-diam Au and $Au_{60}Pd_{40}$ rings was measured in a perpendicular magnetic field at temperatures as low as 5 mK in search of simple, periodic resistance oscillations that would be evidence of flux quantization in norm
very complex structure developed in the magneto
data did not reveal convincing evidence for flux
that observed in the rings was also found in th
lines. This structure appears to be associated wit



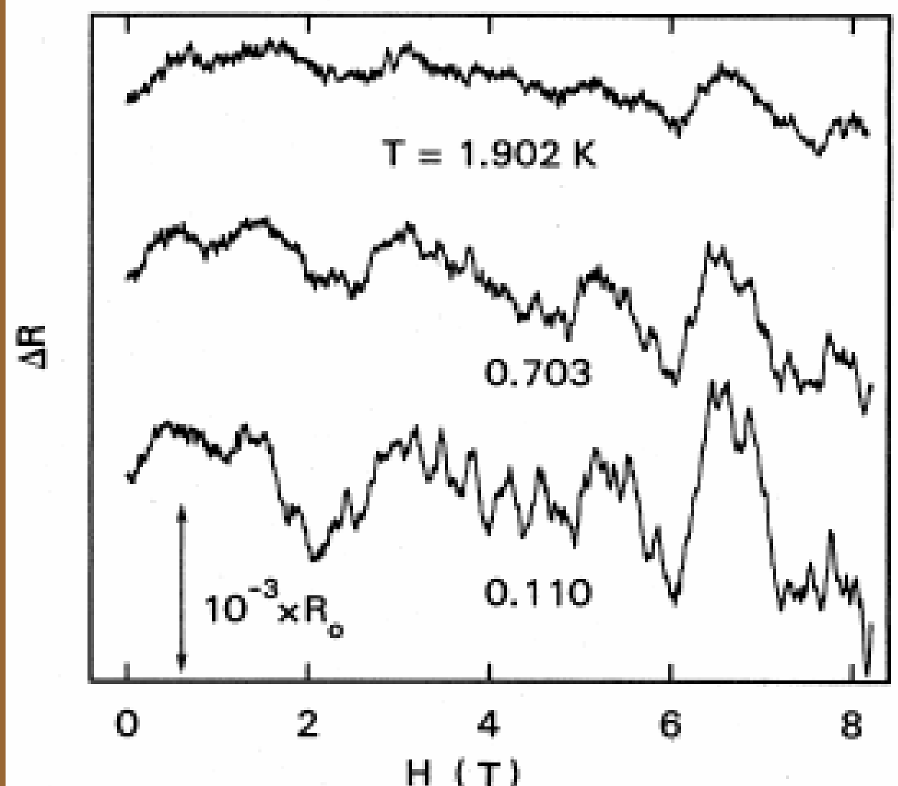
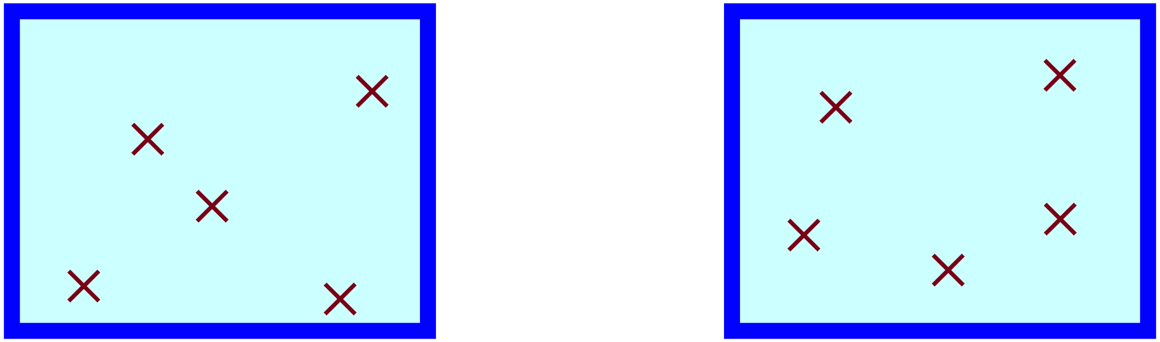


FIG. 4. Temperature dependence of the magnetoresistance from 0–8 T of a 60-nm-diam by 790-nm-long $Au_{60}Pd_{40}$ line. The zero-field resistance of the line, R_0, was 101.7 Ω.

Mesoscopic fluctuations

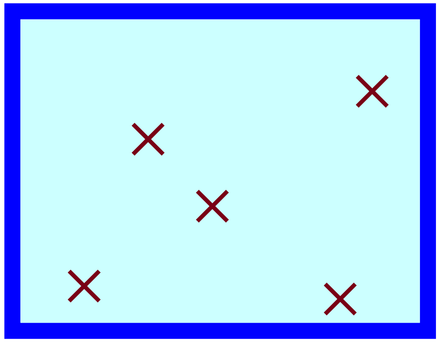
Mesoscopic Fluctuations.



Properties of systems with identical set of macroscopic parameters but different realizations of disorder **are different**!

$$g_1 \neq g_2$$

Mesoscopic Fluctuations.



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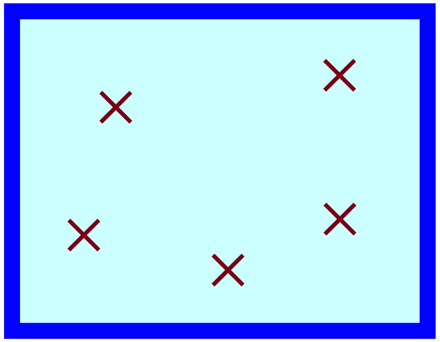

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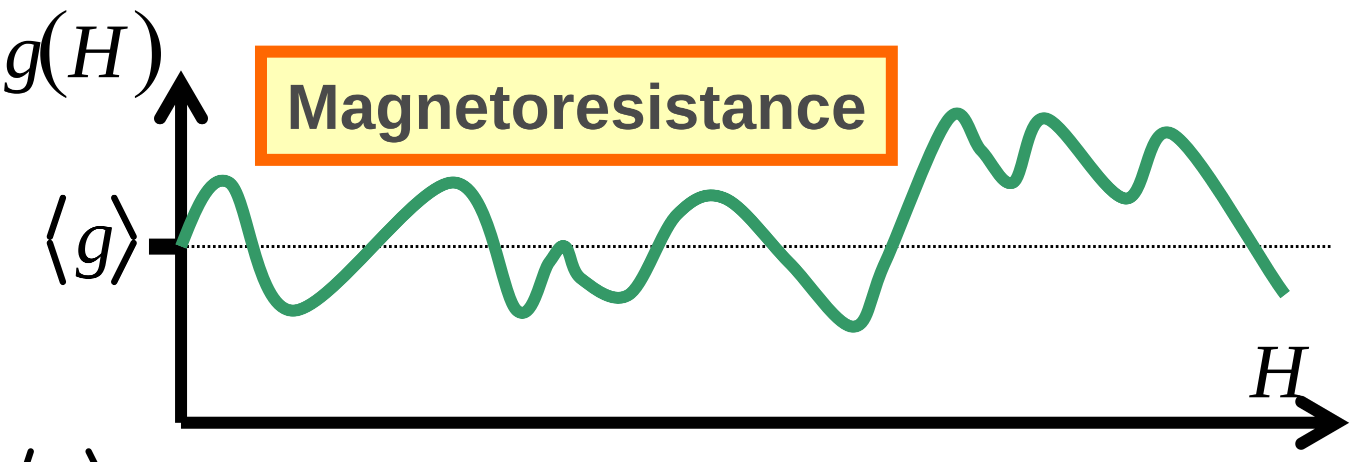


$g(H)$ is sample -dependent

$\langle \ldots \rangle$ - **ensemble averaging** $\qquad \langle g \rangle >> 1$

Before Einstein:

Correct question would be: **describe** $\vec{r}(t)$

OK, **maybe you can restrict yourself by** $\langle \vec{r}(t) \rangle$

Before Einstein:

Correct question would be: describe $\vec{r}(t)$

OK, maybe you can restrict yourself by $\langle \vec{r}(t) \rangle$

Einstein: What is $\left\langle \left[\vec{r}(0) - \vec{r}(t) \right]^2 \right\rangle$?

$$\left\langle \left[\vec{r}(0) - \vec{r}(t) \right]^n \right\rangle = ?$$

Before Einstein:

Correct question would be: **describe** $\vec{r}(t)$

OK, **maybe you can restrict yourself by** $\langle \vec{r}(t) \rangle$

Einstein:

What is $\left\langle \left[\vec{r}(0) - \vec{r}(t) \right]^2 \right\rangle$ **?**

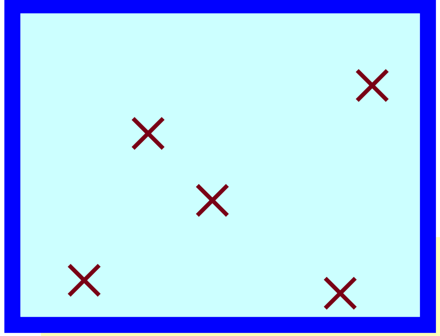
$$\left\langle \left[\vec{r}(0) - \vec{r}(t) \right]^n \right\rangle = ?$$

Mesoscopic physics:

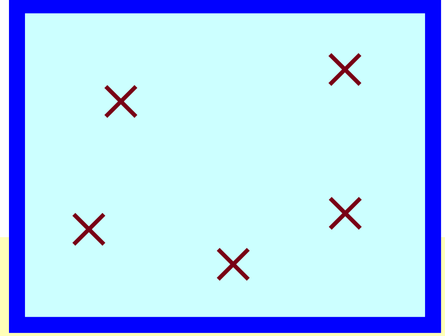
Not only $\langle g(H) \rangle$

But also $\left\langle \left[g(H) - g(H+h) \right]^2 \right\rangle$

	Brownian motion	Conductance fluctuations
ensemble	Set of brownian particles	Set of small conductors
observables	Position of each particle $\vec{r}$	Conductance of each sample g
evolves as function of	Time t	Magnetic field H or any other external tunable parameter
Interested in	Statistics of $\vec{r}(t)$	Statistics of $g(H)$
Example	$\left\langle \left[\vec{r}(t_1) - \vec{r}(t_2) \right]^2 \right\rangle$	$\left\langle \left[g(H_1) - g(H_2) \right]^2 \right\rangle$

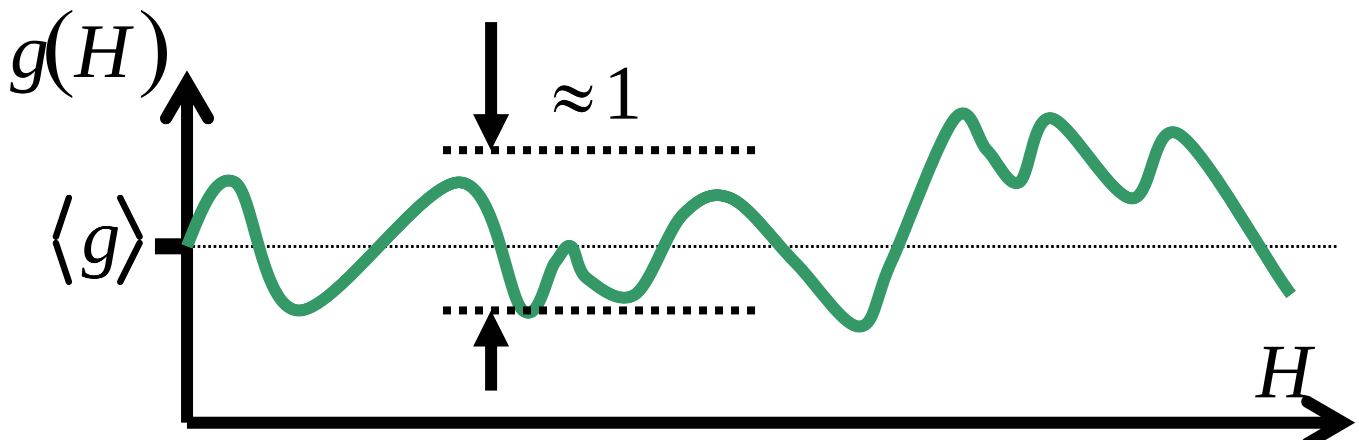


$g_1 \neq g_2$



$g_1 - g_2 \cong 1$ $G_1 - G_2 \cong e^2/\hbar$

Magnetoresistance



Statistics of the functions of $g(H)$ are universal

B.A.(1985);
Lee & Stone (1985)

Statistics of random function(s) $g(H)$ are universal !!!

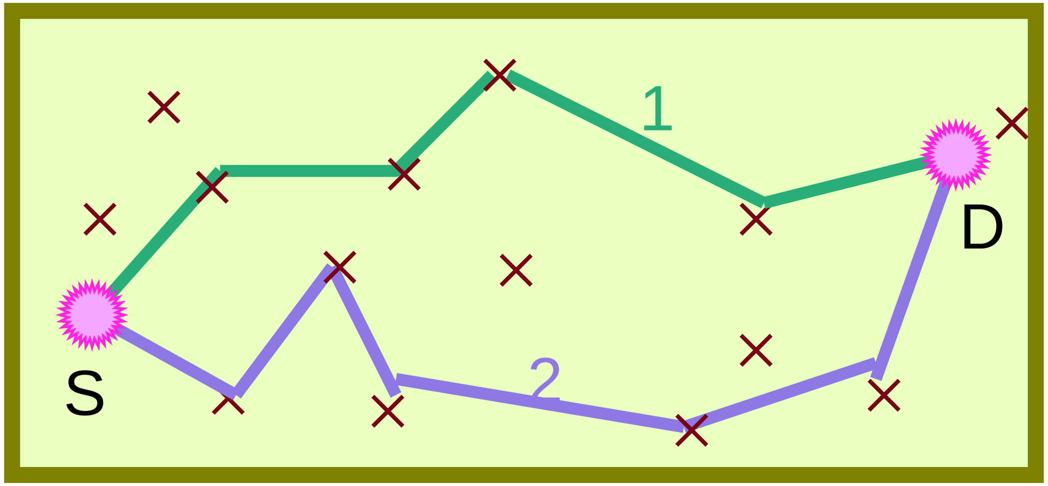
In particular, $\left\langle (\delta g)^2 \right\rangle \sim 1$

$$g \propto L^{d-2} \quad \rightarrow \quad \frac{\left\langle (\delta g)^2 \right\rangle}{g^2} \propto L^{4-2d} >> L^{-d}$$

$\Downarrow$

Fluctuations are large and nonlocal

Waves in Random Media



W_1, W_2 **probabilities**

A_1, A_2 **probability amplitudes**

$$W_{1,2} = |A_{1,2}|^2$$

$$A_{1,2} = |A_{1,2}| e^{i\varphi_{1,2}}$$

Total probability

$$\boxed{W = |A_1 + A_2|^2 = W_1 + W_2 + 2\,\mathrm{Re}\left(A_1 A_2^*\right)}$$

interference term:

$$2\,\mathrm{Re}\left(A_1 A_2^*\right) = 2\sqrt{W_1 W_2}\cos(\varphi_1 - \varphi_2)$$

$$\boxed{W = \left|A_1 + A_2\right|^2 = W_1 + W_2 + 2\,\mathrm{Re}\left(A_1 A_2^{\,*}\right)}$$

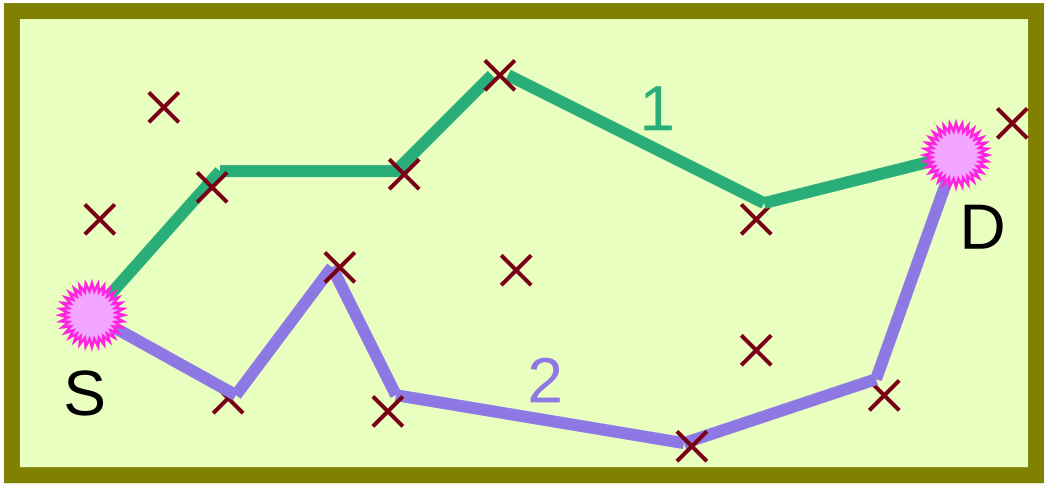
$$2\,\mathrm{Re}\left(A_1 A_2^{\,*}\right) = 2\sqrt{W_1 W_2}\cos(\varphi_1 - \varphi_2)$$

1. $A_{1,2} = \sqrt{W_{1,2}}\,\exp\left(i\varphi_{1,2}\right)$
2. Phases $\varphi_{1,2}$ are random
3. $\left|\varphi_1 - \varphi_2\right| >> 2\pi$

The interference term disappears after averaging

$$\left\langle \cos(\varphi_1 - \varphi_2) \right\rangle = 0$$

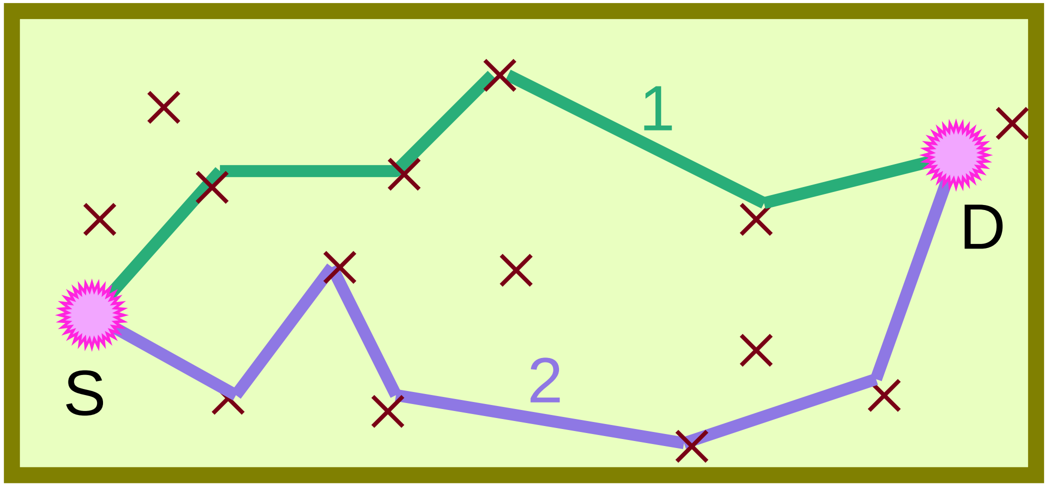
$$\boxed{\langle W \rangle = \langle W_1 \rangle + \langle W_2 \rangle}$$



$$W = \left| A_1 + A_2 \right|^2 = W_1 + W_2 + 2\,\mathrm{Re}\left(A_1 A_2^{\,*} \right)$$

Classical result for average probability:

$$\langle W \rangle = W_1 + W_2$$



Consider now square of the probability

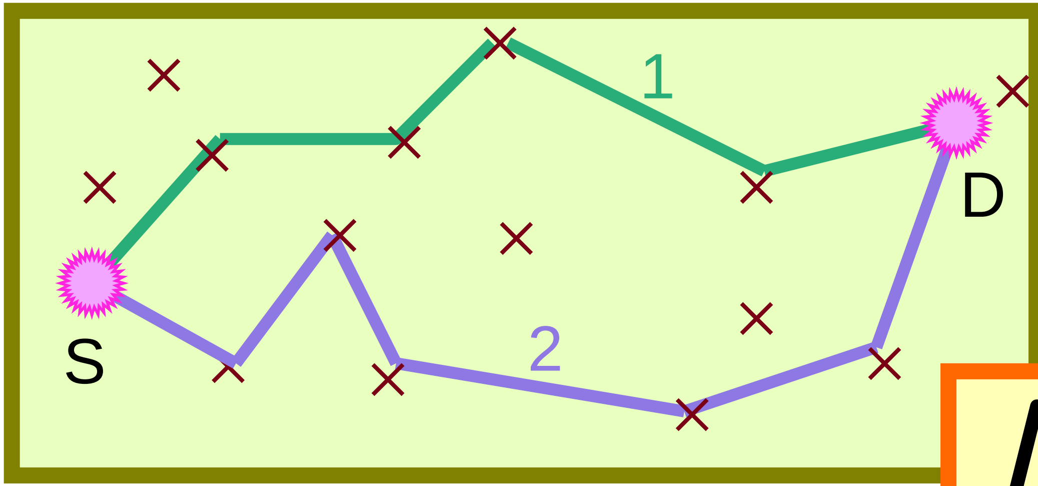
$$\left\langle W^2 \right\rangle = \left(W_1 + W_2\right)^2 + 2W_1 W_2$$

Reason:

$$\left\langle \cos(\varphi_1 - \varphi_2) \right\rangle = 0$$

$$\left\langle \cos^2(\varphi_1 - \varphi_2) \right\rangle = 1/2$$

$$\left\langle W^2 \right\rangle \neq \left\langle W \right\rangle^2$$



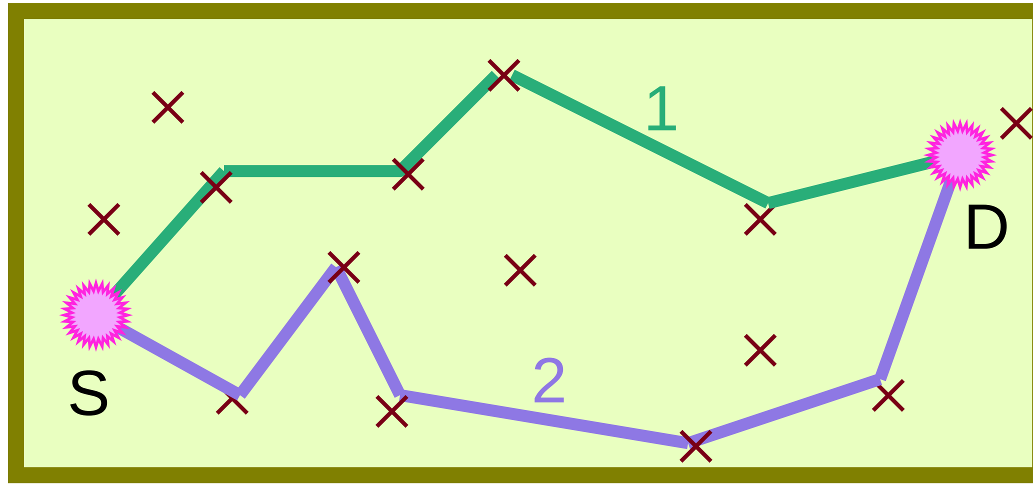
$$\left\langle W^2 \right\rangle \neq \left\langle W \right\rangle^2$$

CONCLUSIONS:

1. There are fluctuations!

2. Effect is nonlocal.

Now let us try to understand the effect of magnetic field. Consider the correlation function



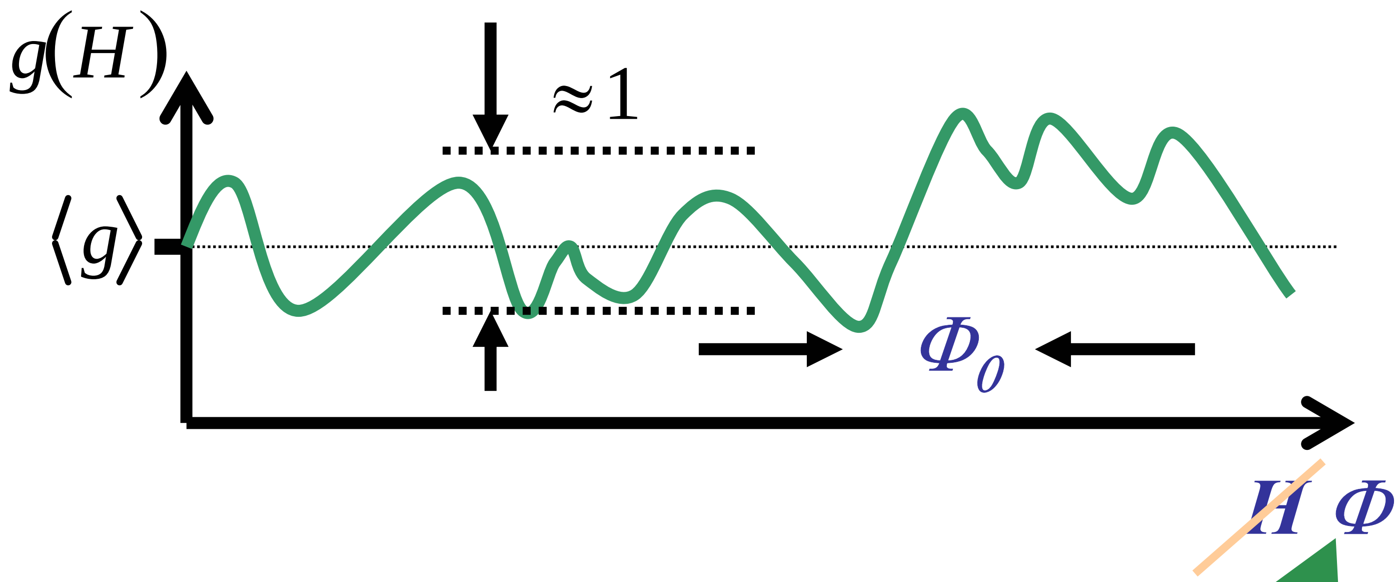
$$\langle W(H)W(H+h)\rangle = \langle W(H)\rangle\langle W(H+h)\rangle$$
$$+2W_1W_2\left\langle \cos\left(\delta\varphi(H)\right)\cos\left(\delta\varphi(H+h)\right)\right\rangle$$

$$\delta\varphi \equiv \varphi_1 - \varphi_2$$

$$\left\langle \cos\left(\delta\varphi(H)\right)\cos\left(\delta\varphi(H+h)\right)\right\rangle \Rightarrow \begin{array}{ll} \frac{1}{2} & \text{for } h \to 0 \;\; \left(\Phi(h) \ll \Phi_0\right) \\ 0 & \text{for } \Phi(h) \gg \Phi_0 \end{array}$$

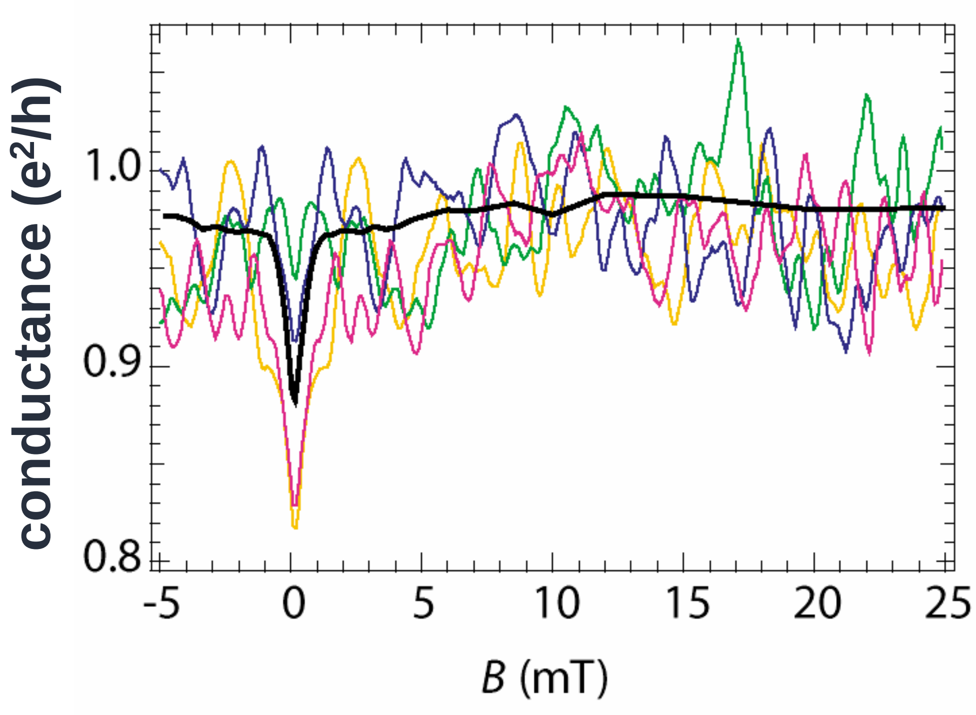
$$\Phi(h) = h \bullet (\text{area of the loop})$$

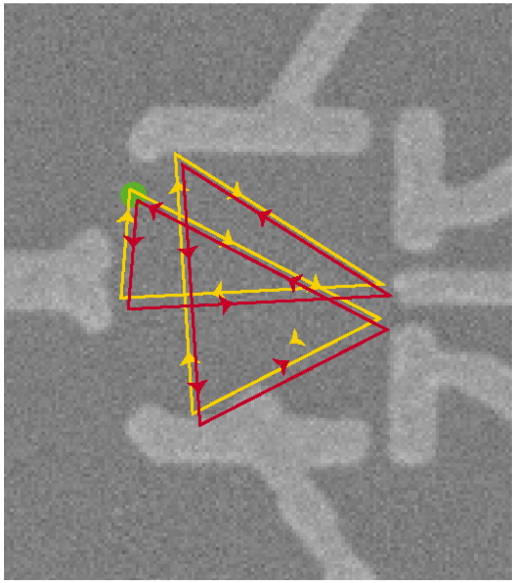
Magnetoresistance

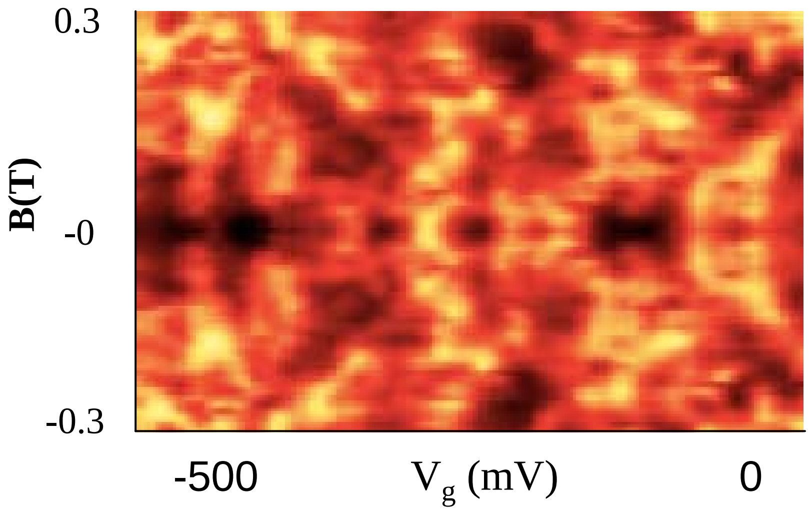


Flux through the whole system

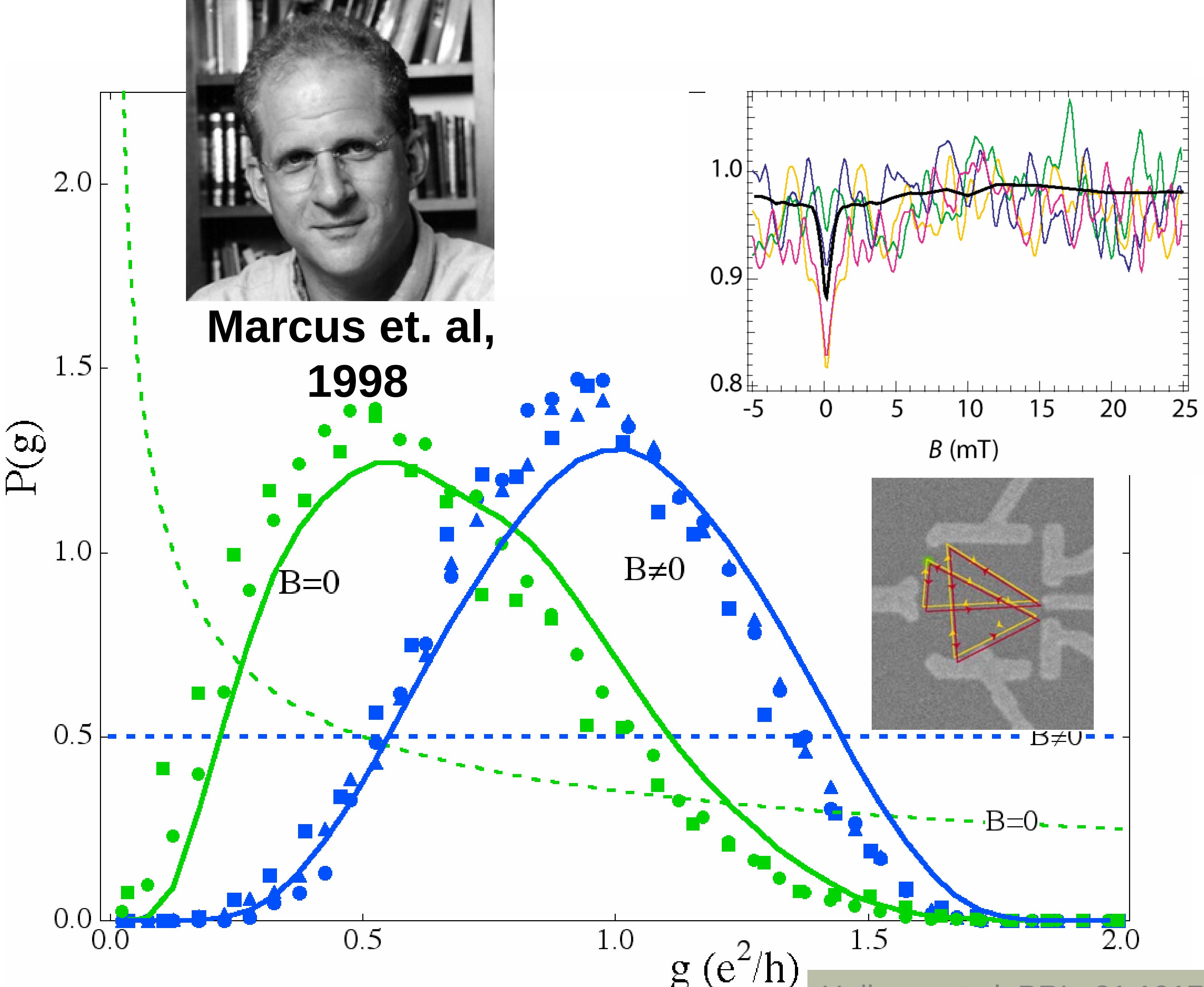
Quantum Chaos





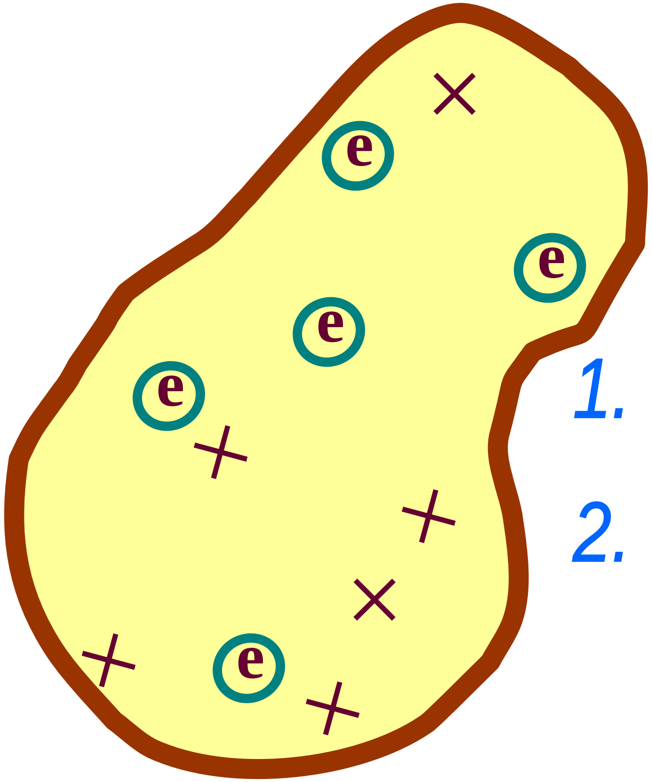


Marcus et al



Marcus et. al, 1998

Huibers, et al. PRL, 81 1917(1998).



1. *Disorder (× – impurities)*

2. *Complex geometry*

How to deal with disorder?

- ~~Solve the Shrodinger equation exactly~~
- Make statistical analysis

What if there in no disorder?

Part 2:

Random Matrix Theory And Quantum Chaos

RANDOM MATRIX THEORY

Spectral statistics

$N \times N$ *ensemble of Hermitian matrices with random matrix element* $N \to \infty$

E_α - spectrum (set of eigenvalues)

$\delta_1 \equiv \langle E_{\alpha+1} - E_\alpha \rangle$ - mean level spacing

$\langle \ldots\ldots \rangle$ - ensemble averaging

$s \equiv \dfrac{E_{\alpha+1} - E_\alpha}{\delta_1}$ - spacing between nearest neighbors

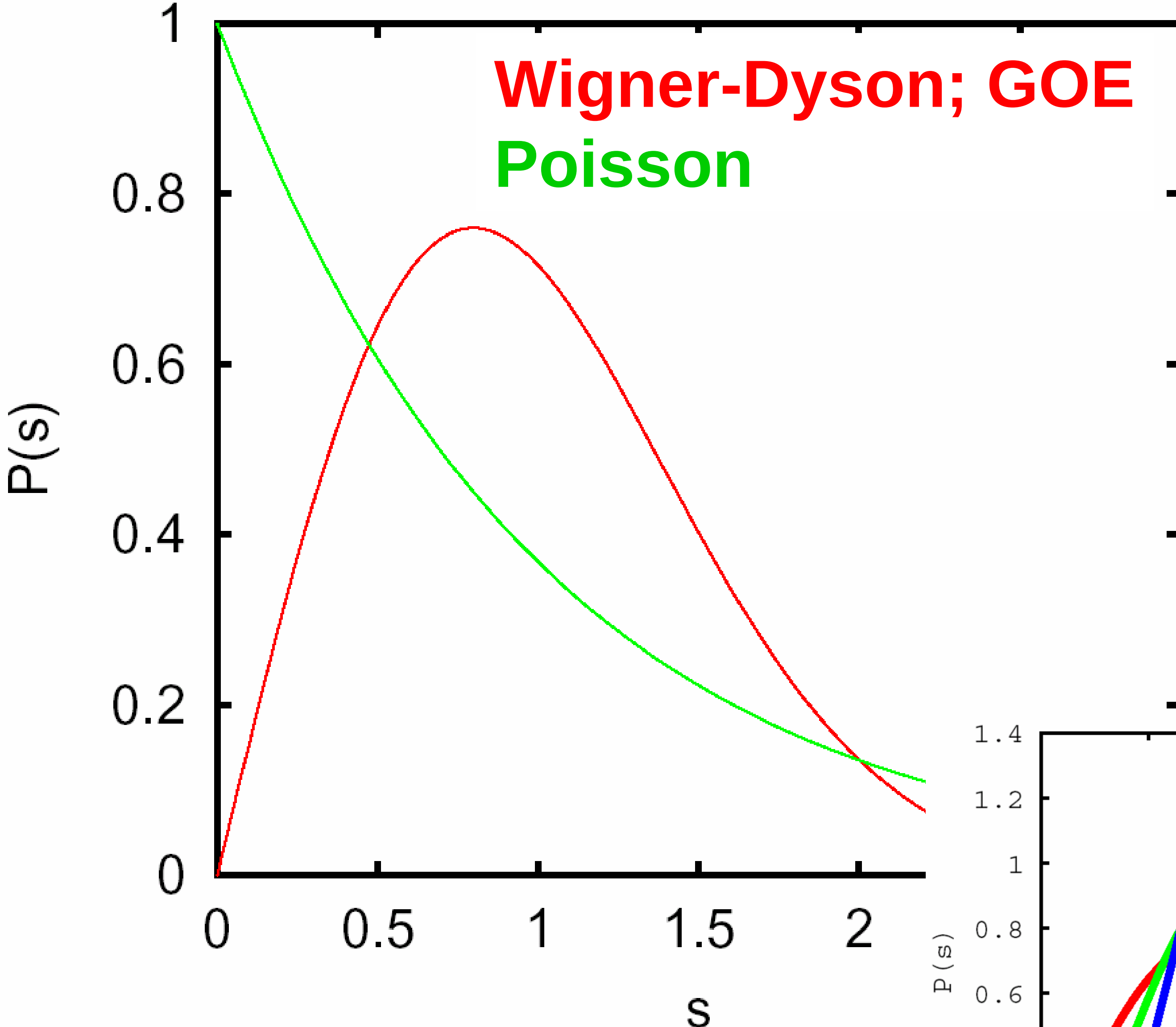
$P(s)$ - distribution function of nearest neighbors spacing between

Spectral Rigidity

Level repulsion

$$P(s=0)=0$$

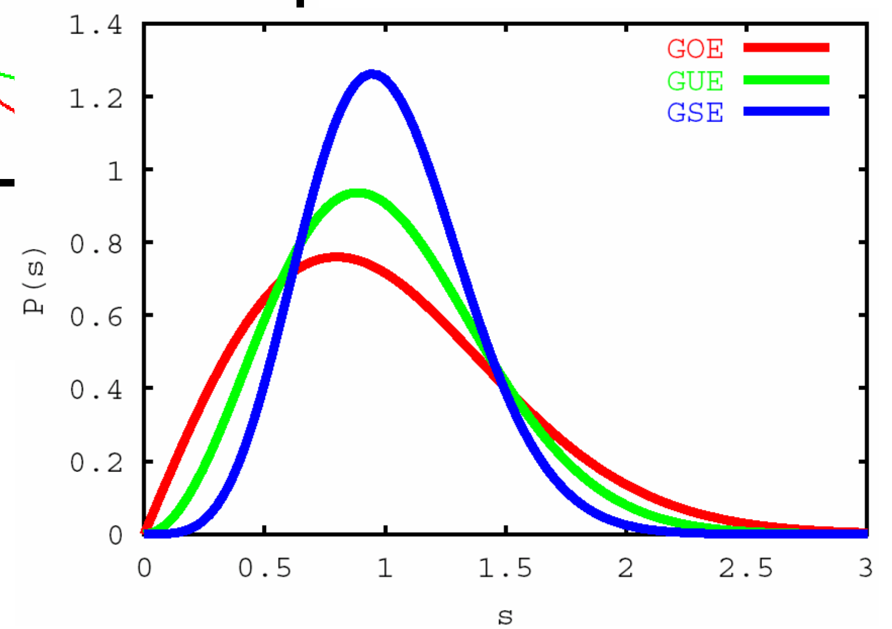
$$P(s<<1) \propto s^\beta \qquad \beta=1,2,4$$



Gaussian
Orthogonal
Ensemble

Unitary
$\beta=2$

Simplectic
$\beta=4$



Poisson – completely uncorrelated levels

Reason for $P(s) \to 0$ when $s \to 0$:

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(H_{22} - H_{11})^2 + |H_{12}|^2}$$

small

small

small

1. The assumption is that the matrix elements are statistically independent. Therefore probability of two levels to be degenerate vanishes.

2. If H_{12} is real (orthogonal ensemble), then for s to be small two statistically independent variables ($(H_{22}$- $H_{11})$ and H_{12}) should be small and thus $P(s) \propto s$ $\quad \beta = 1$

3. Complex H_{12} (unitary ensemble) ⟹ both $Re(H_{12})$ and $Im(H_{12})$ are statistically independent ⟹ three independent random variables should be small ⟹ $P(s) \propto s^2$ $\quad \beta = 2$

RANDOM MATRICES

$N \times N$ matrices with random matrix elements. $N \to \infty$

Dyson Ensembles

Matrix elements	Ensemble	β	realization
real	orthogonal	1	T-inv potential
complex	unitary	2	broken T-invariance (e.g., by magnetic field)
2×2 matrices	simplectic	4	T-inv, but with spin-orbital coupling

Finite size quantum physical systems

Atoms

Nuclei

Molecules $\left.\begin{array}{l} \\ . \\ . \\ . \end{array}\right\}$ Quantum Dots

.

.

.

ATOMS

Main goal is to classify the eigenstates in terms of the quantum numbers

NUCLEI

For the nuclear excitations this program does not work

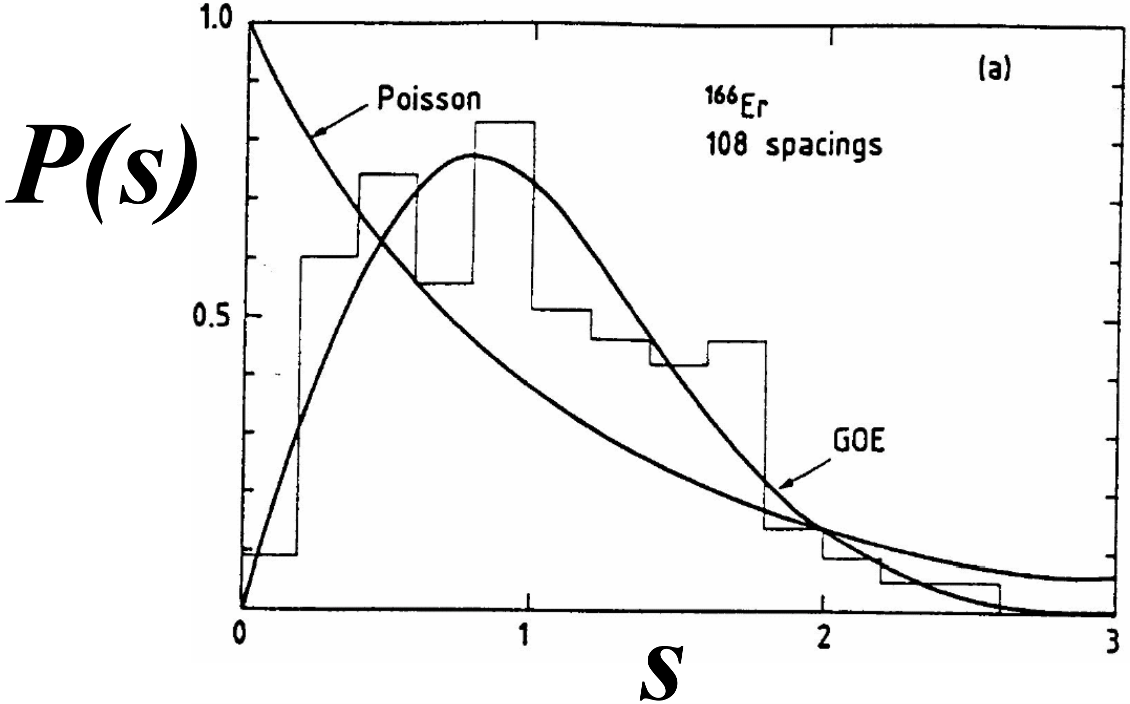
E.P. Wigner:

Study spectral **statistics** of a **particular** quantum system – a given nucleus

Random Matrices	Atomic Nuclei
• ***Ensemble***	• ***Particular quantum system***
• ***Ensemble averaging***	• ***Spectral averaging (over α)***

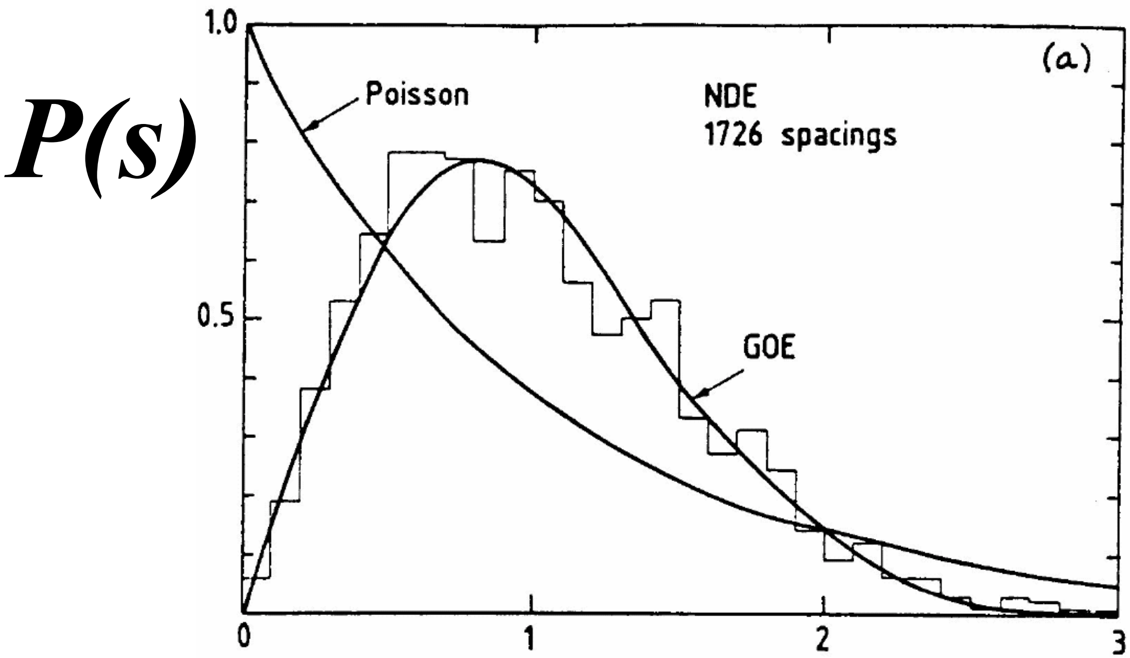
Nevertheless

Statistics of the nuclear spectra are almost exactly the same as the Random Matrix Statistics



Particular nucleus

^{166}Er



Spectra of several nuclei combined (after rescaling by the mean level spacing)

E.P. Wigner, **Conference on Neutron Physics by Time of Flight, November 1956**

P.W. Anderson, **"*Absence of Diffusion in Certain Random Lattices*"; Phys.Rev., 1958, v.109, p.1492**