

Introduction to Theory of Mesoscopic Systems

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Introduction

Einstein's Miraculous Year - 1905

Six papers:

1. The light-quantum and the photoelectric effect.
Completed March 17.
2. A new determination of molecular dimensions.
Completed April 30. Published in 1906
Ph.D. thesis.
3. Brownian Motion.
Received by Annalen der Physik May 11.
- 4,5. The two papers on special relativity.
Received June 30 and September 27
6. Second paper on Brownian motion.
Received December 19.

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Einstein's Miraculous Year - 1905

Diffusion and Brownian Motion:

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Q: Are these papers indeed important enough to stay in the same line with the relativity and photons. **?**
Why

Einstein's Miraculous Year - 1905

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*Nobel
Prize*

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By far the
largest number
of citations

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Brownian Motion - history



**Robert Brown
(1773-1858)**



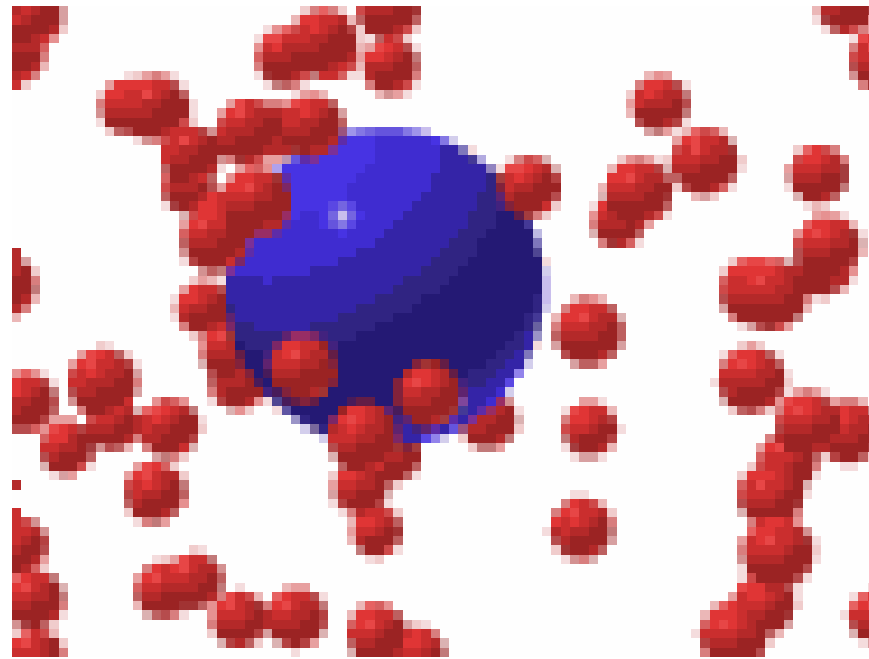
The instrument with which Robert Brown studied Brownian Motion and which he used in his work on identifying the nucleus of the living cell. This instrument is preserved at the Linnean Society in London.

Brownian Motion - history

Robert Brown, *Phil.Mag.* 4,161(1828); 6,161(1829)

Random motion of particles suspended in water ("dust or soot deposited on all bodies in such quantities, especially in London")

Action of water molecules pushing against the suspended object?

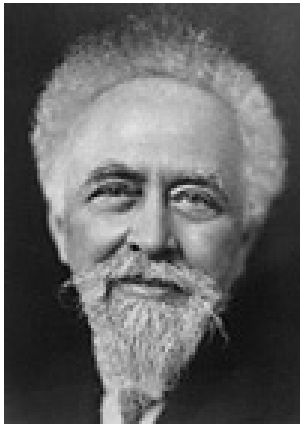


Giovanni Cantoni (Pavia). *N.Cimento*, 27,156(1867).



The Nobel Prize in Physics 1926

"for his work on the discontinuous structure of matter, and especially for his discovery of sedimentation equilibrium"



**Jean Baptiste
Perrin**

France

b. 1870

d. 1942

**... measurements on the
Brownian movement
showed that Einstein's
theory was in perfect
agreement with reality.
Through these
measurements a new
determination of
Avogadro's number was
obtained.**

The Nobel Prize in Physics 1926

**From the Presentation Speech by Professor
C.W. Oseen, member of the Nobel Committee
for Physics of The Royal Swedish Academy of
Sciences on December 10, 1926**

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Random motion of particles suspended in water ("dust or soot deposited on all bodies in such quantities, especially in London")

Action of water molecules pushing against the suspended object

Problems:

1. Each molecule is too light to change the momentum of the suspended particle.
2. Does Brownian motion violate the second law of thermodynamics ?



Jules Henri Poincaré
(1854-1912)

"We see under our eyes now motion transformed into heat by friction, now heat changes inversely into motion. This is contrary to Carnot's principle."

H. Poincaré, "The fundamentals of Science",
p.305, Scientific Press, NY, 1913

Problems:

1. Each molecules is too light to change the momentum of the suspended particle.
2. Does Brownian motion violate the second law of thermodynamics ?
3. Do molecules exist as real objects and are the laws of mechanics applicable to them?

Kinetic theory



Ludwig Boltzmann
1844 - 1906

$$S = k \log W + \text{const}$$

entropy

probability



Kinetic theory



Ludwig Boltzmann
1844 - 1906

$$S = k \log W + \text{const}$$

entropy

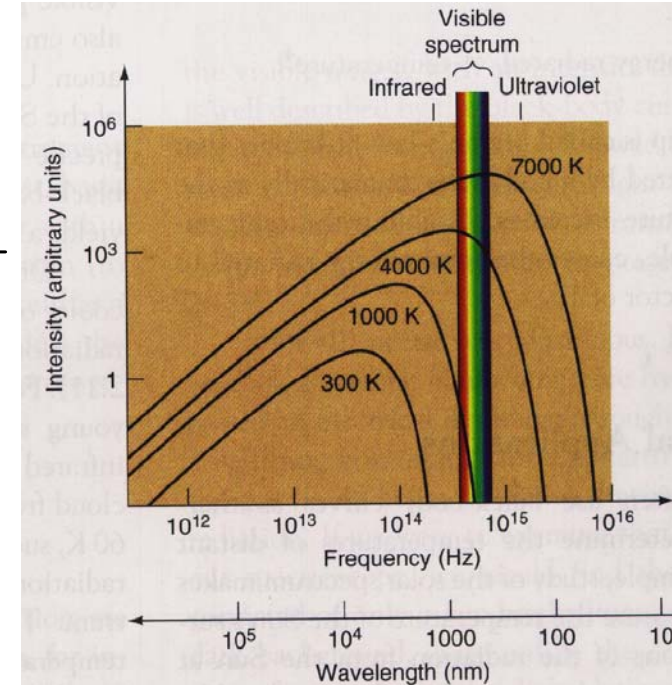
probability

k is Boltzmann constant



Max Planck
1858 - 1947

$$\rho(\nu, T) = \frac{8\pi h \nu^3}{c^3 \left[\exp\left(\frac{h\nu}{kT}\right) - 1 \right]}$$





From Micro
to Macro



$$S = k \log W + \textit{const}$$



From Macro
to Micro

"It is of great importance since it permits exact computation of Avogadro number The great significance as a matter of principle is, however ... that one sees directly under the microscope part of the heat energy in the form of mechanical energy."

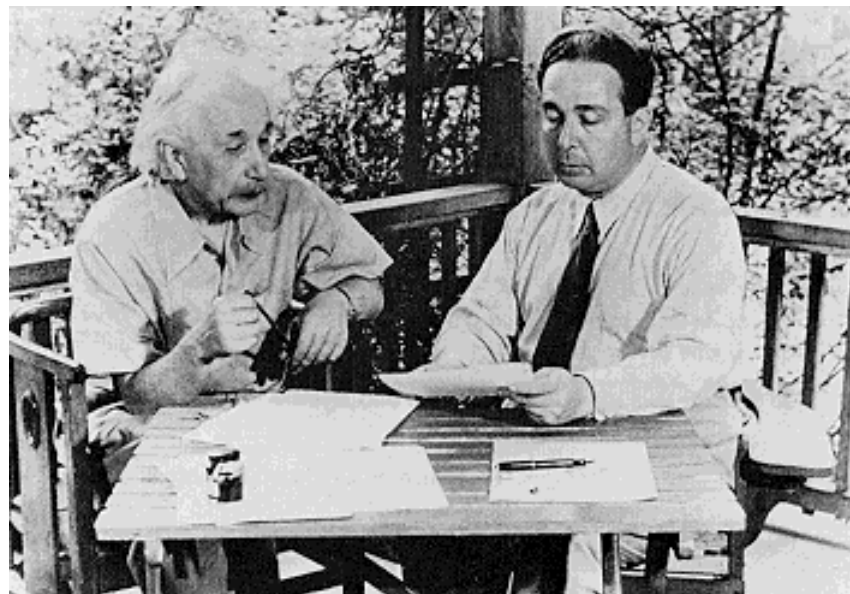
Einstein, 1915

Brownian Motion - history

Einstein **was not** the first to:

1. Attribute the Brownian motion to the action of water molecules pushing against the suspended object
2. Write down the diffusion equation
3. Saved Second law of Thermodynamics

L. Szilard, Z. Phys, 53, 840(1929)



Brownian Motion - history

Einstein **was not** the first to:

1. Attribute the Brownian motion to the action of water molecules pushing against the suspended object
2. Write down the diffusion equation
3. Saved Carnot's principle [L. Szilard, Z. Phys, 53, 840(1929)]

Einstein **was the first** to:

1. Apply the diffusion equation to the probability
2. Derive the diffusion equation from the assumption that the process is **markovian** (before Markov) and take into account nonmarkovian effects
3. Derived the relation between diffusion const and viscosity (conductivity), i.e., connected fluctuations with dissipation

By studying large molecules in solutions sugar in water or suspended particles Einstein **made molecules visible**

Diffusion Equation

$$\frac{\partial \rho}{\partial t} - D \nabla^2 \rho = 0$$

Diffusion constant

Einstein-Sutherland Relation for electric conductivity σ



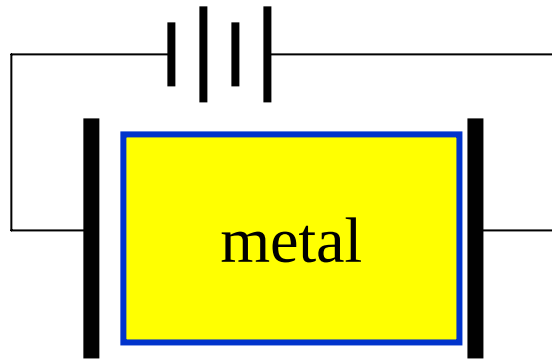
William Sutherland
(1859-1911)

$$\sigma = e^2 D \nu \quad \nu \equiv \frac{dn}{d\mu}$$

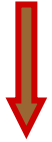
If electrons would be degenerate and form a classical ideal gas

$$\nu = \frac{1}{T n_{tot}}$$

Einstein-Sutherland Relation for electric conductivity σ



No current



$$eD \frac{dn}{dx} = \sigma E$$

$$n = n(\mu)$$

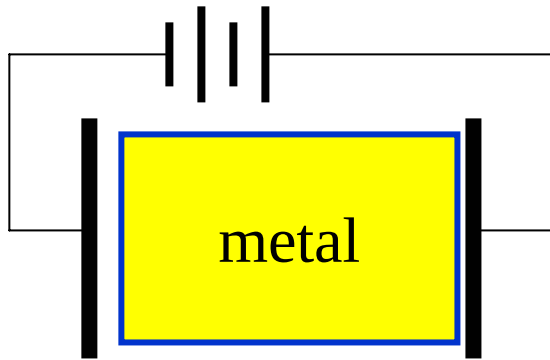
Density of
electrons

Chemical
potential

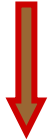
$$\frac{dn}{dx} = \frac{dn}{d\mu} \frac{d\mu}{dx} = eE \frac{dn}{d\mu}$$

Electric
field

Einstein-Sutherland Relation for electric conductivity σ



No current



$$eD \frac{dn}{dx} = \sigma E$$

Conductivity

$$n = n(\mu)$$

Density of electrons

$$\frac{dn}{dx} = \frac{dn}{d\mu} \frac{d\mu}{dx} = eE \frac{dn}{d\mu}$$

Chemical potential

Electric field

$$\sigma = e^2 D \nu \quad \nu \equiv \frac{dn}{d\mu}$$

Density of states

Diffusion Equation

$$\frac{\partial \rho}{\partial t} - D \nabla^2 \rho = 0$$

Lessons from the Einstein's work:

- **Universality**: the equation is valid as long as the process is marcovian
- Can be applied to the **probability** and thus describes both fluctuations and dissipation
- There is a universal relation between the diffusion constant and the viscosity
- Studies of the diffusion processes brings information about micro scales.

What is a Mesoscopic System?

- Statistical description
- Can be effected by a **microscopic** system and the effect can be **macroscopically** detected

Meso can serve as a microscope to study micro

Brownian particle was the first mesoscopic device in use

**Brownian particle was the first
mesoscopic device in use**

**First paper on Quantum Theory of
Solid State (Specific heat)**

Annalen der Physik, 22, 180, 800 (1907)

First paper on Mesoscopic Physics

Annalen der Physik, 17, 549 (1905)

Finite size quantum physical systems

Atoms

Nuclei

Molecules

.

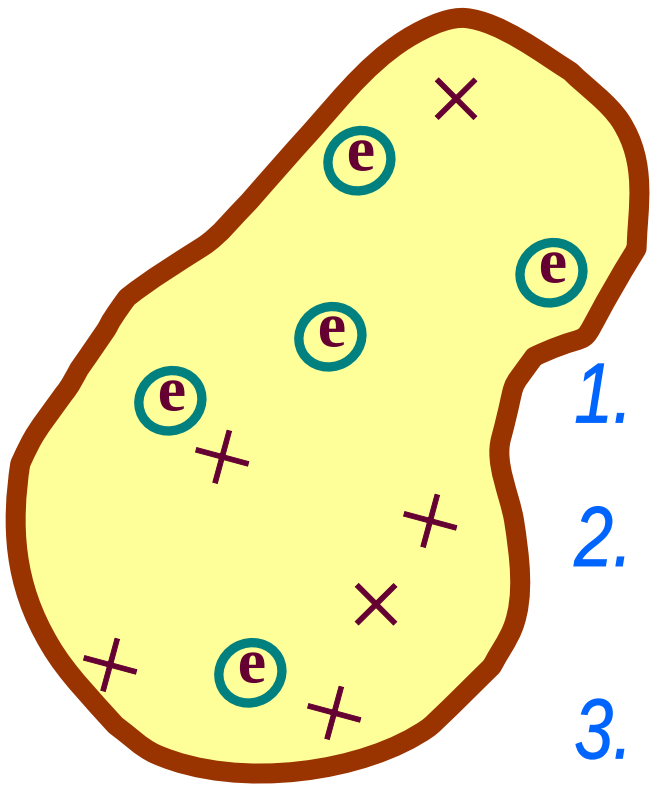
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Quantum
Dots

Quantum Dot

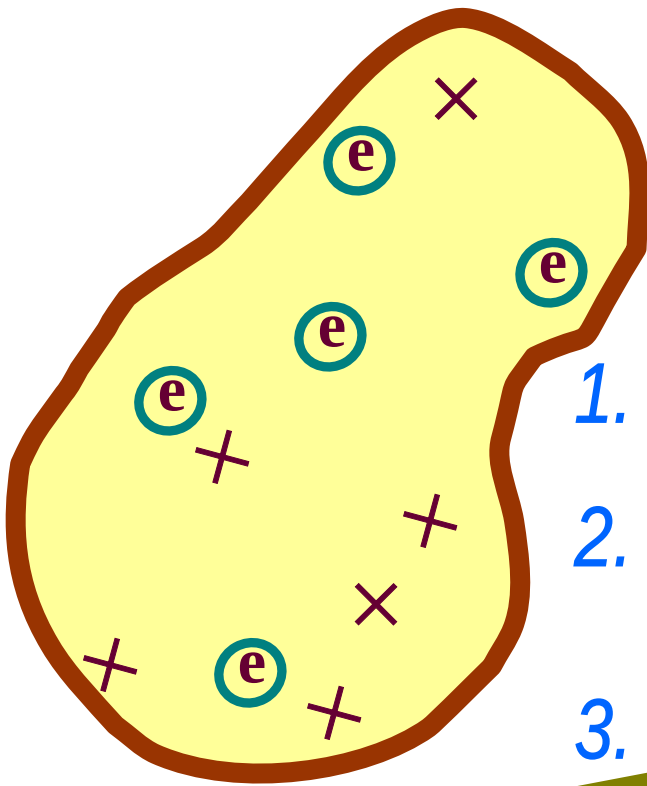


1. Disorder ($\times$ – impurities)

2. Complex geometry

3. e - e interactions

Quantum Dot



1. Disorder ($\times$ – impurities)

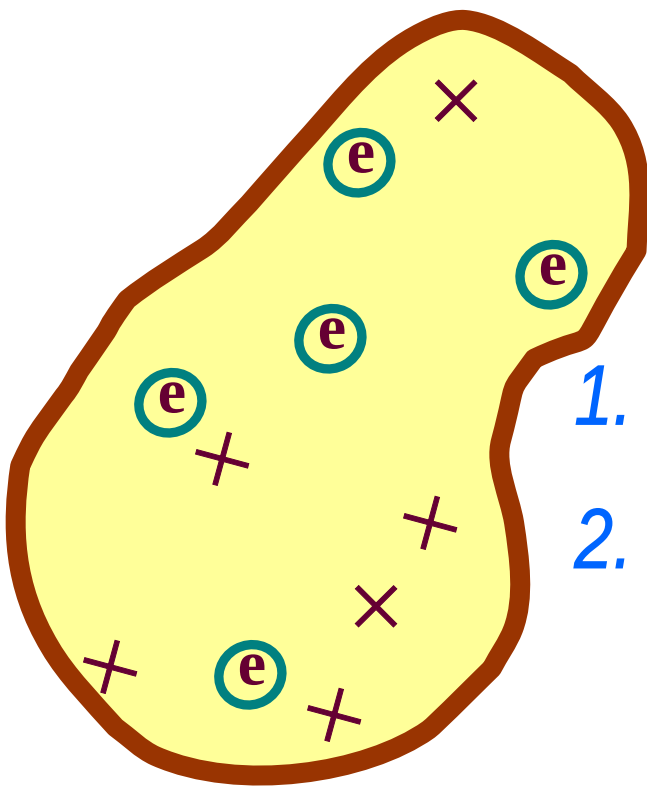
2. Complex geometry

3. ~~e - e interactions~~

for a while

Realizations:

- Metallic clusters
- Gate determined confinement in 2D gases (e.g. **GaAs/AlGaAs**)
- Carbon nanotubes
-
-



1. Disorder ($\times$ – impurities)

2. Complex geometry

How to deal with disorder?

~~• Solve the Shrodinger equation exactly~~

• Start with plane waves, introduce the mean free path, and . . .

How to take quantum interference into account



Electrons in nanostructures

Clean systems without boundaries:

- Electrons are characterized by their momenta or quasimomenta
⇒ electronic wave functions are plane waves
- Physics is essentially local
- Interaction between electrons is often apparently not important

In mesoscopic systems:

- Due to the scattering of the electrons off disorder (impurities) and/or boundaries the momentum is not a good quantum number
- Response to external perturbation is usually nonlocal
- Interaction between electrons is often crucial

Lesson 1:

Beyond Markov chains:

Anderson Localization
and

Magnetoresistance

ОБ ИЗМЕНЕНИИ ЭЛЕКТРИЧЕСКОГО СОПРОТИВЛЕНИЯ ТЕЛЛУРА В МАГНИТНОМ ПОЛЕ ПРИ НИЗКИХ ТЕМПЕРАТУРАХ

Р. А. Ченцов

R.A. Chentsov “*On the variation of electrical conductivity of tellurium in magnetic field at low temperatures*”, Zh. Exp. Theor. Fiz. v.18, 375-385, (1948).

Таблица 2

Уменьшение сопротивления теллура
в магнитном поле

Образец	Температура (°K)	Максимальное уменьшение сопро- тивления
Te-1	2,13	$0,7 \cdot 10^{-3}$
Te-2	2,15	$1,0 \cdot 10^{-3}$
Te-4	1,96	$1,1 \cdot 10^{-3}$
Te-5	1,96	$0,5 \cdot 10^{-3}$

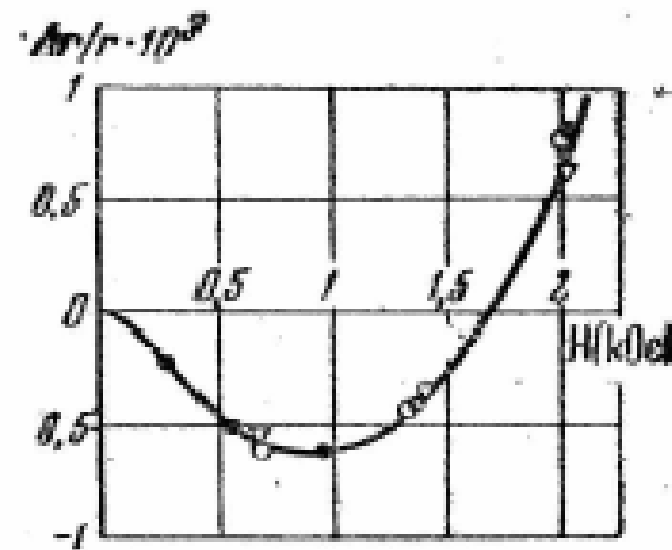


Рис. 2

Quantum particle in random quenched potential

Quantum particle in random quenched potential

PHYSICAL REVIEW

VOLUME 109, NUMBER 5

MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON

Bell Telephone Laboratories, Murray Hill, New Jersey

(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.



Quantum particle in random quenched potential

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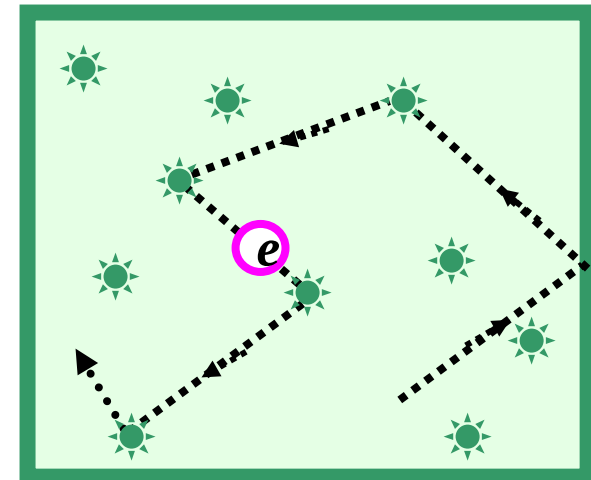
Absence of Diffusion in Certain Random Lattices

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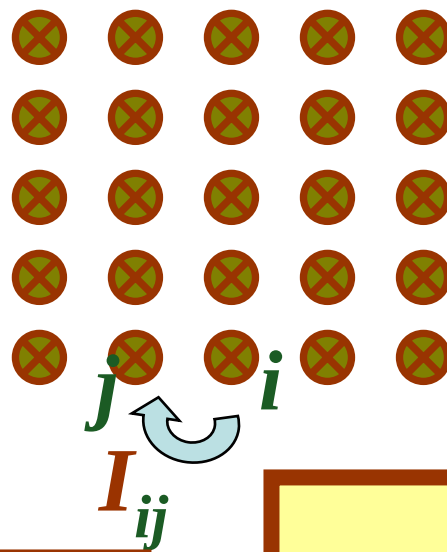
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Anderson Model



- Lattice - tight binding model
- Onsite energies ε_i - **random**
- Hopping matrix elements I_{ij}

$$-W < \varepsilon_i < W$$

uniformly distributed

$$I_{ij} = \begin{cases} I & \mathbf{i} \text{ and } \mathbf{j} \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

Anderson Transition

$$I < I_c$$

Insulator

All eigenstates are **localized**
Localization length ξ

$$I > I_c$$

Metal

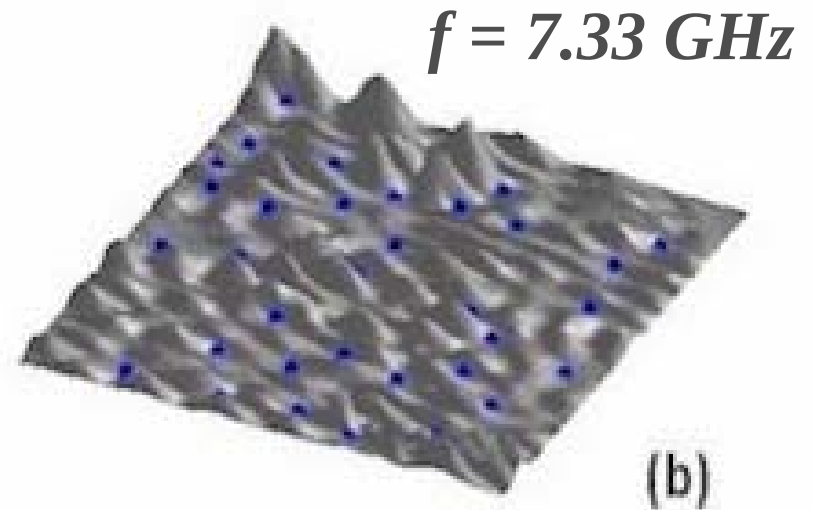
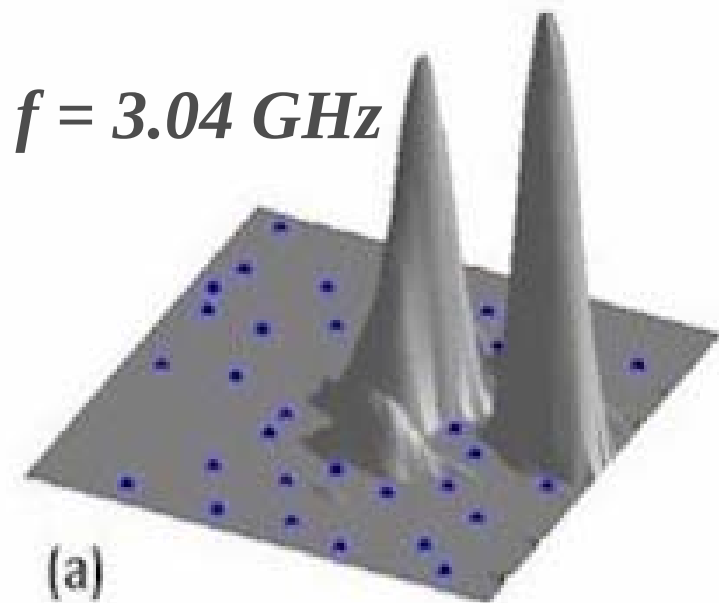
There appear states **extended**
all over the whole system

Correlations due to Localization in Quantum Eigenfunctions of Disordered Microwave Cavities

Prabhakar Pradhan and S. Sridhar

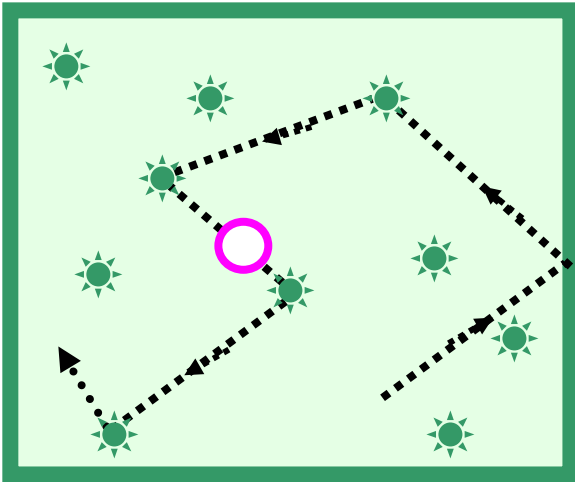
Department of Physics, Northeastern University, Boston, Massachusetts 02115

(Received 28 February 2000)



Anderson Insulator

Anderson Metal



1 particle - random walk

Density of the particles ρ

Density fluctuations $\rho(\mathbf{r}, t)$ at a given point in space $\mathbf{r}$ and time t .

$$\frac{\partial \rho}{\partial t} - D \nabla^2 \rho = 0$$

Diffusion
Equation

D - Diffusion constant

$$D = \frac{l^2}{d\tau}$$

l mean free path

τ mean free time

d # of dimensions

Einstein - Sutherland Relation for electric conductivity σ

$$\sigma = e^2 D \nu \quad \nu \equiv \frac{dn}{d\mu}$$

Conductance

$$G = \sigma L^{d-2}$$

for a cubic sample
of the size L

$$G = \frac{e^2}{h} \underbrace{\left(\nu L^d \right)}_{g(L)} \frac{Dh}{L^2}$$

$$g(L) = \frac{hD/L^2}{1/\nu L^d}$$

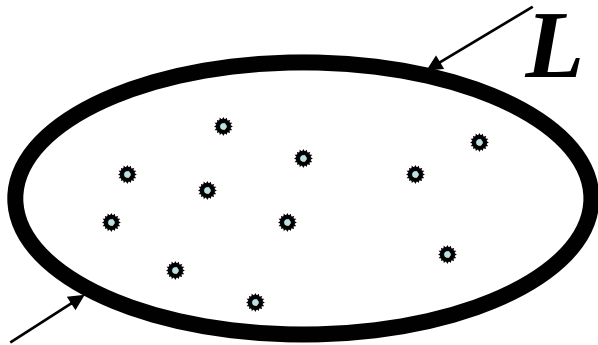
$$= \frac{\text{Thouless energy}}{\text{mean level spacing}}$$

**Dimensionless
Thouless
conductance**

Energy scales (Thouless, 1972)

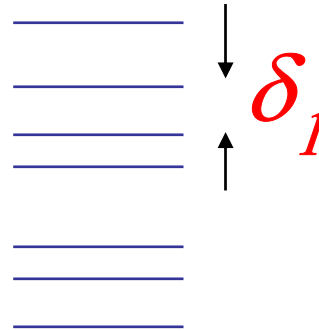


1. Mean level spacing



$$\delta_1 = 1/\nu \times L^d$$

energy



L is the system size;

d is the number of dimensions

2. Thouless energy

$$E_T = \hbar D / L^2$$

D is the diffusion constant

E_T has a meaning of the *inverse diffusion time* of the traveling through the system or the *escape rate* (for open systems)

$$g = E_T / \delta_1$$

dimensionless
Thouless
conductance

$$g = G \hbar / e^2$$

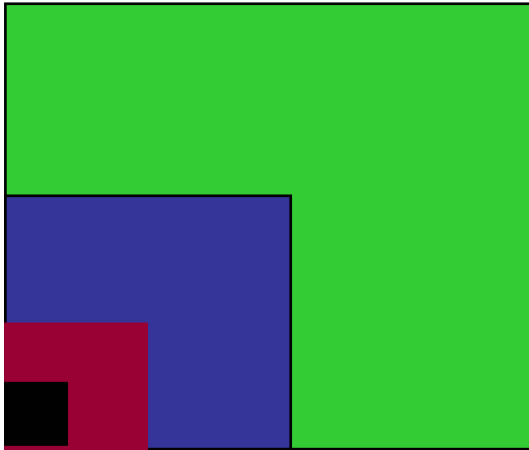
Scaling theory of Localization

(Abrahams, Anderson, Licciardello and Ramakrishnan
1979)

$$g = E_T / \delta_1$$

Dimensionless *Thouless*
conductance

$$g = Gh/e^2$$



$$L = 2L = 4L = 8L \dots$$

without quantum corrections

$$E_T \propto L^{-2} \quad \delta_1 \propto L^{-d}$$

$$\begin{array}{cccc} E_T & E_T & E_T & E_T \\ \rightarrow & \rightarrow & \rightarrow & \\ \delta_1 & \delta_1 & \delta_1 & \delta_1 \end{array}$$

$$g \rightarrow g \rightarrow g \rightarrow g$$

$$\frac{d(\log g)}{d(\log L)} = \beta(g)$$

$$\frac{d(\log g)}{d(\log L)} = \beta(g)$$

Universal, i.e., material independent

But

β – function is

It depends on the global symmetries, e.g., it is different with and without ***T*-invariance** (in orthogonal and unitary ensembles)

Limits:

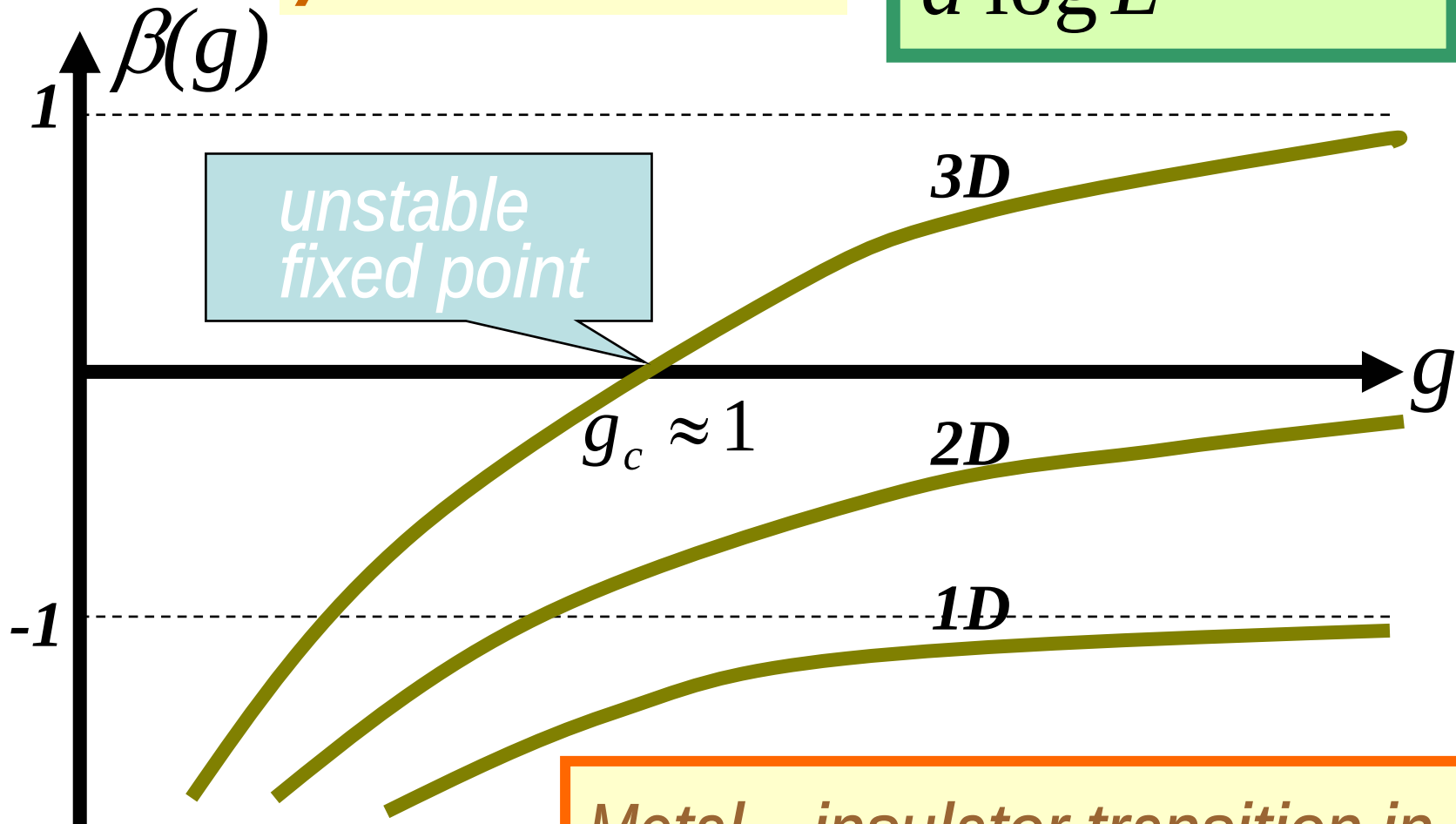
$$g \gg 1 \quad g \propto L^{d-2} \quad \beta(g) = (d-2) + O\left(\frac{1}{g}\right)$$

$$g \ll 1 \quad g \propto e^{-L/\xi} \quad \beta(g) \approx \log g < 0$$

$$\begin{array}{ll} > 0 & d > 2 \\ ?? & d = 2 \\ < 0 & d < 2 \end{array}$$

β - function

$$\frac{d \log g}{d \log L} = \beta(g)$$



Metal – insulator transition in **3D**
All states are localized for **$d=1,2$**

Questions:

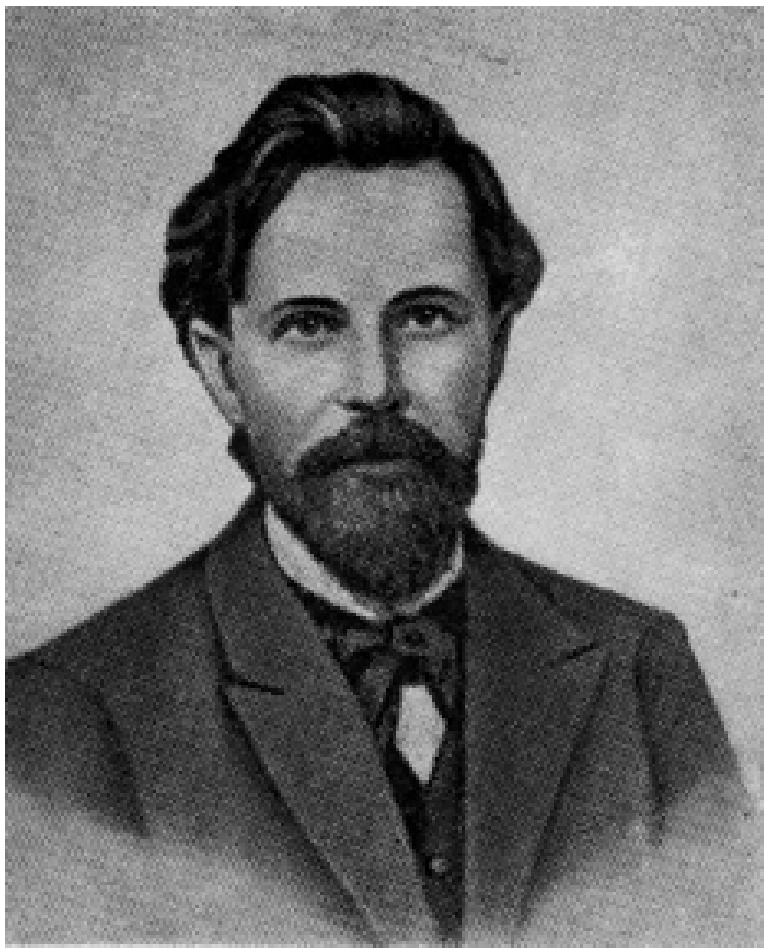
Why

- the scaling theory is correct?
- the corrections of the diffusion constant and conductance are negative?

Why diffusion description fails at **large** scales ?

Diffusion description fails at **large** scales
Why?

Einstein: there is no diffusion at too **short** scales - there is memory, i.e., the process is **not markovian**.



Andrei Markov
1856-1922

- A. A. Markov. « Rasprostranenie zakona bol'shih chisel na velichiny, zavisyaschie drug ot druga ». *Izvestiya Fiziko-matematicheskogo obschestva pri Kazanskom universitete*, 2-ya seriya, tom 15, pp 135-156, **1906**.
- A. A. Markov. « Extension of the limit theorems of probability theory to a sum of variables connected in a chain ». reprinted in Appendix B of: R. Howard. *Dynamic Probabilistic Systems, volume 1: Markov Chains*. John Wiley and Sons, 1971.

Diffusion description fails at **large** scales
Why?

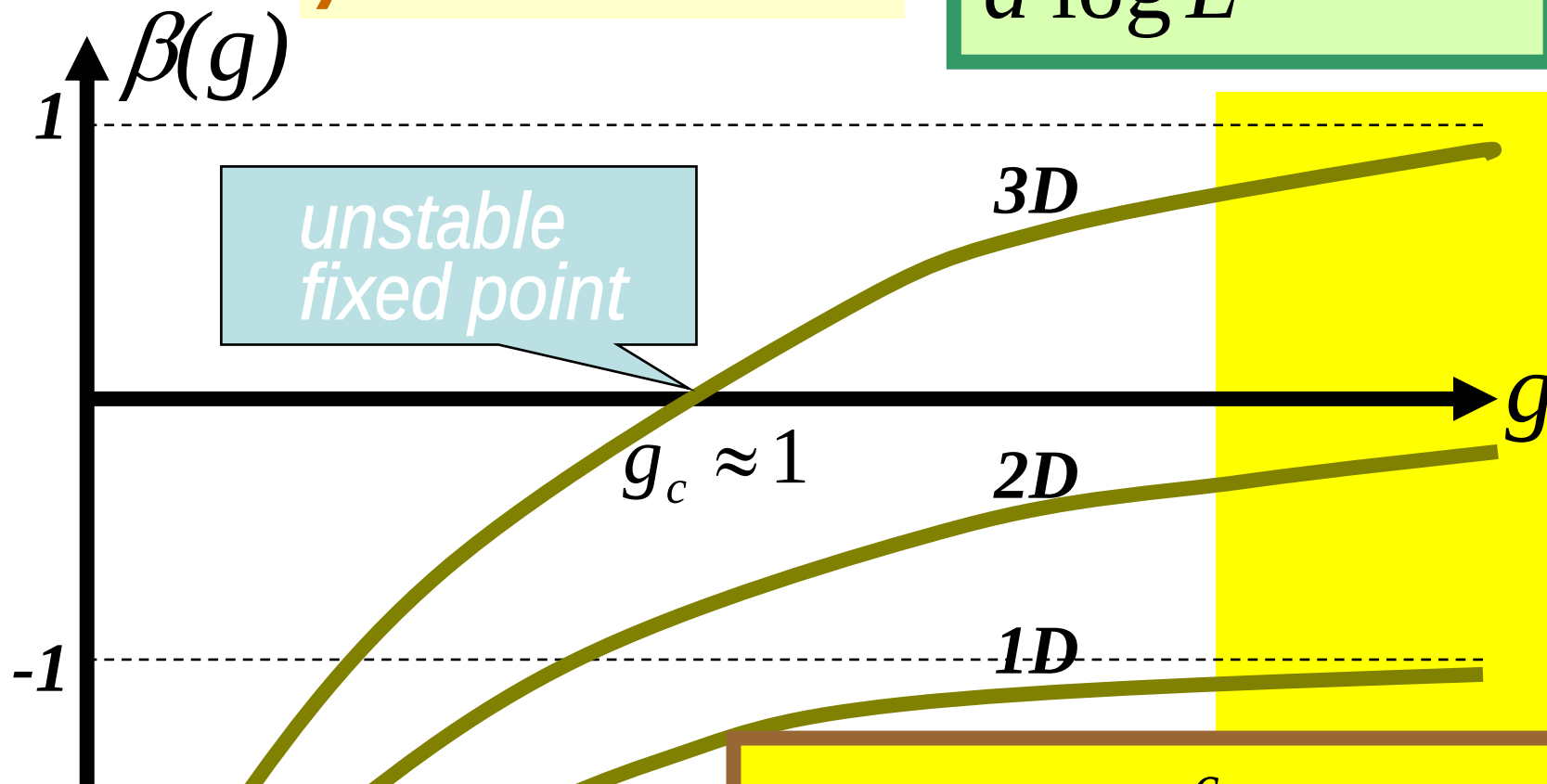
Einstein: there is no diffusion at too **short** scales - there is memory, i.e., the process is **not marcovian**.

Why there is memory at large distances in quantum case ?

Quantum corrections at large Thouless conductance - **weak localization**
Universal description

β - function

$$\frac{d \log g}{d \log L} = \beta(g)$$



$$\beta(g) = d - 2 + \frac{c_d}{g}$$

$$c_d = ? \quad \pm ?$$

$$g(L) = \sigma_{cl} L^{d-2} - \frac{c_d}{d-2} \quad d \neq 2$$

$$c_2 \log\left(\frac{L}{l}\right) \quad d = 2$$