# BSS2026: Problems for lectures on computational methods for elastic sheets

1. **Connections between the mesh-spring model and continuum elasticity.** Consider the regular hexagonal mesh introduced in the lectures, in the $xy$-plane, with a spacing of $s$ between nodes. For a continuum sheet, the strain energy density[1] is

$$u_s = \tfrac{1}{2}\boldsymbol{\epsilon} \cdot \mathbf{C} \cdot \boldsymbol{\epsilon} \tag{1}$$

where $\mathbf{C}$ is the two-dimensional (2D) stiffness tensor, and $\boldsymbol{\epsilon}$ is the two-dimensional strain tensor. Recall from the lectures that for two neighboring nodes at $\mathbf{x}_i$ and $\mathbf{x}_j$, the energy stored in the spring between them is

$$E_s(\mathbf{x}_i, \mathbf{x}_j) = \frac{k_s}{2}(\|\mathbf{x}_i - \mathbf{x}_j\| - s)^2, \tag{2}$$

where $k_s$ is a spring constant.

   (a) Assume that the node positions as deformed according to an infinitesimal strain $\boldsymbol{\epsilon}$. Compute the strain energy density $u_s$ to leading order, associated with the hexagonal lattice.

   (b) Using your expression from (a), compute the components of the 2D elasticity tensor. Use your results to evaluate the 2D Young's modulus and Poisson ratio.

   (c) *Extra credit.* Repeat your computations for the Warren girder mesh introduced in the lectures.

2. **Accuracy of computations of 2D packing fraction.**

   (a) In $\mathbb{R}^2$, consider a circular particle of diameter 1 centered at $(c, d)$. Calculate its area in the region $\{(x, y) : x < 0 \ \& \ y < 0\} \subseteq \mathbb{R}^2$.

   (b) Using your result from part (a), write a program that evaluates the area of a particle centered at $(c, d)$ within the box $|x| < L/2$, $|y| < L/2$.

   (c) Consider a randomly positioned and oriented hexagonal lattice. Determine a procedure for creating such a lattice.

   (d) Consider box sizes of $L$ between 2 and 10 with increments of 0.1. For each value of $L$, calculate $N$ different random lattices where $N \geq 100$ is a number that you choose. Evaluate the packing fraction $\phi$ of the particles based on the two methods:

      i. The full area of the particles whose centers are within the box.

      ii. The partial areas of all particles within the box, using your code from part (b)

   For methods i and ii, calculate the root mean squared error between the computed packing fraction $\phi$ and its exact asymptotic value of $\pi/2\sqrt{3}$. Determine how the standard deviation scales as a function of $L$.

   (e) *Extra credit.* For $L = 3$, compute the minimum and maximum packing fractions that are possible using the two different methods. Plot the arrangements of the particles for these cases.

[1]This is the elastic stored energy per unit area.

3. **Adaptive resolution mesh using a weighted Lloyd's algorithm.** Consider a set of $n$ particles located at $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ within the unit square $[0,1]^2$. Let $V_i$ be the associated Voronoi cell of particle $i$. The centroid of this Voronoi cell is given by

$$\mathbf{c}_i = \frac{\int_{V_i} \mathbf{x} \, d^2\mathbf{x}}{\int_{V_i} d^2\mathbf{x}}. \tag{3}$$

As discussed in the lectures, Lloyd's algorithm works by iteratively moving the particles to the centroids of their Voronoi cells. After many iterations, the particles even out in position, and the associated Delaunay mesh has favorable properties.

(a) The program directory `lloyd` contains a Python program that interfaces with VORO++ to perform Lloyd's algorithm with 100 iterations. Compile and run the program, and plot the output.

(b) Consider a triangular domain $T$ with vertices at $\mathbf{t}_1$, $\mathbf{t}_2$, and $\mathbf{t}_3$. Define the midpoints

$$\mathbf{s}_1 = \tfrac{\mathbf{t}_2+\mathbf{t}_3}{2}, \qquad \mathbf{s}_2 = \tfrac{\mathbf{t}_3+\mathbf{t}_1}{2}, \qquad \mathbf{s}_3 = \tfrac{\mathbf{t}_1+\mathbf{t}_2}{2}. \tag{4}$$

Demonstrate that the relation

$$\int_T f(\mathbf{x}) d^2\mathbf{x} = \frac{f(\mathbf{s}_1) + f(\mathbf{s}_2) + f(\mathbf{s}_3)}{3} \tag{5}$$

holds exactly for the functions $f(\mathbf{x}) = 1, x, y, x^2, xy, y^2$. *[Hint: first consider the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$, and then show that any $T$ can be mapped onto this triangle.]*

(c) Consider computing a weighted centroid

$$\mathbf{C}_i = \frac{\int_{V_i} \rho(\mathbf{x}) \mathbf{x} \, d^2\mathbf{x}}{\int_{V_i} \rho(\mathbf{x}) d^2\mathbf{x}} \tag{6}$$

where $\rho(\mathbf{x}) = a + bx + cy$ for constants $a, b, c \in \mathbb{R}$. Adapt the programs from part (a) to implement Lloyd's algorithm but moving particles to the weighted centroid instead. To evaluate the integrals in Eq. (6), use the relation given in Eq. (5). Plot the resulting mesh points after 100 iterations.

(d) *Extra credit.* Instead of performing a fixed number of 100 iterations, modify the code to have a stopping criterion based on when the particles stop moving.

(e) *Extra credit.* Modify the code to work with the Gaussian density,

$$\rho(\mathbf{x}) = \exp\left(-20[(x - 1/2)^2 + (y - 1/2)^2]\right). \tag{7}$$

In this case, it may no longer be possible to evaluate the integrals in Eq. (6) exactly, but you should implement a quadrature rule that computes the integrals sufficiently accurately.