Homework Problems for James' Lectures

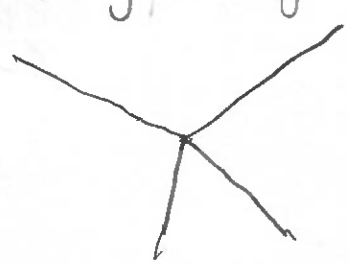
1) Before the lectures begin, answer the questions in bold in "Notes and Examples from Tensor Analysis for Continuum Mechanics".

2) (Lecture 1, p. 9). Let $\underset{\sim}{F}$ have polar decomposition $\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U}$, $\underset{\sim}{R} \in SO(3)$, $\underset{\sim}{U} = \underset{\sim}{U}^T > 0$. Show that the closest tensor to $\underset{\sim}{F}$ in $SO(3)$ is $\underset{\sim}{R}$ (Euclidean norm)

3) (Lecture 2, p. 14). Find a piece of paper and a protractor. Crumple up the piece of paper (not too severely) and then push it down onto the table so it is as flat as possible. Then, unfold it and look at the crease pattern. Let $\Omega \subset \mathbb{R}^2$ be the initial, unfolded piece of paper, and let $\underset{\sim}{y} : \Omega \to \mathbb{R}^2$ be the deformation to the flat folded state. Explain why it is reasonable to describe $\nabla \underset{\sim}{y}(\underset{\sim}{x})$ as lying on the two circles pictured at the top of p. 15, i.e.,

$$\nabla \underset{\sim}{y}(\underset{\sim}{x}) \in O(2) = SO(2) \cup SO(2)\underset{\sim}{P}$$

You should also see several places on the paper where four (nearly straight) creases come together



Notice that in each of the sectors the paper appears undisturbed. This is consistent with Liouville's theorem (Ω open, connected, y smooth):

$$\nabla y(x) = R(x) \in SO(n) \text{ on } \Omega \qquad n \geq 1$$

$$\Rightarrow y(x) = R_0 x + c, \quad R_0 \in SO(n), \quad c \in \mathbb{R}^n$$

(you can work out the proof: some hints below)

$$R^T_{ij} R_{jk} = \delta_{ik} \text{ in RCC}$$

$$\rightarrow \underset{\text{switch}}{R_{ji,l}} R_{jk} + R_{ji} \underset{\text{switch}}{R_{jk,l}} = 0$$

$(R_{ji,l} = y_{j,il})$

$$\rightarrow (R_{jl} R_{jk})_{,i} - R_{jl} \underset{\text{switch}}{R_{jk,i}} + (R_{ji} R_{jl})_{,k} - R_{ji,k} R_{jl} = 0$$

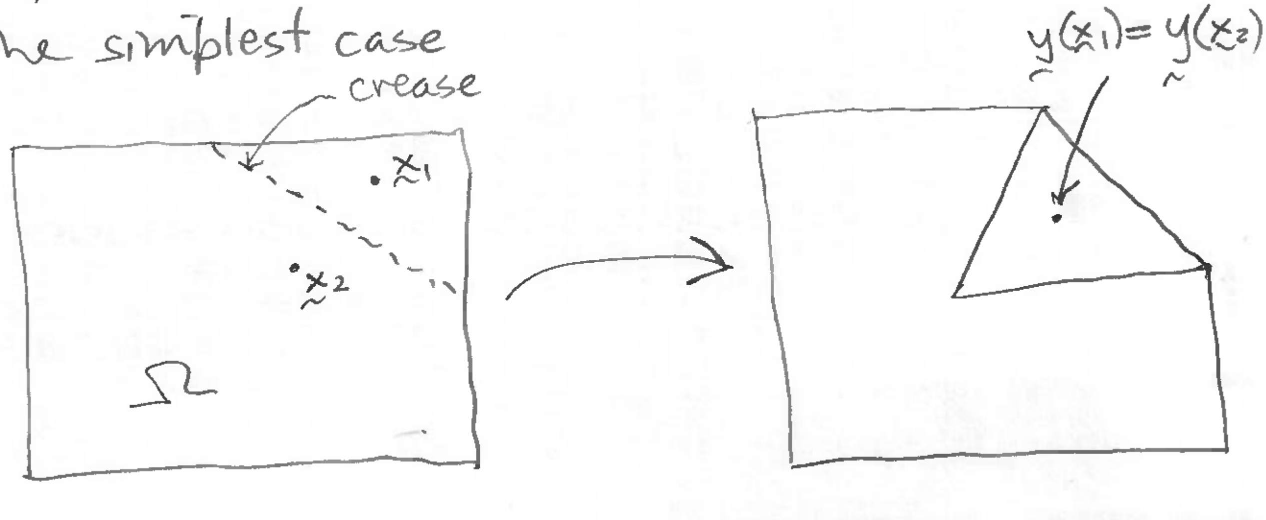
$$R_{jl} R_{ji,k} = 0 \Rightarrow R_{ji,k} = 0 \Rightarrow R_{ji} = \text{const.}$$

Obviously, it is also true (same smoothness, connected Ω) that

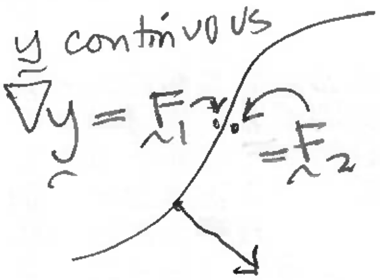
$$\nabla y(x) \in SO(2) P \qquad P = e_1 \otimes e_1 - e_2 \otimes e_2$$

$$\Rightarrow \nabla y = \text{const.}$$

See p. 15 of the class notes. We can picture in the simplest case



So, unlike classical elasticity $\underset{\sim}{y}(\underset{\sim}{x})$ is not invertible. The compatibility condition for two gradients meeting at a smooth curve is:

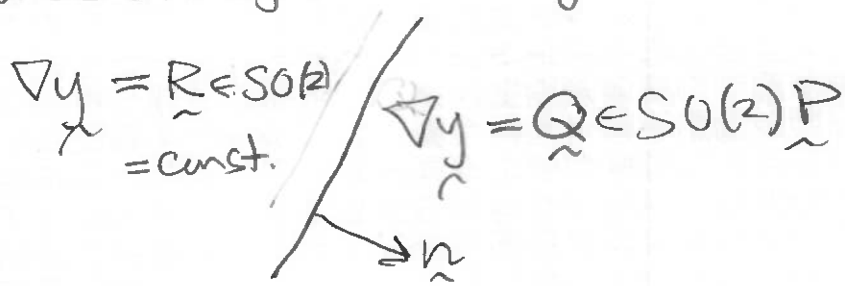


$\Longleftrightarrow$ $\underbrace{\underset{\sim}{F}_2 - \underset{\sim}{F}_1}_{\text{limiting values at the interface}} = \underset{\sim}{a} \otimes \underset{\sim}{n}$

$\left(F_{2_{ij}} - F_{1_{ij}} = a_i n_j \right)$

You can prove this by writing a Taylor expansion for $\underset{\sim}{y}(\underset{\sim}{x})$ parallel to the interface on each side, and then imposing continuity of $\underset{\sim}{y}$.

Putting these facts together, we expect creases to be straight line segments,



where

$$\underset{\sim}{R} - \underset{\sim}{Q} = \underset{\sim}{a} \otimes \underset{\sim}{n} \quad \text{for some } \underset{\sim}{a} \in \mathbb{R}^2,$$

(Note that this implies the <u>direction</u> of $\underset{\sim}{n}$ is constant, so creases have to be straight lines)

So, the question becomes, given $\underset{\sim}{R} \in SO(2)$, what are all $\underset{\sim}{Q} \in SO(2)\underset{\sim}{P}$ such that

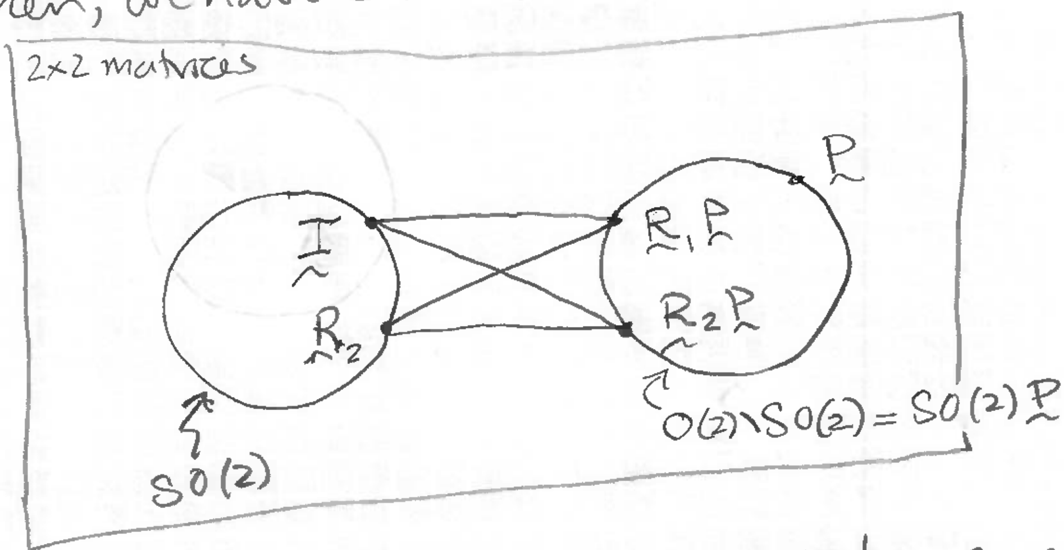
$$\underset{\sim}{R} - \underset{\sim}{Q} = \underset{\sim}{a} \otimes \underset{\sim}{n} \quad *$$

→ Prove that given $\underset{\sim}{R} \in SO(2)$, <u>every</u> $\underset{\sim}{Q} \in SO(2)P$ satisfies * for some choice of $\underset{\sim}{a}$ and $\underset{\sim}{n}$ (!)

We can represent this diagramatically

means that $\mathbf{B}-\mathbf{A}$ is a tensor of rank 1 (i.e., $=\mathbf{a}\otimes\mathbf{n}$)

Then, we have shown



we can start at $\mathbf{I}$ without loss of generality

We have shown that there exists $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ and $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4$ such that all the conditions of compatibility hold:

$$\left.\begin{aligned}
\mathbf{R}_1\mathbf{P}-\mathbf{I} &= \mathbf{a}_1\otimes\mathbf{n}_1\\
\mathbf{R}_2-\mathbf{R}_1\mathbf{P} &= \mathbf{a}_2\otimes\mathbf{n}_2\\
\mathbf{R}_3\mathbf{P}-\mathbf{R}_2 &= \mathbf{a}_3\otimes\mathbf{n}_3\\
\mathbf{I}-\mathbf{R}_3\mathbf{P} &= \mathbf{a}_4\otimes\mathbf{n}_4
\end{aligned}\right\} *$$

Are we done? It seems we are done.... NO!

$*$ might imply some geometric restrictions, i.e., some restrictions on $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4$.

In fact, that's true.

To bring out these restrictions, we list some facts about $SO(2)$:

i) $\underset{\sim}{R}_1, \underset{\sim}{R}_2 \in SO(2) \Rightarrow \underset{\sim}{R}_1\underset{\sim}{R}_2 = \underset{\sim}{R}_2\underset{\sim}{R}_1$ (not true in $SO(3)$)

ii) $\underset{\sim}{R} \in SO(2)$, $\underset{\sim}{P}$ as above,

$$\underset{\sim}{R}\,\underset{\sim}{P} = \underset{\sim}{P}\,\underset{\sim}{R}^T$$

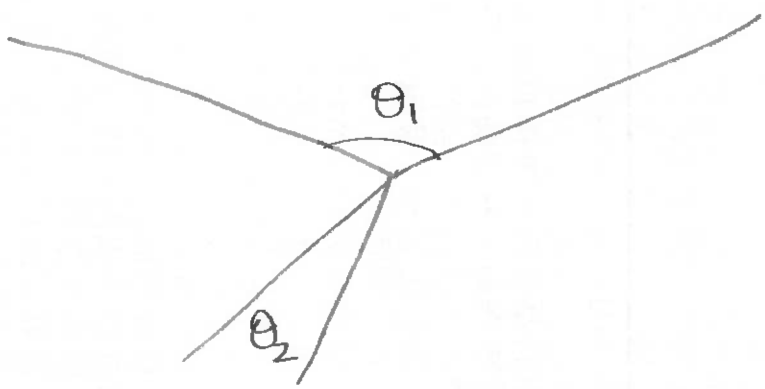
$\underset{\sim}{P}$ as above

iii) The normal uniquely determines the rotation; that is, $\underset{\sim}{R}_1, \underset{\sim}{R}_2 \in SO(2)$

$$\underset{\sim}{R}_1\underset{\sim}{P} - \underset{\sim}{I} = \underset{\sim}{a} \otimes \underset{\sim}{n} \text{ and } \underset{\sim}{R}_2\underset{\sim}{P} - \underset{\sim}{I} = \underset{\sim}{b} \otimes \underset{\sim}{n}$$

$$\Rightarrow \underset{\sim}{R}_1 = \underset{\sim}{R}_2$$

Using these facts, and labeling the four-fold intersection as follows,



show that $\theta_1 + \theta_2 = \pi$ (Kawasaki's condition)

(one can give a "recreational math" proof of this fact, but the above reasoning generalizes to a broad class of physical problems: phase transformations, magnetoelasticity, the folding of insect wings, pleated clothing, the construction of Frank Gehry buildings, ...)

F.

4) Lecture 3 (p. 32ff)

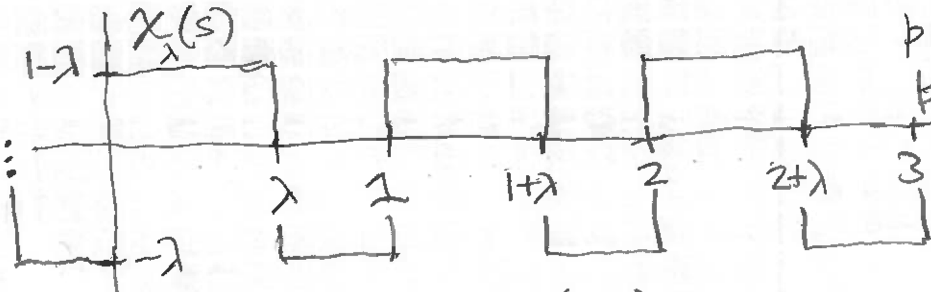
We have seen (problem 3) that if $y : \Omega \to \mathbb{R}^3$, $\Omega \subset \mathbb{R}^3$, is continuous and piecewise differentiable, then across interfaces where ∇y has a jump discontinuity,

$$\nabla y^+ - \nabla y^- = a \otimes n$$

Let A, B be tensors on $\mathbb{R}^3$ with $\det A > 0$, $\det B > 0$ (i.e., deformation gradients) and suppose

$$B - A = a \otimes n$$

Let $0 < \lambda < 1$ and define $\chi_\lambda : \mathbb{R} \to \mathbb{R}$ by



periodic of period 1

Define a deformation $y^{(k,\lambda)}$ by

$$y^{(k,\lambda)}(x) = (\lambda B + (1-\lambda) A)x + \frac{1}{k}\left(\int_0^{k(x\cdot n)} \chi_\lambda(s)\,ds\right) a$$

This is called the laminate formula. Show that

$$y^{(k,\lambda)}(x) \to (\lambda B + (1-\lambda)A)x \text{ as } k \to \infty.$$ uniformly

and calculate $\nabla y^{(k,\lambda)}$: show that $\nabla y^{(k,\lambda)}$ represents a "laminate" of alternating deformation gradients with "fineness" k and alternating volume fraction

$\ldots \lambda, 1-\lambda, \lambda, 1-\lambda \ldots$

Now consider a deformation $\bar{\underset{\sim}{y}} : \Omega \to \mathbb{R}^3$. and a free energy φ and suppose that $\bar{\underset{\sim}{y}}$ minimizes the total free energy

$$\int_\Omega \varphi(\nabla \underset{\sim}{y}(\underset{\sim}{x}))\, d\underset{\sim}{x} + \mathcal{L}[\underset{\sim}{y}] \qquad *$$

(p. 7 of the lectures). $\mathcal{L}$ is the energy of the loading device and we assume that it only depends on the boundary values of $\underset{\sim}{y}$.

(In fact, we could also assume $\bar{\underset{\sim}{y}}$ minimizes the energy with respect to $\underset{\sim}{y} : \Omega \to \mathbb{R}^3$ such that $|\underset{\sim}{y} - \bar{\underset{\sim}{y}}| < \delta$ for some $\delta > 0$.) Choose $\underset{\sim}{x}_0 \in \Omega$ where

$$\underset{\sim}{F}_0 = \nabla \bar{\underset{\sim}{y}}(x_0)$$

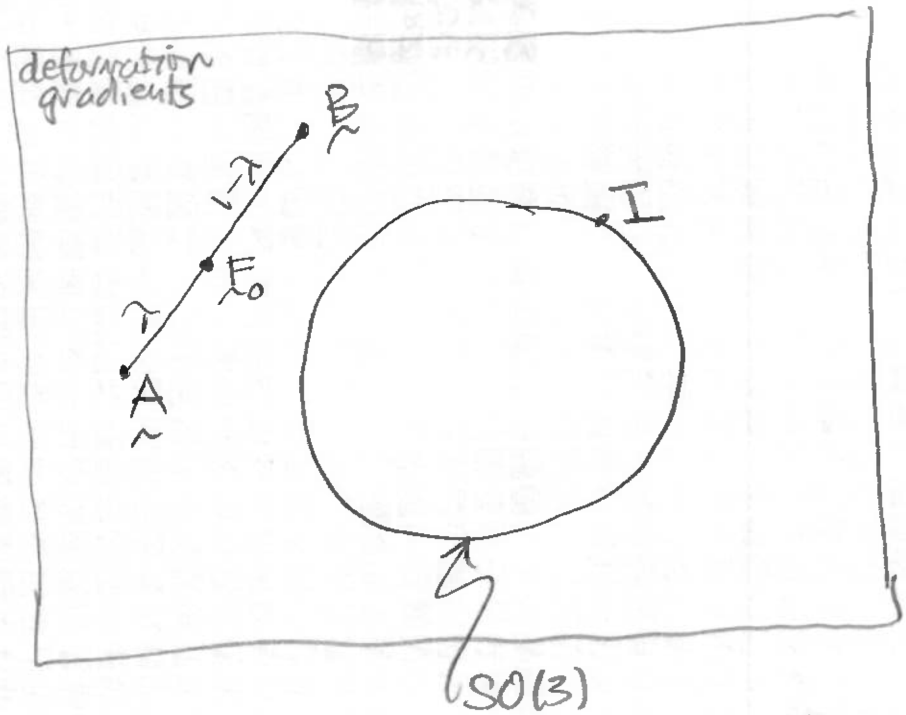
(for students with limited experience in analysis you can assume $\nabla y(\underset{\sim}{x}) = \underset{\sim}{F}_0$ also in a small neighborhood of $\underset{\sim}{x}_0$). Consider a small ball $B_r(\underset{\sim}{x}_0)$ centered at $\underset{\sim}{x}_0$ with radius r.

As a "test function" for the minimization of the total free energy $*$ above, we

Choose the laminate formula with

$$\lambda \underset{\sim}{B} + (1-\lambda)\underset{\sim}{A} = \underset{\sim}{F}_0 \quad 0 \le \lambda \le 1. \quad \underset{\sim}{B} - \underset{\sim}{A} = \underset{\sim}{a} \otimes \underset{\sim}{n}$$

Of course, there are lots of choices of $\lambda, \underset{\sim}{B}, \underset{\sim}{A}$ that satisfy this. We can add this to our diagram of the domain of the free energy

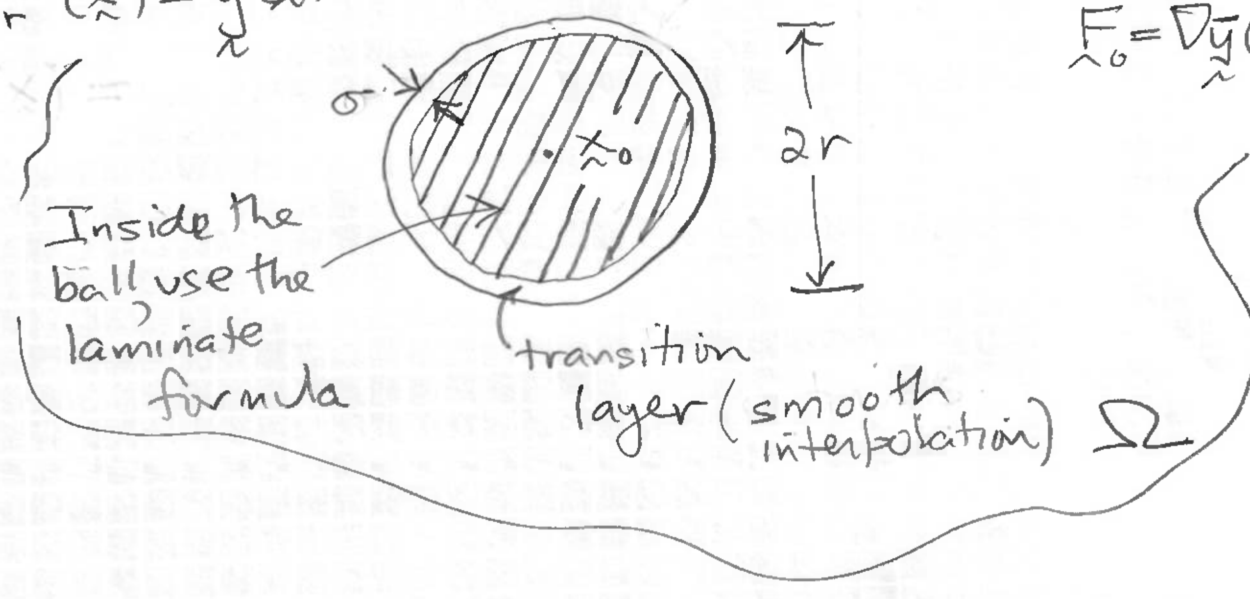


$$\underset{\sim}{B} - \underset{\sim}{A} = \underset{\sim}{a} \otimes \underset{\sim}{n}$$

Now we make a test function, $\underset{\sim}{y}_\sigma^{(\lambda,k)}(\underset{\sim}{x})$

$\underset{\sim}{y}_r^{(\lambda,k)}(\underset{\sim}{x}) = \bar{\underset{\sim}{y}}(\underset{\sim}{x})$ outside the ball

$$\underset{\sim}{F}_0 = \nabla \bar{\underset{\sim}{y}}(\underset{\sim}{x}_0)$$



Since $\bar{\underset{\sim}{y}}(\underset{\sim}{x})$ is a minimizer, and assuming we ignore the transition layer (choose $k = k(\sigma, r)$ appropriately, Then since $\underset{\sim}{y}_0$ is a minimizer

$$\int_{B_{r-\sigma}} \varphi(\nabla \underset{\sim}{y}_\sigma^{(\lambda,k)})\,dx \geq \int_{B_r} \varphi(\nabla \underset{\sim}{y})\,d\underset{\wedge}{x}$$

the assumed minimizer

Divide by volume of B_r and let $r \to 0$, $k(\sigma, r) \to \infty$, $\sigma \to 0$ to eliminate the contribution from the transition layer. You should get:

use Lebesgue's theorem

$$\boxed{\begin{gathered} \varphi(\underbrace{\lambda \underset{\sim}{B} + (1-\lambda)\underset{\wedge}{A}}_{= \underset{\sim}{F}_\infty}) \leq \lambda \varphi(\underset{\sim}{B}) + (1-\lambda)\varphi(\underset{\wedge}{A}) \qquad 0 \leq \lambda \leq 1. \\ \text{for all } \underset{\wedge}{A}, \underset{\sim}{B},\ 0 \leq \lambda \leq 1, \text{ such that} \\ \operatorname{rank}(\underset{\sim}{B} - \underset{\wedge}{A}) = 1 \qquad (\underset{\sim}{B} - \underset{\wedge}{A} = \underset{\wedge}{a} \otimes \underset{\wedge}{n}) \end{gathered}}$$

This is the condition of rank-1 convexity, one of the most important restrictions on the free energy of elasticity. Some remarks:

1) Based on the argument, one could call rank-1 convexity, "stability with respect to infinitesimal lamination".

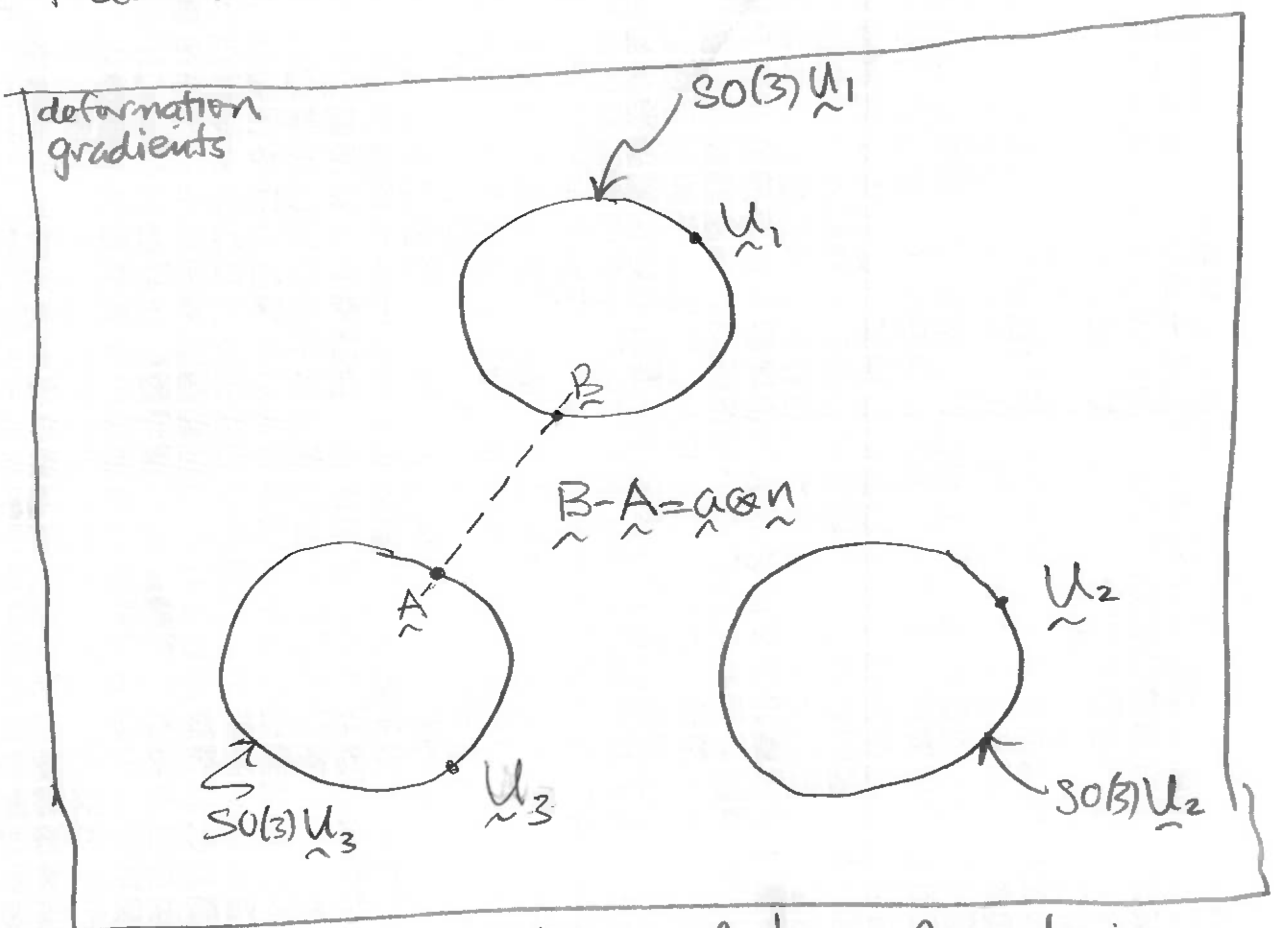
2) For rubber, metals, successful free energies satisfy rank-1 convexity.

3) Actually, the St. Venant/Kirchhoff model fails rank-1 convexity a certain large strains. It is nevertheless quite successful because φ grows very rapidly away from $SO(3)$

J.

and the deformations where rank-1 convexity fails are rarely encountered in simulations.

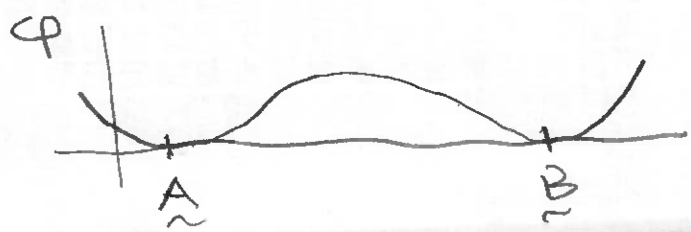
4) Models of materials that undergo phase transformations <u>do fail</u> the condition of rank-1 convexity, almost universally! In such models the generic domain of the free energy density has multiple "energy wells" (i.e., minimizers) and there are tensors on these wells that differ by a matrix of rank-1:



(a cubic-to-tetragonal transformation below transformation temperature)

Plot of the free energy over the dashed line (for appropriate choices of $\underset{\sim}{A}$, $\underset{\sim}{B}$):

clearly nonconvex!



(The laminate formula then gives many layered minimizers)

What should one expect? Lots of laminates, some shown below in a CuAlNi alloy (but that is only the beginning of the story!) K.

See the next 3 pages