Boulder Lectures 2026

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1 Tensors and basic notation

Typeface conventions. Tensors are multi-index spatial fields and (differential) operators that, when multiplied, have particular rules for their combination. We distinguish tensors of different rank by their typeface:

- *Scalars*: plain italic, e.g. ρ, p, t, λ.

- *Vectors* (rank-1 tensors): bold italic, e.g. $\boldsymbol{u}$, $\boldsymbol{x}$, $\boldsymbol{r}$, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$.
- *Rank-2 tensors*: bold Roman, e.g. $\mathbf{E}$, $\mathbf{W}$, $\mathbf{F}$, $\mathbf{I}$, $\mathbf{\Sigma}$.
- *Rank-3 tensors*: calligraphic, e.g. $\mathcal{A}$. The rank-3 Levi-Civita tensor $\boldsymbol{\epsilon}$ follows common convention.
- *Rank-4 tensors*: blackboard bold, e.g. $\mathbb{A}$.

Position and velocity vectors are expressed in components as

$$\boldsymbol{x} = (x, y, z) = (x_1, x_2, x_3), \qquad \boldsymbol{u} = (u, v, w) = (u_1, u_2, u_3).$$

Index notation and Einstein summation. A repeated index in a multiplicative expression implies summation over that index. Index notation is used to define tensorial structures explicitly, or when a calculation is clearest in component form.

Gradient. The gradient of a vector field $\boldsymbol{u}$ is the rank-2 tensor

$$[\nabla \boldsymbol{u}]_{ij} = \frac{\partial u_j}{\partial x_i},$$

i.e. the gradient index i comes first, the vector component index j second. More generally, for a rank-n tensor T, $[\nabla \mathsf{T}]_{i\,j_1 \cdots j_n} = \partial T_{j_1 \cdots j_n} / \partial x_i$.

Contraction and multiplication. We adopt a *left-multiply* convention throughout:

$$[\boldsymbol{r} \cdot \mathbf{A}]_j = r_i\, A_{ij},$$
$$[\mathbf{A} \cdot \mathbf{B}]_{ij} = A_{ik}\, B_{kj}.$$

The *double contraction* of two rank-2 tensors contracts the inner pair of indices first:

$$\mathbf{A} : \mathbf{B} = A_{ij}\, B_{ji} \quad \text{(scalar)}.$$

The Frobenius norm of a rank-two tensor (or matrix) is defined by

$$\|\mathbf{A}\|_F = \sqrt{A_{ij} A_{ij}} = \sqrt{\mathbf{A} : \mathbf{A}^\top}$$

For a rank-3 tensor $\boldsymbol{\mathcal{A}}$ doubly contracted with a rank-2 tensor $\mathbf{B}$ we have:

$$[\boldsymbol{\mathcal{A}} : \mathbf{B}]_i = \mathcal{A}_{ijk}\, B_{kj}.$$

Divergence. Consistent with the left-multiply convention:

$$[\nabla \cdot \mathbf{A}]_j = \frac{\partial A_{ij}}{\partial x_i}.$$

Material derivative.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla.$$

Levi-Civita tensor and vorticity. The Levi-Civita tensor $\boldsymbol{\epsilon}$ (rank-3) has components ϵ_{ijk}, totally antisymmetric with $\epsilon_{123} = +1$. The curl of a velocity vector field $\boldsymbol{u}$ is written $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ in symbolic form:

$$[\boldsymbol{\omega}]_i = [\nabla \times \boldsymbol{u}]_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j} = -[\boldsymbol{\epsilon} : \nabla \boldsymbol{u}]_i,$$

since $[\boldsymbol{\epsilon} : \nabla \boldsymbol{u}]_i = \epsilon_{ijk} [\nabla \boldsymbol{u}]_{kj} = \epsilon_{ijk} \partial u_j / \partial x_k = -\epsilon_{ijk} \partial u_k / \partial x_j$.

Eulerian and Lagrangian variables. The independent variables of the Eulerian frame will be Roman, e.g. $\boldsymbol{x}$ or $\boldsymbol{y}$, etc., and those of the Lagrangian frame Greek, $\boldsymbol{\alpha}$ (or $\boldsymbol{\beta}$, etc.. Gradient with respect to Lagrangian variables will always be subscripted, i.e. $\nabla_{\boldsymbol{\alpha}}$, while gradients with respect to Eulerian variables are not unless it is needed to remove ambiguity.

2 The Basics of Continuum and Fluid Mechanics

First, some useful reference texts:

- Incompressible Fluid Dynamics:
  George Batchelor – *An Introduction to Fluid Dynamics*
  D.J. Acheson – *Elementary Fluid Dynamics*
  C. Pozrikidis – *Boundary Integral and Singularity Methods for Linearized Viscous Flow*
- Complex Fluids and Solids
  R.G. Larson – *The Structure and Rheology of Complex Fluids*
  G.A. Holzapfel – *Nonlinear Solid Mechanics*
  Doi & Edwards – *The Theory of Polymer Dynamics*
- Bio Mechanics, Fluids, Locomotion
  S. Childress – *Mechanics of Swimming and Flying*
  S. Vogel – *Life in Moving Fluids*
  S. Vogel – *Comparative Biomechanics*
  R.M. Alexander – *Principles of Animal Locomotion*
- Active Fluids / Materials
  S. Spagnolie, *Complex Fluids in Biological Systems*

2.1 Local structure of flows

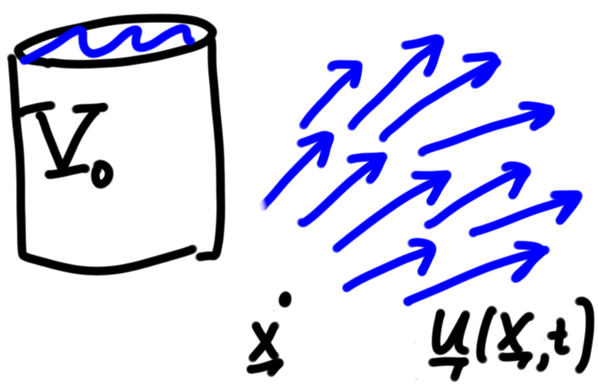

Figure 1: A fluid domain $\mathcal{V}_0 \subseteq \mathbb{R}^3$, with a zoom-in showing the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ at a point $\boldsymbol{x} \in \mathcal{V}$.

Consider a volume V filled with a fluid or continuous material. At each time t and at each point $\boldsymbol{x}$ the fluid has a velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ and density $\rho(\boldsymbol{x}, t)$. Here we describe the fluid flow as passing through a fixed (lab) coordinate frame, traditionally called the *Eulerian frame.*

Notation:

$$\boldsymbol{x} = (x, y, z) = (x_1, x_2, x_3)$$
$$\boldsymbol{u} = (u, v, w) = (u_1, u_2, u_3)$$

The basic constituents of the velocity field – translation, deformation, rotation

Consider a steady velocity field $\boldsymbol{u}(\boldsymbol{x})$, fixing a point $\boldsymbol{x}$ and considering a nearby point $\boldsymbol{x} + \boldsymbol{r}$. Then,

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) &= \boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{r} \cdot \nabla \boldsymbol{u}(\boldsymbol{x}) + O\left(|\boldsymbol{r}|^2\right) \\ &\approx \boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{r} \cdot \mathbf{E} + \boldsymbol{r} \cdot \mathbf{W}\end{aligned}$$

where $(\nabla \boldsymbol{u})_{ij} = \partial u_j / \partial x_i$ is called the *rate-of-strain tensor*, which we have re-expressed in terms of its symmetric and antisymmetric parts $E_{ij} = (\partial u_j/\partial x_i + \partial u_i/\partial x_j)/2$, the *symmetric rate-of-strain tensor*, and $W_{ij} = (\partial u_j/\partial x_i - \partial u_i/\partial x_j)/2$, the *rotation tensor*. The velocity field can be decomposed as

1. A *translation* $\boldsymbol{u}(\boldsymbol{x})$.

2. A *pure straining flow* $\boldsymbol{r} \cdot \mathbf{E}$: The tensor $\mathbf{E}$ is symmetric and thus has 3 real eigenvalues λ_i with 3 associated, mutually orthogonal eigenvectors $\boldsymbol{p}_i$. Here,

   the λ_i are called the principal rates-of-strain, and

   the $\boldsymbol{p}_i$ are called the principal axes (or directions) of strain

   Consider the trajectory $\dot{\boldsymbol{r}} = \mathbf{r} \cdot \mathbf{E}$. Writing $\boldsymbol{r} = \sum_k r_k \boldsymbol{p}_k$, we obtain $\dot{\boldsymbol{r}} = \sum_k \lambda_k r_k \boldsymbol{p}_k$, having the solution

   $$r(t) = r_{10} e^{\lambda_1 t} \boldsymbol{p}_1 + r_{20} e^{\lambda_2 t} \boldsymbol{p}_2 + r_{30} e^{\lambda_3 t} \boldsymbol{p}_3 \ .$$

   Taking $\boldsymbol{p}_i = \boldsymbol{e}_i$ (the Cartesian basis) and $\lambda_{1,2} < 0$, and $\lambda_3 > 0$, the pure straining flow deforms a ball centered at $\boldsymbol{r} = \mathbf{0}$ into an ellipsoid whose principal axes are the principal axes of strain through compression (λ_1, λ_1) and expansion (λ_3).

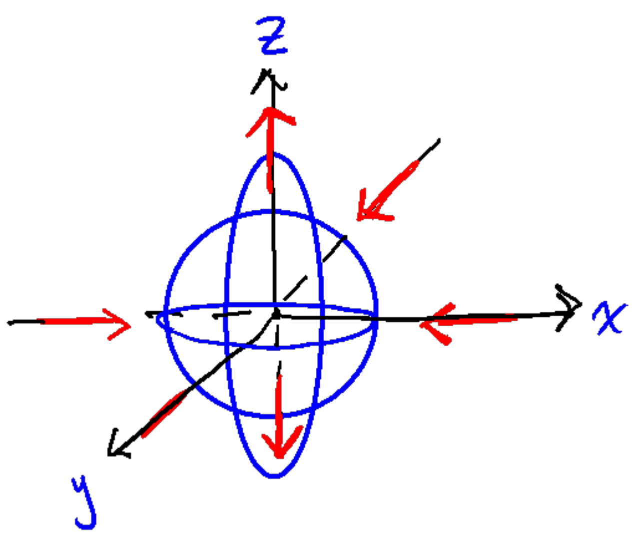

Figure 2: The deformation of an initially spherical fluid volume, by an straining flow, into an ellipsoid.

We can further decompose $\mathbf{E}$ as

$$\mathbf{E} = \frac{1}{3} tr\left(\mathbf{E}\right)\mathbf{I} + \left(\mathbf{E} - \frac{1}{3} tr\left(\mathbf{E}\right)\mathbf{I}\right)$$

The isotropic part ($tr\left(\mathbf{E}\right) < 0$: compression; $tr\left(\mathbf{E}\right) > 0$: expansion) is called isotropic since all directions dilate equally. The second contribution is trace-free, which corresponds to a divergence-free (incompressible) flow: Given $\mathbf{A}$ with $tr\left(\mathbf{A}\right) = 0$ then $\nabla_{\boldsymbol{r}} \cdot \left(\boldsymbol{r} \cdot \mathbf{A}\right) = tr\left(\mathbf{A}\right) = 0$.

3. A *rotation*: The rotation tensor $\mathbf{W}$ is antisymmetric with purely imaginary eigenvalues, and can be expressed as

$$\mathbf{W} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ is called the *vorticity*. Vorticity is a fundamental quantity in incompressible fluid dynamics.

$$\boldsymbol{r} \cdot \mathbf{W} = \frac{1}{2}\begin{pmatrix} \omega_2 r_3 - \omega_3 r_2 \\ \omega_3 r_1 - \omega_1 r_3 \\ \omega_1 r_2 - \omega_2 r_1 \end{pmatrix} = \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{r}$$

The velocity field $\boldsymbol{r} \cdot \mathbf{W}$ is a rigid-body rotation (and is divergence free), with angular velocity $\frac{1}{2}\boldsymbol{\omega}$. Note

$$\begin{aligned} \dot{\boldsymbol{r}} &= \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{r} \Rightarrow \\ \frac{d}{dt}\left(\boldsymbol{r} \cdot \boldsymbol{r}\right) &= 0 \;\; \text{fixed length} \\ \frac{d}{dt}\left(\boldsymbol{r} \cdot \boldsymbol{\omega}\right) &= 0 \;\; \text{fixed angle} \end{aligned}$$

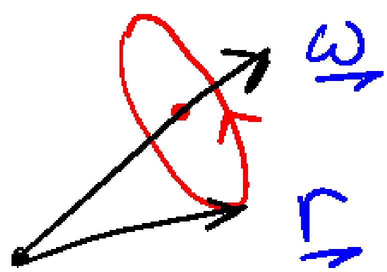

Figure 3: Vorticity $\boldsymbol{\omega}$ generates a cone upon which $\boldsymbol{r}(t)$ moves.

Example: *Solid body rotation.* Let $\boldsymbol{\omega} = \omega \boldsymbol{e}_3$. Then

$$\boldsymbol{r} \cdot \mathbf{W} = \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{r} = \frac{1}{2}\omega\left(-r_2, r_1, 0\right); \quad \left|\boldsymbol{r} \cdot \mathbf{W}\right| = \frac{1}{2}\left|\omega\right| r$$

This is solid body rotation about $\boldsymbol{\omega}$.

In summary: The local flow is composed of (i) a translation; (ii) a pure straining flow, itself decomposable into an incompressible part, and an isotropic expansion or compression; (iii) a rigid body rotation.

2.2 Conservation of Mass

Consider a *fixed* subvolume $V_0 \subseteq V$ with outward normal $\hat{\boldsymbol{n}}$. The mass of V_0 at time t is:

$$M[V_0, t] = \int_{V_0} dV_x \, \rho(\boldsymbol{x}, t)$$

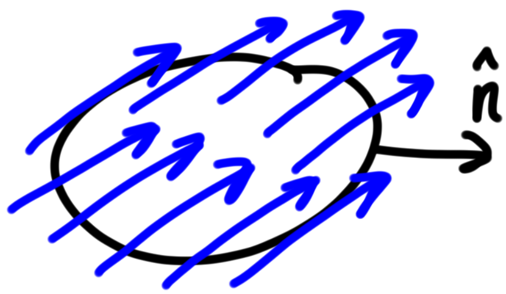

Figure 4: The flux of mass through an Eulerian volume V_0.

The rate of change of $M[V_0, t]$ is balanced by the flux of mass through its boundary ∂V_0, or

$$\frac{d}{dt} \int_{V_0} dV_x \, \rho = - \int_{\partial V_0} dS_x \, (\rho \, \boldsymbol{u}) \cdot \hat{\boldsymbol{n}} \tag{1}$$

This is the *integral form* of mass conservation. Using the divergence theorem we can write

$$\int_{V_0} dV_x \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \, \boldsymbol{u}) \right] = 0$$

As V_0 was arbitrary, this gives the *continuity equation:*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \, \boldsymbol{u}) = 0$$

which is a PDE governing the evolution of material density in a moving fluid or continuous material, and is called the *differential form* of mass conservation. $\boldsymbol{j} = \rho \, \boldsymbol{u}$ is called the *mass density flux.*

2.3 The Lagrangian formulation

The quantities $\boldsymbol{u}$ and ρ have been expressed in the *Eulerian frame*, e.g., ρ is measured at a fixed point $\boldsymbol{x}$ in the lab frame. In the *Lagrangian frame* a quantity, say ρ, is measured in the frame of moving fluid. Let $\boldsymbol{X}(t)$ satisfy

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{u}(\boldsymbol{X}(t), t) \text{ with } \boldsymbol{X}(0) = \boldsymbol{X}_0$$

The function $\boldsymbol{X}(t)$ is called the *Lagrangian*, *material*, or *particle path.* Consider $\rho(\boldsymbol{X}(t), t)$, that is, the evolution of fluid density along a Lagrangian path.

Figure 5: The Lagrangian path of a particle $\boldsymbol{X}(t)$.

Then

$$\frac{d}{dt}\rho\left(\boldsymbol{X}\left(t\right),t\right) = \left[\frac{\partial\rho}{\partial t}\left(\boldsymbol{x},t\right) + \dot{\boldsymbol{X}}\cdot\nabla\rho\left(\boldsymbol{x},t\right)\right]_{\boldsymbol{x}=\boldsymbol{X}(t)}$$
$$= \left[\frac{\partial\rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho\right]_{\boldsymbol{x}=\boldsymbol{X}(t)}$$

The substantial derivative: The operator $\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla$ is called the *Lagrangian, material,* or *substantial derivative*. It is the Eulerian expression for the time-rate-of-change of quantities along Lagrangian paths. And so

$$\frac{D\rho}{Dt} = -\rho\left(\nabla\cdot\boldsymbol{u}\right)$$

Some properties of the substantial derivative:

- $\frac{D}{Dt}\left(fg\right) = f\frac{Dg}{Dt} + g\frac{Df}{Dt}$ plus other usual aspects of a derivative
- $\frac{Df}{Dt} = 0 \Longleftrightarrow f\left(\boldsymbol{X}\left(t\right),t\right) = f\left(\boldsymbol{X}_0, 0\right)$, i.e., f is conserved along particle paths.

Previously we had considered a fixed, or Eulerian volume V_0. Now, let $\Omega\left(t\right)$ be a time dependent volume moved by the flow from $\Omega\left(0\right)$:

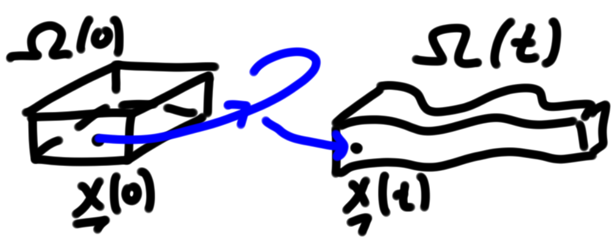

Figure 6: The deformation of the material or Lagrangian volume Ω by the flow.

That is, solve

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{u}\left(\boldsymbol{X}\left(t\right),t\right) \text{ with } \boldsymbol{X}\left(0\right) = \boldsymbol{X}_0 \;\forall \boldsymbol{X}_0 \subseteq \Omega_0$$

$\Omega(t)$ is the set of all consequent $\boldsymbol{X}\left(t\right)$, and is called a *Lagrangian* or *material volume.*

Lagrangian flow-map: A *Lagrangian variable* is one that stays constant along a Lagrangian path. The key idea of the Lagrangian formulation is to use the set of initial coordinates $\boldsymbol{X}_0 \subseteq \Omega_0$ as independent spatial coordinates. So, consider the time-dependent transformation of spatial coordinates

$$\boldsymbol{\alpha} \longmapsto \boldsymbol{X}(\boldsymbol{\alpha}, t)$$

found by solving

$$\frac{\partial \boldsymbol{X}}{\partial t}(\boldsymbol{\alpha}, t) = \boldsymbol{u}(\boldsymbol{X}(\boldsymbol{\alpha}, t), t) \text{ with } \boldsymbol{X}(\boldsymbol{\alpha}, 0) = \boldsymbol{\alpha}$$

(i.e., $\boldsymbol{\alpha} = \boldsymbol{X}_0$). $\boldsymbol{X}(\boldsymbol{\alpha}, t)$ is the *Lagrangian flow-map* and $\boldsymbol{\alpha}$ is the *Lagrangian variable.*

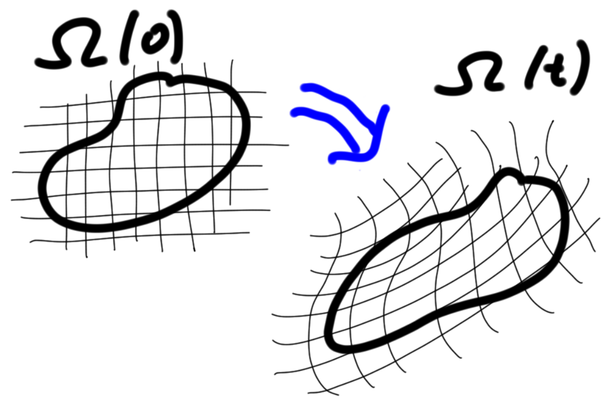

Figure 7: The evolution of the Lagrangian flow-map.

The Lagrangian flow map has many important properties: **(1)**

$$\begin{aligned}
\frac{\partial}{\partial t} f(\boldsymbol{X}(\boldsymbol{\alpha}, t), t) &= \left[\frac{\partial f}{\partial t} + \frac{\partial \boldsymbol{X}}{\partial t} \cdot \nabla f\right]_{\boldsymbol{x}=\boldsymbol{X}(\boldsymbol{\alpha}, t)} \\
&= \left[\frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \nabla f\right]_{\boldsymbol{x}=\boldsymbol{X}(\boldsymbol{\alpha}, t)} \\
&= \left[\frac{Df}{Dt}\right]_{\boldsymbol{x}=\boldsymbol{X}(\boldsymbol{\alpha}, t)}
\end{aligned}$$

Hence, the substantial derivative relates the Eulerian and Lagrangian frames.

The deformation tensor: A fundamental object defined by the Lagrangian flow-map is the deformation tensor $\mathbf{F}$, defined as the Jacobian of the flow-map:

$$\mathbf{F} = \nabla_{\boldsymbol{\alpha}} \boldsymbol{X} \text{ or } F_{ij} = \frac{\partial X_j}{\partial \alpha_i}$$

$\mathbf{F}$ encodes the deformations of the Lagrangian flow-map relative to the initial state. Let $\boldsymbol{V}(\boldsymbol{\alpha}, t) = \boldsymbol{u}(\boldsymbol{X}(\boldsymbol{\alpha}, t), t)$. Then $\mathbf{F}$ evolves by

$$\begin{aligned}
\frac{\partial F_{ij}}{\partial t} &= \frac{\partial}{\partial t}\frac{\partial X_j}{\partial \alpha_i} = \frac{\partial}{\partial \alpha_i}\frac{\partial X_j}{\partial t} = \frac{\partial V_j}{\partial \alpha_i} \\
&= \frac{\partial}{\partial \alpha_i} u_j(\boldsymbol{X}(\boldsymbol{\alpha}, t), t) = \frac{\partial u_j}{\partial X_k}\frac{\partial X_k}{\partial \alpha_i} = F_{ik}\left[\nabla_x \boldsymbol{u}\right]_{kj},
\end{aligned}$$

or

$$\frac{\partial \mathbf{F}}{\partial t} = \mathbf{F} \cdot \nabla_x \boldsymbol{u} \text{ with } \mathbf{F}(\boldsymbol{\alpha}, 0) = \mathbf{I}$$

or, in Eulerian variables:

$$\frac{D\mathbf{F}}{Dt} = \mathbf{F} \cdot \nabla \boldsymbol{u} \text{ with } \mathbf{F}(\boldsymbol{x}, 0) = \mathbf{I}.$$

Side Note: If the flow is incompressible, then the deformation tensor $\mathbf{F}$ satisfies $\nabla \cdot \mathbf{F}^\top = \mathbf{0}$ at all times.

Proof: In Eulerian coordinates, using that $\partial u_j / \partial x_j = 0$, $\mathbf{F}$ satisfies

$$\begin{aligned} \partial_t F_{ij} + u_k \frac{\partial F_{ij}}{\partial x_k} &= \frac{\partial u_j}{\partial x_k} F_{ik} \\ \Rightarrow \partial_t \frac{\partial F_{ij}}{\partial x_i} + \underline{\frac{\partial u_k}{\partial x_i} \frac{\partial F_{ij}}{\partial x_k}} + u_k \frac{\partial}{\partial x_k} \frac{\partial F_{ij}}{\partial x_i} &= \underline{\frac{\partial u_j}{\partial x_k} \frac{\partial F_{ik}}{\partial x_i}} \end{aligned}$$

The underlined terms are identical under interchange of k and i. Hence

$$\begin{aligned} \partial_t \frac{\partial F_{ij}}{\partial x_i} + u_k \frac{\partial}{\partial x_k} \frac{\partial F_{ij}}{\partial x_i} &= 0 \text{ or} \\ \frac{D}{Dt} \left(\nabla \cdot \mathbf{F}^\top \right) &= \mathbf{0} \text{ where } \mathbf{F}_0 = \mathbf{I} \end{aligned}$$

Side Note: The concept of material curves and surfaces are frequently useful. For example, consider the 1d curve $\Gamma(t)$ given by $\boldsymbol{X}(\beta, t)$ with β a scalar Lagrangian coordinate β. The curve is then defined by

$$\boldsymbol{X}_t(\beta, t) = \boldsymbol{u}(\boldsymbol{X}(\beta, t), t) \text{ with } \boldsymbol{X}(\beta, 0) = \boldsymbol{X}_0(\beta).$$

An unnormalized tangent vector to $\Gamma(t)$ is given by $\boldsymbol{X}_\beta(\beta, t)$, which evolves according to

$$\partial_t \boldsymbol{X}_\beta(\beta, t) = \boldsymbol{X}_\beta(\beta, t) \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u}(\boldsymbol{X}(\beta, t), t)$$

Writing $\nabla_{\boldsymbol{x}} \boldsymbol{u} = \mathbf{E} + \mathbf{W}$, we see that $\boldsymbol{X}_\beta$ is stretched by the $\mathbf{E}$ tensor and rotated by $\mathbf{W}$.

Structurally, this dynamics is identical to the vorticity transport equation, $\omega_t = \omega \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u}$, for the Euler equation in the Lagrangian frame. This structural similarity establishes that the vorticity lines can be identified with material curves moving in the fluid. If we now consider the unit tangent vector $\boldsymbol{s} = \boldsymbol{X}_\beta / |\boldsymbol{X}_\beta|$ we can show that

$$\partial_t \boldsymbol{s} = (\mathbf{I} - \boldsymbol{s}\boldsymbol{s}) \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u}^\top \cdot \boldsymbol{s}.$$

This the structurally identical to Jeffrey's equation for the dynamics of a thin rigid rod in low Reynolds number flow.

The Jacobian and the Liouville formula: Let J be the Jacobian determinant of the deformation tensor, that is,

$$J(\boldsymbol{\alpha}, t) = \det [\mathbf{F}] = \det [\boldsymbol{F}_1, \ldots, \boldsymbol{F}_n] = \det [\nabla_\alpha X_1, \nabla_\alpha X_2, \nabla_\alpha X_3]$$

($J(\boldsymbol{\alpha}, 0) \equiv 1$). We have the following important and standard result from dynamical systems theory for its evolution: *Liouville's Formula:*

$$\frac{\partial}{\partial t} J(\boldsymbol{\alpha}, t) = (\nabla_x \cdot \boldsymbol{u}) |_{\boldsymbol{X}(\boldsymbol{\alpha}, t)} \cdot J(\boldsymbol{\alpha}, t)$$

Proof: In $\mathbb{R}^n$

$$\mathbf{F} = [\nabla_\alpha X_1, \ldots, \nabla_\alpha X_n] = [\boldsymbol{F}_1, \ldots, \boldsymbol{F}_n]$$

The Jacobian can be expressed in terms of the multi-linear operator, the *wedge product*:

$$J = \boldsymbol{F}_1 \wedge \boldsymbol{F}_2 \wedge \cdots \wedge \boldsymbol{F}_n$$

which has the properties:

1. $\boldsymbol{F}_1 \wedge \cdots \wedge (\alpha \boldsymbol{U} + \beta \boldsymbol{W}) \wedge \cdots \wedge \boldsymbol{F}_n = \alpha (\boldsymbol{F}_1 \wedge \cdots \wedge \boldsymbol{U} \wedge \cdots \wedge \boldsymbol{F}_n) + \beta (\boldsymbol{F}_1 \wedge \cdots \wedge \boldsymbol{W} \wedge \cdots \wedge \boldsymbol{F}_n)$
2. $\boldsymbol{F}_i \subseteq \operatorname{span}[\boldsymbol{F}_j, j \neq i] \Rightarrow J = 0$
3. $\frac{d}{dt} J = \left(\dot{\boldsymbol{F}}_1 \wedge \boldsymbol{F}_2 \wedge \cdots \wedge \boldsymbol{F}_n\right) + \left(\boldsymbol{F}_1 \wedge \dot{\boldsymbol{F}}_2 \wedge \cdots \wedge \boldsymbol{F}_n\right) + \cdots + \left(\boldsymbol{F}_1 \wedge \boldsymbol{F}_2 \wedge \cdots \wedge \dot{\boldsymbol{F}}_n\right)$

Now,

$$\begin{aligned}
\dot{\boldsymbol{F}}_i &= \frac{\partial}{\partial t} \nabla_\alpha X_i = \nabla_\alpha u_i \left(\boldsymbol{X}(\boldsymbol{\alpha}, t), t\right) \\
&= \frac{\partial u_i}{\partial X_1} (\nabla_\alpha X_1) + \cdots + \frac{\partial u_i}{\partial X_i} (\nabla_\alpha X_i) + \cdots + \frac{\partial u_i}{\partial X_n} (\nabla_\alpha X_n) \\
&= \frac{\partial u_i}{\partial X_i} \boldsymbol{F}_i + \boldsymbol{T}_i \text{ with } \boldsymbol{T}_i \subseteq \operatorname{span}[\boldsymbol{F}_j, j \neq i]
\end{aligned}$$

Then

$$\begin{aligned}
\frac{d}{dt} J &= \left(\left(\frac{\partial u_1}{\partial X_1} \boldsymbol{F}_1 + \boldsymbol{T}_1\right) \wedge \boldsymbol{F}_2 \wedge \cdots \wedge \boldsymbol{F}_n\right) \\
&\quad + \left(\boldsymbol{F}_1 \wedge \left(\frac{\partial u_2}{\partial X_2} \boldsymbol{F}_2 + \boldsymbol{T}_2\right) \wedge \cdots \wedge \boldsymbol{F}_n\right) + \cdots \\
&\quad + \left(\boldsymbol{F}_1 \wedge \boldsymbol{F}_2 \wedge \cdots \wedge \left(\frac{\partial u_n}{\partial X_n} \boldsymbol{F}_n + \boldsymbol{T}_n\right)\right) \\
&= \frac{\partial u_1}{\partial X_1} J + \frac{\partial u_2}{\partial X_2} J + \cdots + \frac{\partial u_n}{\partial X_n} J \\
&= (\nabla_x \cdot \boldsymbol{u}) J
\end{aligned}$$

The upper-convected derivative: Consider the Eulerian frame evolution operator

$$\mathbf{A}^\nabla = \frac{D\mathbf{A}}{Dt} - \left(\mathbf{A} \cdot \nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top \cdot \mathbf{A}\right)$$

on a symmetric tensor $\mathbf{A}$. $\mathbf{A}^\nabla$ is called the *upper convected time derivative* of $\mathbf{A}$ (in the Lagrangian frame we simply replace D/Dt by $\partial/\partial t$ for fixed α.) This time operator is intimately related to conservation principles in the Lagrangian frame. To make the point consider a vector field $\boldsymbol{a}$ that evolves in the Lagrangian frame as

$$\boldsymbol{a}_t = \boldsymbol{a} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u}.$$

If $\boldsymbol{a} = \boldsymbol{\omega}$ this is the vorticity evolution equation for the 3D incompressible Euler equations. This is sometimes written, as we shall do here, as $\boldsymbol{a}^\nabla = \mathbf{0}$ with $\boldsymbol{a}^\nabla$ called the upper-convected derivative of the vector field $\boldsymbol{a}$. This can be trivially rewritten as

$$\boldsymbol{a}_t = \boldsymbol{a} \cdot \mathbf{F}^{-1} \cdot \mathbf{F} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u} = \boldsymbol{a} \cdot \mathbf{F}^{-1} \cdot \mathbf{F}_t.$$

Using that $\partial_t(\mathbf{F}^{-1} \cdot \mathbf{F}) = \mathbf{0}$ yields

$$\left(\boldsymbol{a} \cdot \mathbf{F}^{-1}\right)_t = \mathbf{0} \Rightarrow \boldsymbol{a} = \boldsymbol{a}_0 \cdot \mathbf{F},$$

which uses that $\mathbf{F}_0 = \mathbf{I}$. For the vorticity vector field of the Euler equations this is the famous *Result of Cauchy*. Thus, we see that $\boldsymbol{a}^{\nabla} = \mathbf{0}$ is both related to a temporal invariance and that the evolution of $\boldsymbol{a}$ is determined by the deformation tensor.

Consider now the dyadic tensor $\mathbf{A} = \boldsymbol{a}\boldsymbol{a} = \mathbf{F}^{\top} \cdot (\boldsymbol{a}_0 \boldsymbol{a}_0) \cdot \mathbf{F} = \mathbf{F}^{\top} \cdot \mathbf{A}_0 \cdot \mathbf{F}$. It can then be checked directly that $\mathbf{A}^{\nabla} = \mathbf{0}$. *In general*, we have the result that the symmetric tensor $\mathbf{Z}$ satisfies the conservation law

$$\left(\mathbf{F}^{-\top} \cdot \mathbf{Z} \cdot \mathbf{F}^{-1}\right)_t = \mathbf{0} \text{ or } \mathbf{Z} = \mathbf{F}^{\top} \cdot \mathbf{Z}_0 \cdot \mathbf{F}$$

if and only if

$$\mathbf{Z}^{\nabla} \equiv \mathbf{Z}_t - \left(\mathbf{Z} \cdot \nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\top} \cdot \mathbf{Z}\right) = \mathbf{0}$$

Note that if $\mathbf{Z}_0 = \mathbf{I}$ then $\mathbf{Z} = \mathbf{F}^{\top} \cdot \mathbf{F}$ which is also known as the left Cauchy-Green or Finger tensor (typically labelled $\mathbf{B}$). The symmetric positive-definite Finger tensor arises commonly in the modeling of elastic solids.

2.4 Mass Conservation in the Lagrangian frame

The mass of a Lagrangian volume does not change in time, that is

$$M\left[\Omega\left(t\right)\right] = M\left[\Omega\left(0\right)\right]$$

where M can be expressed as:

$$\begin{aligned} M\left[\Omega\left(t\right)\right] &= \int_{\Omega(t)} dV_x\, \rho(\boldsymbol{x}, t) \text{ tf. to Lagrangian coords} \\ &= \int_{\Omega(0)} dV_\alpha\, \rho(\boldsymbol{X}\left(\boldsymbol{\alpha}, t\right), t) J\left(\boldsymbol{\alpha}, t\right) \end{aligned}$$

Then

$$0 = M\left[\Omega\left(t\right)\right] - M\left[\Omega\left(0\right)\right] = \int_{\Omega(0)} dV_\alpha \left[\rho(\boldsymbol{X}\left(\boldsymbol{\alpha}, t\right), t) J\left(\boldsymbol{\alpha}, t\right) - \rho(\boldsymbol{\alpha}, 0)\right]$$

As $\Omega\left(0\right)$ was arbitrary we have

$$\rho(\boldsymbol{X}\left(\boldsymbol{\alpha}, t\right), t) J\left(\boldsymbol{\alpha}, t\right) = \rho(\boldsymbol{\alpha}, 0)$$

This is one Lagrangian form of mass conservation. As J also represents the measure of an infinitesimal volume, it says that if the volume increases, then the density must decrease. This now gives sense to incompressibility of a fluid or material. Incompressibility means that material volumes, infinitesimal or otherwise, do not change their volume. That is, $J\left(\boldsymbol{\alpha}, t\right) \equiv 1$, and consequently $\rho(\boldsymbol{X}\left(\boldsymbol{\alpha}, t\right), t) = \rho(\boldsymbol{\alpha}, 0)$. This has two consequences, following Liouville's formula:

$$\nabla \cdot \boldsymbol{u} = 0 \text{ and } \frac{D\rho}{Dt} = 0.$$

Hence, incompressible fluids have divergence free velocity fields, and the density is conserved along Lagrangian paths.

To recover the Eulerian form, take a time-derivative and use the relation with the substantial derivative and Liouville's formula:

$$
\begin{aligned}
0 &= \frac{\partial}{\partial t}\left(\rho(\boldsymbol{X}\left(\boldsymbol{\alpha}, t\right), t) J\left(\boldsymbol{\alpha}, t\right)\right) \\
&= \left[\frac{D\rho}{Dt}(\boldsymbol{X}\left(\boldsymbol{\alpha}, t\right), t)\, J\left(\boldsymbol{\alpha}, t\right) + \rho(\boldsymbol{X}\left(\boldsymbol{\alpha}, t\right), t)\left(\nabla_x \cdot \boldsymbol{u}\right)\left(\boldsymbol{X}\left(\boldsymbol{\alpha}, t\right), t\right) J\left(\boldsymbol{\alpha}, t\right)\right]
\end{aligned}
$$

If the flow is smooth, then $J \neq 0$, and we have

$$
\frac{D\rho}{Dt}(\boldsymbol{x}, t) = -\rho(\boldsymbol{x}, t)\nabla \cdot \boldsymbol{u}(\boldsymbol{x}, t)
$$

which we have already proved.

The Lagrangian statement of mass conservation yields the following fundamental result:

2.5 The Transport Theorem

For any smooth $f(\boldsymbol{x}, t)$

$$
\frac{d}{dt}\int_{\Omega(t)} dV_x\, \rho f = \int_{\Omega(t)} dV_x\, \rho \frac{Df}{Dt}
$$

Proof: Use that $\partial_t\left(\rho(\boldsymbol{X}\left(\boldsymbol{\alpha}, t\right), t) J\left(\boldsymbol{\alpha}, t\right)\right) = 0$ and that $\partial_t\left(f\left(\boldsymbol{X}\left(\boldsymbol{\alpha}, t\right), t\right)\right) = \left(Df/Dt\right)\left(\boldsymbol{X}\left(\boldsymbol{\alpha}, t\right), t\right)$:

$$
\begin{aligned}
\frac{d}{dt}\int_{\Omega(t)} dV_x\, \rho f &= \frac{d}{dt}\int_{\Omega(0)} dV_\alpha\, J\rho f = \int_{\Omega(0)} dV_\alpha\, \frac{\partial}{\partial t}\left(J\rho f\right) \\
&= \int_{\Omega(0)} dV_\alpha\, J\rho \frac{\partial f}{\partial t} = \int_{\Omega(t)} dV_x\, \rho \frac{Df}{Dt}
\end{aligned}
$$

2.6 Balance of momentum and forces in a fluid or deformable material

The acceleration of a fluid particle is given by

$$
\begin{aligned}
\boldsymbol{a}\left(t\right) &= \frac{d^2}{dt^2}\boldsymbol{X}\left(t\right) = \frac{d}{dt}\boldsymbol{u}\left(\boldsymbol{X}\left(t\right), t\right) \\
&= \left(\frac{\partial \boldsymbol{u}}{\partial t} + \left(\boldsymbol{u} \cdot \nabla_x\right)\boldsymbol{u}\right) = \left(\frac{D\boldsymbol{u}}{Dt}\right)\left(\boldsymbol{X}\left(t\right), t\right)
\end{aligned}
$$

where notationally $\left[\left(\boldsymbol{u} \cdot \nabla_x\right)\boldsymbol{u}\right]_j = u_i \partial_{x_i} u_j$.

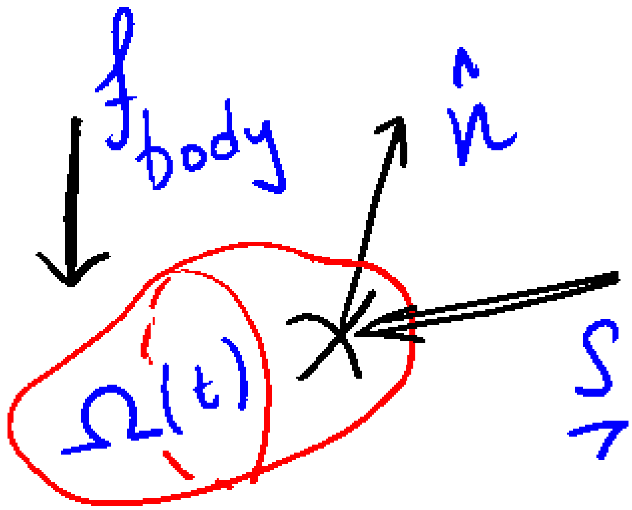

Figure 8: A Lagrangian subvolume $\Omega(t)$ being acted upon by body forces $\boldsymbol{f}_{body}$ and forces of stresses $\boldsymbol{s}\delta A$.

Now, let's develop Newton's 2^{nd} Law for balance of forces in a fluid. The momentum carried by a Lagrangian volume of fluid $\Omega(t)$ is

$$\boldsymbol{m}(t) = \int_{\Omega(t)} dV_x \, \rho \boldsymbol{u}$$

Generally, forces come in two flavors, *body forces* and *stresses:*

- *Body forces* – externally imposed forces such as gravity or electro-magnetic fields, that exert a force/unit mass. Let $\boldsymbol{g}(\boldsymbol{x}, t)$ be such a force/unit mass. The total body force exerted upon $\Omega(t)$ is:

$$\boldsymbol{f}_{body} = \int_{\Omega(t)} dV_x \, \rho \boldsymbol{g}$$

- *Forces of stress* – Forces arising from the mechanical contact of the volume Ω, across $\partial\Omega$, with the rest of the fluid or material. According to Cauchy, the (Cauchy) stress $\boldsymbol{s}$ (units of force/unit area) across a surface with outward normal $\hat{\boldsymbol{n}}$, at a point $\boldsymbol{x}$, has the form

$$\boldsymbol{s} = \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma} \text{ or } s_j = \hat{n}_i \sigma_{ij}$$

$\boldsymbol{\sigma}$ is called the *Cauchy stress tensor* and it is a central focus of nearly all modeling of complex fluids and deformable materials. Conservation of angular momentum implies that the stress tensor is symmetric. The total force of stress exerted upon $\Omega(t)$ is:

$$\boldsymbol{f}_{stress} = \int_{\partial\Omega(t)} dS_x \, \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}$$

Newton's 2^{nd} law then gives

$$\frac{d}{dt}\boldsymbol{m} = \boldsymbol{f}_{body} + \boldsymbol{f}_{stress}$$

Applying the transport theorem to the expression for $(d/dt)\,\boldsymbol{m}$ and the divergence theorem to the expression of $\boldsymbol{f}_{stress}$ gives:

$$\int_{\Omega(t)} dV_x \, \rho \frac{Du_j}{Dt} = \int_{\Omega(t)} dV_x \, \rho g_j + \int_{\Omega(t)} dV_x \, \frac{\partial \sigma_{ij}}{\partial x_i}$$

or

$$\rho \frac{D\boldsymbol{u}}{Dt} = \nabla_x \cdot \boldsymbol{\sigma} + \rho \boldsymbol{g}$$

2.7 Momentum balance in the Lagrangian frame

What is the analogous expression for momentum balance in the Lagrangian frame? For this, we need *Nanson's formula.* This crucial identity allows a change of surface variables between Eulerian and Lagrangian descriptions (Holzapfel, Eq. 2.55):

$$\hat{\boldsymbol{n}}\, dS_x = J\, \mathbf{F}^{-\top} \cdot \hat{\boldsymbol{N}}\, dS_\alpha$$

Here $\hat{\boldsymbol{n}}$ is the normal to a patch of surface of area dS_x in the Eulerian frame, while $\hat{\boldsymbol{N}}$ is the surface normal to the originating Lagrangian surface of size dS_α. Now, here is the proof *not* given by Holzapfel of Nanson's equality in differential form:

$$\nabla_x \cdot \boldsymbol{\sigma} = J^{-1} \nabla_\alpha \cdot \left(\boldsymbol{\sigma} \cdot \mathbf{F}^{-\top} J \right)$$

Proof: Consider the (stress) tensor $\boldsymbol{\sigma}$ as a set of vectors $\boldsymbol{\sigma}^j$ indexed by j, the column index, i.e., $\sigma_{ij} \to \sigma_i^j$. Then, $[\nabla_x \cdot \boldsymbol{\sigma}]_j = \frac{\partial \sigma_{ij}}{\partial x_i}$. Now

$$\frac{\partial}{\partial \alpha_p} \sigma_{ij} = \frac{\partial \sigma_{ij}}{\partial x_q} \frac{\partial x_q}{\partial \alpha_p} = \frac{\partial \sigma_{ij}}{\partial x_q} F_{pq} \text{ or } \nabla_\alpha \boldsymbol{\sigma}^j = \mathbf{F} \cdot \nabla_x \boldsymbol{\sigma}^j$$

$$\text{and hence, } \nabla_x \boldsymbol{\sigma}^j = \mathbf{F}^{-1} \cdot \nabla_\alpha \boldsymbol{\sigma}^j \text{ or taking a trace: } \frac{\partial \sigma_{ij}}{\partial x_i} = \frac{\partial \sigma_{ij}}{\partial \alpha_p} F_{ip}^{-1}$$

We now make use of the following identity:

$$\frac{\partial}{\partial \alpha_p} \left(F_{ip}^{-1} J \right) = \left[\nabla_\alpha \cdot \left(\mathbf{F}^{-\top} J \right) \right]_i = 0$$

(Do this as an exercise.) And so,

$$[\nabla_x \cdot \boldsymbol{\sigma}]_j = J^{-1} \frac{\partial}{\partial \alpha_p} \left(\sigma_{ij} F_{ip}^{-1} J \right) = J^{-1} \left[\nabla_\alpha \cdot \left(\boldsymbol{\sigma} \cdot \mathbf{F}^{-\top} J \right) \right]_j$$

Hence, we have

$$\int_{\Omega(t)} \nabla_x \cdot \boldsymbol{\sigma}\, dV_x = \int_{\Omega(0)} \nabla_\alpha \cdot \left(\boldsymbol{\sigma} \cdot \mathbf{F}^{-\top} J \right) dV_\alpha$$

or, setting $\mathbf{P} = \boldsymbol{\sigma} \cdot \mathbf{F}^{-\top} J \Longleftrightarrow \boldsymbol{\sigma} = J^{-1} \mathbf{P} \cdot \mathbf{F}^\top$, we have

$$\int_{\Omega(t)} \nabla_x \cdot \boldsymbol{\sigma}\, dV_x = \int_{\Omega(0)} \nabla_\alpha \cdot \mathbf{P}\, dV_\alpha$$

Here $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor, and $\mathbf{P}$ is called the *first Piola-Kirchoff* stress tensor. Since $\boldsymbol{\sigma}$ is symmetric,

$$\mathbf{P} \cdot \mathbf{F}^\top = \mathbf{F} \cdot \mathbf{P}^\top$$

The tensor $\mathbf{P}$ is itself not generally symmetric. We also have

$$\int_{\partial\Omega(t)} \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}\, dS_x = \int_{\partial\Omega(0)} \hat{\boldsymbol{N}} \cdot \mathbf{P}\, dS_\alpha$$

which defines the two stress vectors:

$$\boldsymbol{s}(\boldsymbol{x},t,\hat{\boldsymbol{n}}) = \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}(\boldsymbol{x},t) \text{ and } \boldsymbol{S}(\boldsymbol{\alpha},t,\hat{\boldsymbol{N}}) = \hat{\boldsymbol{N}} \cdot \mathbf{P}(\boldsymbol{\alpha},t)$$

where $\boldsymbol{s}$ is the (Cauchy) stress vector relative to the current configuration, and $\boldsymbol{S}$ is the (first Piola-Kirchoff) stress vector relative to the reference configuration.

Now, reconsidering balance of momentum,

$$\frac{d}{dt}\int_{\Omega(t)} dV_x\, \rho(\boldsymbol{x},t)\, \boldsymbol{u}(\boldsymbol{x},t) = \int_{\partial\Omega(t)} dS_x\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}(\boldsymbol{x},t) + \int_{\Omega(t)} dV_x\, \rho(\boldsymbol{x},t)\boldsymbol{g}(\boldsymbol{x},t)$$

or, rewriting everything in Lagrangian variables, i.e. letting $\boldsymbol{V}(\boldsymbol{\alpha},t) = \boldsymbol{u}(\boldsymbol{X}(\boldsymbol{\alpha},t),t)$, $\boldsymbol{G}(\boldsymbol{\alpha},t) = \boldsymbol{g}(\boldsymbol{X}(\boldsymbol{\alpha},t),t)$, and using $\rho(\boldsymbol{\alpha},t)\, J(\boldsymbol{\alpha},t) = \rho_0(\boldsymbol{\alpha})$:

$$\frac{d}{dt}\int_{\Omega(0)} dV_\alpha\, \rho_0(\boldsymbol{\alpha})\, \boldsymbol{V}(\boldsymbol{\alpha},t) = \int_{\partial\Omega(0)} dS_\alpha\, \hat{\boldsymbol{N}}(\boldsymbol{\alpha}) \cdot \mathbf{P}(\boldsymbol{\alpha},t) + \int_{\Omega(0)} dV_\alpha\, \rho_0(\boldsymbol{\alpha})\, \boldsymbol{G}(\boldsymbol{\alpha},t)$$

And applying the divergence theorem yields:

$$\int_{\Omega(0)} dV_\alpha\, \rho_0(\boldsymbol{\alpha})\, \boldsymbol{V}_t(\boldsymbol{\alpha},t) = \int_{\Omega(0)} dV_\alpha\, \nabla_\alpha \cdot \mathbf{P}(\boldsymbol{\alpha},t) + \int_{\Omega(0)} dV_\alpha\, \rho_0(\boldsymbol{\alpha})\, \boldsymbol{G}(\boldsymbol{\alpha},t)$$

Now using the arbitrariness of $\Omega(0)$, we have

$$\rho_0 \frac{\partial \boldsymbol{V}}{\partial t} = \nabla_\alpha \cdot \mathbf{P} + \rho_0 \boldsymbol{G}$$

Very nice.

3 Some models of flow and deformation

Consider stresses written in the form

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mathbf{d}$$

where $\mathbf{d}$ is the deviatoric stress, customarily assumed to be trace-free (though we do not do that here), and p is (usually) the thermodynamic or mechanical pressure (in the case of incompressible flow). *Newtonian fluids* are those that have a linear relation between the *deviatoric stress* $\mathbf{d}$ and the rate-of-strain tensor $\nabla \boldsymbol{u}$. All others are termed non-Newtonian.

3.1 The Euler equations

Set $\mathbf{d} = \mathbf{0}$ and take the stress to be simply

$$\boldsymbol{\sigma} = -p\,(\boldsymbol{x}, t)\,\mathbf{I}.$$

The scalar function $p(\boldsymbol{x}, t)$ is the pressure and is compressive for $p > 0$. Hence,

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \rho \boldsymbol{g} \text{ (L. Euler, 1755)}$$

If the fluid is compressible, p is a thermodynamic variable and will generally depend upon other thermodynamic variables, such as the fluid density ρ and the entropy density s, whose dynamics are specified.

We are interested in the case when when the fluid is incompressible. We then have a closed set of evolution equations

$$\begin{aligned} \rho \frac{D\boldsymbol{u}}{Dt} &= -\nabla p + \rho \boldsymbol{g} \\ \frac{D\rho}{Dt} &= 0 \text{ and } \nabla \cdot \boldsymbol{u} = 0 \end{aligned}$$

Very Important Note: Here the pressure plays the role of a Lagrange multiplier that enforces incompressibility, adjusting itself at each time to ensure that velocity remains divergence free. This system is *nonlinear* due to the nature of the substantial derivative, but also *nonlocal* as the divergence free condition yields an elliptic character to the equations. To wit, we simplify by assuming that $\rho = \bar{\rho}$, a constant – this is preserved by the dynamics – and neglect the body force. So, using $\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{u}_t = 0$

$$\bar{\rho}\,(\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) = -\nabla p \implies \Delta p = -\bar{\rho} \nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u})$$

which is a "pressure-Poisson" equation. Further,

$$\nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = \partial_i\,(u_j \partial_j u_i) = \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} = (E_{ij} + W_{ij})\,(E_{ij} - W_{ij})$$

Given any symmetric tensor $\mathbf{S}$ and antisymmetric $\mathbf{A}$, $\mathbf{S} : \mathbf{A} = 0$. So we have

$$\Delta p = \bar{\rho} \left(\|\mathbf{W}\|_F^2 - \|\mathbf{E}\|_F^2 \right),$$

which I find interesting, if not useful.

3.2 The incompressible Navier-Stokes equations

Here, the deviatoric stress tensor $\mathbf{d}$ is linearly proportional to the symmetric rate-of-strain tensor:

$$\boldsymbol{\sigma} = -p(\boldsymbol{x}, t)\mathbf{I} + 2\mu\mathbf{E}(\boldsymbol{x}, t),$$

where μ is a material coefficient called the *bulk* or *shear viscosity.* (For compressible fluids there is typically another viscous stress contribution proportional to $(\nabla \cdot \boldsymbol{u})\mathbf{I}$.) Note that the stress tensor is symmetric. Using that $\nabla \cdot \mathbf{E} = \frac{1}{2}\Delta\boldsymbol{u}$ (which uses $\nabla \cdot \boldsymbol{u} = 0$) yields the famed Navier-Stokes equations

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\nabla p + \mu\Delta\boldsymbol{u} + \rho\boldsymbol{g}$$
$$\frac{D\rho}{Dt} = 0 \text{ and } \nabla \cdot \boldsymbol{u} = 0$$

In the case of constant density the pressure satisfies the same pressure-Poisson equation as for the constant density Euler equations.

3.3 The Oldroyd-B equations

One of the simplest models of a viscoelastic fluid is given by the (incompressible) Oldroyd-B equations. There, immersed polymer chains are modeled as extensible dumbbells that have a linear force response to distension by the flow. The dumbbells store elastic energy in their stretch, which is released back into a flow on a polymer relaxation time-scale τ. The Oldroyd-B momentum balance equation combines a Newtonian viscous stress tensor with an elastic stress tensor arising from the immersed dumbbells. That is,

$$\boldsymbol{\sigma} = (-p\mathbf{I} + 2\mu\mathbf{E}) + \boldsymbol{\sigma}_e$$

which yields

$$\bar{\rho}\frac{D\boldsymbol{u}}{Dt} = -\nabla p + \mu\Delta\boldsymbol{u} + \nabla \cdot \boldsymbol{\sigma}_e \text{ and } \nabla \cdot \boldsymbol{u} = 0.$$

The elastic stress evolves according to the advection-relaxation equation

$$\tau\boldsymbol{\sigma}_e^{\nabla} = -(\boldsymbol{\sigma}_e - G\mathbf{I})$$

where τ is a relaxation time and G is an elastic modulus. The equation above, like $\mathbf{A}^{\nabla} = \mathbf{0}$, can be integrated in the Lagrangian frame (see Hohenegger & Shelley, Les Houches Lecture Notes, 2009). Typically, these equations are solved in the zero Reynolds number limit as a Stokes equations forced by an evolving elastic stress.

3.4 The Neo-Hookean elastic solid

This model does not describe the motion of a fluid but of a deformable and incompressible elastic solid. Here the deviatoric stress does not depend upon rates-of-strain, but rather upon strains, that is the deformation of the material. This particular model is of a so-called Neo-Hookean solid, and has the general stress tensor

$$\boldsymbol{\sigma} = -p\mathbf{I} + GJ^{-1}\mathbf{B}$$

where G is a modulus, $\mathbf{B} = \mathbf{F}^{\top}\mathbf{F}$ is the Finger tensor, and we take $J \equiv 1$ as a consequence of incompressibility. Again, pressure adjusts itself to maintain that condition. Then we have

$$\bar{\rho}\frac{D\boldsymbol{u}}{Dt} = -\nabla p + G\nabla \cdot \mathbf{B} \text{ and } \nabla \cdot \boldsymbol{u} = 0 \ (J = 1)$$
$$\mathbf{B}^{\nabla} = \mathbf{0} \text{ with } \mathbf{B}(\boldsymbol{x}, 0) = \mathbf{I}.$$

These are closed equations, giving the simplest model of a perfectly elastic solid that dissipates no energy. The slow relaxation time limit of the Oldroyd-B equations describes a Neo-Hookean solid with viscous dissipation.

For small displacements: $\boldsymbol{u} \to \varepsilon \boldsymbol{u}$, $\mathbf{B} = \mathbf{I} + \varepsilon \mathbf{C}$, and expanding to first-order in ε:

$$\begin{aligned}
\rho_0 \boldsymbol{u}_t &= -\nabla p + G\, \nabla \cdot \mathbf{C} \\
\mathbf{C}_t &= \nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top = 2\mathbf{E} \\
\nabla \cdot \boldsymbol{u} &= 0
\end{aligned}$$

Taking a time derivative of the first equation and setting $q = p_t$, we have

$$\rho_0 \boldsymbol{u}_{tt} = -\nabla q + G\, \nabla \cdot (\nabla \boldsymbol{u}) = -\nabla q + G \Delta \boldsymbol{u} \text{ and } \nabla \cdot \boldsymbol{u} = 0$$

That is, an "incompressible" wave equation.

The model is somewhat more linear in the Lagrangian frame. We introduce the Piola-Kirchoff tensor $\mathbf{P}$:

$$\boldsymbol{\sigma} = -p\mathbf{I} + GJ^{-1}\mathbf{B} \Longleftrightarrow \mathbf{P} = -p\mathbf{F}^{-\top} + G\mathbf{F}$$

and the Lagrangian-frame velocity $\boldsymbol{V}(\boldsymbol{\alpha}, t)$. This yields

$$\begin{aligned}
\rho_0 \frac{\partial \boldsymbol{V}}{\partial t} &= -\nabla_{\boldsymbol{\alpha}} \cdot (p\mathbf{F}^{-\top}) + G\, \nabla_\alpha \cdot \mathbf{F} \text{ and } J = 1 \\
\frac{\partial \mathbf{F}}{\partial t} &= \nabla_\alpha \boldsymbol{V} \text{ with } \mathbf{F}(\boldsymbol{\alpha}, 0) = \mathbf{I}.
\end{aligned}$$

This is a generalized wave equation.

4 The Navier-Stokes Eqns and its properties.

The stress tensor for the NS equations is $\boldsymbol{\sigma}_N = -p\mathbf{I} + 2\mu\mathbf{E}$, where N stands for Newtonian. Assume constant density $\rho \equiv \bar{\rho}$, so that

$$\bar{\rho}\frac{D\boldsymbol{u}}{Dt} = \nabla \cdot \boldsymbol{\sigma}_N = -\nabla p + \mu \Delta \boldsymbol{u} + \bar{\rho}\boldsymbol{g}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

The N-S equations have a symmetric stress tensor: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\top$. This guarantees conservation of angular momentum.

Boundary conditions on the N-S equations:

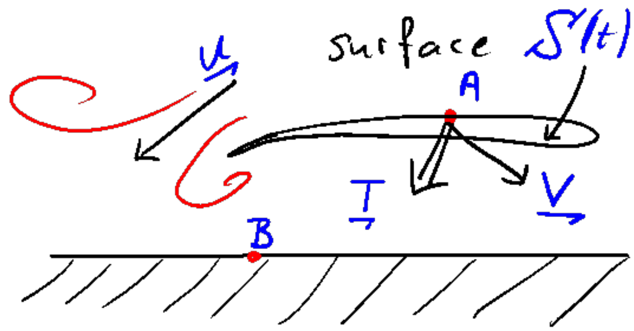

Figure 9: A body of time dependent shape with surface $S(t)$ moving above a solid wall.

- On a solid boundary, as at point B, we require, for a viscous fluid, the *no-slip condition:* $\boldsymbol{u}|_B = \mathbf{0}$. For an inviscid fluid, we impose the no-penetration conditions, $\boldsymbol{u}|_B \cdot \hat{\boldsymbol{n}}_{wall} = 0$, so that no fluid penetrates the wall.
- On an impenetrable body with time-dependent surface $S(t)$, surface velocity $\boldsymbol{V}$, and which exerts a stress $\boldsymbol{T}$ on the fluid, we require that $\boldsymbol{u}|_{A \in S} = \boldsymbol{V}$ and $\hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}|_{A\in S} = -\boldsymbol{T}$.

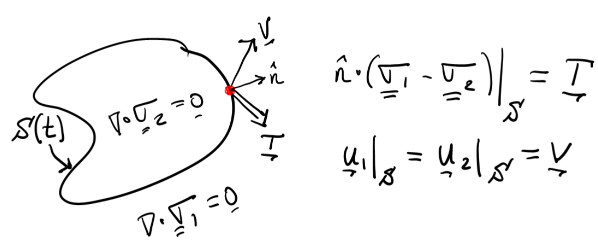

Figure 10: A deformable surface separating two fluids and across which there is a traction jump $\boldsymbol{T}$.

- At an impermeable interface $S(t)$ between two fluids, or at least two continuum materials, with stress tensors $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, we require $\hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}_1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}_2 = \boldsymbol{T}$ where $\boldsymbol{T}$ is the surface traction. Typically, $\boldsymbol{T} = \gamma\kappa\hat{\boldsymbol{n}}$ for surface tension.

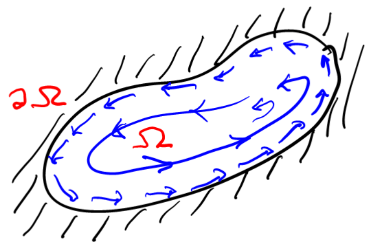

Figure 11: A fixed bounded domain Ω filled with a viscous fluid.

The Kinetic Energy and Viscous Dissipation: The kinetic energy of the fluid in a domain $\Omega(t)$ is

$$K(t) = \int_{\Omega(t)} \tfrac{1}{2}\rho\, \boldsymbol{u}\cdot\boldsymbol{u}\, dV.$$

Consider Ω a closed, fixed, and with rigid walls (i.e. $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}$). By the Transport Theorem and the NS momentum equation:

$$\begin{aligned}\dot{K} &= \int_\Omega \tfrac{1}{2}\rho \frac{D}{Dt}(\boldsymbol{u}\cdot\boldsymbol{u})\, dV = \int_\Omega \rho\, \boldsymbol{u}\cdot\frac{D\boldsymbol{u}}{Dt}\, dV \\ &= \int_\Omega \boldsymbol{u}\cdot(-\nabla p + \mu\Delta\boldsymbol{u} + \rho\boldsymbol{g})\, dV.\end{aligned}$$

Treating each term in turn (using $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$ and $\nabla\cdot\boldsymbol{u} = 0$):

$$\int_\Omega \boldsymbol{u}\cdot\nabla p dV = \oint_{\partial\Omega} p\, \boldsymbol{u}\cdot\hat{\boldsymbol{n}}\, dS - \int_\Omega p\, \nabla\cdot\boldsymbol{u}\, dV = 0,$$

$$\begin{aligned}\int_\Omega u_i \frac{\partial}{\partial x_j}(\mu E_{ij})\, dV &= \oint_{\partial\Omega} \mu\, u_i E_{ij}\hat{n}_j\, dS - \int_\Omega \mu\frac{\partial u_i}{\partial x_j} E_{ij}\, dV = -\mu\int_\Omega \frac{\partial u_i}{\partial x_j}E_{ij}\, dV, \\ &= -\mu\int_\Omega [\nabla\boldsymbol{u}]_{ji}\, E_{ij}\, dV = -\mu\int_\Omega \nabla\boldsymbol{u} : \mathbf{E}\, dV \\ &= -\mu\int_\Omega (\mathbf{E}+\mathbf{W}) : \mathbf{E}\, dV = -\mu\int_\Omega \mathbf{E}:\mathbf{E}\, dV,\end{aligned}$$

where the boundary integral vanishes by no-slip ($\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$), and the last step uses $\mathbf{W} : \mathbf{E} = W_{ij}E_{ji} = 0$, which follows since $W_{ij} = -W_{ji}$ and $E_{ji} = E_{ij}$ imply $W_{ij}E_{ji} = -W_{ij}E_{ji}$. Hence:

$$\dot{K} = -2\mu\int_\Omega \mathbf{E}:\mathbf{E}\, dV + \int_\Omega \rho\, \boldsymbol{g}\cdot\boldsymbol{u}\, dV.$$

The first term, $-\mu\int_\Omega \mathbf{E}:\mathbf{E}\, dV \le 0$, is the rate of viscous dissipation (production of heat). The second term is the power (rate of work) done by external body forces. In the absence of external forcing, the kinetic energy decays monotonically through viscous dissipation.

Galilean Invariance: The N-S equations retain their form under the Galilean transformation

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{U}, \qquad \boldsymbol{y} = \boldsymbol{x} + \boldsymbol{U}t,$$

where $\boldsymbol{U}$ is a constant velocity. This is left as an exercise.

Breaking Galilean invariance: If a linear drag body force $\boldsymbol{f} = -\gamma \boldsymbol{u}$ (modelling, e.g., drag on a fluid moving over a rough wall) is included:

$$\begin{cases} \rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \mu \Delta \boldsymbol{u} - \gamma \boldsymbol{u}, \\ \nabla \cdot \boldsymbol{u} = 0, \end{cases}$$

then Galilean invariance is broken, since the drag term $-\gamma \boldsymbol{u}$ is not invariant under $\boldsymbol{u} \mapsto \boldsymbol{u} + \boldsymbol{U}$. The basic lesson is that the presence of bodies or walls (whose effect is modeled here by the drag term) will break Galilean invariance.

4.1 The Reynolds Number

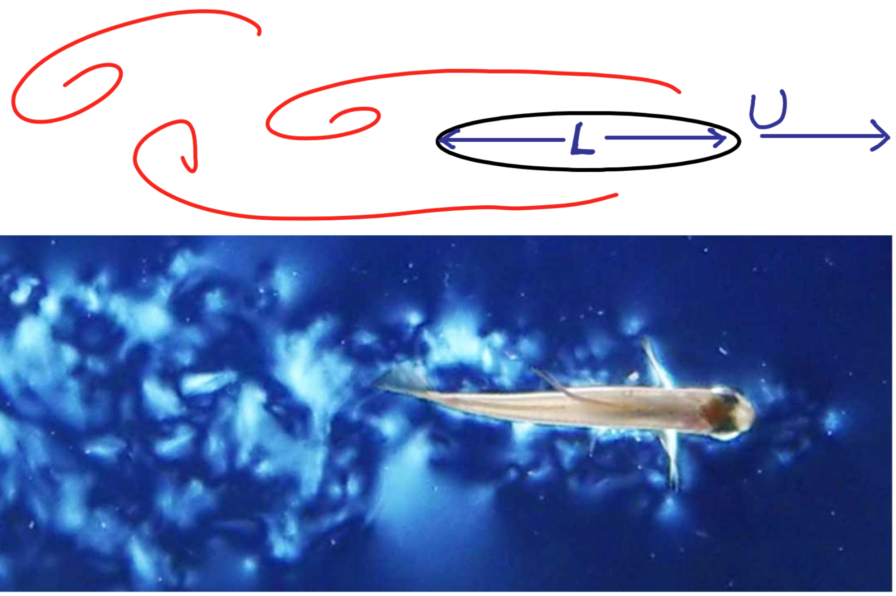

Figure 12: A body of length-scale L moving with speed U through a fluid of density ρ and viscosity μ.

The **Reynolds number** is the most important (or at least the most famous; Wikipedia lists ~70) dimensionless constant in fluid dynamics. Its magnitude quantifies the relative balance of inertial and viscous forces in a fluid. Consider a body of characteristic size L moving with speed U through a Newtonian fluid. This also defines a characteristic time $T = L/U$. Rescale variables as

$$x \to L\,x,\; t \to (L/U)\,t,\; u \to U\,u, \text{ and } p \to P\,p$$

Then the incompressible NS eqns become:

$$Re\,\frac{D\boldsymbol{u}}{Dt} = -\left(\frac{P/L}{\mu U/L^2}\right)\nabla p + \Delta \boldsymbol{u} \text{ and } \nabla \cdot \boldsymbol{u} = 0$$

where all variables are without dimension. The dimensionless constant

$$Re = \frac{\rho U^2 L^2}{\mu U L} = \frac{\text{“inertial” force}}{\text{“viscous” force}} = \frac{\rho U L}{\mu}$$

is the (famous) Reynolds number. We have left the pressure scale to be determined. Because of the role the pressure plays in satisfying the divergence free condition it is simply scaled to keep it in the dynamics, regardless of what limiting system in Reynolds number is considered. Consider two extreme, but centrally important cases:

$Re \gg 1$**:** This means that the fluid flow is dominated by the inertial forces of the fluid. This is typical for the locomotion of most birds, fish, whales, etc. In this case, choose $P = Re \cdot \mu U/L$, and we have

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p + \frac{1}{Re}\Delta \boldsymbol{u} \text{ and } \nabla \cdot \boldsymbol{u} = 0$$

Taking the *formal limit* $Re \to \infty$, we get the Euler equations:

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p \text{ and } \nabla \cdot \boldsymbol{u} = 0$$

We emphasize that this is a formal limit because in the presence of boundaries, static or dynamic, the no-slip condition is a singular perturbation and makes that limit a possibly singular one; there can be a persistent shedding of vorticity produced at the wall even in the limit of infinite Reynolds number. While the Euler equations retain the convective nonlinearity of the NS equations, their lack of diffusion gives their dynamics a great deal of geometric structure that is useful in understanding the structure of solutions, as well as giving special tools, such as potential theory, for constructing sapecial classes of solutions.

$Re \ll 1$**:** This means that the fluid flow is dominated by the viscous forces of the fluid. This is the typical situation for micro-organismal locomotion, transport of small particles of any sort, flows inside of biological cells, and indeed any dynamics that takes place on either a sufficiently slow time-scale, or at a sufficiently small spatial scale. In this case, we choose $P = \mu U/L$, which scales the stress tensor uniformly, $\boldsymbol{\sigma} \to (\mu U/L)\,\boldsymbol{\sigma}$, and gives

$$Re\,\frac{D\boldsymbol{u}}{Dt} = -\nabla p + \Delta \boldsymbol{u} \text{ and } \nabla \cdot \boldsymbol{u} = 0$$

and the formal limit $Re \to 0$ yields the *Stokes equations*:

$$-\nabla p + \Delta \boldsymbol{u} = \boldsymbol{0} \text{ and } \nabla \cdot \boldsymbol{u} = 0$$

Note that the Stokes equations are linear, constant coefficient PDEs. For the Stokes equations there is no loss of boundary conditions, unlike the Euler equations, since the highest order spatial term is retained. Unlike either the NS or Euler equations, the Stokes equations are not necessarily solved as an initial value problem as the equations do not contain any time derivatives. They are typically solved as a boundary value problem, where any dynamics devolves from time dependence in boundary data or in solution domain (e.g. as in free boundary problems).

Note that if there are free bodies in the fluid, then the low Reynolds number scaling requires that they exert zero net force and torque upon the surrounding fluid. To see this, a body in the fluid moves through Newton's 2nd law as

$$\begin{aligned}
m_b \ddot{\boldsymbol{X}}_c &= \int_\Gamma dS_x\,(\hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma})\,, \text{ or in dimensionless units} \\
\frac{m_b U^2}{L}\ddot{\boldsymbol{X}}_c &= L^2 \int_\Gamma dS_x \left(\frac{\mu U}{L}\hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}\right), \text{ which can be rearranged to yield} \\
Re\frac{m_b}{m_f}\ddot{\boldsymbol{X}}_c &= \int_\Gamma dS_x\,(\hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}) \;\text{ where } m_f = \rho L^3
\end{aligned}$$

Hence, if $Re \ll 1$, then the inertial term can be dropped so long as m_b/m_f is not large, and we will generate the constraints

$$\boldsymbol{F} = \int_\Gamma dS_x\,(\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}) = \boldsymbol{0} \text{ and}$$
$$\boldsymbol{T} = \int_\Gamma dS_x\,(\boldsymbol{X}-\boldsymbol{X}_c)\times(\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}) = \boldsymbol{0}$$

Intermediate Reynolds number: In this neither high nor low regime both inertial and viscous forces are important, and this is a regime that has come under increasing scrutiny, for example in studies of small insect locomotion, and efficient mixing in micro-fluidic devices. In the low and high Reynolds regimes there have been many tools – asymptotic reductions, special numerical methods – that have greatly aided in understanding the fluid dynamics. All of these tools fail in the moderate Reynolds number regime, or must be used at best perturbatively, and theoretical studies have been almost exclusively computational in nature.

The physical regimes. Re $\ll 1$ when motions are slow, objects are small, or viscosity is high. Example: a bacterium swimming at $U = 30\,\mu$m/s, $L = 10\,\mu$m in water gives Re $= 3\times 10^{-5}$. Flows in biological cells: Re $\sim 10^{-8}$.

A self-propelled organism	**Reynolds number**
A large whale swimming at 10 m/s	300,000,000
A tuna swimming at the same speed	30,000,000
A duck flying at 20 m/s	300,000
A large dragonfly going 7 m/s	30,000
A copepod in a speed burst of 0.2 m/s	300
Flapping wings of the smallest flying insects	30
An invertebrate larva, 0.3 mm long, at 1 mm/s	0.3
A sea urchin sperm advancing the species at 0.2 mm/s	0.03
A bacterium swimming at 0.01 mm/s	0.00003

Table 1: Reynolds numbers of self-propelled organisms (swimmers and flyers). Annotated ranges: Re $\gg 1$ (whale to dragonfly); Re $\sim \mathcal{O}(1)$ (copepod to smallest insects); Re $\ll 1$ (invertebrate larva to bacterium).

4.2 Vorticity dynamics for NS and Euler

The vorticity field, $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$, is a fundamental quantity for incompressible flows and has distinctly different dynamics in two and three dimensions, and physical regime. Here I'll discuss the evolution of vorticity for the Navier-Stokes and Euler equations, as the presence of inertial forces make these cases very distinct from the Stokes limit. First we trivially rewrite the Navier-Stokes equations, assuming that $\boldsymbol{g} = \boldsymbol{0}$:

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} = -\frac{1}{\bar\rho}\nabla p + \nu\Delta\boldsymbol{u},$$

where $\nu = \mu/\bar\rho$ is called the *kinematic viscosity.*

Vorticity dynamics in the Eulerian frame: To take a curl of this equation, and so derive a vorticity evolution equation, we make use of the vector identity:

$$\nabla \times [(\boldsymbol{u} \cdot \nabla) \boldsymbol{u}] = (\boldsymbol{u} \cdot \nabla) \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla) \boldsymbol{u} + (\nabla \cdot \boldsymbol{u}) \boldsymbol{\omega}$$

, while recalling that $\nabla \cdot \boldsymbol{u} = 0$. And so we have

$$\boldsymbol{\omega}_t + (\boldsymbol{u} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \boldsymbol{u} + \nu \Delta \boldsymbol{\omega},$$

which is called the *vorticity transport equation.*

The term $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$ is the so-called *vorticity stretching term*, and is the term that controls how the vorticity vector field can be amplified or diminished by the local straining flows of the fluid flow (in addition to being advected and diffused). To see this, we recall that $\nabla \boldsymbol{u} = \mathbf{E} + \mathbf{W}$, where $\boldsymbol{f} \cdot \mathbf{W} = \frac{1}{2} \boldsymbol{\omega} \times \boldsymbol{f}$ for any vector $\boldsymbol{f}$. Hence we have

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \mathbf{E} + \nu \Delta \boldsymbol{\omega}$$

Recall that the symmetric rate-of-strain tensor $\mathbf{E}$ is trace-free. If $\boldsymbol{\omega}$ is aligned with a principal direction of positive (negative) rate-of-strain, then the magnitude of $\boldsymbol{\omega}$ will be increased (decreased) (neglecting diffusion). What we shall see shortly is that vorticity actually induces the velocity field, and hence the straining flow in which it evolves. This coupling of vorticity stretching/depletion to the vorticity dynamics itself makes the understanding of the 3-d Navier-Stokes equations especially difficult.

This vorticity transport equation simplifies considerably in two dimensions. In the x-y plane, $\boldsymbol{u} = (u(x,y), v(x,y), 0)$ and $\boldsymbol{\omega} = (v_x - u_y) \hat{\boldsymbol{z}} = \omega(x,y) \hat{\boldsymbol{z}}$. $\omega(x,y,t)$ is called the *scalar vorticity.* Then note that

$$\boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = \begin{bmatrix} u_x & u_y & 0 \\ v_x & v_y & 0 \\ 0 & 0 & 0 \end{bmatrix}^\top \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = \mathbf{0} \Rightarrow$$

$$\Rightarrow \frac{D\omega}{Dt} = \nu \Delta \omega$$

Thus, in two dimensions there is no vorticity stretching term and vorticity is only transported and diffused.

Inviscid vorticity dynamics in the Lagrangian frame: It is worth examining vorticity transport in the Lagrangian frame in the absence of viscosity. Both the two- and three-dimensional equations reflect fundamental conservation laws of the Euler equations.

2d: In two dimensions the statement $D\omega/Dt = 0$ simply becomes, in the Lagrangian frame:

$$\omega_t = 0 \Rightarrow \omega(\boldsymbol{\alpha}, t) = \omega_0(\boldsymbol{\alpha})$$

or that vorticity is conserved in the Lagrangian frame, that is, along Lagrangian particle paths.

3d: The 3d statement is similar, but more complicated. We manipulate the vorticity advection equation in the Lagrangian frame using the evolution equation for the deformation tensor $\mathbf{F}$:

$$\boldsymbol{\omega}_t = \boldsymbol{\omega} \cdot \nabla_x \boldsymbol{u} = \boldsymbol{\omega} \cdot \mathbf{F}^{-1} \cdot \mathbf{F} \cdot \nabla_x \boldsymbol{u} = \boldsymbol{\omega} \cdot \mathbf{F}^{-1} \cdot \frac{D\mathbf{F}}{Dt}$$

giving the conservation law:

$$\left(\boldsymbol{\omega} \cdot \mathbf{F}^{-1}\right)_t = \mathbf{0} \Rightarrow \boldsymbol{\omega}(\boldsymbol{\alpha}, t) = \boldsymbol{\omega}_0(\boldsymbol{\alpha}) \cdot \mathbf{F}(\boldsymbol{\alpha}, t)$$

This is the so-called *Result of Cauchy*, already referred to, which states that vorticity is stretched or depleted by the action of the deformation tensor.

The vorticity-stream formulation: This establishes the fundamental relation between fluid velocity and vorticity. We work this out for two-dimensions; there is a similar but more complex relation in 3d.

In two-dimensions the condition $\nabla \cdot \boldsymbol{u} = 0 \Longrightarrow \exists\, \psi$ such that $\boldsymbol{u} = \nabla^{\perp}\psi = (-\psi_y, \psi_x) \Longrightarrow \omega = v_x - u_y = \psi_{xx} + \psi_{yy} = \Delta\psi$. For an open flow this then yields the Biot-Savart law.

$$\psi(\boldsymbol{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} dA_{x'} \ln\left|\boldsymbol{x} - \boldsymbol{x}'\right| \omega\left(\boldsymbol{x}'\right) \Longrightarrow$$
$$\boldsymbol{u}(\boldsymbol{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} dA_{x'} \frac{\left(\boldsymbol{x} - \boldsymbol{x}'\right)^{\perp}}{\left|\boldsymbol{x} - \boldsymbol{x}'\right|^2}\omega\left(\boldsymbol{x}'\right)$$

This is generally true. But, for an inviscid fluid in the Lagrangian frame we have

$$\boldsymbol{u}(\boldsymbol{X}(\boldsymbol{\alpha}, t), t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} dA_{\alpha'} \frac{\left(\boldsymbol{X}(\boldsymbol{\alpha}, t) - \boldsymbol{X}(\boldsymbol{\alpha}', t)\right)^{\perp}}{\left|\boldsymbol{X}(\boldsymbol{\alpha}, t) - \boldsymbol{X}(\boldsymbol{\alpha}', t)\right|^2}\omega\left(\boldsymbol{X}\left(\boldsymbol{\alpha}', t\right), t\right)$$

and using the definition of the Lagrangian frame, and conservation of vorticity along particle paths, we have

$$\boldsymbol{X}_t(\boldsymbol{\alpha}, t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} dA_{\alpha'} \frac{\left(\boldsymbol{X}(\boldsymbol{\alpha}, t) - \boldsymbol{X}(\boldsymbol{\alpha}', t)\right)^{\perp}}{\left|\boldsymbol{X}(\boldsymbol{\alpha}, t) - \boldsymbol{X}(\boldsymbol{\alpha}', t)\right|^2}\omega_0\left(\boldsymbol{\alpha}'\right)$$

which is a closed set of equations for evolving the Lagrangian flow map (sometimes expressed as "vorticity moves itself").

4.3 Special Solutions: Unidirectional Flows

In general the nonlinearity of the NS equations, via the term $\boldsymbol{u} \cdot \nabla\boldsymbol{u}$, prevents finding analytical solutions, and most known solutions are those for which $\boldsymbol{u} \cdot \nabla\boldsymbol{u} = \boldsymbol{0}$. Most such solutions are *unidirectional flows.*

Unidirectional flows with straight streamlines: As an example, consider flows of the form $\boldsymbol{u} = (u(y, z, t), 0, 0)$. Then $\nabla \cdot \boldsymbol{u}$ automatically, and $(\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u} = (u\partial_x)\,\boldsymbol{u} = \boldsymbol{0}$, since u is independent of x. The y- and z-momentum equations reduce to

$$p_y = p_z = 0 \Longrightarrow p = p(x, t).$$

The x-momentum equation is

$$\rho u_t = -p_x + \mu\left(u_{yy} + u_{zz}\right).$$

Self-consistency now forces p_x to be independent of x, i.e. $p_x = p_x(t)$ alone: the pressure gradient acts as a specified body force on the fluid (exactly like gravity). To close the problem we need to also specify consistent boundary conditions on u.

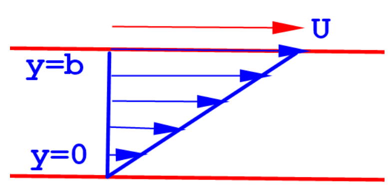

Figure 13: Linear shear (Couette) flow in a channel.

2d Couette flow: Consider 2d flow between two plates at $y = 0, b$ where the bottom plate is stationary and the top plate moves rightwards with speed U. We set the pressure to a constant and it does not enter the solution. We have then the diffusion equation

$$\rho\, u_t(y,t) = \mu\, u_{yy}, \text{ with } u(0,t) = 0 \text{ and } u(b,t) = U$$
$$u(y,0) = u_0(y).$$

The steady-state solution is the linear shear profile $\bar{u}(y) = \frac{U}{b} y$.

- While this solution is independent of the viscosity μ, the force required to push the upper plate at speed U is not. Using the stress tensor we calculate the stress (force/area) applied by the plate on the fluid to be $\boldsymbol{s} = \frac{\mu U}{b}\hat{\boldsymbol{x}}$. Thus, the required stress scales linearly with μ, and diverges as $b \to 0$.
- Note that if $\mu = 0$ we would be unable to apply the boundary conditions. That is, the boundary value problem would be ill-posed.
- It is straightfoward to show that $u(y,t) \to \bar{u}(y)$ as $t \to \infty$ for any initial data $u_0(y)$.
- A Reynolds number here is $Re = \frac{\rho U b}{\mu}$. For the full 2d Navier-Stokes equations the linear shear solution is linearly stable for all Reynolds number, but transitions to turbulence at rather modest value of Re via nonlinear instability.

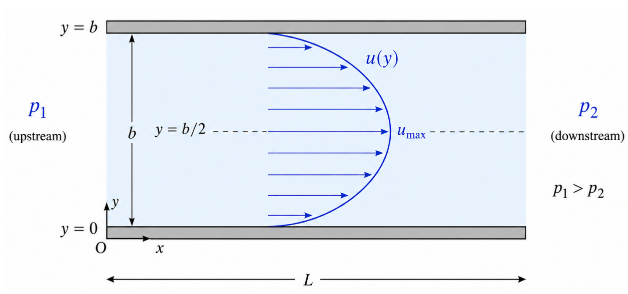

Figure 14: Parabolic flow in a channel under the action of a constant pressure gradient.

2d parabolic channel flow (Poiseuille): Here is another classic example. Consider a channel between two plates at $y = 0, b$ of length L, with $p = p_1$ at $x = 0$ and $p = p_2$ at $x = L$, with $p_1 > p_2$ so the fluid moves left to right. Seek a solution of the form

$$\boldsymbol{u} = (u(y,t), 0)\,, \quad p = \frac{\Delta p}{L} x, \quad \Delta p \equiv p_2 - p_1 < 0.$$

Note that at the walls we satisfy the no-penetration condition $\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = 0$.

Inviscid case ($\mu = 0$): With no viscosity to balance the constant forcing, u grows without bound,

$$u(y,t) = u_0(y) - \frac{\Delta p}{L}\, t,$$

the analog of a free particle accelerating under a constant force.

Viscous case ($\mu > 0$): The x-momentum equation becomes a forced heat equation,

$$\rho\, u_t(y,t) = -\frac{\Delta p}{L} + \mu\, u_{yy}, \quad u|_{y=0,b} = 0$$
$$u(y,0) = u_0(y).$$

The steady state satisfies

$$-\frac{\Delta p}{L} + \mu\, u_{yy} = 0 \Longrightarrow u = A + By + \frac{1}{2\mu}\frac{p_1 - p_2}{L}\, y^2,$$

and applying the boundary conditions gives the *Poiseuille* parabolic profile

$$u(y) = \frac{1}{2\mu}\frac{p_1 - p_2}{L}\, y(b-y).$$

This steady parabolic flow is an attracting state for unidirectional flow.

The volume flux: The flux through the channel (per unit depth) is

$$Q = \int_0^b \rho\, u\, dy = \frac{\rho}{2\mu}\frac{p_1 - p_2}{L}\int_0^b \left(by - y^2\right) dy = \frac{\rho}{12\mu}\frac{p_1 - p_2}{L}\, b^3,$$

so $Q \sim O(b^3)$: the flux is extremely sensitive to the gap width.

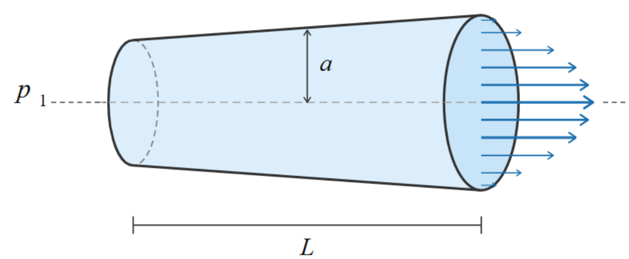

Figure 15: Parabolic flow in a circular pipe under the action of a constant pressure gradient.

Pipe flow (Hagen–Poiseuille): For flow driven by a constant axial pressure gradient $G = (p_1 - p_2)/L$ through a circular pipe of radius a, the axisymmetric unidirectional analog of the channel calculation gives the parabolic profile

$$u(r) = \frac{G}{4\mu}\left(a^2 - r^2\right),$$

with volume flux

$$Q = \frac{\pi G}{8\mu}\, a^4,$$

so that $Q \sim O(a^4)$: pipe flow is even more sensitive to the tube radius than channel flow is to the gap width, giving very high hydraulic impedance for narrow tubes (e.g. hypodermic needles).

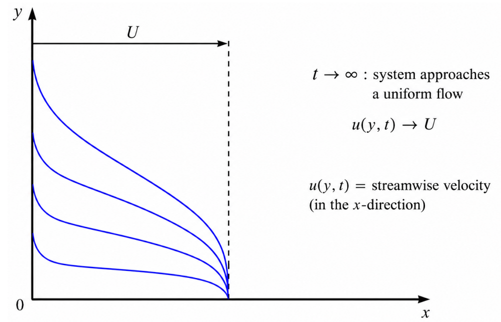

Figure 16: The horizontal, unidirectional flow above a wall that is impulsively started.

The impulsively started plate (Rayleigh's problem) Consider a semi-infinite 2d fluid $y > 0$ above a rigid plate at $y = 0$, initially at rest, with $\boldsymbol{u} = (u(y,t), 0, 0)$. At $t = 0$ the plate is impulsively set into motion at constant velocity U:

$$u(y,0) = 0 \text{ for } y > 0, \qquad u(0,t) = U \text{ for } t > 0.$$

Then $\nabla \cdot \boldsymbol{u} = 0$ and $(\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u} = \boldsymbol{0}$ automatically; with no pressure gradient, the x-momentum equation is the pure diffusion equation

$$u_t = \nu\, u_{yy}.$$

We look directly for a similarity solution $u(y,t) = f(\eta)$ with $\eta = y/t^{\beta}$, for some exponent β to be determined. Then

$$u_t = -\beta\, \frac{y}{t^{\beta+1}} f_\eta, \qquad u_{yy} = t^{-2\beta} f_{\eta\eta},$$

so the PDE becomes

$$-\beta\, t^{-1} \eta\, f_\eta = \nu\, t^{-2\beta} f_{\eta\eta}.$$

Choosing β to eliminate t requires $\beta = \frac{1}{2}$, giving the reduced ODE

$$-\frac{\eta}{2} f_\eta = \nu f_{\eta\eta} \Longrightarrow \frac{f_{\eta\eta}}{f_\eta} = -\frac{\eta}{2\nu}.$$

Integrating,

$$\partial_\eta \ln f_\eta = \partial_\eta \left(-\frac{\eta^2}{4\nu}\right) \Longrightarrow f_\eta = K\, e^{-\eta^2/4\nu},$$

and hence

$$f(\eta) = A + K \int_0^\eta e^{-\eta'^2/4\nu}\, d\eta' = A + K' \operatorname{erf}\!\left(\frac{\eta}{2\sqrt{\nu}}\right).$$

Applying the boundary conditions: $u(0,t) = U \Rightarrow f(0) = A = U$, and $\lim_{y\to\infty} u(y,t) = \lim_{\eta\to\infty} f(\eta) = A + K' = 0 \Rightarrow K' = -U$. Hence

$$\boxed{u(y,t) = U\left(1 - \operatorname{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right).}$$

Consequences. As $t \to \infty$, the system relaxes to uniform flow $u \equiv U$ everywhere: the plate's motion diffuses outward through the fluid. There is no vertical mass flux ($v \equiv 0$ throughout); rather, the plate injects *vorticity* into the fluid, which then spreads diffusively. (An alternative mechanism for vorticity generation, not relevant here, is baroclinic production via density gradients.)

Unidirectional flows with circular streamlines Streamlines need not be straight for a flow to be unidirectional. Consider axisymmetric, purely azimuthal flow in cylindrical coordinates (r, θ, z),

$$\boldsymbol{u} = u_\theta(r,t)\,\hat{\boldsymbol{e}}_\theta \quad \Rightarrow \quad \omega(r,t) = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta).$$

Continuity is satisfied automatically. The radial momentum equation reduces to a centripetal balance,

$$\frac{\partial p}{\partial r} = \rho\,\frac{u_\theta^2}{r},$$

which fixes the pressure but does not feed back into the dynamics of u_θ. Note that we immediately have that the pressure is at its minimum at $r = 0$. This is the root of the statement that pressure is low inside of a vortex.

In the azimuthal momentum equation, the nonlinear self-advection term $(u_\theta/r)\,\partial u_\theta/\partial\theta$ vanishes identically since $\partial_\theta u_\theta = 0$. So, despite the curved streamlines, the equation for u_θ is linear:

$$\frac{\partial u_\theta}{\partial t} = \nu\,\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial\,(r u_\theta)}{\partial r}\right].$$

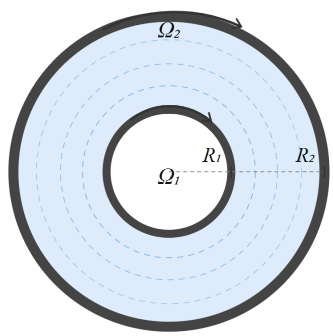

Figure 17: The flow geometry of a Taylor-Couette cell.

Steady circular Taylor-Couette flow. Consider fluid confined between two concentric cylinders of radii $R_1 < R_2$, rotating at constant angular velocities Ω_1, Ω_2. In steady state,

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d\,(r u_\theta)}{dr}\right] = 0 \Longrightarrow \frac{1}{r}\frac{d\,(r u_\theta)}{dr} = 2A \Longrightarrow u_\theta(r) = Ar + \frac{B}{r}.$$

Applying the no-slip conditions $u_\theta(R_1) = \Omega_1 R_1$, $u_\theta(R_2) = \Omega_2 R_2$ gives

$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, \qquad B = \frac{(\Omega_1 - \Omega_2)\,R_1^2 R_2^2}{R_2^2 - R_1^2}.$$

This is the rotational analog of channel flows: a rigid-body-rotation term Ar plus a point-vortex term B/r, in place of the linear and parabolic profiles found there. In the narrow-gap limit $R_2 - R_1 \ll R_1$, expanding in the small gap recovers plane Couette flow, a linear shear profile between two walls moving at the local surface speeds $\Omega_1 R_1$ and $\Omega_2 R_2$ – consistent with curvature effects becoming subdominant once the local geometry looks flat.

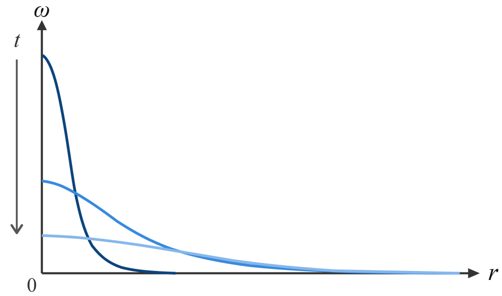

Figure 18: Beginning from a δ-function at the origin, the spreading of vorticity under the action of viscosity in the 2d Navier-Stokes equations.

The decaying line vortex. Now consider an initial condition concentrated as a δ-function at the origin, which is a line vortex of circulation Γ_0, for which $u_\theta(r,0) = \frac{\Gamma_0}{2\pi}\frac{1}{r}$ for $r > 0$. This is the circular-streamline analog of the impulsively started plate.

It is easiest to work with the (scalar) vorticity which satisfies the axisymmetric diffusion equation

$$\frac{\partial \omega}{\partial t} = \nu \, \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial \omega}{\partial r}\right).$$

The initial line vortex corresponds to all the vorticity concentrated at the origin, $\omega(r,0) = \Gamma_0\,\delta^2(\boldsymbol{x})$, with total circulation $\oint \boldsymbol{u}\cdot d\boldsymbol{s} = \int \omega\, dA = \Gamma_0$. This is precisely the fundamental (point-source) solution of the 2D heat equation, giving the *Lamb–Oseen vortex*:

$$\omega(r,t) = \frac{\Gamma_0}{4\pi\nu t}\exp\!\left(-\frac{r^2}{4\nu t}\right).$$

Recovering u_θ from $r u_\theta = \int_0^r r'\omega(r',t)\,dr'$ and substituting $s = r'^2/4\nu t$:

$$r\,u_\theta(r,t) = \frac{\Gamma_0}{4\pi\nu t}\int_0^r r' e^{-r'^2/4\nu t}\,dr' = \frac{\Gamma_0}{2\pi}\left[1 - \exp\!\left(-\frac{r^2}{4\nu t}\right)\right],$$

so that

$$\boxed{u_\theta(r,t) = \frac{\Gamma_0}{2\pi r}\left[1 - \exp\!\left(-\frac{r^2}{4\nu t}\right)\right].}$$

Consequences. For fixed $r > 0$ as $t \to 0^+$, the exponential vanishes and $u_\theta \to \Gamma_0/(2\pi r)$, the ordinary irrotational point-vortex flow. Near the core, expanding the exponential for small $r^2/4\nu t$ gives $u_\theta \approx \Gamma_0 r/(8\pi\nu t)$: solid-body rotation, with no singularity at $r = 0$ despite the singular initial data. The vortex core has a viscous length scale $\sqrt{\nu t}$, entirely analogous to the diffusive boundary-layer thickness in the impulsively started plate. As $t \to \infty$, $u_\theta \to 0$ everywhere: the vortex spreads and weakens while the total circulation Γ_0 is conserved, just as total momentum flux is conserved (and merely redistributed) in the Rayleigh problem.

Comment: Note that if we have $\nu = 0$ for an open axisymmetric flow, then $\partial_t \omega(r,t) = 0$. This implies that axisymmetric flows for the 2d Euler equations are steady-states.

5 The Stokes Equations

The Stokes equations have considerable analytic structure. Again,

$$-\nabla p + \mu \Delta \boldsymbol{u} = \boldsymbol{0} \text{ and } \nabla \cdot \boldsymbol{u} = 0$$

It is often useful to write them as:

$$\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{0} \text{ and } \nabla \cdot \boldsymbol{u} = 0 \text{ with } \boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{E}$$

Taking a divergence of the momentum equation gives that

$$\Delta p = 0$$

so the pressure is harmonic (in the absence of an external force). Taking a curl of the momentum equation gives that

$$\Delta \boldsymbol{\omega} = \boldsymbol{0}$$

so the vorticity is also harmonic. In three dimensions the divergence-free condition on velocity implies the existence of a vector stream function $\boldsymbol{\Psi}$ which satisfies $\Delta\boldsymbol{\Psi} = -\boldsymbol{\omega}$. Hence

$$\Delta^2 \boldsymbol{\Psi} = \boldsymbol{0}$$

and so the stream function is biharmonic.

All of these results point to a fundamental character of the Stokes equations: they are solved as a boundary value problem. Taking two dimensions as an example, there is a scalar stream function ψ satisfying $\Delta^2\psi = 0$. If we have velocity boundary conditions on a body then we are specifying $\partial\psi/\partial s$ and $\partial\psi/\partial n$ (i.e. normal and tangential velocity components, respectively). These BCs determine ψ up to a constant which determines the gradients of ψ, the velocities, uniquely.

5.1 Fundamental Solutions of the Stokes Equations

Because the Stokes equations are constant coefficient linear PDEs, solutions to them can be represented in terms of Green's functions. There are several important fundamental solutions for the Stokes equations, such as the Stokeslet, Rotlet, and Stresslet.

Formal Construction of the Stokeslet: Find a solution to the equation

$$\nabla \cdot \tilde{\boldsymbol{\sigma}} = -\nabla q + \mu \Delta \boldsymbol{v} = -\hat{\boldsymbol{e}}\,\delta(\boldsymbol{x}) \text{ and } \nabla \cdot \boldsymbol{v} = 0$$

where $\hat{\boldsymbol{e}}$ is an arbitrary unit vector, and δ is the 3d δ-function. Recall that the 3d free-space Green's function for the Laplacian is

$$G = \frac{-1}{4\pi}\frac{1}{|\boldsymbol{x}|}$$

i.e., $\Delta G = \delta(\boldsymbol{x})$. Taking a divergence gives $\Delta q = \hat{\boldsymbol{e}} \cdot \nabla\delta = \hat{\boldsymbol{e}} \cdot \nabla\Delta G = \Delta(\hat{\boldsymbol{e}} \cdot \nabla G)$ and so we choose

$$q = \hat{\boldsymbol{e}} \cdot \nabla G = \frac{1}{4\pi}\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} \cdot \hat{\boldsymbol{e}} = \frac{1}{4\pi}\frac{\hat{\boldsymbol{x}}}{|\boldsymbol{x}|^2} \cdot \hat{\boldsymbol{e}} = P_k \hat{e}_k$$

Hence, the fundamental solution for the pressure is

$$P_k = \frac{1}{4\pi}\frac{\hat{x}_k}{|\boldsymbol{x}|^2}$$

We then have

$$\nabla q = \hat{\boldsymbol{e}} \cdot \frac{\mathbf{I} - 3\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}{4\pi\,|\boldsymbol{x}|^3}$$
$$\Rightarrow \mu\Delta\boldsymbol{v} = -\hat{\boldsymbol{e}}\,\delta + \nabla\left(\hat{\boldsymbol{e}}\cdot\nabla G\right)$$

Now we construct two functions B_1, B_2 that satisfy

$$\Delta B_1 = \delta \text{ and } \Delta B_2 = G$$

and let $\boldsymbol{v} = \frac{1}{\mu}\left(-\hat{\boldsymbol{e}}\,B_1 + \nabla\left(\hat{\boldsymbol{e}}\cdot\nabla B_2\right)\right)$. Taking only the radially symmetric particular solutions for $B_{1,2}$ gives

$$B_1 = G \text{ and } B_2 = \frac{-1}{8\pi}\,|\boldsymbol{x}|$$

and further calculation gives

$$\boldsymbol{v} = \hat{\boldsymbol{e}} \cdot \left(\frac{1}{8\pi\mu}\right)\left[\frac{\mathbf{I} + \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}{|\boldsymbol{x}|}\right] = \hat{\boldsymbol{e}}\cdot\mathbf{S}\left(\boldsymbol{x}\right)$$

The rank-2 tensor

$$\mathbf{S} = \frac{1}{8\pi\mu}\frac{\mathbf{I} + \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}{|\boldsymbol{x}|} \quad \text{or} \quad S_{ij} = \frac{1}{8\pi\mu}\frac{\delta_{ij} + \hat{x}_i\hat{x}_j}{|\boldsymbol{x}|}$$

is called the *Stokeslet* or the *Oseen tensor*. It has a long-range $|\boldsymbol{x}|^{-1}$ decay and is symmetric positive definite. It can be used to construct other relevant fundamental solutions.

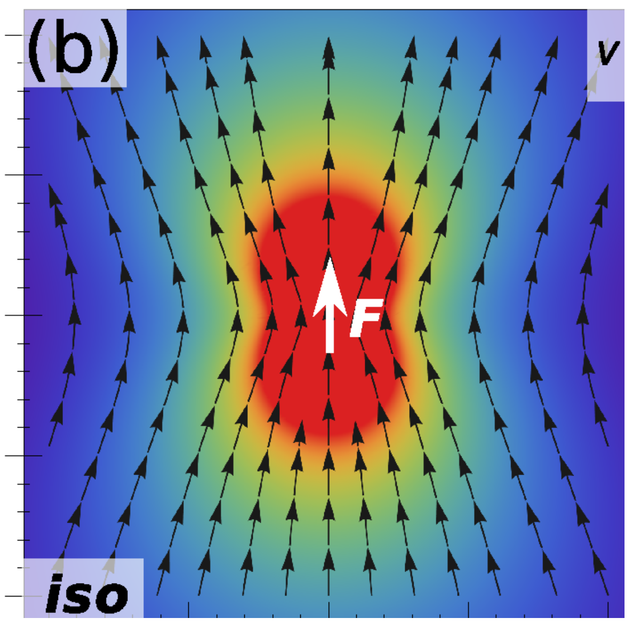

Figure 19: The flow field induced by a Stokeslet.

We define the *Stresslet* as the rank-3 tensor $\boldsymbol{\mathcal{T}}$ with components $\mathcal{T}_{ijk}$ satisfying $\tilde{\sigma}_{ij} = \mathcal{T}_{ijk}\hat{e}_k$, or

$$\mathcal{T}_{ijk} = -P_k\delta_{ij} + \mu\left(\frac{\partial S_{jk}}{\partial x_i} + \frac{\partial S_{ik}}{\partial x_j}\right)$$
$$= -\frac{3}{4\pi}\frac{\hat{x}_i\hat{x}_j\hat{x}_k}{|\boldsymbol{x}|^2}$$

The Stokes doublet: Let's calculate the flow response to a force dipole. First, we write the Stokeslet as satisfying a linear functional equation where $\boldsymbol{F}\delta(\boldsymbol{x})$ is the force applied **to the fluid** (this distinction is important for interpretation). Then

$$\mathsf{L}\left[\boldsymbol{v}=\boldsymbol{F}\cdot\mathbf{S}(\boldsymbol{x})\right]+\boldsymbol{F}\ \delta(\boldsymbol{x})=\mathbf{0}\Leftrightarrow\mathsf{L}\left[v_j=F_iS_{ij}(\boldsymbol{x})\right]=-F_j\delta(\boldsymbol{x})$$

To construct a force dipole we place oppositely oriented forces $\pm\boldsymbol{F}$ at positions $\pm\frac{1}{2}\boldsymbol{h}$, respectively. The doublet exerts zero net force upon the fluid, but does exert a torque: $\boldsymbol{T}=\boldsymbol{h}\times\boldsymbol{F}$. By linearity, the consequent flow is then

$$\mathsf{L}\left[v_j=F_i\left(S_{ij}\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{h}\right)-S_{ij}\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{h}\right)\right)\right]=-F_j\left(\delta\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{h}\right)-\delta\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{h}\right)\right).$$

Fix $\boldsymbol{x}\neq\mathbf{0}$, take $|\boldsymbol{h}|<<|\boldsymbol{x}|$. At leading order we have

$$\mathsf{L}\left[v_j=-h_kF_i\frac{\partial S_{ij}}{\partial x_k}(\boldsymbol{x})\right]=\frac{\partial\delta}{\partial x_i}(\boldsymbol{x})h_iF_j=\left[\nabla\cdot(\delta(\boldsymbol{x})\boldsymbol{h}\boldsymbol{F})\right]_j$$

Hence, the consequent velocity $\boldsymbol{v}$ can be interpreted as being in response to the extra-stress $\boldsymbol{\sigma}^e=-\delta(\boldsymbol{x})\mathbf{D}$ where $\mathbf{D}=\boldsymbol{h}\boldsymbol{F}$ is the so-called dipole tensor, which has both symmetric and antisymmetric contributions.

We next calculate

$$\begin{aligned}\frac{\partial S_{ij}}{\partial x_k}(\boldsymbol{x})&=\frac{1}{8\pi\mu}\frac{\partial}{\partial x_k}\left[\frac{\delta_{ij}}{|\boldsymbol{x}|}+\frac{x_ix_j}{|\boldsymbol{x}|^3}\right]\\&=\frac{1}{8\pi\mu}\left[-\frac{\delta_{ij}\hat{x}_k}{|\boldsymbol{x}|^2}+\frac{\delta_{ik}\hat{x}_j+\hat{x}_i\delta_{jk}}{|\boldsymbol{x}|^2}-\frac{3\hat{x}_i\hat{x}_j\hat{x}_k}{|\boldsymbol{x}|^2}\right].\end{aligned}$$

The Rotlet arises from the antisymmetric part of the dipole tensor $D^A_{ki}=\frac{1}{2}(h_kF_i-F_kh_i)$ (Note $\boldsymbol{T}=-\boldsymbol{\epsilon}:\mathbf{D}^A$). Using the identity $\boldsymbol{\epsilon}\cdot\boldsymbol{T}=\boldsymbol{h}\boldsymbol{F}-\boldsymbol{F}\boldsymbol{h}$ gives $\mathbf{D}^A=\frac{1}{2}\boldsymbol{\epsilon}\cdot\boldsymbol{T}$ ($D^A_{ik}=\frac{1}{2}\epsilon_{ikl}T_l$). We define $v^R_j=-D^A_{ki}\partial_kS_{ij}$ (R for Rotlet) and note that two symmetric k,i components of ∂_kS_{ij} are anniliated by the D^A_{ki} contraction. This yields

$$v^R_j(\boldsymbol{x})=-\frac{1}{8\pi\mu}\frac{\epsilon_{jkl}\hat{x}_k}{|\boldsymbol{x}|^2}T_l\Leftrightarrow\boldsymbol{v}^R(\boldsymbol{x})=-\frac{1}{8\pi\mu}\frac{1}{|\boldsymbol{x}|^2}(\hat{\boldsymbol{x}}\times\boldsymbol{T}).$$

This the Rotlet flow. It's associated pressure field is $p^R\equiv0$, and the flow itself is two-dimensional in planes orthoginal to the $\boldsymbol{T}$ axis. It can be interpreted as arising from the antisymmetric extra stress $\boldsymbol{\sigma}^A=-\delta(\boldsymbol{x})\mathbf{D}^A$.

Dipolar flows: The symmetric part of $\mathbf{D}$ gives rise to an extensile or contractile dipolar flow. This is most easily found by directly aligning displacement and force: $\boldsymbol{h}=h\boldsymbol{e}$ and $\boldsymbol{F}=F\boldsymbol{e}$, where $\boldsymbol{e}$ is an orientation vector. In this case there is no net torque, and the dipole tensor is $\boldsymbol{D}=\tilde{F}\boldsymbol{e}\boldsymbol{e}$ is directly symmetric and $\tilde{F}=hF$. The consequent velocity is

$$\boldsymbol{v}^S(\boldsymbol{x})=-\frac{\tilde{F}}{8\pi\mu}\frac{1-3\left(\hat{\boldsymbol{e}}\cdot\hat{\boldsymbol{x}}\right)^2}{|\boldsymbol{x}|^2}\hat{\boldsymbol{x}}$$

For $\tilde{F}<0$, this is a contractile dipolar flow streaming **inwards** along the $\hat{\boldsymbol{e}}$-axis, consistent with the form of the extra-stress and of the forcing.

5.2 Some exact solutions to the Stokes equations:

The Stokes solution for a sphere: Consider a sphere of radius a moving at velocity $\boldsymbol{U}$ under an applied force $\boldsymbol{F}$. Setting the sphere at the origin, the external flow is given by

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{F} \cdot \left[\mathbf{S}(\boldsymbol{x}) + \frac{a^2}{6}\mathbf{H}(\boldsymbol{x})\right]$$

where $\mathbf{S}$ is the Stokeslet tensor and $\mathbf{H}(\boldsymbol{x}) = (\mathbf{I} - 3\hat{\boldsymbol{x}}\hat{\boldsymbol{x}})/(4\pi\mu|\boldsymbol{x}|^3)$ is the source doublet. Evaluating upon the surface of the sphere, i.e. $\boldsymbol{x} = a\hat{\boldsymbol{x}}$, yields Stokes' famous result

$$\boldsymbol{U} = \frac{1}{6\pi\mu a}\boldsymbol{F}.$$

Further, the fluid stress everywhere on the sphere is uniform, and given by

$$\boldsymbol{s} = \hat{\boldsymbol{x}} \cdot \boldsymbol{\sigma} = \mu\frac{3U}{2a}\hat{\boldsymbol{\zeta}}$$

where the polar axis of the sphere is taken in the direction $\hat{\boldsymbol{\zeta}}$ pointing along $\boldsymbol{F}$ (the far-field pressure has also been taken as zero). From this we again have Stokes formula:

$$\boldsymbol{F} = \int_S dA\, \boldsymbol{s} = 6\pi\mu a\boldsymbol{U}$$

The Jeffrey equation for ellipsoidal particles: Consider an axisymmetric ellipsoid of length l and diameter d rotating in a linear flow $\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{x} \cdot \mathbf{A}$ so that $\nabla\boldsymbol{u} = \mathbf{A} = \mathbf{E} + \mathbf{W}$. Let the unit vector $\boldsymbol{p}\,(t)$ point in the direction of the major axis, $\boldsymbol{X}_c\,(t)$ be the ellipsoid center, and assume that no force or torque acts upon the ellipsoid. Then (Jeffrey, 1922)

$$\begin{aligned}
\dot{\boldsymbol{X}}_c &= \boldsymbol{U} + \boldsymbol{X}_c \cdot \mathbf{A} \\
\dot{\boldsymbol{p}} &= \boldsymbol{p} \cdot \mathbf{W} + \frac{\lambda^2 - 1}{\lambda^2 + 1}\,(\mathbf{I} - \boldsymbol{pp}) \cdot \mathbf{E} \cdot \boldsymbol{p} \\
&= (\mathbf{I} - \boldsymbol{pp}) \cdot \left[-\mathbf{W} + \frac{\lambda^2 - 1}{\lambda^2 + 1}\mathbf{E}\right] \cdot \boldsymbol{p}
\end{aligned}$$

with $\lambda = l/d$. (The second line rewrites the first using $\boldsymbol{p} \cdot \mathbf{X} = \mathbf{X}^\top \cdot \boldsymbol{p}$ for any rank-2 tensor $\mathbf{X}$, so the whole expression reads as an unambiguous left-to-right chain; $\mathbf{E}^\top = \mathbf{E}$ and $\mathbf{W}^\top = -\mathbf{W}$ account for the sign change on the $\mathbf{W}$ term.)

- *Sphere:* $\lambda = 1 \Rightarrow \dot{\boldsymbol{p}} = \boldsymbol{p} \cdot \mathbf{W} = \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{p}$. Rotation of the director about the vorticity vector. The strain flow contributes nothing to the rotation of the sphere.

  Sidenote: Consider the tensor $\mathbf{Q} = \boldsymbol{pp}$. Then

  $$\begin{aligned}
  \dot{\mathbf{Q}} &= \dot{\boldsymbol{p}}\boldsymbol{p} + \boldsymbol{p}\dot{\boldsymbol{p}} = \mathbf{Q} \cdot \mathbf{W} - \mathbf{W} \cdot \mathbf{Q}, \text{ or} \\
  \dot{\mathbf{Q}} &- \mathbf{Q} \cdot \mathbf{W} + \mathbf{W} \cdot \mathbf{Q} = \mathbf{0}
  \end{aligned}$$

  $\overset{\triangle}{\mathbf{Q}} = \dot{\mathbf{Q}} - \mathbf{Q} \cdot \mathbf{W} + \mathbf{W} \cdot \mathbf{Q}$ is called the Jaumann or co-rotational derivative.

- *Slender rod:* $\lambda = \infty \Rightarrow \dot{\boldsymbol{p}} = (\mathbf{I} - \boldsymbol{pp}) \cdot (\nabla\boldsymbol{u})^\top \cdot \boldsymbol{p}$. Rotated by the flow, but constrained from stretching.

- *Plate:* $\lambda = 0 \Rightarrow \dot{\boldsymbol{p}} = -\,(\mathbf{I} - \boldsymbol{pp}) \cdot \nabla\boldsymbol{u} \cdot \boldsymbol{p}$

5.3 The Lorentz Identity

A fundamental identity satisfied by any two solutions $(\boldsymbol{\sigma}, \boldsymbol{u})$ and $(\tilde{\boldsymbol{\sigma}}, \boldsymbol{v})$ of the Stokes equations is the Lorentz identity:

$$\nabla \cdot (\boldsymbol{v} \cdot \boldsymbol{\sigma} - \boldsymbol{u} \cdot \tilde{\boldsymbol{\sigma}}) = 0 \text{ or } \frac{\partial}{\partial x_i} (v_j \sigma_{ji} - u_j \tilde{\sigma}_{ji}) = 0$$

Using symmetry of the stress tensor, we can write:

$$\nabla \cdot (\boldsymbol{v} \cdot \boldsymbol{\sigma} - \boldsymbol{u} \cdot \tilde{\boldsymbol{\sigma}}) = \boldsymbol{\sigma} : \nabla \boldsymbol{v} - \tilde{\boldsymbol{\sigma}} : \nabla \boldsymbol{u}$$
$$= [-p\mathbf{I} + 2\mu \mathbf{E}_u] : [\mathbf{E}_v + \mathbf{W}_v] - [-q\mathbf{I} + 2\mu \mathbf{E}_v] : [\mathbf{E}_u + \mathbf{W}_u] = 0$$

Integrating this identity in the exterior of a moving body with boundary $\Gamma(t)$ and exterior normal $\hat{\boldsymbol{n}}$, and using the divergence theorem yields the integral form

$$\int_{\Gamma(t)} dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{v} = \int_{\Gamma(t)} dS\, \hat{\boldsymbol{n}} \cdot \tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{u} \Leftrightarrow \int_{\Gamma(t)} dS\, \boldsymbol{s} \cdot \boldsymbol{v} = \int_{\Gamma(t)} dS\, \tilde{\boldsymbol{s}} \cdot \boldsymbol{u},$$

The Lorentz identity is extremely useful:

The speed of a Blake squirmer (Stone & Samuel *PRL* '96): Consider a freely-moving spherical body upon which a surface velocity $\boldsymbol{u}_b(\boldsymbol{y}, t)$ (body frame) is prescribed so that $\boldsymbol{u}^\Gamma = \boldsymbol{U}(t) + \boldsymbol{u}_b(\boldsymbol{y}, t)$ (in the lab frame) for $\boldsymbol{y} \in \Gamma$. Its total force is zero. Let's say the body is a sphere of radius a. Then, use Stokes solution $(\tilde{\boldsymbol{s}}, \tilde{\boldsymbol{v}})$ for a sphere to derive:

$$\boldsymbol{U} = -\frac{1}{4\pi a^2} \int_\Gamma dS_y\, \boldsymbol{u}_b(\boldsymbol{y}, t)$$

Blake squirmer models (1971): the surface velocity $\boldsymbol{u}_b$ is tangential and axisymmetric, with $\boldsymbol{u}_b(\phi) = u_\phi \hat{\boldsymbol{\phi}}$, with ϕ the polar angle and

$$u_\phi = B_1 \sin\phi + B_2 \sin 2\phi.$$

$\beta = B_2/B_1$ and $\boldsymbol{U} = \alpha B_1 \hat{\boldsymbol{z}}$ (independent of B_2). This is originally a model for paramecia and now of Janus particles, and gives a trivial example of Stokes reversibility.

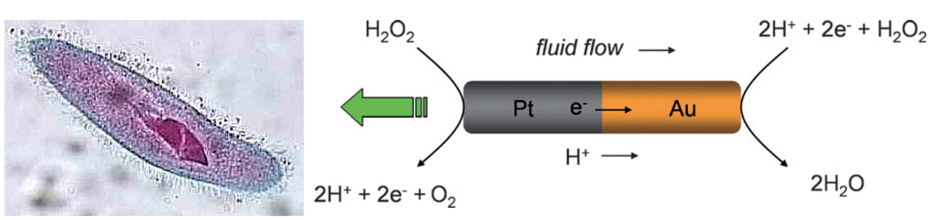

Figure 20: Squirmers in the real world: Left, a paramecium. Right: A synthetic particle propelled by electrophoresis.

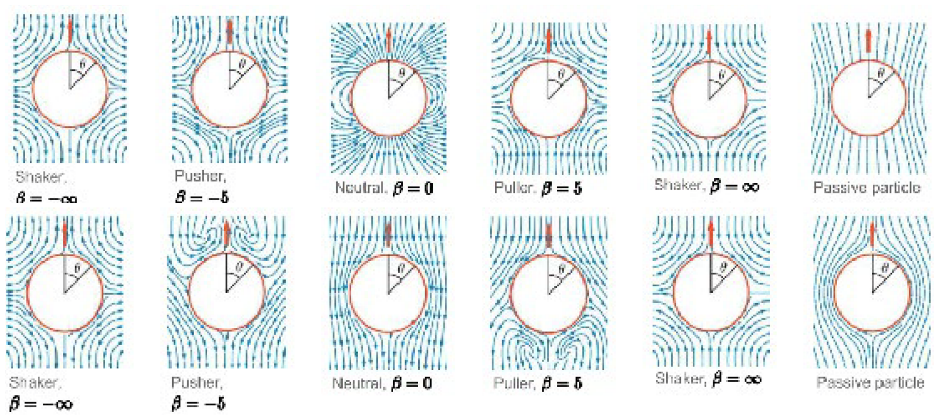

Figure 21: Top: Blake squirmers in the lab frame. Bottom: Blake squirmers in the swimmer frame (the tangential flows).

The Classical Boundary Integral Formulation: The Stokeslet and Stresslet can be used to construct a boundary integral representation for solutions of the Stokes equations, which we very roughly outline (see Pozrikidis for a more detailed derivation). Consider an infinite fluid domain Ω containing a closed body B with surface Γ and outer normal $\hat{\boldsymbol{n}}$, and with a surface stress distribution $\boldsymbol{\zeta}$ (acting upon the fluid) and surface velocity $\boldsymbol{u}_\Gamma$. Let $(\sigma, \boldsymbol{u})$ be the Stokes solution satisfying $\hat{\boldsymbol{n}} \cdot \sigma|_\Gamma = -\boldsymbol{\zeta}$ and $\boldsymbol{u}|_\Gamma = \boldsymbol{u}^\Gamma$.

Now let $(\tilde{\sigma}, \boldsymbol{v})$ be the Stresslet/Stokeslet pair translated to place their singularities at a point $\boldsymbol{y}$ in the fluid domain. Then, integrate the Lorentz identity over the punctured fluid domain $\Omega / D_\varepsilon(\boldsymbol{y})$ (with normal into the domain) where $D_\varepsilon(\boldsymbol{y})$ is the ε-ball about $\boldsymbol{y}$, hence excluding the singular point from the domain. The divergence theorem then gives

$$0 = \int_{\Gamma + \{|\boldsymbol{x}'-\boldsymbol{y}|=\varepsilon\}} \left[v_j(\boldsymbol{x}')\sigma_{ij}\left(\boldsymbol{x}'\right) n_i(\boldsymbol{x}') - u_j(\boldsymbol{x}')\tilde{\sigma}_{ij}\left(\boldsymbol{x}'\right) n_i(\boldsymbol{x}') \right] dA_{x'}$$

(1) On $|\boldsymbol{x}-\boldsymbol{y}| = \varepsilon$: Note that on the boundary of $D_\varepsilon(\boldsymbol{y})$, $\widehat{\boldsymbol{x}'-\boldsymbol{y}} = \hat{\boldsymbol{n}}$,

$$\begin{aligned} &\int_{|\boldsymbol{x}'-\boldsymbol{y}|=\varepsilon} v_j(\boldsymbol{x}')\sigma_{ij}\left(\boldsymbol{x}'\right) n_i(\boldsymbol{x}') dA_{x'} \\ &= \frac{1}{8\pi\mu} \int_{|\boldsymbol{x}'-\boldsymbol{y}|=\varepsilon} \frac{\delta_{ij} + n_i n_j}{\varepsilon} \sigma_{ij}\left(\boldsymbol{x}'\right) n_i(\boldsymbol{x}') dA_{x'} \to 0 \text{ as } \varepsilon \to 0 \end{aligned}$$

since the area element scales as ε^2. Now the second term is given by:

$$\begin{aligned} &- \int_{\Gamma + \{|\boldsymbol{x}-\boldsymbol{y}|=\varepsilon\}} u_j(\boldsymbol{x}')\tilde{\sigma}_{ij}\left(\boldsymbol{x}'\right) n_i(\boldsymbol{x}') dA_{x'} \\ &= \frac{3}{4\pi\varepsilon^2} \int_{|\boldsymbol{x}-\boldsymbol{y}|=\varepsilon} u_j n_j n_i \, dA_{x'} \to u_j(\boldsymbol{y}) \end{aligned}$$

This yields

$$\begin{aligned} 0 &= u_k(\boldsymbol{y}) + \int_\Gamma \left[S_{lk}(\boldsymbol{x}'-\boldsymbol{y})\sigma_{il}\left(\boldsymbol{x}'\right) n_i(\boldsymbol{x}') - u_j^\Gamma(\boldsymbol{x}')\mathcal{T}_{ijk}\left(\boldsymbol{x}'-\boldsymbol{y}\right) n_i(\boldsymbol{x}') \right] dA_{x'} \\ &= u_k(\boldsymbol{y}) - \int_\Gamma \left[S_{lk}(\boldsymbol{x}'-\boldsymbol{y})\zeta_l(\boldsymbol{x}') + u_j^\Gamma(\boldsymbol{x}')\mathcal{T}_{ijk}\left(\boldsymbol{x}'-\boldsymbol{y}\right) n_i(\boldsymbol{x}') \right] dA_{x'} \end{aligned}$$

One can thus show that:

$$\boldsymbol{u}(\boldsymbol{y}) = \int_\Gamma \left[\boldsymbol{\zeta}(\boldsymbol{x}') \cdot \mathbf{S}(\boldsymbol{x}' - \boldsymbol{y}) + \boldsymbol{u}^\Gamma(\boldsymbol{x}') \cdot \boldsymbol{\mathcal{T}}\left(\boldsymbol{x}' - \boldsymbol{y}\right) \cdot \hat{\boldsymbol{n}}(\boldsymbol{x}')\right] dA_x'$$

Note that $\mathbf{S}$ is even with respect to its argument, while $\boldsymbol{\mathcal{T}}$ is odd. So, in a more standard convolution form we would have

$$\boldsymbol{u}(\boldsymbol{y}) = \int_\Gamma \left[\boldsymbol{\zeta}(\boldsymbol{x}') \cdot \mathbf{S}(\boldsymbol{y} - \boldsymbol{x}') - \boldsymbol{u}^\Gamma(\boldsymbol{x}') \cdot \boldsymbol{\mathcal{T}}\left(\boldsymbol{y} - \boldsymbol{x}'\right) \cdot \hat{\boldsymbol{n}}(\boldsymbol{x}')\right] dA_{x'}$$

Hence, we have expressed the velocity at every point in the fluid as a function of the surface stress and velocity. Of course the surface velocity and the fluid velocity are related by the no-slip condition, and so it remains to take the limit $\boldsymbol{y} \to \boldsymbol{x} \in \Gamma$. The hard one is the stresslet, so let's do that one first. The dominant part of the limit to the surface should arise from this integral:

$$J_k = \int_{|\boldsymbol{x}-\boldsymbol{x}_0| \leq \varepsilon} u_j^\Gamma(\boldsymbol{x}) \mathcal{T}_{ijk}\left(\boldsymbol{x} - \boldsymbol{y}\right) n_i(\boldsymbol{x}) dA_x$$

where $\boldsymbol{x}_0$ is the closest point to $\boldsymbol{y}$. Let's replace the ε-patch with a flat disk, and assume that $\boldsymbol{y} - \boldsymbol{x} = r\,\hat{\boldsymbol{n}}_0 + \rho\,\mathbf{R}_0 \boldsymbol{e}\left(\theta\right)$ where $\mathbf{R}_0$ is a rotation matrix ($\mathbf{R}_0 \hat{\boldsymbol{z}} = \hat{\boldsymbol{n}}_0$) and $\boldsymbol{e}\left(\theta\right) = (\cos\theta, \sin\theta, 0)$. That is, this is a little (ρ, θ) coordinate system on the patch. Then

$$\widehat{\boldsymbol{y} - \boldsymbol{x}} = \frac{r\,\hat{\boldsymbol{n}}_0 + \rho\,\mathbf{R}_0 \boldsymbol{e}}{\left(r^2 + \rho^2\right)^{1/2}}$$

and so

$$\begin{aligned} \boldsymbol{J} &= -\frac{3}{4} r \int_0^\varepsilon d\rho\,\rho\, \frac{2r^2\left(\boldsymbol{u}^\Gamma \cdot \hat{\boldsymbol{n}}_0\right)\hat{\boldsymbol{n}}_0 + \rho^2\left(\boldsymbol{u}^\Gamma - \left(\boldsymbol{u}^\Gamma \cdot \hat{\boldsymbol{n}}_0\right)\hat{\boldsymbol{n}}_0\right)}{\left(r^2+\rho^2\right)^{5/2}} \\ &\to -\frac{1}{2}\boldsymbol{u}^\Gamma \end{aligned}$$

And so, in this limit we have

$$\frac{1}{2}\boldsymbol{u}^\Gamma(\boldsymbol{x}) = \int_\Gamma \boldsymbol{\zeta}(\boldsymbol{x}') \cdot \mathbf{S}(\boldsymbol{x}' - \boldsymbol{x}) dA_x' + P\int_\Gamma \boldsymbol{u}^\Gamma(\boldsymbol{x}') \cdot \boldsymbol{\mathcal{T}}\left(\boldsymbol{x}' - \boldsymbol{x}\right) \cdot \hat{\boldsymbol{n}}(\boldsymbol{x}') dA_x'$$

where $P\int$ represents a principal value integral. As an integral equation in convolution form we then have

$$\frac{1}{2}\boldsymbol{u}^\Gamma(\boldsymbol{x}) + P\int_\Gamma \boldsymbol{u}^\Gamma(\boldsymbol{x}') \cdot \boldsymbol{\mathcal{T}}\left(\boldsymbol{x} - \boldsymbol{x}'\right) \cdot \hat{\boldsymbol{n}}(\boldsymbol{x}') dS_x' = \int_\Gamma \boldsymbol{\zeta}(\boldsymbol{x}') \cdot \mathbf{S}(\boldsymbol{x} - \boldsymbol{x}') dS_x'$$

or a Fredholm integral equation of the second kind for the surface velocity, or a first-kind equation for the surface stress. If we are given the surface stress, then this equation can in principle be solved for the surface velocity, and then used to give the fluid velocity everywhere in the fluid domain. This is one of the fundamental relations of the Stokes equations.

Purcell's Scallop Theorem: Establishing this linear integral relation between the surface velocity and the surface stress establishes Purcell's Scallop Theorem (1977), which states that if a freely moving body goes through a set of surface deformations (which may not lead to shape changes – this is a statement about the motion of material points of the surface) on the time interval $[0, T]$

and then exactly reverses that set of surface deformations on $[T, 2T]$, the body will end up just where it started (that is, in the same position and orientation). Further, the rate at which the deformation takes place is not important. One could speed up time on the second part of the cycle, and displacement would still be zero. Establishing this theorem is intuitively obvious but technically subtle (see K. Ishimoto, SIAP 2012).

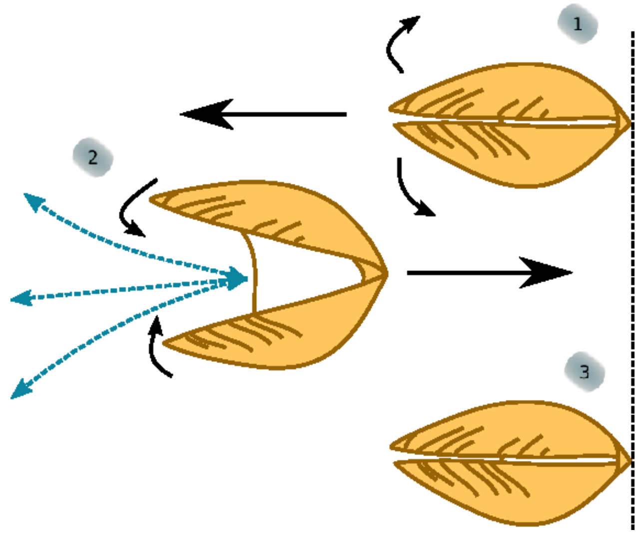

Figure 22: An illustration of Purcell's Scallop Theorem.

"Freely moving" means here that there are no external forces, and so in this regime we have the constraints:

$$\int_\Gamma dS_x \, \boldsymbol{\zeta}\left(\boldsymbol{x}\right) = \int_\Gamma dS_x \left(\boldsymbol{x} - \boldsymbol{X}_c\right) \times \boldsymbol{\zeta}\left(\boldsymbol{x}\right) = \mathbf{0}$$

which become linear equations for translational and rotational velocities. The scallop-theorem is broken by nonlinear fluid rheology, multiple bodies, and fluid elasticity. Still, it establishes why one doesn't typically see time reversible motion of swimmers at low Reynolds number.

Now, for further illustration, consider a rigid body moving under an applied force $\boldsymbol{F}$ and torque $\boldsymbol{L}$, that is

$$\int_\Gamma dS_x \, \boldsymbol{\zeta}\left(\boldsymbol{x}\right) = \boldsymbol{F} \text{ and } \int_\Gamma dS_x \left(\boldsymbol{x} - \boldsymbol{X}_c\right) \times \boldsymbol{\zeta}\left(\boldsymbol{x}\right) = \boldsymbol{L}$$

and being rigid means that for the surface velocity $\boldsymbol{u}^\Gamma = \boldsymbol{U} + \left(\boldsymbol{x} - \boldsymbol{X}_c\left(t\right)\right) \times \boldsymbol{\Omega}\left(t\right)$. Before inserting this into the integral equation we note two identities for $\boldsymbol{x} \in \Omega$ (the fluid domain):

$$\int_\Gamma \boldsymbol{v} \cdot \boldsymbol{\mathcal{T}}\left(\boldsymbol{x} - \boldsymbol{x}'\right) \cdot \hat{\boldsymbol{n}}(\boldsymbol{x}') dS_{x'} = \mathbf{0} \text{ for any constant vector } \boldsymbol{v}.$$

$$\int_\Gamma x_l' \mathcal{T}_{ijk}\left(\boldsymbol{x} - \boldsymbol{x}'\right) n_i(\boldsymbol{x}') dS_{x'} = 0$$

or

$$\boldsymbol{u}(\boldsymbol{x}) = \int_\Gamma \boldsymbol{\zeta}(\boldsymbol{x}') \cdot \mathbf{S}(\boldsymbol{x} - \boldsymbol{x}') dS_x'$$

which means that for rigid bodies, taking the limit $\boldsymbol{x} \to \Gamma$, we have:

$$\boldsymbol{U} + \left(\boldsymbol{x} - \boldsymbol{X}_c\left(t\right)\right) \times \boldsymbol{\Omega}\left(t\right) = \int_\Gamma \boldsymbol{\zeta}(\boldsymbol{x}') \cdot \mathbf{S}(\boldsymbol{x} - \boldsymbol{x}') dS_x'$$

which is an integral equation for $\boldsymbol{\zeta}$ in terms of the two unknowns $\boldsymbol{U}$ and $\boldsymbol{\Omega}$. The system is closed by the specification of the force and torque. The body is then evolved via

$$\dot{\boldsymbol{X}}_c = \boldsymbol{U} \text{ and } \dot{\boldsymbol{\theta}} = \boldsymbol{\Omega}$$

Note however that this is essentially a first-kind integral equation for the surface stress $\boldsymbol{\zeta}$, and is widely used but ill-conditioned.

5.4 Slender-Body Theory

See Tornberg & Shelley (J. Comp. Phys. 196, 8–40 (2004); TS2004) for discussion and references (most especially Keller & Rubinow (JFM 1976), Johnson (JFM 1980), and Gotz (PhD thesis 2000)) on nonlocal slender-body theory.

The leading-order Local SBT: Consider a slender rod of length l, with center-line $\boldsymbol{X}(s,t)$, and radius $r(s) = 2\varepsilon\sqrt{s^2 - l^2/4}$ (ellipsoidal ends) so that $r(s=0) = \varepsilon l$. Take $\varepsilon = a/l \ll 1$. It moves in a background flow $\boldsymbol{u}(\boldsymbol{x},t)$ and has center-line velocity $\boldsymbol{V}(s)$. Let $\boldsymbol{f}(s) = r(s)\int_0^{2\pi} d\theta\, \boldsymbol{\xi}(r(s),\theta)$ be the force per unit length exerted upon the fluid by the fiber. Then

$$\eta\left[\boldsymbol{V}(s) - \boldsymbol{u}(\boldsymbol{X}(s))\right] = (\mathbf{I} + \boldsymbol{X}_s\boldsymbol{X}_s)\cdot\boldsymbol{f}(s) + [\text{nonlocal HOTs}]$$

where $\eta = 8\pi\mu/|c|$, with $c = \ln\left(e\varepsilon^2\right)$. This is a local balance between applied and drag forces. In many cases the arclength s is a material parameter.

A small rigid rod moving freely in a background flow: For a rigid rod we can write

$$\begin{aligned}
\boldsymbol{X}(s,t) &= \boldsymbol{X}_c(t) + s\boldsymbol{p}(t) \Rightarrow \\
\boldsymbol{X}_t &= \boldsymbol{V} = \dot{\boldsymbol{X}}_c + s\dot{\boldsymbol{p}} \\
\boldsymbol{X}_s &= \boldsymbol{p}
\end{aligned}$$

Assume that the length of the rod is very small relative to the length-scale of the flow, i.e., $\boldsymbol{u}(\boldsymbol{x}) \approx \boldsymbol{u}(\boldsymbol{X}_c) + (\boldsymbol{x} - \boldsymbol{X}_c)\cdot\nabla\boldsymbol{u}(\boldsymbol{X}_c)$

$$\eta\left[\dot{\boldsymbol{X}}_c - \boldsymbol{u}(\boldsymbol{X}_c) + s\left(\dot{\boldsymbol{p}} - \boldsymbol{p}\cdot\nabla\boldsymbol{u}(\boldsymbol{X}_c)\right)\right] = (\mathbf{I} + \boldsymbol{p}\boldsymbol{p})\cdot\boldsymbol{f}$$

$$\text{with } \int_{-L/2}^{L/2} ds\, \boldsymbol{f}(s) = \mathbf{0} \text{ and}$$

$$\int_{-L/2}^{L/2} ds\, \left(\boldsymbol{X}(s,t) - \boldsymbol{X}_c(t)\right)\times\boldsymbol{f}(s) = \boldsymbol{p}\times\int_{-L/2}^{L/2} ds\, s\boldsymbol{f}(s) = \mathbf{0}$$

we have

$$\boldsymbol{f} = \eta\left(\mathbf{I} - \frac{1}{2}\boldsymbol{p}\boldsymbol{p}\right)\cdot\left[\dot{\boldsymbol{X}}_c - \boldsymbol{u}(\boldsymbol{X}_c) + s\left(\dot{\boldsymbol{p}} - \boldsymbol{p}\cdot\nabla\boldsymbol{u}(\boldsymbol{X}_c)\right)\right]$$

and from the force-free condition and oddness of $\boldsymbol{f}$ gives

$$\dot{\boldsymbol{X}}_c(t) = \boldsymbol{u}(\boldsymbol{X}_c) \text{ and hence } \boldsymbol{f} = \eta s\left(\mathbf{I} - \frac{1}{2}\boldsymbol{p}\boldsymbol{p}\right)\cdot\left(\dot{\boldsymbol{p}} - \boldsymbol{p}\cdot\nabla\boldsymbol{u}(\boldsymbol{X}_c)\right)$$

Zero torque gives:

$$\begin{aligned}
\boldsymbol{p}\times\left(\mathbf{I} - \frac{1}{2}\boldsymbol{p}\boldsymbol{p}\right)\cdot\left(\dot{\boldsymbol{p}} - \boldsymbol{p}\cdot\nabla\boldsymbol{u}(\boldsymbol{X}_c)\right) = \mathbf{0} \Longleftrightarrow \\
\boldsymbol{p}\times\left(\dot{\boldsymbol{p}} - \boldsymbol{p}\cdot\nabla\boldsymbol{u}(\boldsymbol{X}_c)\right) = \mathbf{0}
\end{aligned}$$

Now we use the identity $\boldsymbol{p}\times(\boldsymbol{p}\times\boldsymbol{g}) = -(\mathbf{I}-\boldsymbol{pp})\cdot\boldsymbol{g}$ and that $\boldsymbol{p}\cdot\dot{\boldsymbol{p}} = 0$ to get

$$\dot{\boldsymbol{p}} = (\mathbf{I}-\boldsymbol{pp})\cdot\left(\nabla\boldsymbol{u}\left(\boldsymbol{X}_c\right)\right)^\top\cdot\boldsymbol{p}$$

i.e. **Jeffery's equation for rods**. Finally, we calculate the force itself:

$$\boldsymbol{f} = -s\eta\left(\mathbf{I}-\frac{1}{2}\boldsymbol{pp}\right)\cdot\left(\boldsymbol{pp}:\nabla\boldsymbol{u}\right)\boldsymbol{p} = -s\frac{\eta}{2}\left(\boldsymbol{pp}:\nabla\boldsymbol{u}\right)\boldsymbol{p}$$

Note that the force is entirely in the $\boldsymbol{p}$ direction and is linear in s. This force is the so-called constraint force that maintains the rigidity of the rod against the flow. If we write $\boldsymbol{f} = T_s\boldsymbol{p}$ then the tension $T = -\frac{\eta}{4}\left(\boldsymbol{pp}:\nabla\boldsymbol{u}\right)\left(s^2-1/4\right)$, which will be negative for a compressive flow along the rod. This constraint force generates a stress in the surrounding fluid. The single particle "extra-stress" density is given by **Batchelor's formula**

$$\boldsymbol{\Sigma}_p = -\int_{-l/2}^{+l/2} ds\,\boldsymbol{f}(s)\,\boldsymbol{X}(s) = -\int_{-l/2}^{+l/2} ds\,s\boldsymbol{f}(s)\,\boldsymbol{p}(t) = \frac{l^3}{24}\eta\left(\boldsymbol{pp}:\nabla\boldsymbol{u}\right)\boldsymbol{pp}$$

This term is generally relaxational, contributing an anisotropic viscosity term.